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A MODEL CATEGORY FOR LOCAL PO-SPACES

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(*communicated by J.F. Jardine*)

Abstract

Locally partial-ordered spaces (local po-spaces) have been used to model concurrent systems. We provide equivalences for these spaces by constructing a model category containing the category of local po-spaces. We show that the category of simplicial presheaves on local po-spaces can be given Jardine's model structure, in which we identify the weak equivalences between local po-spaces. In the process, we give an equivalence between the category of sheaves on a local po-space and the category of étale bundles over a local po-space. Finally, we describe a localization that should provide a good framework for studying concurrent systems.

1. Introduction

The motivation for this paper stems from the study of concurrent processes accessing shared resources. Such systems were originally described by discrete models based on graphs, possibly equipped with some additional information [**Mil80**]. The precision of these models suffers, however, from an inaccuracy in distinguishing between concurrent and non-deterministic executions. It turned out that a satisfactory way to organize this information can be based on cubical sets, giving rise to the notion of *Higher-Dimensional Automata* or HDA's [**Gou96, Gou02**]. HDA's live in slice categories of **cSet**, the category of cubical sets and their morphisms.

A different view, which has its origins in Dijkstra's notion of *progress graphs* [**Dij68**], takes the flow of time into account. The difficulty here is to adequately model the fact that time is irreversible as far as computation is concerned. On the other hand, one would like to identify execution paths corresponding to (at least) the same sequence of acquisitions of shared resources. However, in order not to lose precision, this notion of homotopy is also subject to the constraint above of the irreversibility of time. There are two distinct approaches, both based on topological spaces.

One approach, advocated by Gaucher, is to topologize the sets of paths between the states of an automaton, which technically amounts to an enrichment with no

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units [**Gau03**]. The intuition behind the setup is to distinguish between *spatial* and *temporal* deformations of computational paths. The related framework of *Flows* has clear technical advantages from a (model-)categorical point of view.

The other approach, advocated by Fajstrup, Goubault, Raussen and others, is to topologize partially ordered states of automata. Such objects are called partially-ordered spaces or *po-spaces* (also *pospaces*)[1]. The advantage of using po-spaces is that there is a very simple and intuitive way to express directed homotopy or *dihomotopy* [**Gou03, FGR99**].

However, the price paid is that po-spaces cannot model executions of (concurrent) programs with loops. The solution is to order the underlying topological space only *locally*. Such objects are called *local po-spaces* and the notion of dihomotopy becomes more intricate in this context. Nevertheless, practical reasons like tractability call for a good notion of equivalence in the category of local po-spaces. Put differently, it would be useful to be able to replace a given local po-space model with a simpler local po-space which nevertheless preserves the relevant computer-scientific properties.

In this paper, we study these questions in the framework of Quillen's (closed) model categories [**Qui67, Hov99, Hir03**]. Briefly, a model category is a category with all small limits and colimits and three distinguished classes of morphisms called *weak equivalences*, *cofibrations* and *fibrations*. Weak equivalences that are also cofibrations or fibrations are called *trivial cofibrations* and *trivial fibrations*, respectively. These morphisms satisfy four axioms that allow one to apply the machinery of homotopy theory to the category. This machinery allows a rigorous study of equivalences. We remark that there are other frameworks for studying equivalence. However, model categories have the most developed theory, and have succeeded in illuminating many diverse subjects.

Our aim is to construct a model category of locally partial-ordered spaces as a foundation for the study of concurrent systems. This is technically difficult because locally partial-ordered spaces are not known to be closed under taking colimits. We will define a category **LPS** of local po-spaces, which embeds into the category **sPre**(**LPS**) of simplicial presheaves on local po-spaces. The objects of **sPre**(**LPS**) are contravariant functors from **LPS** to the category of simplicial sets and the morphisms are the natural transformations. This embedding is given by a Yoneda embedding (see Definition 2.17),

$$\bar{y}\colon \mathbf{LPS} \to \mathbf{sPre}(\mathbf{LPS}).$$

We now briefly describe some technical conditions on model categories which strengthen our theorems. For more details, see Definitions 8.2 and 8.4 and [**Hov99, Hir03**]. A model category is *proper* if the weak equivalences are closed under both pushouts with cofibrations and pullbacks with fibrations. It is *left proper* if the first condition holds. A model category is *cofibrantly generated* if the model category structure is induced by a set of generating cofibrations and a set of generating trivial cofibrations, both of which permit the small object argument. A *cellular* model category is a cofibrantly generated model category in which the cell complexes are

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[1] Grandis uses a related approach [**Gra03**] in which the underlying topological space comes with a class of directed paths. However, these spaces are not partially-ordered, even locally.

well behaved. A *simplicial* model category $\mathcal{M}$ is a model category enriched over simplicial sets, which for any $X \in \mathcal{M}$ and any simplicial set K has objects $X \otimes K$ and X^K which satisfy various compatibility conditions.

Theorem 1.1. *The category* $\mathbf{sPre}(\mathbf{LPS})$ *has a proper, cellular, simplicial model structure in which*

- *the cofibrations are the monomorphisms,*
- *the weak equivalences are the* stalkwise equivalences, *and*
- *the fibrations are the morphisms which have the right lifting property with respect to all trivial cofibrations.*

Furthermore among morphisms coming from $\mathbf{LPS}$ (*using the Yoneda embedding* $\mathbf{LPS} \hookrightarrow \mathbf{sPre}(\mathbf{LPS})$), *the weak equivalences are precisely the isomorphisms.*

The model structure on $\mathbf{sPre}(\mathbf{LPS})$ is Jardine's model structure [**Jar87, Jar96**] on the category of simplicial presheaves on a small *Grothendieck site*. We show that $\mathbf{Shv}(\mathbf{LPS})$ is a *Grothendieck topos* which has *enough points*. Under this condition, Jardine showed that the weak equivalences are the *stalkwise equivalences*.

This model category can be thought of as a localization of the universal injective model category of local po-spaces [**Joy84, Dug01, DHI04**]. While, in general, the weak equivalences are interesting and non-trivial [**Jar87**], this is not true for those coming from **LPS**. To obtain a more interesting category from the point of view of concurrency, we would like to localize with respect to directed homotopy equivalences. In [**Bub04**], it is argued that the relevant equivalences are the directed homotopy equivalences relative to some *context*. The context is a local po-space A and the directed homotopy equivalences rel A are a set of morphisms in $\mathbf{A} \downarrow \mathbf{LPS}$.

We combine this approach with Theorem 1.1 as follows. First, we remark that A embeds in $\mathbf{sPre}(\mathbf{LPS})$ as $\bar{y}(A)$. Next, the model structure on $\mathbf{sPre}(\mathbf{LPS})$ induces a model structure on the coslice category $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$. Finally, one can take the left Bousfield localization of this model category with respect to the directed homotopy equivalences rel A.

Theorem 1.2. *Let* $\mathcal{I} = \{\bar{y}(f) \mid f$ *be a directed homotopy equivalence rel* $A\}$. *Then the category* $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$ *has a left proper, cellular model structure in which*

- *the cofibrations are the monomorphisms,*
- *the weak equivalences are the* $\mathcal{I}$*-local equivalences, and*
- *the fibrations are those morphisms which have the right lifting property with respect to monomorphisms which are* $\mathcal{I}$*-local equivalences.*

Recall that, given a topological space Z, *étale bundles over* Z are maps $W \to Z$ which are local homeomorphisms. Let $\mathcal{O}(Z)$ be Z's locale of open subsets and recall that sheaves over Z are functors $\mathcal{O}(Z)^{op} \to \mathbf{Set}$ that enjoy a good gluing property. There is a well-known correspondence between étale bundles and sheaves. We establish a directed version of this correspondence, which may be of independent interest.

Theorem 1.3. *Let $Z \in$ **LPS**. Let* **Etale**(Z) *be the category of di-étale bundles over Z, i.e. the category of bundles which are local dihomeomorphisms. Let $\mathcal{O}(Z)$ be the category of open subobjects of Z. There is an equivalence of categories:*

$$\Gamma\colon \mathbf{Etale}(Z) \leftrightarrows \mathbf{Shv}(\mathcal{O}(Z))\colon \Lambda.$$

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Contents

2. Background

This section contains some known definitions and facts we build on. We start by stating the definition of a model category in Section 2.1. Next, we review the basics on presheaves in Section 2.2 and on sheaves in Section 2.3. We then recall the notions of topoi and geometric morphisms in Section 2.4 and of stalks in Section 2.5. Our main reference for this material is [**MLM92**]. Section 2.6 is devoted to some important model structures on $\mathbf{sSet}^{\mathbb{C}^{op}}$, the category of simplicial presheaves over a category $\mathbb{C}$. The material is drawn from [**Jar87, Jar96, DHI04**].

2.1. Model categories

Recall that a morphism $i\colon A \to B$ has the left lifting property with respect to a morphism $p\colon X \to Y$ if in every commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] & Y
\end{tikzcd}$$

there is a morphism $h\colon B \to X$ making the diagram commute. Also, f is a retract of g if there is a commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr] \arrow[ddd, "f"'] & & A \arrow[ddd, "f"] \\
& X \arrow[ur] \arrow[ddd, "g"'] & \\
& & \\
B \arrow[rr, equal] \arrow[dr] & & B \\
& Y \arrow[ur] &
\end{tikzcd}$$

Definition 2.1. A *model category* is a category with all small limits and colimits that has three distinguished classes of morphisms: $\mathcal{W}$, called the *weak equivalences*; $\mathcal{C}$, called the *cofibrations*; and $\mathcal{F}$, called the *fibrations*, which together satisfy the axioms below. We remark that morphisms in $\mathcal{W} \cap \mathcal{C}$, and $\mathcal{W} \cap \mathcal{F}$, are called trivial cofibrations and trivial fibrations, respectively.

1. Given composable morphisms f and g if any of the two morphisms f, g, and $g \circ f$ are in $\mathcal{W}$, then so is the third.
2. If f is a retract of g and g is in $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$, then so is f.
3. Cofibrations have the left-lifting property with respect to trivial fibrations, and trivial cofibrations have the left-lifting property with respect to fibrations.
4. Every morphism can be factored as a cofibration followed by a trivial fibration, and as a trivial cofibration followed by a fibration. These factorizations are functorial.

2.2. Presheaves

Recall that a presheaf P on $\mathbb{C}$ is just a functor $P \in \mathbf{Set}^{\mathbb{C}^{op}}$. In particular, "homing"

$$\begin{array}{rccc} \mathbb{C}(\_, C)\colon & \mathbb{C}^{op} & \to & \mathbf{Set} \\ & X & \mapsto & \mathbb{C}(X, C) \end{array}$$

gives rise to a presheaf and further to the Yoneda embedding

$$\begin{array}{rccc} y\colon & \mathbb{C} & \rightarrowtail & \mathbf{Set}^{\mathbb{C}^{op}} \\ & C & \mapsto & \mathbb{C}(\_, C). \end{array}$$

This embedding is *dense*, i.e.

$$P \cong \operatorname{colim}(y \circ \pi),$$

canonically for any presheaf P, where $\pi\colon (y \downarrow P) \to \mathbb{C}$ is the projection from the comma-category $y \downarrow P$. Recall that a presheaf in the image of the Yoneda-embedding (up to equivalence) is called *representable*.

