

ON REDUCTION MAP FOR ÉTALE K -THEORY OF CURVES

G. BANASZAK, W. GAJDA AND P. KRASOŃ

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Abstract

In this paper we investigate reduction of nontorsion elements in the étale K -theory of a curve X over a global field F . We show that the reduction map can be well understood in terms of Galois cohomology of l -adic representations.

Dedicated to Victor Snaith on the occasion of his 60-th birthday.

0. Introduction

Assume that X is a smooth, proper and geometrically irreducible curve of genus g , defined over a global field F . Let l be an odd rational prime different from the characteristic of the field F . In the function field case we fix the set of places at infinity. Let $n > 0$ and let S_l be the set of places of F which consists of places of bad reduction of X , places at infinity, and in the number field case, primes lying over l . Denote by \mathcal{X} a smooth and proper model of X over the ring of S_l -integers of F . Let $K_i^{et}(Y)$ denote the étale K -theory group of a scheme Y over $\text{Spec } \mathbb{Z}[\frac{1}{l}]$ cf. [DF]. For places $v \notin S_l$, we consider the reduction map:

$$K_{2n}^{et}(\mathcal{X}) \rightarrow K_{2n}^{et}(\mathcal{X}_v)$$

induced on étale K -theory by the injection $\mathcal{X}_v \hookrightarrow \mathcal{X}$ of the special fiber at v .

Theorem . *Let J be the Jacobian variety of X and assume that $\text{End}(J) = \mathbb{Z}$. Let $P_1, P_2, \dots, P_r \in K_{2n}^{et}(\mathcal{X})$ be nontorsion elements which are linearly independent over \mathbb{Z}_l . Fix a partition $\{1, 2, \dots, r\} = I \cup J$ and let l^M be a power of l . There exist infinitely many primes $v \notin S_l$ such that P_i , for each $i \in I$, reduces in $K_{2n}^{et}(\mathcal{X}_v)$ to an element of order at least l^M and P_j , for each $j \in J$, reduces in $K_{2n}^{et}(\mathcal{X}_v)$ to zero.*

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The assumption that $\text{End}(J) = \mathbb{Z}$, in the theorem, is implied by the fact that $K_{2n}^{et}(\mathcal{X})$ does not support, for $n > 0$, any suitable action of $\text{End}(J)$ bigger than \mathbb{Z} . In our previous work on the reduction map for abelian varieties [BGK3] we investigated in the present context the maps

$$J(F) \rightarrow J_v(k_v)$$

for $v \notin S_l$ where J_v is the Jacobian of \mathcal{X}_v and $J(F)$ (resp. $J_v(k_v)$) is the group of F (resp. k_v) points of J (resp. J_v). This map is an essential part of the map

$$K_0(\mathcal{X}) \rightarrow K_0(\mathcal{X}_v).$$

Note that according to the Beilinson conjectures the Quillen K-group $K_m(\mathcal{X})$ is expected to be finitely generated. The Quillen-Lichtenbaum conjecture (cf. [FW] p. 57) predicts that the Dwyer-Friedlander map [DF]:

$$K_m(X; \mathbb{Z}/l^k) \rightarrow K_m^{et}(X; \mathbb{Z}/l^k)$$

is an isomorphism for $m > 3$ since the cohomological dimension of X is no bigger than 4 by [Mi], Corollary 1.4, p 221. Hence by the localization sequences in K-theory the conjectures imply that for $m > 3$ there is an isomorphism:

$$K_m(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \rightarrow K_m^{et}(\mathcal{X}).$$

Provided these conjectures hold true, our Theorem implies the corresponding statement for the even dimensional Quillen K-groups of \mathcal{X} . Note that if X is a curve over an algebraically closed field then the Quillen-Lichtenbaum conjecture is supposed to hold for $m \geq 2$ since the cohomological dimension of X is no bigger than 2 cf. [Mi] Remark 1.5, p.222.

In section 1 of the paper we introduce intermediate Jacobians and Kummer theory for l -adic representations which will be used in section 2 in the proof of the Theorem.

1. Kummer Theory and intermediate Jacobians

Let \bar{F} be a fixed separable closure of F . For a finite place v in F and a finite extension L/F contained in \bar{F} , let w be a finite prime in L over v . To indicate that w is not over any place in the set S_l , we will write $w \notin S_l$.

