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FIBRATIONS OVER ASPHERICAL MANIFOLDS

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(*communicated by James Stasheff*)

Abstract

Let $f\colon E \to B$ be a map between closed connected orientable manifolds. In this note, we give a necessary condition for f to be a manifold fibration. In particular, we show that if $F \hookrightarrow E \xrightarrow{f} B$ is a fibration where $F = f^{-1}(b)$, E and B are closed connected triangulated orientable manifolds and B is aspherical, then $f|_{E^{(n)}}\colon E^{(n)} \to B$ is surjective, where $E^{(n)}$ denotes the n-th skeleton of E and $n = \dim B$.

Dedicated to Professor Ed Fadell

Fibrations are fundamental objects of study in topology and in other areas of mathematics. Given a surjective map $f\colon E \to B$, it is natural to ask if the homotopy class $[f]$ contains a representative f' that is a fibration. Farrell in [**4**], upon improving his previous results, gave an obstruction for a smooth map $f\colon M \to S^1$ to be a fiber bundle over the circle S^1. Our objective here is to consider this problem from the point of view of coincidence theory involving aspherical manifolds as the target space.

Let $f, g\colon M \to N$ be maps between two closed connected manifolds. Coincidence theory is concerned with the study of the coincidence set $C(f,g) = \{x \in M \mid f(x) = g(x)\}$. When $g = \bar{b}$ is the constant map at a point $b \in N$, the coincidence problem becomes the study of preimages of a point under f. In [**5**], the primary obstruction to deforming f and g to be coincidence free on the n-th skeleton of M, $n = \dim N$, was defined and studied. In this note, we study the preimage problem, i.e., when $g = \bar{b}$. In particular, we show that this primary obstruction is non-trivial when f is a fibration and $B = N$ is aspherical. This implies that a typical fiber must intersect the n-th skeleton of the total space E, regardless of the cellular decomposition of E. This provides more information about (the fiber of) such fibrations.

1. Primary obstruction to deformation

Let $f, g\colon M \to N$ be maps between two closed connected orientable manifolds with $\dim M \geqslant \dim N = n$. We write $\pi_M = \pi_1(M)$, $\pi_N = \pi_1(N)$. Then, $[\pi_n(N^\times)]_{ab}$ is a local system on $N \times N$, where $N^\times$ denotes the pair $(N \times N, N \times N - \Delta_N)$ and

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$[G]_{ab}$ denotes the abelianization of a group G. Similar to the definition in [**5**], we define the *twisted Thom class* $\tilde{\tau}_N \in H^n(N^\times; [\pi_n(N^\times)]_{ab})$ to be the cohomology class of the cocycle $u_N \in C^n(N^\times; [\pi_n(N^\times)]_{ab})$ given by $\langle u_N, \sigma\rangle = [\sigma] \in \pi_n(N^\times; \sigma(v_0))$ for $\sigma\colon (\Delta_n, \partial\Delta_n) \to N^\times$, where v_0 is the leading vertex of the standard n-simplex Δ_n. Denote by τ_N the ordinary Thom class of the normal bundle of the diagonal Δ_N in $N \times N$ using $\mathbb{Z}$ coefficients. If $n \geqslant 3$, then $[\pi_n(N^\times)]_{ab} \cong \pi_n(N^\times) \cong \mathbb{Z}\pi_N$. When $n = 2$, N is an orientable surface of genus $g \geqslant 0$. Note that $[\pi_n(N^\times)]_{ab} \cong [\pi_n(N, N - b)]_{ab}$, where $b \in N$. Using the universal covering space of N and the same argument as in [**2**], one can show that $[\pi_2(N, N - b)]_{ab} \cong \mathbb{Z}\pi_N$. Then the augmentation map $\mathbb{Z}\pi_N \to \mathbb{Z}$ induces $\epsilon^*\colon H^n(N^\times; \mathbb{Z}\pi_N) \to H^n(N^\times; \mathbb{Z})$ such that $\epsilon^*(\tilde{\tau}_N) = \tau_N$.