2.3. Sheaves

Definition 2.2. A *sieve* on $M \in \mathbb{C}$ is a subfunctor $S \subseteq \mathbb{C}(\_, M)$. A *Grothendieck topology* J on $\mathbb{C}$ assigns to each $M \in \mathbb{C}$ a collection $J(M)$ of sieves on M such that

(i) (maximal sieve) $\mathbb{C}(\_, M) \in J(M)$ for all $M \in \mathbb{C}$;

(ii) (stability under pullback) if $g\colon M \to N$ and $S \in J(N)$, then $(g \circ \_)^*(S) \in J(M)$ as given by

$$\begin{tikzcd} (g \circ \_)^*(S) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail] \\ \mathbb{C}(\_, M) \arrow[r, "(g\circ\_)"'] & \mathbb{C}(\_, N) \end{tikzcd}$$

(iii) (transitivity) if $S \in J(M)$ and R is a sieve on M such that $(f \circ \_)^*(R) \in J(U)$ for all $f\colon U \to M$ in the image of S, then $R \in J(M)$;

We say that a sieve S on M is a *covering sieve* or a *cover of* M whenever $S \in J(M)$.

Remark 2.3. Unwinding Definition 2.2 pinpoints a sieve as a right ideal, i.e. a set of arrows S with codomain M such that $f \in S \implies f \circ h \in S$ whenever the codomain of h, $\mathrm{cod}(h) = \mathrm{dom}(f)$, the domain of f. From this point of view, pulling back a sieve S on M by an arrow $N \xrightarrow{f} M$ amounts to building the set

$$f^*(S) \stackrel{def}{=} \{h |\ \mathrm{cod}(h) = N,\ f \circ h \in S\}.$$

It is then immediate how to rephrase a Grothendieck topology in terms of right ideals.

Definition 2.4. Let J be a Grothendieck topology on $\mathbb{C}$. A presheaf $P \in \mathbf{Set}^{\mathbb{C}^{op}}$ is a *sheaf* with respect to J provided any natural transformation $\theta\colon S \Rightarrow P$ uniquely extends through $y(M)$ as in

$$\begin{tikzcd} S \arrow[r, "\theta"] \arrow[d, tail] & P \\ y(M) \arrow[ur, dotted] & \end{tikzcd}$$

for all $S \in J(M)$ and all $M \in \mathbb{C}$. J is *subcanonical* if the representable presheaves are sheaves.

Remark 2.5. Let $\theta\colon S \to P$ be a natural transformation from a sieve S to a presheaf P. If one sees S as a right ideal $S = \{u_j \colon M_j \to M\}$, then θ amounts to a function that assigns to every $u_j\colon M_j \to M \in S$ an element $a_j \in P(M_j)$ such that

$$P(v)(a_j) = a_k$$

for all $v\colon M_k \to M_j$ and for all $u_k = u_j \circ v \in S$. Such a function is called a *matching family* for S of elements of P. A matching family $a_j \in P(M_j)$ admits an *amalgamation* $a \in P(M)$ if

$$P(u_j)(a) = a_j$$

for all $u_j\colon M_j \to M \in S$. From this point of view, the Yoneda Lemma characterizes a *sheaf* as a presheaf such that every matching family has a unique amalgamation for all $S \in J(M)$ and all $M \in \mathbb{C}$.

A Grothendieck topology is a huge object. In practice, a generating device is used.

Definition 2.6. A *basis* K for a Grothendieck topology assigns to each object M a collection $K(M)$ of families of morphisms with codomain M such that

(i) all isomorphisms $f\colon U \to M$ are contained in $K(M)$,

(ii) given a morphism $g\colon N \to M \in \mathbb{C}$ and $\{f_i\colon U_i \to M\} \in K(M)$, then the family of pullbacks $\{\pi_2\colon U_i \times_M N \to N\} \in K(N)$, and

(iii) given $\{f_i\colon U_i \to M\} \in K(M)$ and for each i, $\{h_{ij}\colon A_{ij} \to U_i\} \in K(U_j)$, then the family of composites $\{f_i \circ h_{ij}\colon A_{ij} \to M\} \in K(M)$.

Remark 2.7. Given a basis K for a Grothendieck topology, one generates the corresponding Grothendieck topology J by defining

$$V \in J(M) \iff \text{there is } U \in K(M) \text{ such that } U \subset V.$$

As expected, the sheaf condition can be rephrased in terms of a basis.

As an example, consider the case $\mathbb{C} = \mathcal{O}(X)$ with X a topological space and $\mathcal{O}(X)$ its locale of opens. The basis of the *open-cover* (Grothendieck) topology is, as expected, given by open coverings of the opens.

Theorem 2.8. *Let* $\mathbf{Shv}(\mathbb{C}, J)$ *be the full subcategory of* $\mathbf{Set}^{\mathbb{C}^{op}}$ *whose objects are sheaves for* J*. The inclusion functor* $i\colon \mathbf{Shv}(\mathbb{C}, J) \to \mathbf{Set}^{\mathbb{C}^{op}}$ *has a left adjoint a called the* associated sheaf functor *or* sheafification*. This left adjoint preserves finite limits.*

Theorem 2.8 is listed as Theorem III.5.1 in [**MLM92**]. There are several equivalent ways to construct the associated sheaf functor, the most classical one being the "plus-construction" applied twice.

Remark 2.9. A cover on M amounts to a cocone in $\mathbb{C}$ with vertex M. The associated sheaf functor maps these cocones onto colimiting ones. Moreover, it is universal with respect to this property.

2.4. Topoi

Definition 2.10. A category $\mathcal{E}$ *has exponentials* provided that for all $X \in \mathcal{E}$, the functor $\_ \times X : \mathcal{E} \to \mathcal{E}$ has a right adjoint denoted $(\_)^X$, so that

$$\mathcal{E}(Y \times X, Z) \cong \mathcal{E}(Y, Z^X).$$

Suppose now $\mathcal{E}$ has a terminal object 1, and has finite limits. A *subobject classifier* is a monomorphism true: $1 \rightarrowtail \Omega$ such that for every monomorphism $s\colon S \rightarrowtail X$, there is a unique morphism ϕ_S such that pullback of true along ϕ_S yields s:

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\text{true}"] \\ X \arrow[r, "\phi_S"'] & \Omega \end{tikzcd}$$

The category $\mathcal{E}$ is a *topos* if it has exponentials and a subobject classifier.

A subobject classifier is obviously unique (up to isomorphism). Furthermore, a topos has all finite colimits, though this is not easy to prove. It would take pages to enumerate all the remarkable features of a topos, see [**Joh77**] for an introduction to the lore of the material. Let us just say that topoi as introduced by Grothendieck and his collaborators had a very strong algebro-geometrical flavor [**AGV72**], yet the rich structure is relevant not only for algebraic geometers, but for logicians as well [**Law63, Law64, Law73**].

Definition 2.11. A site $(\mathbb{C}, J)$ is a small category $\mathbb{C}$ equipped with a Grothendieck topology J. A *Grothendieck topos* is a category equivalent to the category $\mathbf{Shv}(\mathbb{C}, J)$ of sheaves on $(\mathbb{C}, J)$.

The following are well known.

Proposition 2.12. 1. *A Grothendieck topos is a topos;*

2. **Set** *is a topos;*
3. $\mathbf{Set}^{\mathbb{C}^{op}}$ *is a topos for any* $\mathbb{C}$.

Definition 2.13. Let $\mathcal{E}$ and $\mathcal{F}$ be topoi. A *geometric morphism* $g\colon \mathcal{F} \to \mathcal{E}$ is a pair of adjoint functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "g^*"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, shift left, "g_*"] \end{tikzcd}$$

such that the left adjoint g^* is left-exact (that is, it preserves finite limits). The right adjoint is called *direct image* and the left one *inverse image*.

As an example, $i\colon \mathbf{Shv}(\mathbb{C}, J) \hookrightarrow \mathbf{Set}^{\mathbb{C}^{op}}$ is the direct image part of a geometric morphism. Notice that the convention for a geometric morphism is to have the direction of its direct image part.

Definition 2.14. A (geometric) *point* in a topos $\mathcal{E}$ is a geometric morphism

$$p\colon \mathbf{Set} \to \mathcal{E}$$

(we write $p \in \mathcal{E}$ by abuse of notation). A topos $\mathcal{E}$ *has enough points* if given $f \neq g\colon P \to Q \in \mathcal{E}$, there is a point $p \in \mathcal{E}$ such that $p^* f \neq p^* g \in \mathbf{Set}$.

2.5. Stalks and Germs

Definition 2.15. Let $(\mathbb{C}, J)$ be a site, $a\colon \mathbf{Set}^{\mathbb{C}^{op}} \to \mathbf{Shv}(\mathbb{C}, J)$ the associated sheaf functor and $x \in \mathbf{Shv}(\mathbb{C}, J)$ a point. The *stalk functor* at x is given by

$$stalk_x \overset{def}{=} x^* \circ a\colon \mathbf{Set}^{\mathbb{C}^{op}} \to \mathbf{Set}.$$

Given a presheaf F, we say that $stalk_x(F)$ is the stalk of F at x. As an example, consider again the case $\mathbb{C} = \mathcal{O}(X)$ with X a (this time) Hausdorff topological space and $\mathcal{O}(X)$ its locale of opens equipped with the open-cover topology. Let $\mathbf{Shv}(X)$ be the corresponding topos of sheaves. It can be shown that any geometric point $x\colon \mathbf{Set} \to \mathbf{Shv}(X)$ corresponds to a "physical" point $x' \in X$. The stalk of $F \in \mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$ at x is then given by

$$stalk_x(F) := \operatorname*{colim}_{U \in \mathcal{O}(X), x' \in U} F(U).$$

Write $\mathrm{germ}_{x,U}\colon F(U) \to stalk_x(F)$ for the canonical map at U (germ_x when U is clear from the context). We call the equivalence class $\mathrm{germ}_{x,U}(s)$ of s in $stalk_x(F)$ the *germ of s at x*. Obviously,

$$stalk_x(F) = \{\mathrm{germ}_{x,U}(s) \mid U \in \mathcal{O}(X),\ x' \in U,\ s \in F(U)\}.$$

2.6. Simplicial Presheaves

For the rest of this section, let $\mathbb{C}$ be a small category with a Grothendieck topology J such that $\mathbf{Shv}(\mathbb{C}, J)$ has *enough points*.

Let Δ be the simplicial category which has objects $[n] = \{0, 1, \ldots, n\}$ for $n \geqslant 0$, and whose morphisms are the maps such that $x \leqslant y$ implies that $f(x) \leqslant f(y)$. Then $\mathbf{sSet}$ is the category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$. This category has a well-known model structure (e.g. see [**Hov99**]) where $\mathcal{W}_{\mathbf{sSet}}$ are the morphisms whose geometric realization is a weak homotopy equivalence and $\mathcal{C}_{\mathbf{sSet}}$ are the monomorphisms.