Let T_l be a free \mathbb{Z}_l -module of finite rank d . Let $V_l = T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and $A_l = V_l/T_l$. Consider an l -adic representation $\rho_l : G_F \rightarrow GL(T_l)$ unramified outside S_l . For $w \notin S_l$ we denote by G_w the absolute Galois group $G(\bar{L}_w/L_w)$ of the completion L_w of L at w . Let k_w be the residue field for the place w and $g_w = G(\bar{k}_w/k_w)$. The Galois group g_w has a topological generator $Fr_w : k_w \rightarrow k_w$ defined by the formula $Fr_w(x) = x^{q_w}$, where $q_w = |k_w|$. The inertia group I_w is defined as the kernel of the natural map $G_w \rightarrow g_w$. We put $H_f^1(G_w; T_l) = i_w^{-1} H_f^1(G_w; V_l)$, where

$$i_w : H^1(G_w; T_l) \rightarrow H^1(G_w; V_l)$$

and the group $H_f^1(G_w; V_l)$ is defined as follows:

$$H_f^1(G_w; V_l) = \text{Ker} (H^1(G_w; V_l) \xrightarrow{\text{Res}_w} H^1(I_w; V_l))$$

where I_w is the inertia subgroup of G_w . Define

$$H_{f,S_l}^1(G_L; T_l)$$

to be the kernel of the natural map:

$$H^1(G_L; T_l) \rightarrow \prod_{w \notin S_l} H^1(G_w; T_l) / H_f^1(G_w; T_l)$$

Note that as far as H^1 is concerned, the l -adic cohomology is the same as the continuous cohomology. This follows from the simple "lim¹ argument", since $T_l/l^k T_l$ is finite.

As in [BGK2, Definition 2.3] we define the intermediate Jacobian:

$$J_{f,S_l}(T_l) = \varinjlim_{L/F} H_{f,S_l}^1(G_L; T_l).$$

To understand the action of G_F on $J_{f,S_l}(T_l)$ note that for L/F Galois we have a standard G_F -action on continuous cochains defining $H^n(G_L, T_l)$ which induces the action on $H^n(G_L, T_l)$, in particular on $H^1(G_L, T_l)$. Similarly, with a little more care, one sees that G_F acts on $\prod_{w \notin S_l} H^1(G_w; T_l) / H_f^1(G_w; T_l)$. These actions are compatible in the sense that the map defining $H_{f,S_l}^1(G_L; T_l)$ is G_F -equivariant. In addition, the corresponding G_F -actions are compatible with direct limits yielding a G_F -action on the intermediate Jacobian.

Proposition 1.1. *Assume that for any finite extension L/F and any place w of \mathcal{O}_L , $w \notin S_l$, we have $T_l^{Fr_w} = 0$. The following statements hold.*

- 1 *The group $J_{f,S_l}(T_l)$ is divisible.*
- 2 *There is an isomorphism of G_F -modules $J_{f,S_l}(T_l)_{\text{tor}} \cong A_l$*
- 3 *$H^1(G_L; T_l) \cong J(T_l)^{G_L}$,*

Proof. For the number field case it is [BGK2, Prop. 2.14, Prop. 2.16]. In the function field case the same argument works. □

Proposition 1.1 shows that we can do Kummer theory in the G_F -module $J_{f,S_l}(T_l)$. Note that by definition of A_l there is a compatible in k , G_F - isomorphism $T_l/l^k T_l \cong A_l[l^k]$. For any $k > 0$ we consider the residual representation

$$\overline{\rho}_{l^k} : G_F \rightarrow GL(A_l[l^k]),$$

induced by ρ_l . Let $F_{l^k} = F(A_l[l^k])$ denote the global field $\overline{F}^{ker \overline{\rho}_{l^k}}$. For $\hat{P} \in H_{f,S_l}^1(G_F; T_l)$ we have the Kummer maps [BGK3, (2.3)]:

$$\phi_{\hat{P}}^{(k)} : G(\overline{F}/F_{l^k}) \longrightarrow A_l[l^k] \tag{1.2}$$

$$\phi_{\hat{P}}^{(k)}(\sigma) = \sigma\left(\frac{1}{l^k}\hat{P}\right) - \frac{1}{l^k}\hat{P},$$