The primary obstruction to deforming f and g to be coincidence free on the n-th skeleton is given by

$$o_n(f, g) := [j(f \times g)d]^*(\tilde{\tau}_N) \in H^n(M; \mathbb{Z}\pi_N{}^*),$$

where d is the diagonal map $M \to M \times M$ and j is the inclusion of pairs $(N \times N, \emptyset) \to N^\times$. Here, $\mathbb{Z}\pi_N{}^*$ denotes the local system over M induced by pulling back the local system $\mathbb{Z}\pi_N$ along the map $j(f \times g)d$. Thus, for $n \geqslant 3$, $o_n(f, g)$ coincides with the primary obstruction defined in [**5**] whereas for $n = 2$, $o_2(f, g)$ is the abelianized obstruction which coincides with that in [**3**] when g is the identity and $M = N$. Note that if $M^{(n)}$ denotes the n-th skeleton of M and $o_n(f, g) \neq 0$, then for any $f' \sim f$, $g' \sim g$ we have $M^{(n)} \cap C(f', g') \neq \emptyset$.

2. Main results

Let $p\colon E \to B$ be a fibration and let $F = p^{-1}(b)$, $b \in B$ be a typical fiber. Suppose that F, E, and B are 0-connected closed triangulated orientable manifolds. We call such a fibration a *manifold fibration*.

Theorem 2.1. *Let $p\colon E \to B$ be a manifold fibration with $\dim E = n + k$ and $\dim B = n$. Suppose that $\pi_q(B) = 0$ for $2 \leqslant q \leqslant k + 1$. Then for any $b \in B$, $p^{-1}(b) \cap E^{(n)} \neq \emptyset$. In other words, $p|_{E^{(n)}}\colon E^{(n)} \to B$ is surjective. For $n \geqslant 2$, the primary obstruction $o_n(p, \bar{b})$ is a non-zero integer.*

Proof. Case 1: $n = 1$.

Here, $B = S^1$, the unit circle. If p were deformable to a map p' such that p' does not have preimages in the 1-skeleton of E, then p', which we may assume is cellular, has the property that $p'|E^{(1)}$ factors through $S^1 - b$. Now, $(p'|E^{(1)})_\#\colon \pi_1(E^{(1)}) \to \pi_1(S^1)$ is surjective. This is because the fiber is path connected so that $p'_\#\colon \pi_1(E) \to \pi_1(S^1)$ is surjective, and $(p'|E^{(1)})_\#\colon \pi_1(E^{(1)}) \to \pi_1(E)$ is also surjective. However, the factoring of p' through the contractible subspace $S^1 - b$ yields a contradiction to the surjectivity of $(p'|E^{(1)})_\#$. Hence, the assertion follows.

Case 2: $n > 1$.

Since $\pi_2(B) = 0$, from the long exact sequence of homotopy groups of the fibration, we have a short exact sequence

$$0 \to \pi_1(F) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{p_\#} \pi_1(B) \to 1,$$

where $i\colon F \hookrightarrow E$ is the inclusion.

Let $\eta\colon E_0 \to E$ be the covering space corresponding to the subgroup $i_\#(\pi_1(F)) \lhd \pi_1(E)$. It follows from the topological analog for homology of Shapiro's Lemma [**6**] (see also the cohomology version in [**1**]) that

$$H_*(E_0;\mathbb{Z}) \cong H_*(E;\mathbb{Z}[\pi_B{}^*]) = H_*(E;\mathbb{Z}[p^*\pi_B]),$$

where $\pi_B \cong \pi_1(E)/i_\#(\pi_1(F))$ is the local coefficient system on B.

Now, the inclusion $i\colon F \hookrightarrow E$ lifts to $\tilde{i}\colon F \to E_0$ so that $i = \eta \circ \tilde{i}$. Since $\pi_q(B) = 0$ for $2 \leqslant q \leqslant k+1$, it follows that $\tilde{i}\colon F \to E_0$ induces isomorphisms on π_q for $1 \leqslant q \leqslant k$ and an epimorphism on π_{k+1}. Hence, $\tilde{i}_*\colon H_k(F;\mathbb{Z}) \to H_k(E_0;\mathbb{Z})$ is an isomorphism. Now,

$$H_k(E;\mathbb{Z}[p^*\pi_B]) \cong H_k(E_0;\mathbb{Z}) \xrightarrow{\tilde{i}_*^{-1}} H_k(F;\mathbb{Z}) \cong \mathbb{Z}. \tag{1}$$