Objects of $\mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}}$ are called *simplicial presheaves* on $\mathbb{C}$ since

$$\mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}} = \left(\mathbf{Set}^{\Delta^{op}}\right)^{\mathbb{C}^{op}} \cong \mathbf{Set}^{\Delta^{op} \times \mathbb{C}^{op}} \cong \left(\mathbf{Set}^{\mathbb{C}^{op}}\right)^{\Delta^{op}}.$$

There is an embedding

$$\begin{array}{rccc} \kappa\colon & \mathbf{Set}^{\mathbb{C}^{op}} & \to & \mathbf{sSet}^{\mathbb{C}^{op}} \\ & F & \mapsto & \kappa_F, \end{array}$$

where κ_F is constant *levelwise* i.e. $(\kappa_F)(C)_n \overset{def}{=} F(C)$ for all $n \in \mathbb{N}$, and all the face and degeneracy maps are the identity. There is a further embedding

$$\begin{array}{rccc} \gamma\colon & \mathbf{sSet} & \to & \mathbf{sSet}^{\mathbb{C}^{op}} \\ & K & \mapsto & \gamma_K, \end{array}$$

where γ_K is constant *objectwise* i.e. $\gamma_K(C) \overset{def}{=} K$ for all $C \in \mathbb{C}$.

Recall that for $C \in \mathbb{C}$ and $F \in \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$, the Yoneda Lemma gives the isomorphism $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}(y(C), F) \cong F(C)$, where y is the Yoneda embedding (see Section 2.2). In the simplicial case, we have the following variation, which can be proved using the same idea used in the proof of the Yoneda Lemma.

Proposition 2.16. *(Bi–Yoneda) Let $C \in \mathbb{C}$ and $F \in \mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}}$. There is an isomorphism*

$$\mathbf{sSet}^{\mathbb{C}^{op}}\left(\kappa_{y(C)} \times \gamma_{\Delta[n]}, F\right) \cong F(C)_n\,,$$

natural in all variables.

Definition 2.17. Using the Yoneda embedding $y\colon \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ for presheaves one can define an embedding

$$\bar{y}\colon \mathbb{C} \xrightarrow{y} \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} \xrightarrow{\kappa} \mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}}$$

for simplicial presheaves. The functor $\bar{y}$ is also called a Yoneda embedding.

There are two Quillen equivalent model structures on $\mathbf{sSet}^{\mathbb{C}^{op}}$ which are in a certain sense *objectwise:*

- the *projective* model structure $\mathbf{sSet}^{\mathbb{C}^{op}}_{prj}$ where $\mathcal{W}_{prj}$ and $\mathcal{F}_{prj}$ are objectwise (that is, $f\colon P \to Q \in \mathcal{W}_{prj}(\mathcal{F}_{prj})$ if and only if for all $C \in \mathbb{C}$, $f(C)\colon P(C) \to Q(C) \in \mathcal{W}_{\mathbf{sSet}}(\mathcal{F}_{\mathbf{sSet}})$), and
- the *injective* model structure $\mathbf{sSet}^{\mathbb{C}^{op}}_{inj}$ where $\mathcal{W}_{inj}$ and $\mathcal{C}_{inj}$ are objectwise.

These were studied by Bousfield and Kan [**BK72**] and Joyal [**Joy84**], respectively.

Proposition 2.18. *Both $\mathbf{sSet}^{\mathbb{C}^{op}}_{prj}$ and $\mathbf{sSet}^{\mathbb{C}^{op}}_{inj}$ are proper, simplicial, cellular model categories. All objects are cofibrant in the latter. The identity functor is a left Quillen equivalence from the projective model structure to the injective model structure.*

The injective one is more handy when it comes down to calculating homotopical localizations, yet the fibrant objects are easier to grasp in the projective one[2].

Using the stalk functor for presheaves, one can define a simplicial stalk functor for simplicial presheaves.

Definition 2.19. The *simplicial stalk functor* at a point p in $\mathbf{Shv}(\mathbb{C})$ is given by

$$\begin{array}{llll} (\_)_p\colon & \mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}} & \to & \mathbf{sSet} \\ & P & \mapsto & \{stalk_p(P_n)\}_{n\geqslant 0}. \end{array}$$

A morphism $f\colon P \to Q \in \mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}}$ is a *stalkwise equivalence* if $f_p\colon P_p \to Q_p \in \mathbf{sSet}$ is a weak equivalence for all points p in $\mathbf{Shv}(\mathbb{C})$.

Jardine [**Jar87**] proved the existence of a local version of Joyal's injective model structure. Since we will only be interested in the special case where $\mathbf{Shv}(\mathbb{C})$ has enough points, we will not recall the definition of local weak equivalences.

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[2] They are objectwise Kan.

Theorem 2.20 ([**Jar87, Jar96**]). *Let $\mathbb{C}$ be a small category with a Grothendieck topology. Then $\mathbf{sSet}^{\mathbb{C}^{\mathrm{op}}}$ the category of simplicial presheaves on $\mathbb{C}$ has a proper, simplicial, cellular model structure in which*

- *the cofibrations are the monomorphisms, i.e. the levelwise monomorphisms of presheaves,*
- *the weak equivalences are the* local weak equivalences, *and*
- *the fibrations are the morphisms which have the right lifting property with respect to all trivial cofibrations.*

Furthermore, if the Grothendieck topos $\mathbf{Shv}(\mathbb{C})$ *has enough points, then the local weak equivalences are the stalkwise equivalences.*

Jardine's model structure can be seen to be cellular since it can also be constructed as a left Bousfield localization of the injective model structure [**DHI04**].

3. Local po-spaces

The focus of this section is to provide the reader with the main definitions and constructions. We define a small category of local po-spaces **LPS** and state some of the properties, most of which are proved in the later sections. We show that Theorem 1.1 follows from these properties and a theorem of Jardine.

To simplify the analysis, we will only work with topological spaces which are subspaces of $\mathbb{R}^n$ for some n, since this provides more than enough generality for studying concurrent systems. The main technical advantage of this setting is that we obtain small categories.

Definition 3.1. *(i)* Let **Spaces** be the category whose objects are subspaces of $\mathbb{R}^n$ for some n, and whose morphisms are continuous maps.

(ii) Let **PoSpaces** be the category whose objects are *po-spaces*: that is $U \in$ **Spaces** together with a *partial order* (a reflexive, transitive, anti-symmetric relation) $\leqslant$ such that $\leqslant$ is a closed subset of $U \times U$ in the product topology.

(iii) For any $M \in$ **Spaces** define an *order-atlas* on M to be an open cover[3] $U = \{U_i\}$ of M indexed by a set I, where $U_i \in$ **PoSpaces**. These partial orders are compatible: $\leqslant_i$ agrees with $\leqslant_j$ on $U_i \cap U_j$ for all $i, j \in I$. We will usually omit the index set from the notation.

(iv) Let U and U' be two order atlases on M. Say that U' is a *refinement* of U if for all $U_i \in U$, and for all $x \in U_i$, there exists a $U'_j \in U'$ such that $x \in U'_j \subseteq U_i$ and for all $a, b \in U'_j$, $a \leqslant_{j'} b$ if and only if $a \leqslant_i b$.

(v) Say that two order atlases are *equivalent* if they have a common refinement. This is an equivalence relation: reflexivity and symmetry follow from the definition. For transitivity, if U and U' have a refinement $V = \{V_i\}$ and U' and U'' have a refinement $W = \{W_j\}$, let $T = \{V_i \cap W_j\}$. One can check that T is an order atlas of M and that is a refinement of U' and U''.

[3]That is, for all i, U_i is open as a subspace of M and $M = \cup_i U_i$.

Any po-space $(U, \leqslant)$ is a local po-space with the equivalence class of order atlases generated by the order atlas $\{U\}$. As a further example, we remark that any discrete space has a unique equivalence class of order-atlases.

Definition 3.2. Let **LPS** be the category of local po-spaces described as follows. The objects, called *local po-spaces*, are all pairs $(M, \mathcal{U})$ where M is an object in **Spaces** and $\mathcal{U}$ is an equivalence class of *order-atlases* of M. The morphisms, called *dimaps* are described as follows. $f \in \mathbf{LPS}((M, \mathcal{U}), (N, \mathcal{V}))$ if and only if $f \in \mathbf{Spaces}(M, N)$ and for all $V = \{V_j\}_{j \in J} \in \mathcal{V}$, there is a $U = \{U_i\}_{i \in I} \in \mathcal{U}$ such that for all $i \in I$, $j \in J$, for all $x, y \in U_i \cap f^{-1}(V_j)$,

$$x \leqslant_{U_i} y \implies f(x) \leqslant_{V_j} f(y).$$

Remark 3.3. This condition is not necessarily true for arbitrary $U \in \mathcal{U}$. For example, take $M = \{-1, 1\}$ with $\mathcal{U}$ the unique equivalence class of order atlases generated by the order atlas $U = \{\{-1\}, \{1\}\}$. Let $f = \mathrm{Id}_M : (M, \mathcal{U}) \to (M, \mathcal{U})$. Now let M_+ be the po-space on M with the ordering $-1 \leqslant 1$ and let M_- be the po-space on M with the ordering $1 \leqslant -1$. Then $\{M_+\} \in \mathcal{U}$ and $\{M_-\} \in \mathcal{U}$ (both have U as a common refinement). However, even though $-1, 1 \in M_+ \cap f^{-1}(M_-)$,

$$-1 \leqslant_{M_+} 1, \text{ but } f(-1) \nleqslant_{M_-} f(1).$$

Remark 3.4. It is easy to check that a dimap of po-spaces is also a dimap of local po-spaces. Thus **PoSpaces** the category of po-spaces is a subcategory of **LPS**.

Remark 3.5. Subobjects in **LPS**.

If $(M, \mathcal{U}) \in \mathbf{LPS}$, then a subspace $L \subseteq M \in \mathbf{Spaces}$ inherits local po-space structure as follows. Let $U = \{U_i\} \in \mathcal{U}$ and let $W = \{W_i\}$ where $W_i = L \cap U_i$ and W_i has the partial order inherited from U_i. Then W is an open cover of L and the partial orders are compatible. That is, W is an order atlas. Let $\mathcal{W}$ be the equivalence class of W.

We claim that $\mathcal{W}$ does not depend on the choice of U. Let $\tilde{U} = \{\tilde{U}_i\} \in \mathcal{U}$, let $\tilde{W}_i = L \cap \tilde{U}_i$, and let $\tilde{W} = \{\tilde{W}_i\}$. U and $\tilde{U}$ have a common refinement $\hat{U} = \{\hat{U}_i\}$. Let $\hat{W}_i = L \cap \hat{U}_i$ and let $\hat{W} = \{\hat{W}_i\}$. Then one can check that $\hat{W}$ is a common refinement of W and $\tilde{W}$. So the equivalence class of $\tilde{W}$ is also $\mathcal{W}$.

Next, we claim that there is a dimap $\iota : (L, \mathcal{W}) \to (M, \mathcal{U})$ given by the inclusion $\iota : L \hookrightarrow M$. Let $U = \{U_k\} \in \mathcal{U}$, let $W_k = L \cap U_k$, and let $W = \{W_k\}$. Then $W \in \mathcal{W}$. Let $x, y \in W_j \cap \iota^{-1}(U_k) = W_j \cap L \cap U_k = W_j \cap W_k$. Note that $\iota(x) = x$ and $\iota(y) = y$. Then

$$x \leqslant_{W_j} y \iff x \leqslant_{W_k} y \iff x \leqslant_{U_k} y.$$

Therefore, when $L \subseteq M \in \mathbf{Spaces}$, then there is an induced inclusion $(L, \mathcal{W}) \subseteq (M, \mathcal{U}) \in \mathbf{LPS}$.