which are well defined due to Proposition 1.1 (1), (2) and due to the fact that $G(\bar{F}/F_{l^k})$ acts trivially on $A_l[l^k]$ by definition of F_{l^k} . These maps are compatible with the natural maps $G(\bar{F}/F_{l^{k+1}}) \rightarrow G(\bar{F}/F_{l^k})$, therefore they induce a map:

$$\phi_{\hat{P}}^{(\infty)} : G(\bar{F}/F_{l^\infty}) \longrightarrow T_l,$$

where $F_{l^\infty} = \bigcup F_{l^k}$. Denote by $\hat{\Lambda}$ the \mathbb{Z}_l -submodule of $H_{f,S_l}^1(G_F; T_l)$, generated by $\hat{P}_1, \dots, \hat{P}_r$ and define

$$\Phi_{\hat{\Lambda}}^{(k)} : G(\bar{F}/F_{l^k}) \longrightarrow \bigoplus_{i=1}^r A_l[l^k]$$

$$\Phi_{\hat{\Lambda}}^{(k)} = (\phi_{\hat{P}_1}^k, \dots, \phi_{\hat{P}_r}^k).$$

Similarly define

$$\Phi_{\hat{\Lambda}}^{(\infty)} : G(\bar{F}/F_{l^\infty}) \longrightarrow \bigoplus_{i=1}^r T_l$$

$$\Phi_{\hat{\Lambda}}^{(\infty)} = (\phi_{\hat{P}_1}^\infty, \dots, \phi_{\hat{P}_r}^\infty).$$

We use the notation $T_l^r = \bigoplus_{i=1}^r T_l$. Let \mathcal{O} be a ring with identity which is a finitely generated \mathbb{Z} -module. Assume that the ring \mathcal{O} acts on T_l and the actions of \mathcal{O} and G_F on T_l commute. Assume also that $\hat{\Lambda}$ is \mathcal{O} -equivariant and that $\hat{P}_1, \dots, \hat{P}_r$ are independent over $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_l$. In [BGK3, Lemmas 2.12 and 2.13] we proved the following generalization of the theorem of Bertrand [Be, Theorem 2].

Proposition 1.3. *Let $G_{l^\infty} = G(F_{l^\infty}/F)$ and V_l be a semisimple $\mathbb{Q}_l[G_{l^\infty}]$ -module such that the group $H^1(G_{l^\infty}; V_l)$ vanishes. Then the image of $\Phi_{\hat{\Lambda}}^{(\infty)}$ is an open subset of T_l^r in the l -adic topology.*

Observe that by definitions G_F acts on V_l via its quotient G_{l^∞} .

Note that the proofs of Lemmas 2.12 and 2.13 in [BGK3] were done for the number field case, but the function field case can be treated analogously.

2. Proof of the theorem

Let G_{F,S_l} be the Galois group of the maximal, unramified outside S_l , extension of F . The diagrams (3.1) and (3.2) of [BGK1], for $v \notin S_l$ give the following commutative diagram:

$$\begin{array}{ccc} K_{2n}^{et}(\mathcal{X}) & \longrightarrow & K_{2n}^{et}(\mathcal{X}_v) \\ \downarrow & & \simeq \downarrow \\ H^1(G_{F,S_l}; T_l(J)(n)) & \xrightarrow{r_v} & H^1(g_v; T_l(J)(n)) \end{array} \quad (2.1)$$