By Poincaré duality (see [**7**]), we have

$$H^n(E;\mathbb{Z}[p^*\pi_B]) \cong H_k(E;\mathbb{Z}[p^*\pi_B]).$$

It has been shown in [**5**] that the primary obstruction $o_n(f,g)$ is Poincaré dual to the twisted homology class corresponding to the coincidence submanifold of f and g (without loss of generality, we may assume that f and g are transverse). Although it was assumed in [**5**] that $n \geqslant 3$, the same argument there holds when $n = 2$ provided the coefficients form a local system and thus we replace $\pi_2(N^\times)$ with its abelianization $[\pi_2(N^\times)]_{ab}$. With $f = p$ and $g = \bar{b}$, the primary obstruction $o_n(p,\bar{b}) \in H^n(E;\mathbb{Z}[p^*\pi_B])$ is Poincaré dual to the class $[z_{i(F)}^{p^*\pi_B}] \in H_k(E;\mathbb{Z}[p^*\pi_B])$, which, by (1), corresponds to $[z_F^{i^*p^*\pi_B}]$, the fundamental class $[z_F]$ of F in $H_k(F;\mathbb{Z})$, since the coefficient system $i^*p^*\pi_B$ is trivial. Hence, $o_n(p,\bar{b})$ corresponds to $[z_F] \in \mathbb{Z} - \{0\}$. □

Remark 2.2. Theorem 2.1 shows that $p^{-1}(b)$ must intersect the n-th skeleton $E^{(n)}$ of E, regardless of the cellular decomposition of E. In fact, any map homotopic to p has the same property, that is, the set of preimages of b must also intersect $E^{(n)}$. Furthermore, this result shows that the non-vanishing of the obstruction is a necessary condition for a map to be (homotopic to) a fibration.

Remark 2.3. In Theorem 2.1, the hypotheses on the higher homotopy groups of B cannot be relaxed. The Hopf bundles $S^{2n+1} \to \mathbb{C}P^n$ and $S^{4n+3} \to \mathbb{H}P^n$ are easy counter-examples.

The following is an equivalent formulation of Theorem 2.1 when B is aspherical.

Theorem 2.4. *Given a map $f\colon E \to B$ from a closed connected triangulable oriented manifold to a closed connected triangulable oriented aspherical manifold with $n = \dim B \geqslant 2$, if the primary obstruction $o_n(f,\bar{b}) = 0$, then f cannot be a manifold fibration.*

For $n \geqslant 3$, $o_n(f,g) = 0$ iff $\exists f' \sim f$, $g' \sim g$ such that $C(f',g') \cap E^{(n)} = \emptyset$. For $n = 2$, $o_2(f,g)$ is just the abelianized obstruction so that $o_2(f,g) = 0$ does not guarantee that f and g are deformable to be coincidence free on the n-th skeleton. Therefore, as already pointed out in Remark 2.2, the following result is a consequence of and not equivalent to Theorem 2.4. We present a simpler and direct proof of this result, as suggested by the generous anonymous referee.

Corollary 2.5. *Given a manifold fibration $f\colon E \to B$ where B is aspherical with $n = \dim B \geqslant 2$, $f|_{E^{(n)}}\colon E^{(n)} \to B$ is surjective.*

Proof. Let $\eta\colon B' \to B$ be the universal cover and $f'\colon E' \to B'$ be the pull-back of $f\colon E \to B$ by η. Since B is aspherical, B' is contractible and hence E' is equivalent to the trivial fibration $B' \times F$. By duality, $H^n_c(B') \cong \mathbb{Z}$ and $f'^*\colon H^n_c(B') \to H^n_c(E')$ is injective, where H^*_c denotes integral cohomology with compact support. If K' denotes the n-th skeleton of E' then the inclusion $i'\colon K' \to E'$ induces an injective homomorphism $H^n_c(E') \to H^n_c(K')$. It follows that $f' \circ i'$ induces a non-zero homomorphism on H^n_c. On the other hand, if $f \circ i\colon K \to B$ is not surjective, where i is the inclusion of the n-th skeleton K of B, then $\eta \circ f' \circ i'$ is not onto which in turn implies that $f \circ i$ must factor through some subspace B'' of B' with $B' \setminus B'' \cong int(D^n)$. By duality, it is easy to see that $H^n_c(B'') \cong 0$. It follows that $(f' \circ i')^*\colon H^n_c(B') \to H^n_c(K')$ is zero, a contradiction. □

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