The remark above will be used implicitly and without reference in Section 6.

Definition 3.6. A collection of dimaps $\{\phi_j : (M_j, \mathcal{U}^j) \to (M, \mathcal{U})\}$ **LPS** is an *open dicover* if

(i) $\{\phi_j : M_j \to M\}$ is an open cover, and

(ii) for each j, $\mathcal{U}^j$ is the local po-space structure inherited from $(M,\mathcal{U})$.

Remark 3.7. The local po-space structures inherited by the subspaces of $(M,\mathcal{U})$ are compatible. So if $\{\phi_j : (M_j,\mathcal{U}^j) \to (M,\mathcal{U})\}$ is an open cover, then for each j, there is a $U^j = \{U_k^j\} \in \mathcal{U}^j$ such that $U' = \{U_k^j\}_{j,k}$ is an order atlas for M and $U' \in \mathcal{U}$.

The following is easy to check.

Lemma 3.8. **Spaces** *and* **LPS** *are small categories.*

Define U: **LPS** $\to$ **Spaces** to be the forgetful functor defined on objects and morphisms as follows $(M,U) \mapsto M$ and $\varphi \mapsto \varphi$.

Define F: **Spaces** $\to$ **LPS** as follows. If M is an object in **Spaces**, then let $F(M) = (M, \bar{M}_\phi)$, where $\bar{M}_\phi$ is the equivalence class of $M_\phi = \{M\}$ with $x \leqslant_M y \iff x = y$. If $f\colon M \to N \in$ **Spaces**, then $F(f) = f\colon (M,\bar{M}_\phi) \to (N,\bar{N}_\phi)$. This is a dimap since for any $V = \{V_j\} \in \bar{N}_\phi$ with $x, y \in f^{-1}V_j$, $x \leqslant_M y \implies x = y \implies f(x) = f(y) \implies f(x) \leqslant_{V_j} f(y)$.

Remark 3.9. Note that U is faithful and F includes **Spaces** as a full subcategory of **LPS**.

Proposition 3.10. F: **Spaces** $\rightleftarrows$ **LPS**: U *is an adjunction.*

Proof. Let M be an object in **Spaces** and $(N,\bar{V}) \in$ **LPS**. We claim that there is a natural bijection

$$\mathbf{LPS}(F(M),(N,\bar{V})) \cong \mathbf{Spaces}(M,U(N,\bar{V})).$$

We need to show that there is a natural bijection

$$\theta\colon \mathbf{Spaces}(M,N) \xrightarrow{\cong} \mathbf{LPS}((M,\bar{M}_\phi),(N,\bar{V})).$$

If $f \in \mathbf{LPS}((M,\bar{M}_\phi),(N,\bar{V}))$, then $f \in \mathbf{Spaces}(M,N)$ such that for any $V = \{V_j\} \in \bar{V}$, for all j, $f|_{f^{-1}(V_j)}$ satisfies $x \leqslant_M y \implies f(x) \leqslant_{V_j} f(y)$. Since $x \leqslant_M y$ if and only if $x = y$, this last condition is vacuous. Thus, the bijection is simply $\theta\colon f \mapsto f$.

To show naturality, let $\alpha\colon (N,\bar{V}) \to (N',\bar{V}') \in$ **LPS** and $\xi\colon M' \to M \in$ **Spaces**. Then

$$\theta(U(\alpha) \circ f \circ \xi) = \alpha \circ f \circ \xi = \alpha \circ \theta(f) \circ \xi.$$

□

4. The open-dicover topology

We define the open cover Grothendieck topology for **Spaces** and the open dicover Grothendieck topology for **LPS** in the following lemma. The proof of the lemma follows directly from the definition of a basis for a Grothendieck topology.

Lemma 4.1. *1.* **Spaces** *has a* Grothendieck topology *whose basis is given by the open covers. For $M \in$* **Spaces**, *let $K(M) = \{$open covers of $M\}$. Let J be the Grothendieck topology generated by K. Call J the* open cover topology.

2. *Analogously,* **LPS** *has a Grothendieck topology whose basis is given by the open dicovers in* **LPS**. *Let* $K((M,\mathcal{U})) = \{$*open dicovers of* $(M,\mathcal{U})\}$. *Call the Grothendieck topology generated by* K *the* open-dicover topology*.*

In Section 3, we defined a Grothendieck topology to be subcanonical if every representable presheaf is a sheaf. In this section, we will prove that the open-dicover topology is subcanonical.

The following proposition shows that if a Grothendieck topology is generated by a basis K, then to see if a presheaf is a sheaf, it suffices to check the basis. For the definition of matching families and amalgamations, see Remark 2.5.

Proposition 4.2 ([**MLM92**, Proposition III.4.1])**.** *Let* $\mathbb{C}$ *be a small category with a Grothendieck topology* J *generated by a basis* K*. Then a presheaf* $P \in \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ *is a sheaf for* J *if and only if for every* $M \in \mathbb{C}$ *and every cover* $\{\phi_j \colon M_j \to M\} \in K(M)$*, every matching family for* $\{\phi_j\}$ *of elements of* P *has a unique amalgamation.*

Example 4.3. Let $N \in \mathbf{Spaces}$ and $y(N) = \mathbf{Spaces}(-, N) \in \mathbf{Set}^{\mathbf{Spaces}^{\mathrm{op}}}$. Suppose $\phi_j \colon M_j \to M$ is an open cover, and let $\alpha_j \colon M_j \to N$ be a matching family. Then ϕ_j has a unique amalgamation $\phi \colon M \to N$. Therefore, $y(N)$ is a sheaf for the open cover topology, and hence the open cover topology is subcanonical.

Proposition 4.4. *In the open-dicover topology* J *for local po-spaces, every representable presheaf is a sheaf. That is,* J *is subcanonical.*

Proof. Consider the representable presheaf

$$y((N,\bar{V})) = \mathbf{LPS}(-,(N,\bar{V})) \in \mathbf{Set}^{\mathbf{LPS}^{\mathrm{op}}}.$$

By Proposition 4.2, $y((N,\bar{V}))$ is a sheaf if and only if for all open dicovers $\{\phi_j\} \in K((M,\bar{U}))$, any matching family

$$\{\alpha_j \colon (M_j,\bar{U}_j) \to (N,\bar{V})\}$$

has a unique amalgamation $\alpha \colon (M,\bar{U}) \to (N,\bar{V})$. That is, there is a map α such that the diagrams

$$\begin{tikzcd}
(M_j,\bar{U}_j) \arrow[r, "\phi_j"] \arrow[d, "\alpha_j"'] & (M,\bar{U}) \arrow[dl, "\alpha"] \\
(N,\bar{V}) &
\end{tikzcd}$$

commute in **LPS** for all j.

Let $\{\alpha_j\}$ be such a matching family for an open dicover $\{\phi_j\}$. Since $\{\phi_j\}$ is an open dicover, then by Remark 3.7, for each j there is a $U^j = \{U^j_k\} \in \bar{U}_j$ such that $U' = \{U^j_k\}_{j,k}$ is an order atlas and $U' \in \bar{U}$.

By definition, $\{\phi_j \colon M_j \to M\}$ is a cover in **Spaces** and $\{\alpha_j \colon M_j \to N\}$ is a matching family. Therefore, there is a unique amalgamation $\alpha \colon M \to N \in \mathbf{Spaces}$. That is, there is a map α such that

$$\begin{tikzcd} M_j \arrow[r, "\phi_j"] \arrow[d, "\alpha_j"'] & M \arrow[dl, "\alpha"] \\ N & \end{tikzcd}$$

commutes in **Spaces** for all j. It remains to show that α is a dimap. Let $V = \{V_l\} \in \bar{V}$. Since $\alpha_j\colon (M_j, \bar{U}_j) \to (N, \bar{V}) \in \mathbf{LPS}$, there is a $\tilde{U}^j = \{\tilde{U}^j_k\}_k \in \bar{U}^j$ such that for all k, l,

$$\text{for all } x, y \in \tilde{U}^j_k \cap \alpha_j^{-1}(V_l), \quad x \leqslant_{\tilde{U}^j_k} y \implies \alpha_j(x) \leqslant_{V_l} \alpha_j(y).$$

Now, for each j, let $\hat{U}^j = \{\hat{U}^j_k\}_k \in \bar{U}_j$ be a common refinement of $\tilde{U}^j$ and U^j. Then since $\hat{U}^j$ is a refinement of $\tilde{U}^j$,

$$\text{for all } x, y \in \hat{U}^j_k \cap \alpha_j^{-1}(V_l), \quad x \leqslant_{\hat{U}^j_k} y \implies \alpha_j(x) \leqslant_{V_l} \alpha_j(y), \tag{1}$$

and since $\hat{U}^j$ is a refinement of U^j, if we define $U = \{\hat{U}^j_k\}_{j,k}$, then $U \in \bar{U}$.

Since α is an amalgamation of $\{\alpha_j\}$ in **Spaces** if $x \in \hat{U}^j_k \subset M$, then $\alpha(x) = \alpha_j(x)$ and for all l, $\hat{U}^j_k \cap \alpha_j^{-1}(V_l) = \hat{U}^j_k \cap \alpha^{-1}(V_l)$. Therefore, using (1) for all k, l,

$$\text{for all } x, y \in \hat{U}^j_k \cap \alpha^{-1}(V_l), \quad x \leqslant_{\hat{U}^j_k} y \implies \alpha(x) \leqslant_{V_l} \alpha(y).$$

That is, α is a dimap. Therefore, $\alpha\colon (M, \bar{U}) \to (N, \bar{V})$ is a unique amalgamation of $\{\alpha_j\}$. $\square$

5. Equivalence of sheaves and di-étale bundles

In this section, $\mathbb{C}$ is either **Spaces** or **LPS** with the Grothendieck topology generated by open (di)covers.

Notation 5.1. We will use $A \underset{\text{open}}{\subseteq} B$ to denote that A is an open subset of B.

Notation 5.2. Let $Z \in \mathbb{C}$ and let $F \in \mathbf{Set}^{\mathbb{C}^{\text{op}}}$. Choose $x \in U \underset{\text{open}}{\subseteq} Z$ and $s \in F(U)$. Then for open subobjects of U, $L \overset{i}{\hookrightarrow} U$, we have $F(i)\colon F(U) \to F(L)$ and we will use the notation

$$s|_L\colon = F(i)(s).$$

Recall that $stalk_x(F) = \operatorname{colim}_{x \in L \underset{\text{open}}{\subseteq} U} F(L)$ and $\operatorname{germ}_x(s)$ is the equivalence class represented by s in $stalk_x(F)$.

Definition 5.3. Given $Z \in \mathbb{C}$, a *bundle* over Z is just a morphism $p\colon W \to Z \in \mathbb{C}$. An *(di)étale bundle* is a bundle which is a *local (di)homeomorphism*. That is, given $y \in W$, there is some open set $V \subset W$ such that $p(V)$ is open in Z and $p|_V$ is an isomorphism in $\mathbb{C}$.

A morphism of (étale) bundles $p\colon W \to Z$ and $p\colon W' \to Z$ is a morphism $\theta\colon W \to W' \in \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[rr, "\theta"] \arrow[dr, "p"'] & & W' \arrow[dl, "p'"] \\ & Z & \end{tikzcd}$$

Let $\mathbf{Etale}(Z)$ denote the category of (di)étale bundles over Z. In addition, let $\mathcal{O}(Z)$ denote the category of open subobjects of Z, where the objects are open subobjects of Z and the morphisms are the inclusions.