in which the left vertical arrow is surjective with finite kernel and the right vertical arrow is an isomorphism. Let \hat{P}_i denote the image of P_i via the left vertical arrow of (2.1). In order to prove the Theorem it is enough to show the corresponding statement in Galois cohomology for the reduction map r_v of the diagram (2.1). Now we apply the results of section 1 for $T_l = T_l(J)(n)$ and $\mathcal{O} = \mathbb{Z}$. Observe that all assumptions imposed on T_l in section 1 are fulfilled. Namely, by Ogg-Shafarevich criterion cf. [ST], Theorem 1 the set of primes S_l contains primes of bad reduction for J . Moreover, $T_l^{Fr_w} = 0$ for $w \notin S_l$ by the Weil conjectures for J proven in [De]. Note that by the Weil conjectures, all eigenvalues of Fr_w acting on $T_l(J)$ have absolute values $q_w^{\frac{1}{2}}$, and Fr_w acts on $\mathbb{Z}_l(n)$ via q_w^n . Moreover, the $\mathbb{Q}_l[G_F]$ -module V_l is semisimple in the number field case by [Fa] and in the function field case by [Za1], [Za2]. In addition, in both cases, $H^1(G_{l^\infty}; T_l)$ is finite by [Se, Cor. p. 734, and remarque 2, p. 734], for $n = 0$ and [J1, p. 338] for $n > 0$. Note that for $T_l = T_l(J)(n)$ we have $H_{f,S_l}^1(G_L; T_l) = H^1(G_{L,S_l}; T_l)$ for every $F \subset L \subset \bar{F}$. Actually this is true for every representation unramified outside S_l and to see this consider the localization exact sequence

$$0 \longrightarrow H^1(G_{L,S_l}; T_l) \longrightarrow H^1(G_L; T_l) \xrightarrow{\oplus_{w \notin S_l} \partial_w} \oplus_{w \notin S_l} H^0(g_w; T_l(-1))$$

Note that ∂_w can be factored as follows

$$H^1(G_L; T_l) \longrightarrow H^1(G_w; T_l) \xrightarrow{\tilde{\partial}_w} H^0(g_w; T_l(-1))$$

On the other hand for $w \notin S_l$ by Lemma 2.8 (3) [BGK2] there is an exact sequence which comes from the restriction-inflation exact sequence

$$0 \longrightarrow H_f^1(G_w; T_l) \longrightarrow H^1(G_w; T_l) \xrightarrow{Res_w} H^1(I_w; T_l)^{g_w}$$

In addition

$$H^1(I_w; T_l)^{g_w} \cong Hom_{g_w}(I_w; T_l) \cong Hom_{g_w}(\mathbb{Z}_l; T_l(-1)) \cong H^0(g_w; T_l(-1))$$

and we check that the map Res_w can be identified with the boundary map $\tilde{\partial}_w : H^1(G_w; T_l) \rightarrow H^0(g_w; T_l(-1))$. This shows that $H_{f,S_l}^1(G_L; T_l) \cong H^1(G_{L,S_l}; T_l)$. Now the Theorem follows by the next proposition.

Proposition 2.2. *Let $\{1, 2, \dots, r\} = I \cup J$ be a partition, where $I = \{i_1, i_2, \dots, i_s\}$ and $J = \{j_1, j_2, \dots, j_{r-s}\}$. Given a fixed power l^M , there exist infinitely many places $v \notin S_l$ such that $r_v(\hat{P}_i)$ is an element of order at least l^M , for $i \in I$ and $r_v(\hat{P}_j) = 0$, for $j \in J$.*

Proof. We apply the method of the proof of Theorem 3.1 of [BGK3]. For the convenience of the reader we divide the proof into five steps. Define $\hat{\Lambda}_I$ and $\hat{\Lambda}_J$ in a similar way as $\hat{\Lambda}$. We define $\frac{1}{l^k} \hat{\Lambda}_I$ (resp. $\frac{1}{l^\infty} \hat{\Lambda}_I$) to be the submodule of $J_{f,S_l}(T_l)$, generated by all elements \hat{R} such that $l^k \hat{R} = \hat{P}_i$ for some $i \in I$ with k fixed (respectively, by all elements \hat{R} such that $l^k \hat{R} = \hat{P}_i$ for some $i \in I$ and all k). The modules $\frac{1}{l^k} \hat{\Lambda}_I$ and $\frac{1}{l^\infty} \hat{\Lambda}_I$ are well defined by Prop. 1.1 (1). Let N_k (resp. N_∞) be the subgroup of $G(\bar{F}/F_{l^k})$ of such elements that act trivially on the submodule $\frac{1}{l^k} \hat{\Lambda}_I$ (resp. $\frac{1}{l^\infty} \hat{\Lambda}_I$) of $J_{f,S_l}(T_l)$. Set $F_{l^k}(\frac{1}{l^k} \hat{\Lambda}_I) = \bar{F}^{N_k}$ (resp. $F_{l^\infty}(\frac{1}{l^\infty} \hat{\Lambda}_I) = \bar{F}^{N_\infty}$) and put

$$\begin{aligned} \mathcal{D}_k &= G(F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I)/F_{l^k}) \\ \mathcal{D}_\infty &= G(F_{l^\infty}(\frac{1}{l^\infty}\hat{\Lambda}_I)/F_{l^\infty}). \end{aligned}$$