Theorem 5.4 (Theorem 1.3)**.** *Let $Z \in \mathbb{C}$. Then there is an equivalence of categories*

$$\Gamma\colon \mathbf{Etale}(Z) \leftrightarrows \mathbf{Shv}(\mathcal{O}(Z))\colon \Lambda.$$

Proof. It is well known that the statement of Theorem 1.3 is true when $\mathbb{C} = \mathbf{Spaces}$ (see e.g. [**MLM92**, Corollary II.6.3]). We will show that this equivalence between étale bundles on topological spaces and sheaves on topological spaces extends to local po-spaces.

First, we describe the functors Γ and Λ in the case where $\mathbb{C} = \mathbf{Spaces}$. The functor Γ assigns to each bundle $W \xrightarrow{p} Z$ the presheaf of cross-sections:

$$\begin{aligned} P\colon \mathcal{O}(Z)^{\mathrm{op}} &\to \mathbf{Set} \\ U &\mapsto \{s\colon U \to W \in \mathbb{C} \mid p \circ s = \mathrm{Id}_U\} \\ U \overset{\theta}{\hookrightarrow} V &\mapsto \theta^* \quad (\theta^*(t) = t \circ \theta). \end{aligned}$$

One can check that if p is étale, then P is in fact a sheaf [**MLM92**, p. 79]. Thus, Γ restricts to a functor $\Gamma\colon \mathbf{Etale}(Z) \to \mathbf{Shv}(\mathcal{O}(Z))$.

Given a presheaf $P\colon \mathcal{O}(Z)^{\mathrm{op}} \to \mathbf{Set}$, $\Lambda(P)$ is the bundle $W \xrightarrow{p} Z$ where

$$W = \{\mathrm{germ}_x\, s \mid x \in U \underset{\text{open}}{\subseteq} Z, s \in P(U)\} \text{ and } p\colon\ \mathrm{germ}_x\, s \mapsto x.$$

A basis for the topology on W is given by the sets $\dot{s}(U)$, where U is an open set in Z, $s \in P(U)$ and

$$\begin{aligned} \dot{s}\colon U &\to \Lambda(P) \\ x &\mapsto \mathrm{germ}_x\, s. \end{aligned}$$

Using this topology, $p\colon W \to Z$ is a continuous map. Again, one can check that if P is a sheaf, then $W \xrightarrow{p} Z$ is in fact an étale bundle [**MLM92**, p. 85]. So, Λ restricts to a functor $\Lambda\colon \mathbf{Shv}(\mathcal{O}(Z)) \to \mathbf{Etale}(Z)$.

Now, we will show that Γ and Λ can be similarly defined in the case where $\mathbb{C} = \mathbf{LPS}$. Let $p\colon (W, \bar{T}) \to (Z, \bar{U})$ be an étale bundle of local po-spaces. The definition of Γ is exactly the same: $\Gamma((W, \bar{T}) \xrightarrow{p} (Z, \bar{U}))$ is the sheaf of cross-sections.

Given a sheaf P on a local po-space $(Z, \bar{U})$, $\Lambda(P) = (W \xrightarrow{p} Z)$ is an étale bundle of topological spaces. To extend Λ to local po-spaces, it remains to define a local order on W and show that this makes p a dimap.

Lemma 5.5. *W has a canonical local po-space structure such that p is a dimap.*

Proof. Recall that the sets $\dot{s}(U)$ defined above form a basis for the topology of W. Choose an order atlas $\{(U_i, \leqslant_i)\} \in \bar{U}$ for Z. For each open sub-po-space $V \subset U_i$ and each $s \in P(V)$, $\dot{s}(V) \subset W$ is a po-space under the relation

$$\operatorname{germ}_x s \leqslant_{\dot{s}(V)} \operatorname{germ}_y s \text{ if and only } x \leqslant_i y.$$

This is well-defined since $\{U_i\}$ is an order-atlas, and it makes $\dot{s}(V)$ a po-space since $\dot{s}\colon U_i \to \dot{s}(U_i)$ is a homeomorphism.

We claim that

$$T\colon = \{\dot{s}(V) \mid V \underset{\text{open}}{\subset} U_i, s \in P(V)\}$$

is an order atlas on W. First, we need to show that it is an open cover. Each of the sets is open by construction. If $U \in \mathcal{O}(Z)$ and $s \in P(U)$, consider $\operatorname{germ}_x s$. Since $\{U_i\}$ is an open cover of Z, for some i, $x \in U_i$. Let $V = U \cap U_i$. Then $\operatorname{germ}_x s = \operatorname{germ}_x s|_V \in (\dot{s|_V})(V)$. Therefore, T is an open cover of W.

Finally, we need to show that the orders are compatible. For $k = 1, 2$, let $V_k \underset{\text{open}}{\subset} U_{i_k} \underset{\text{open}}{\subset} Z$, and $s_k \in P(V_k)$. Assume $g_1, g_2 \in \dot{s_1}(V_1) \cap \dot{s}(V_2)$. That is, $g_1 = \operatorname{germ}_{x_1} s_1 = \operatorname{germ}_{x_1} s_2$ and $g_2 = \operatorname{germ}_{x_2} s_1 = \operatorname{germ}_{x_2} s_2$. For $k = 1, 2$,

$$g_1 \leqslant_{\dot{s_k}(V_k)} g_2 \iff x_1 \leqslant_{i_k} x_2.$$

Since $\{U_i\}$ is an order-atlas, the order $\leqslant_{i_1}$ and $\leqslant_{i_2}$ are compatible. Therefore, the orders $\leqslant_{\dot{s_1}(V_1)}$ and $\leqslant_{\dot{s_1}(V_1)}$ are compatible, and T is an order-atlas on W.

Let $\bar{T}$ be the equivalence class of order atlases of T. We claim that $\bar{T}$ does not depend on the choice of $U \in \bar{U}$.

Let $U, U' \in \bar{U}$, then U and U' have a common refinement U''. Let T, T', T'' be the corresponding order-atlases for W constructed as above. We will show that T'' is a refinement of T.

Let $A \underset{\text{open}}{\subset} U_j \in U$, $s \in P(A)$ and $\operatorname{germ}_x s \in \dot{s}(A)$. Then there is some $U''_k \in U''$ such that $x \in U''_k$ and U''_k is a sub-po-space of U_j. Let $A'' = A \cap U''_k$. It follows that $(\dot{s|_{A''}})(A'') \subset \dot{s}(A)$, and $\operatorname{germ}_x s = \operatorname{germ}_x(s|_{A''}) \in (\dot{s|_{A''}})(A'') \in T''$. Since U''_k is a sub-po-space of U_j, it follows that $(\dot{s|_{A''}})(A'')$ is a sub-po-space of $\dot{s}(A)$. Thus, T'' is a refinement of T.

Similarly, T'' is a refinement of T' and is hence a common refinement of T and T'. Therefore, $\bar{T}$ does not depend on the choice of $U \in \bar{U}$.

Finally, we will show that the projection $p\colon W \to Z$ given by $\operatorname{germ}_x s \mapsto x$ is a dimap. Let $U \in \bar{U}$ be an order-atlas on Z. Let T be the order-atlas on W constructed above from U. Observe that $T \in \bar{T}$, since $\bar{T}$ does not depend on the choice of $U \in \bar{U}$. Let $U_j \in U$, let $A \underset{\text{open}}{\subset} U_i \in U$, and let $s \in P(A)$. Assume that

$$\operatorname{germ}_{x_1} s, \operatorname{germ}_{x_2} s \in \dot{s}(A) \cap p^{-1}(U_j).$$

Then $x_1, x_2 \in U_i \cap U_j$. By the construction of T and since U is an order atlas,

$$\operatorname{germ}_{x_1} s \leqslant_{\dot{s}(A)} \operatorname{germ}_{x_2} s \iff x_1 \leqslant_{U_i} x_2 \iff x_1 \leqslant_{U_j} x_2.$$

Therefore, Λ can be extended to local po-spaces. □

Thus, we have maps

$$\Gamma\colon \mathbf{Etale}(Z) \leftrightarrows \mathbf{Shv}(\mathcal{O}(Z))\colon \Lambda.$$

To show that they give an equivalence of categories, we will show that for a sheaf P and an étale space $W \xrightarrow{p} Z$, there are natural isomorphisms

$$\epsilon_W\colon \Lambda\Gamma W \to W \text{ and } \eta_P\colon P \to \Gamma\Lambda P.$$

Recall that elements of $\Lambda\Gamma W$ are of the form $\dot{s}(x) = \mathrm{germ}_x s$, where $s\colon U \to W$ satisfies $p \circ s = \mathrm{Id}_U$ and $x \in U$. Define ϵ_W to be the map $\dot{s}x \mapsto sx$. We will show that this is an isomorphism by constructing an inverse θ_W. Let $y \in W$ and let $x = py$. Since W is étale, there exists $y \in V \underset{\text{open}}{\subseteq} W$ such that $p|_V\colon V \xrightarrow{\cong} p(V)$. Let $q = (p|_V)^{-1}$. Then define $\theta_W(y) = \mathrm{germ}_x q = \dot{q}x$. Then we claim θ_W is an inverse for ϵ_W. Indeed

$$\epsilon_W\theta_W y = \epsilon_W \dot{q}x = qx = y.$$

Also for all $\dot{s}x \in \Lambda\Gamma W$, $\theta_W\epsilon_W\dot{s}x = \theta_W sx = \mathrm{germ}_x t$, where t is a restriction of s. So $\mathrm{germ}_x t = \mathrm{germ}_x s = \dot{s}x$.

Finally, we claim that ϵ_W and θ_W are dimaps. First, choose $T = \{T_k\} \in \bar{T}$ and $U = \{U_i\} \in \bar{U}$ such that p satisfies the dimap condition. From T, construct the canonical order atlas of the form $\{\dot{s}V\}$ for $\Lambda\Gamma W$ as in the proof of Lemma 5.5. Now, let $\dot{s}x_1, \dot{s}x_2 \in \dot{s}V \cap \epsilon_W^{-1}(T_k)$. Then by construction,

$$\dot{s}x_1 \leqslant_{\dot{s}V} \dot{s}x_2 \iff x_1 \leqslant_{U_i} x_2.$$

Since s satisfies the dimap condition, this implies that $sx_1 \leqslant_{T_k} sx_2$ which is the same as $\epsilon_W\dot{s}x_1 \leqslant_{T_k} \epsilon_W\dot{s}x_2$. Thus, ϵ_W is a dimap. Next, let $y_1, y_2 \in T_k \cap \theta_W^{-1}(\dot{s}V) = T_k \cap \epsilon_W(\dot{s}V) = T_k \cap sV$. Then there are $x_1, x_2 \in V$ such that $y_1 = sx_1$ and $y_2 = sx_2$. Since p satisfies the dimap condition

$$y_1 \leqslant_{T_k} y_2 \implies py_1 \leqslant_{U_i} py_2.$$

But this is the same as $x_1 \leqslant_{U_i} x_2$ which implies that $\dot{s}x_1 \leqslant_{\dot{s}V} \dot{s}x_2$. Therefore, θ_W is a dimap.

The proof that the morphism η_P is a bijection is the same as the proof in the case of topological spaces [**MLM92**, Theorem II.5.1]. □

6. Points

In this section, $\mathbb{C}$ is either **Spaces** or **LPS** with the Grothendieck topology generated by open (di)covers.

Let $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ and $\mathbf{Shv}(\mathbb{C})$ be the topoi of presheaves and sheaves on $\mathbb{C}$. Recall that the inclusion functor $i\colon \mathbf{Shv}(\mathbb{C}) \to \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ has a right adjoint a called the associated sheaf functor. Recall from Definition 2.15 that if p is a point in $\mathbf{Shv}(\mathbb{C})$ and $\alpha \in \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$, then $stalk_p(F) = p^* \circ a(\alpha)$.

Let $Z \in \mathbb{C}$. Then Z is a topological space or a local po-space and we can choose any point (in the usual sense) $x \in Z$. Define

$$\begin{aligned} p_x^* \colon \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} &\to \mathbf{Set} \\ F &\mapsto \operatorname*{colim}_{x \in L \overset{\subseteq}{\text{open}} Z} F(L), \end{aligned}$$

where the colimit is taken over all open subsets of Z containing x. See Remark 3.5 for a discussion of subobjects in **LPS**.

Given a functor $p^* \colon \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} \to \mathbf{Set}$, there is an induced functor

$$A \colon \mathbb{C} \xrightarrow{y} \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} \xrightarrow{p^*} \mathbf{Set},$$

where y is the Yoneda embedding defined on objects and morphisms by $Z \mapsto \mathbb{C}(-, Z)$ and $\varphi \mapsto \mathbb{C}(-, \varphi)$.

Given a functor $A \colon \mathbb{C} \to \mathbf{Set}$, one can define induced adjoint functors $p^* \colon \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} \to \mathbf{Set}$ and $p_* \colon \mathbf{Set} \to \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ ($p^* = - \otimes_{\mathbb{C}} A$ and $p_* = \mathbb{C}(A, -)$, see [**MLM92**, Section VII.2]).

Definition 6.1. *(i)* The functor $A \colon \mathbb{C} \to \mathbf{Set}$ is *flat* if the corresponding p^* is left exact.

(ii) A is *continuous* if A sends each covering sieve to an epimorphic family of functions. That is, if S is a covering sieve, then the family of functions $\{A(\varphi) | \varphi \in S\}$ is jointly surjective.

Proposition 6.2 ([**MLM92**, Corollary VII.5.4])**.** *Using the correspondence above, p is a point in $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ if and only if A is flat. Furthermore, p descends to a point in $\mathbf{Shv}(\mathbb{C})$ if and only if A is flat and continuous.*

Proposition 6.3. *p_x defined above descends to a point in $\mathbf{Shv}(\mathbb{C})$.*

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} \arrow[r, "p_x^*"] \arrow[d, shift left, "a"] & \mathbf{Set} \\ \mathbf{Shv}(\mathbb{C}) \arrow[u, shift left, "i"] \arrow[ur, dashed] & \end{tikzcd} \tag{2}$$

Proof. Let $A_x = p_x^* \circ y$, where y is the Yoneda embedding.

First, we show that p_x^* is left exact, that is, it preserves finite limits. Let $F \times_G H$ be a pullback in $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$.

$$\begin{aligned} p_x^*(F \times_G H) &= \operatorname*{colim}_{x \in L \subseteq Z} (F \times_G H)(L) \\ &= \operatorname*{colim}_{x \in L \subseteq Z} F(L) \times_{G(L)} H(L) \\ &= \operatorname{colim} F(L) \times_{\operatorname{colim} G(L)} \operatorname{colim} H(L) \\ &= p_x^* F \times_{p_x^* G} p_x^* H. \end{aligned}$$

The third equality holds because colim commutes with pullbacks in **Set**, and the others are by definition. Thus, A is flat and p_x^* is a point in $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$.

Next, we show that A_x is continuous. Let $\{Y_i \xrightarrow{\varphi_i} N\}$ be a covering sieve for N in $\mathbb{C}$. Recall that $A_x = p_x^* \circ y$. Let $(\varphi_i)_*$ denote composition with φ_i. For each arrow in the covering sieve,

$$\begin{aligned} p_x^* \circ y(Y_i \xrightarrow{\varphi_i} N) &= p_x^*(\mathbb{C}(-,Y_i) \xrightarrow{(\varphi_i)_*} \mathbb{C}(-,N)) \\ &= \operatorname*{colim}_{x\in L\subseteq Z}(\mathbb{C}(L,Y_i) \xrightarrow{(\varphi_i)_*} \mathbb{C}(L,N)) \\ &= y(Y_i)_x \xrightarrow{(\varphi_i)_*} y(N)_x. \end{aligned}$$

We claim that this is an epimorphic family of functions in **Set**. Let $f \in y(N)_x$. Then there is an open subspace L such that $x \in L \subseteq Z$ and f is represented by a morphism $f' \in \mathbb{C}(L,N)$. Since $\{Y_i\}$ covers N, $f'(x) \in Y_k$ for some k. Let $K = (f')^{-1}(Y_k)$. Then K is open and $x \in K \subseteq L \subseteq Z$. Furthermore, $f'|_K \in \mathbb{C}(K,Y_k)$ which represents an element $f'' \in y(Y_k)_x$, and $(\varphi_k)_* f'' = f$. Hence, we have an epimorphic family as claimed. Thus, A is continuous and p_x descends to a point in $\mathbf{Shv}(\mathbb{C})$. □

Abusing notation, we will also denote the induced functor in diagram (2) by p_x^*. With this abuse of notation, the stalk of $F \in \mathbf{Set}^{\mathbb{C}^{op}}$ at x is given by $stalk_x(F) = p_x^* a(F) = p_x^*(F)$. Note that $stalk_x(F) = \{\mathrm{germ}_x(s) \mid x \in U \underset{\text{open}}{\subseteq} Z,\ s \in F(U)\}$.

Theorem 6.4. *The points p_x defined above provide enough points for $\mathbf{Shv}(\mathbb{C})$. That is, given $f \neq g\colon P \to Q \in \mathbf{Shv}(\mathbb{C})$, there is a $Z \in \mathbb{C}$ and a $x \in Z$ such that $p_x^* f \neq p_x^* g\colon p_x^* P \to p_x^* Q \in \mathbf{Set}$.*

Proof. Given $Z \in \mathbb{C}$ and either $P \in \mathbf{Shv}(\mathbb{C})$ or $f \in \operatorname{Mor}\mathbf{Shv}(\mathbb{C})$, let P_Z or f_Z denote the restriction to $\mathbf{Shv}(\mathcal{O}(Z))$.

Assume that $f \neq g\colon P \to Q \in \mathbf{Shv}(\mathbb{C})$. Thus, there is some $Z \in \mathbb{C}$ such that $f_Z \neq g_Z\colon P_Z \to Q_Z \in \mathbf{Shv}(\mathcal{O}(Z))$.

By Theorem 1.3, this is equivalent to saying that the corresponding maps between étale spaces are not equal. That is,

$$\Lambda f_Z \neq \Lambda g_Z\colon \Lambda P_Z \to \Lambda Q_Z \in \mathbf{Etale}(Z).$$

Thus, there is some point $y \in \Lambda P_Z$ such that $\Lambda f_Z(y) \neq \Lambda g_Z(y)$.

By the definition of Λ, $y = \mathrm{germ}_x\, s$ for some $x \in U \underset{\text{open}}{\subseteq} Z$ and $s \in P_Z(U)$. That is, $y \in stalk_x(P) = p_x^* P$. Therefore, $p_x^* f \neq g_x^* g\colon p_x^* P \to p_x^* Q$. □

7. Stalkwise Equivalences

Let $(\mathbb{C},\tau)$ be a site with a subcanonical Grothendieck topology such that $\mathbf{Shv}(\mathbb{C})$ has enough points and let $\bar{y}\colon \mathbb{C} \to \mathbf{sSet}^{\mathbb{C}^{op}}$ be the Yoneda embedding. Recall the definition of stalkwise equivalence in Definition 2.19 which uses the simplicial stalk functor $(\cdot)_p$. Also recall the Yoneda embedding $\bar{y}\colon \mathbb{C} \to \mathbf{sSet}^{\mathbb{C}^{op}}$ given in Definition 2.17. Let $\varphi\colon X \to Y \in \mathbb{C}$.

Lemma 7.1. *$\bar{y}(\varphi)$ is a stalkwise equivalence if and only if for all points p in $\mathbf{Shv}(\mathbb{C})$, $p^* a y(\varphi) \in \mathbf{Set}$ is an isomorphism.*

Proof. Let p be a point in $\mathbf{Shv}(\mathbb{C})$. Recall that the simplicial stalk of $\bar{y}(\varphi)$ at p is given by

$$(\bar{y}(\varphi))_p = \{stalk_p(\bar{y}(\varphi)_n)\}_{n\geqslant 0} = \{p^* ay(\varphi)\}_{n\geqslant 0},$$

which is simplicially constant. Thus, $\bar{y}(\varphi)_p \in \mathbf{sSet}$ is an isomorphism if and only if $p^*ay(\varphi) \in \mathbf{Set}$ is an isomorphism. □

Lemma 7.2. *If the Grothendieck topology τ is subcanonical, then the composite functor $\mathbb{C} \xrightarrow{y} \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} \xrightarrow{a} \mathbf{Shv}(\mathbb{C})$ is faithful.*

Proof. By the Yoneda Lemma, y is full and faithful. Since τ is subcanonical $\mathrm{im}(y) \subset \mathbf{Shv}(\mathbb{C})$. Furthermore, $a \circ i\colon \mathbf{Shv}(\mathbb{C}) \to \mathbf{Shv}(\mathbb{C})$ is naturally isomorphic to the identity functor [**MLM92**, Corollary III.5.6]. Thus, ay is naturally isomorphic to y which is faithful. □

Theorem 7.3. *Let $\varphi\colon X \to Y \in \mathbb{C}$ and assume that $\bar{y}(\varphi)$ is a stalkwise equivalence. Then φ is bijective.*

The proof of this theorem is split into the following two propositions.

Proposition 7.4. *Let $\varphi\colon X \to Y \in \mathbb{C}$ and assume that $\bar{y}(\varphi)$ is a stalkwise equivalence. Then φ is epi.*

Proof. For $i = 1, 2$, let $\psi_i\colon Y \to Z \in C$ be a morphism such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi\colon X \to Z$. Then for all points p in $\mathbf{Shv}(\mathbb{C})$, $p^*ay(\psi_1 \circ \varphi) = p^*ay(\psi_2 \circ \varphi)$. From this, it follows that

$$p^*ay(\psi_1) \circ p^*ay(\varphi) = p^*ay(\psi_2) \circ p^*ay(\varphi).$$

But by Lemma 7.1, $p^*ay(\varphi)$ is a set isomorphism, so, in particular, it is epi. Therefore, $p^*ay\psi_1 = p^*ay\psi_2$ for all points p in $\mathbf{Shv}(\mathbb{C})$. Since $\mathbb{C}$ has enough points, $ay\psi_1 = ay\psi_2$. By Lemma 7.2, $a \circ y$ is faithful, and thus $\psi_1 = \psi_2$. Therefore, φ is epi. □

Proposition 7.5. *Let $\varphi\colon X \to Y \in \mathbb{C}$ and assume that $\bar{y}(\varphi)$ is a stalkwise equivalence. Then φ is mono.*

Proof. For $i = 1, 2$, let $\psi_i\colon W \to X \in C$ be a morphism such that $\varphi \circ \psi_1 = \varphi \circ \psi_2\colon W \to Y$. As in the proof of the previous proposition, for all points p in $\mathbf{Shv}(\mathbb{C})$,

$$p^*ay(\varphi) \circ p^*ay(\psi_1) = p^*ay(\varphi) \circ p^*ay(\psi_2).$$

Again by Lemma 7.1, $p^*ay(\varphi)$ is mono. Therefore, $p^*ay\psi_1 = p^*ay\psi_2$ for all points p in $\mathbf{Shv}(\mathbb{C})$. Since $\mathbb{C}$ has enough points, $ay\psi_1 = ay\psi_2$. By Lemma 7.2, $a \circ y$ is faithful, thus $\psi_1 = \psi_2$. Therefore, φ is mono. □

Let $\mathbb{C} = \mathbf{Spaces}$ or $\mathbf{LPS}$ with the open cover topology. By Example 4.3 and Proposition 4.4, this topology is subcanonical.