Step 1. The argument in this step is as in [KP, Lemma 5]. By Proposition 1.3 there is an $m \in \mathbb{N}$ such that $l^m T_l^s \subset \Phi_{\hat{\Lambda}_I}^{(\infty)}(G(\bar{F}/F_{l^\infty})) \subset T_l^s$. Take $k \geq m$ and consider the following diagram:

$$\begin{array}{ccc} \mathcal{D}_\infty & \longrightarrow & T_l^s/l^m T_l^s \\ \downarrow & & \downarrow \cong \\ \mathcal{D}_{k+1} & \longrightarrow & A_l[l^{k+1}]^s/l^m A_l[l^{k+1}]^s \\ \downarrow & & \downarrow \cong \\ \mathcal{D}_k & \longrightarrow & A_l[l^k]^s/l^m A_l[l^k]^s. \end{array} \tag{2.3}$$

Observe that $\ker \Phi_{\hat{\Lambda}_I}^{(k)} = G(\bar{F}/F(\frac{1}{l^k}\hat{\Lambda}_I))$ and $\ker \Phi_{\hat{\Lambda}_I}^{(\infty)} = G(\bar{F}/F(\frac{1}{l^\infty}\hat{\Lambda}_I))$. Hence the horizontal arrows in diagram (2.3), induced by Kummer maps $\Phi_{\hat{\Lambda}_I}^{(\infty)}$, $\Phi_{\hat{\Lambda}_I}^{(k+1)}$, and $\Phi_{\hat{\Lambda}_I}^{(k)}$, are well defined. Now the definitions of these Kummer maps, given in section 1, yield commutativity of the diagram (2.3) immediately. It is clear by an ascending chain argument that for $k \gg 0$ the image of the middle horizontal arrow is isomorphic to the image of the bottom horizontal arrow in diagram (2.3). Hence \mathcal{D}_{k+1} maps onto \mathcal{D}_k via the left bottom vertical arrow in (2.3) because for every k the map $\mathcal{D}_k \rightarrow A_l[l^k]^s$, induced by Kummer map, is injective by definition of \mathcal{D}_k . So the tower of fields

$$\begin{array}{ccc} & F_{l^{k+1}}(\frac{1}{l^{k+1}}\hat{\Lambda}_I) & \\ & \swarrow \quad \searrow & \\ F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I) & & F_{l^{k+1}} \\ & \searrow \quad \swarrow & \\ & F_{l^k} & \end{array} \tag{2.4}$$

shows that for $k \gg 0$

$$F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I) \cap F_{l^{k+1}} = F_{l^k}. \tag{2.5}$$

Step 2. By Proposition 1.3 we see that the upper horizontal arrow in the following commutative diagram has open image:

$$\begin{array}{ccc}
 G(F_{l^\infty}(\frac{1}{l^\infty}\hat{\Lambda}_I, \frac{1}{l^\infty}\hat{\Lambda}_J)/F_{l^\infty}(\frac{1}{l^\infty}\hat{\Lambda}_I)) & \hookrightarrow & T_l^{r-s} \\
 \downarrow & & \downarrow \\
 G(F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)/F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I)) & \hookrightarrow & \bigoplus_{i=1}^{r-s} A_l[l^k].
 \end{array}
 \tag{2.6}$$

Hence the order of cokernel of the lower horizontal map in the diagram (2.6) is bounded independently of k . It follows that for $k \gg 0$ we can pick an element $\sigma \in G(F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)/F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I))$, with the property that all projections onto the $r-s$ summands $A_l[l^k]$ of the image of σ via the bottom horizontal map in (2.6) have orders divisible by l^M .