Recall from Section 6 that if $Z \in \mathbb{C}$ and $x \in Z$, then

$$\begin{aligned} p_x^* \colon \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} &\to \mathbf{Set} \\ F &\mapsto \operatorname*{colim}_{x \in L \subseteq_{\mathrm{open}} Z} F(L) \end{aligned} \tag{3}$$

descends to a point in $\mathbf{Shv}(\mathbb{C})$ (where the colimit is taken over open subspaces of Z which contain x).

Theorem 7.6. *Let $\varphi \colon X \to Y \in \mathbb{C}$. Then $\bar{y}(\varphi)$ is a stalkwise equivalence if and only if φ is an isomorphism in $\mathbb{C}$.*

Proof. ($\Leftarrow$) If φ is an isomorphism, then for all points p in $\mathbf{Shv}(\mathbb{C})$ $p^*ay(\varphi)$ is an isomorphism. Hence, by Lemma 7.1, $\bar{y}(\varphi)$ is a stalkwise equivalence.
($\Rightarrow$) Assume that $\bar{y}(\varphi)$ is a stalkwise equivalence. Then by Theorem 7.3, φ is a bijection.

Let $x \in Y$. Let p_x be the corresponding point defined in (3). Then

$$p_x^*ay(\varphi) \colon \operatorname*{colim}_{x \in L \subseteq Y} \mathbb{C}(L, X) \xrightarrow{\varphi_*} \operatorname*{colim}_{x \in L \subseteq Y} \mathbb{C}(L, Y) \in \mathbf{Set}$$

is a bijection. Let $f \colon Y \to Y$ be given by $f = \mathrm{Id}_Y$. Let $\bar{f} = [f] \in \operatorname{colim}_{x \in L \subseteq Y} \mathbb{C}(L, Y)$. Let $\bar{g} = (\varphi_*)^{-1}(\bar{f})$. Then there is some $x \in W \subseteq Y$ such that $\bar{g}$ has a representative $g \in \mathbb{C}(W, X)$.

Let $f' = \varphi_* g = \varphi \circ g$. Then $[f'] = \varphi_*[g] = [f]$. Therefore, there exists $x \in S \subseteq Y$ such that $S \subset Y \cap W$ and $f'|_S = f|_S = \mathrm{Id}_Y|_S$.

Let $\psi = g|_S$. Therefore, $\varphi\psi = \mathrm{Id}_S$. Let $T = \mathrm{im}(\psi)$. Then $\varphi|_T \circ \psi = \mathrm{Id}_S$ and $\varphi|_T$ is a bijection. Hence, $\varphi|_T \colon T \to S$ is an isomorphism, where $x \in S$.

Finally, this construction can be repeated for all $x \in Y$. For each $x \in Y$, there is a $x \in S_x \subseteq Y$ and there is a map

$$\psi_x \colon S_x \to X \text{ such that } \psi_x = (\varphi|_{\mathrm{im}(\psi_x)})^{-1}.$$

Since φ is a bijection, all local inverses must agree. That is, $\{\psi_x \colon S_x \to X\}$ is a matching family on the open cover $\{S_x\}$ of Y. Since the topology is subcanonical, there is a unique amalgamation $\psi \colon Y \to X$. It remains to be shown that ψ is an inverse for φ.

$$\text{For all } S_x, \quad \varphi \circ \psi|_{S_x} = \varphi \circ \psi_x = \mathrm{Id}_{S_x}.$$

Therefore, φ is an isomorphism in $\mathbb{C}$. $\square$

8. Model Categories for Local Po-spaces

8.1. A model category for local po-spaces

Using our results on **LPS**, Theorem 1.1 will now follow directly from Jardine's model structure (Theorem 2.20).

Proof of Theorem 1.1. The open dicovers induce a Grothendieck topology on the small category **LPS**. Applying Theorem 6.4, the Grothendieck topos **Shv**(**LPS**) has enough points. So by Jardine's Theorem (Theorem 2.20), **sPre**(**LPS**) has a proper, simplicial, cellular model structure in which

- the cofibrations are the monomorphisms, i.e. the levelwise monomorphisms of presheaves,
- the weak equivalences are the stalkwise equivalences, and
- the fibrations are the morphisms which have the right lifting property with respect to all trivial cofibrations.

Finally, by Theorem 7.6, the weak equivalences coming from **LPS** (via the Yoneda embedding) are precisely the isomorphisms. ☐

8.2. Localization

Our main motivation for constructing a model category for local po-spaces was to model concurrent systems. In particular, we would like to be able to define and understand equivalences of concurrent systems using such a model category. However, our model structure on **sPre**(**LPS**) does not have any non-trivial equivalences among the morphisms coming from **LPS**. To obtain a model category more directly useful for studying concurrency, we need to localize with respect to a set of morphisms. In particular, we want morphisms which preserve certain computer-scientific information.

How to best choose such morphisms is an important question and has been studied in [**Bub04**]. For the sake of simplicity, that paper studied only the category **PoSpaces** of po-spaces (a subcategory of **LPS**). There it was shown that the set of morphisms which should be equivalences depends on the *context*. That is, instead of choosing equivalences for **PoSpaces**, one should be choosing equivalences for the coslice category or undercategory $\mathbf{A} \downarrow \mathbf{PoSpaces}$ of po-spaces under a po-space A, where A is called the context.

This result can be easily extended to our setting. First, we remark that if we choose a local po-space A, then the undercategory $\mathbf{A} \downarrow \mathbf{LPS}$ is the category whose objects are dimaps $\iota_M \colon A \to (M, \bar{U})$ and whose morphisms are dimaps $f \colon (M, \bar{U}) \to (N, \bar{V})$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "\iota_M"'] \arrow[dr, "\iota_N"] & \\ (M,\bar{U}) \arrow[rr, "f"] & & (N,\bar{V}) \end{tikzcd}$$

Next, $\bar{y}(A) \in \mathbf{sPre}(\mathbf{LPS})$ and the undercategory $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$ is the category whose objects are morphisms of simplicial presheaves $\iota_\alpha \colon \bar{y}(A) \to \alpha$ and whose morphisms are morphisms of simplicial presheaves $f \colon \alpha \to \beta$ such that the following diagram commutes:

$$\begin{tikzcd} & \bar{y}(A) \arrow[dl, "\iota_\alpha"'] \arrow[dr, "\iota_\beta"] & \\ \alpha \arrow[rr, "f"] & & \beta \end{tikzcd}$$

Since $\bar{y} \colon \mathbf{LPS} \to \mathbf{sPre}(\mathbf{LPS})$ is a functor

$$\bar{y}(\iota_M) \colon \bar{y}(A) \to \bar{y}(M, \bar{U}) \text{ and } \bar{y}(\iota_N) = \bar{y}(f \circ \iota_M) = \bar{y}(f) \circ \bar{y}(\iota_M).$$

Hence, $\mathbf{A} \downarrow \mathbf{LPS}$ embeds as a subcategory of $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$.

Define morphisms in $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$ to be weak equivalences, cofibrations and fibrations if and only if they are weak equivalence, cofibrations and fibrations in $\mathbf{sPre}(\mathbf{LPS})$. Then this makes $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$ into a model category (see [**Hir03**, Theorem 7.6.5]).

We will show that this model category is again proper and cellular. We will need the following definitions and a theorem of Kan.

Definition 8.1. *(i)* Let $\mathbb{C}$ be a category and I be a set of maps in $\mathbb{C}$. A *relative I-cell complex* is a map that can be constructed by a transfinite composition of pushouts of elements of I.

(ii) An object $A \in \mathbb{C}$ is *small relative to a collection of morphisms* $\mathcal{D}$ in $\mathbb{C}$ if there exists a cardinal κ such that for all regular cardinals $\lambda \geqslant \kappa$ and for all λ-sequences

$$X_0 \to X_1 \to X_2 \to \ldots \to X_\beta \to \ldots$$

with $X_\beta \to X_{\beta+1}$ in $\mathcal{D}$ for $\beta + 1 < \lambda$, the set map

$$\operatorname*{colim}_{\beta<\lambda} \mathbb{C}(A, X_\beta) \to \mathbb{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism.

Definition 8.2. A model category $\mathcal{M}$ is cofibrantly generated if there are sets I and J such that

(i) the domains of I are small relative to the relative I-cell complexes,

(ii) the domains of J are small relative to the relative J-cell complexes,

(iii) the fibrations have the right lifting property with respect to J, and

(iv) the trivial fibrations have the right lifting property with respect to I.

We say that $\mathcal{M}$ is cofibrantly generated by I and J.

Definition 8.3. *(i)* Let $\mathcal{M}$ be a model category cofibrantly generated by I and J. An object $A \in \mathcal{M}$ is *compact* if there is a cardinal γ such that for all relative I-cell complexes $f\colon X \to Y$ with a particular presentation, every map $A \to Y$ factors through a subcomplex of size at most γ.

(ii) $f\colon A \to B$ is an *effective monomorphism* if f is the equalizer of the inclusions $B \rightrightarrows B \amalg_A B$.

Definition 8.4. A *cellular* model category is a model category cofibrantly generated by I and J such that

(i) the domains and codomains of elements of I and J are compact,

(ii) the domains of elements of J are small relative to relative I-cell complexes, and

(iii) the cofibrations are effective monomorphisms.

Theorem 8.5 ([**Hir03**, Theorem 11.3.2])**.** *Let $\mathcal{M}$ be a model category cofibrantly generated by the sets I and J, and let $\mathcal{N}$ be a bicomplete category such that there* *exists a pair of adjoint functors* $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon U$. *Define* $FI = \{Fu \mid u \in I\}$ *and* $FJ = \{Fv \mid v \in J\}$. *If*

1. *the domains of* FI *and* FJ *are small relative to* FI*-cell and* FJ*-cell, respectively, and*
2. U *maps relative* FJ*-cell complexes to weak equivalences,*

then $\mathcal{N}$ *has a model category structure cofibrantly generated by* FI *and* FJ *such that* f *is a weak equivalence in* $\mathcal{N}$ *if and only if* Uf *is a weak equivalence in* $\mathcal{M}$*, and* (F, U) *is a Quillen pair.*

Theorem 8.6. *Let* $\mathcal{M}$ *be a model category and let* $A \in \mathcal{M}$*. Then* $\mathbf{A} \downarrow \mathcal{M}$ *has a model structure where a morphism*

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[rr, "f"] & & C \end{tikzcd}$$

is a weak equivalence, cofibration or fibration in $\mathbf{A} \downarrow \mathcal{M}$ *if and only if* f *is a weak equivalence, cofibration or fibration, respectively, in* $\mathcal{M}$*. If* $\mathcal{M}$ *is proper, cofibrantly generated or cellular, then so is* $\mathbf{A} \downarrow \mathcal{M}$*.*

Remark 8.7. For a more detailed proof, we invite the reader to regard Hirschhorn's note [**Hir05**].

Proof. That $\mathbf{A} \downarrow \mathcal{M}$ has the stated model structure follows from the definitions (see [**Hir03**, Theorem 7.6.5]).

Pushouts and pullbacks in $\mathbf{A} \downarrow \mathcal{M}$ can be formed by taking pushouts and pullbacks of the underlying morphisms in $\mathcal{M}$, and then taking the induced maps from A. It thus follows that if $\mathcal{M}$ is proper, so is $\mathbf{A} \downarrow \mathcal{M}$.

Assume $\mathcal{M}$ is cofibrantly generated by I and J. The method for showing that $\mathbf{A} \downarrow \mathcal{M}$ is cofibrantly generated will be to apply Theorem 8.5 to the following adjoint functors:

$$F\colon \mathcal{M} \leftrightarrows (\mathbf{A} \downarrow \mathcal{M})\colon U,$$

where for $B \in \mathcal{M}$ and $f\colon B \to C \in \mathcal{M}$,

$$F(B) = \begin{tikzcd} A \arrow[d, "i_1"] \\ A \amalg B \end{tikzcd}, \quad F(f) = \begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "i_1"] & \\ A \amalg B \arrow[rr, "\mathrm{Id} \amalg f"] & & A \amalg C \end{tikzcd}$$

and U is the forgetful functor

$$U\left(\begin{tikzcd} A \arrow[d, "\iota_B"] \\ B \end{tikzcd}\right) = B, \quad U\left(\begin{tikzcd} & A \arrow[dl, "\iota_B"'] \arrow[dr, "\iota_C"] & \\ B \arrow[rr, "f"] & & C \end{tikzcd}\right) = B \xrightarrow{f} C.$$

Define $FI = \{Fu \mid u \in I\}$ and $FJ = \{Fv \mid v \in J\}$.

The main observation for the proof is that for a morphism u in $\mathcal{M}$, the pushout of Fu is obtained from the pushout of u in $\mathcal{M}$. That is,

$$\begin{tikzcd} A \arrow[dr] \arrow[drr] \arrow[ddr] & & \\ & A \amalg B \arrow[r, "\mathrm{Id} \amalg u"'] \arrow[d, "\iota_X \amalg f"] & A \amalg C \arrow[d] \\ & X \arrow[r] & P \end{tikzcd} \quad \text{where } P \text{ is defined by} \quad \begin{tikzcd} B \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d] \\ X \arrow[r] & P \end{tikzcd}.$$

From this, it follows that for a set of morphisms S in $\mathcal{M}$, the underlying morphisms of a relative FS-complex are a relative S-complex.

Hence, the conditions on $\mathbf{A} \downarrow \mathcal{M}$ in Theorem 8.5 and the definition of a cellular model category (Definition 8.4) are all inherited from the corresponding conditions in $\mathcal{M}$.

Finally, one can check that the model category structure given by Theorem 8.5 coincides with the one in the statement of the theorem. □

Let $\mathcal{M}$ denote the model structure above on $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$. Since $\mathcal{M}$ is cellular, we can apply left Bousfield localization [**Hir03**] to this model structure $\mathcal{M}$ with respect to a set of morphisms which will preserve the computer-scientific properties we are interested in. In [**Bub04**], one inverted the set of *dihomotopy equivalences* in $\mathbf{A} \downarrow \mathbf{PoSpaces}$. So, in our setting, we will let I be the set of *dihomotopy equivalences* in $\mathbf{A} \downarrow \mathbf{LPS}$ defined below. We will invert the set $\mathcal{I} = \{\bar{y}(f) \mid f \in I\} \subset \bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$.

Definition 8.8. *(i)* Let $\vec{I}$ be the po-space $([0,1], \leqslant)$ where $\leqslant$ is the usual total order on $[0,1]$. Given dimaps $f, g\colon (M, \bar{U}) \to (N, \bar{V}) \in \mathbf{A} \downarrow \mathbf{LPS}$, ϕ is a *dihomotopy* from f to g if $\phi\colon (M, \bar{U}) \times \vec{I} \to (N, \bar{V})$, $\phi|_{(M,\bar{U})\times\{0\}} = f$, $\phi|_{(M,\bar{U})\times\{1\}} = g$, and for all $a \in A$, $\phi(\iota_M(a), t) = \iota_N(a)$. In this case, write $\phi\colon f \to g$.

(ii) The symmetric, transitive closure of dihomotopy is an equivalence relation. Write $f \simeq g$ if there is a chain of dihomotopies $f \to f_1 \leftarrow f_2 \to \ldots \leftarrow f_n \to g$.

(iii) A dimap $f\colon (M, \bar{U}) \to (N, \bar{V})$ is a *dihomotopy equivalence* if there is a dimap $g\colon (N, \bar{V}) \to (M, \bar{U})$ such that $g \circ f \simeq \mathrm{Id}_M$ and $f \circ g \simeq \mathrm{Id}_N$.

The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{I}$ provides a model structure on $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$ in which the weak equivalences are the $\mathcal{I}$-local equivalences (see [**Hir03**]), the cofibrations are the cofibrations in $\mathcal{M}$ and the fibrations are morphisms which have the right lifting property with respect to morphisms which are both cofibrations and $\mathcal{I}$-local equivalences.

Theorem 8.9 (Theorem 1.2)**.** *Let $\mathcal{I} = \{\bar{y}(f) \mid f$ is a directed homotopy equivalence rel $A\}$. The category $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathbf{sPre}(\mathbf{LPS})$ has a left proper, cellular model structure in which*

- *the cofibrations are the monomorphisms,*
- *the weak equivalences are the $\mathcal{I}$-local equivalences, and*

- *the fibrations are those morphisms which have the right lifting property with respect to monomorphisms which are $\mathcal{I}$-local equivalences.*

We claim that this model category provides a good model for studying concurrency. An analysis of this model category will be the subject of future research.

Appendix A. Hypercovers

Suppose now $\mathbb{C}$ is small and equipped with a Grothendieck topology, i.e. we have a site $(\mathbb{C}, \tau)$. The Čech structure $\mathbf{sSet}^{\mathbb{C}^{op}}_{\check{c}(\tau)}$ is obtained from the projective structure by homotopically localizing the comparison morphisms given by the Čech covers with respect to τ or, up-to homotopy, from the injective structure by localizing at the same set of morphisms.

Definition A.1. Let $U = \left\{U_i \xrightarrow{u_i} X\right\}_{i\in I} \in J(X)$ be a cover. Let $i_p \in I$ for each $0 \leqslant p \leqslant n$ and $U_{i_0 \dots i_n}$ be the *wide pullback* of the u_{i_p}'s, i.e. the limiting object of the diagram

$$\begin{tikzcd} U_{i_0} \arrow[drr, "u_{i_0}"] & \cdots & U_{i_p} \arrow[d, "u_{i_p}"] & \cdots & U_{i_n} \arrow[dll, "u_{i_n}"'] \\ & & X & & \end{tikzcd}$$

The *Čech* nerve $\check{U}$ of U is the simplicial presheaf given by

$$\check{U}_n \overset{def}{=} \coprod_{i_0,\dots,i_n \in I} y\left(U_{i_0 \dots i_n}\right)$$

Remark A.2. For any $n \in \mathbb{N}$, $X \in \mathbb{C}$ and $U \in J(X)$, there is a morphism

$$u_{i_0 \cdots i_n} : U_{i_0 \dots i_n} \to X$$

and a diagram of presheaves

$$\begin{tikzcd} \check{U}_n \arrow[r, "E_{U,X,n}"] & y(X) \\ y\left(U_{i_0,\dots,i_n}\right) \arrow[u, "in_{y\left(U_{i_0,\dots,i_n}\right)}"] \arrow[ur, "y\left(u_{i_0,\dots,i_n}\right)"'] & \end{tikzcd}$$

where $E_{U,X,n}$ is given by universal property. The $E_{U,X,n}$ assemble to a morphism of simplicial presheaves

$$E_{U,X} : \check{U} \to \kappa_{y(X)}$$

Remark A.3. Given $U \in J(X)$ seen as a subcategory of the slice $\mathbb{C}/X$, there is the evident functor

$$\begin{array}{rccc} \delta_U : & U & \to & \mathbf{sSet}^{\mathbb{C}^{op}} \\ & u_i & \mapsto & \kappa_{y(U_i)} \end{array}$$

Proposition A.4. *Localizing* $\mathbf{sSet}_{inj}^{\mathbb{C}^{op}}$ *at the sets*

(i) $\{E_{U,X} \mid X \in \mathbb{C},\, U \in J(X)\}$ *;*

(ii) $\left\{hocolim\,(\delta_U) \to \kappa_{y(X)} \mid X \in \mathbb{C},\, U \in J(X)\right\}$*;*

(iii) $\{\kappa(\iota_U) \mid X \in \mathbb{C},\, U \in J(X)\}$*, where, given* $X \in \mathbb{C}$ *and* R *a sieve on* X*,* $\iota_R \colon R \hookrightarrow y(X)$ *is the corresponding inclusion of presheaves;*

(iv) $\left\{\eta_F \colon F \to j(F) \mid F \in \mathbf{sSet}^{\mathbb{C}^{op}}\right\}$*, where* $j \colon \mathbf{sSet}^{\mathbb{C}^{op}} \to \mathbf{sSet}^{\mathbb{C}^{op}}$ *is the objectwise sheafification functor;*

yields the same model structure $\mathbf{sSet}_{\check{c}(\tau)}^{\mathbb{C}^{op}}$*. The same holds for the projective version.*

Finally, there is a model structure $\mathbf{sSet}_{hyp(\tau)}^{\mathbb{C}^{op}}$ obtained from the projective structure by homotopically localizing at the set of the comparison morphisms given by hypercovers with respect to τ. This model structure is Quillen equivalent to Jardine's model structure (Theorem 2.20) on $\mathbf{sSet}^{\mathbb{C}^{op}}$ [**DHI04**, Theorem 1.2]. As with the Čech structure, there is also an injective version. Since Čech covers are particular hypercovers, there is the series of inclusions

$$\mathcal{W}_{prj} \subseteq \mathcal{W}_{\check{c}(\tau)} \subseteq \mathcal{W}_{hyp(\tau)}$$

and a similar series for the injective version. It is in general the case that $\mathcal{W}_{\check{c}(\tau)} \subsetneqq \mathcal{W}_{hyp(\tau)}$, yet equality holds in some important particular cases like the smooth Nisnevitch site (c.f. [**DHI04**, Example A10]). It is an interesting question whether or not $\mathcal{W}_{\check{c}(\tau)} = \mathcal{W}_{hyp(\tau)}$ for local po-spaces.

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