Step 3. Consider the tower of fields:

$$\begin{array}{ccccc}
 & & F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J) & & \\
 & \swarrow & | & \searrow & \\
 F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J) & & F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I) & & \\
 \sigma \downarrow & \swarrow & & \searrow & \\
 F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I) & & & & F_{l^{k+1}} \\
 & \searrow \textit{id} & & \swarrow \textit{h} & \\
 & & F_{l^k} & & \\
 & & | & & \\
 & & F & &
 \end{array}
 \tag{2.7}$$

Take $k \gg 0$ such that (2.5) holds and such that there is a nontrivial element $h \in G(\overline{F}/F)$ and such that $\rho_l(h)$ is acting on T_l as a homothety $1 + l^k u_0$, for some $u_0 \in \mathbb{Z}_l^\times$. The element h exists by [Bo, Cor. 1, p.702] in the number field case, and by [Za 3, Thm. 4.1] in the function field case. By abuse of notation we will denote the restriction of h to $F_{l^{k+1}}$ also by h . Using the diagram (2.7) we construct an element

$$\gamma \in G(F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)/F_{l^k}) \subset G(F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)/F),$$

such that $\gamma|_{F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)} = \sigma$, $\gamma|_{F_{l^{k+1}}} = h$, $\gamma|_{F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I)} = id$. By the Chebotarev density theorem [Ne, Theorem 6.4, p.132] there exist infinitely many prime ideals v of F such that γ is equal to the Frobenius element for v in the extension $F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)/F$. For such a v we fix a place w in $F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I, \frac{1}{l^k}\hat{\Lambda}_J)$ above v .

Step 4. In Step 4 we prove that for $i \in I$ the point \hat{P}_i maps to 0 via the bottom

horizontal arrow in the diagram:

$$\begin{array}{ccc}
 H^1(G_{L,S_l}; T_l) & \xrightarrow{r_{w_1}} & H^1(g_{w_1}, T_l) \\
 \uparrow & & \uparrow \\
 H^1(G_{F,S_l}; T_l) & \xrightarrow{r_v} & H^1(g_v, T_l)
 \end{array} \tag{2.8}$$

where $L = F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I)$ and w_1 is the place of $F_{l^{k+1}}(\frac{1}{l^k}\hat{\Lambda}_I)$ below w . The left vertical map in the diagram (2.8) is injective by Prop. 1.1 (3). The right vertical map is obviously injective. Let l^{c_i} be the order of $r_v(\hat{P}_i)$ in $H^1(g_v; T_l)$. The group $G(\bar{F}/F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I))$ acts trivially on $\hat{Q}_i = \frac{1}{l^k}\hat{P}_i \in J_{f,S_l}(T_l)$ so by Prop. 1.1 (3) $\hat{Q}_i \in H^1(G_{F_{l^k}(\frac{1}{l^k}\hat{\Lambda}_I), S_l}; T_l) \subset H^1(G_{L,S_l}; T_l)$. Hence \hat{Q}_i maps to the point $r_{w_1}(\hat{Q}_i) \in H^1(g_{w_1}, T_l)$ of order l^{k+c_i} . By the choice of v we see that

$$h(r_{w_1}(\hat{Q}_i)) = (1 + l^k u_0) r_{w_1}(\hat{Q}_i),$$

where h is the element selected in Step 3. The choice of v cf. diagram (2.7) implies that $r_{w_1}(\hat{Q}_i)$ comes from $H^1(g_v, T_l)$. Hence $h(r_{w_1}(\hat{Q}_i)) = r_{w_1}(\hat{Q}_i)$, so $l^k r_{w_1}(\hat{Q}_i) = 0$ and $c_i = 0$.

Step 5. We prove that for $j \in J$ the order of $r_v(\hat{P}_j)$ in $H^1(g_v; T_l)$ is divisible by l^M by computing its image in the bottom right corner of the following commutative diagram:

$$\begin{array}{ccc}
 H^1(G_{F,S_l}; T_l)/l^k H^1(G_{F,S_l}; T_l) & \longrightarrow & H^1(g_v; T_l)/l^k H^1(g_v; T_l) \\
 \downarrow & & \downarrow \\
 H^1(G_{F_{l^k}, S_l}; T_l)/l^k H^1(G_{F_{l^k}, S_l}; T_l) & \longrightarrow & H^1(g_{u_2}; T_l)/l^k H^1(g_{u_2}; T_l) \\
 \downarrow & & \downarrow \\
 \text{Hom}((G_{F_{l^k}, S_l})^{ab}; A_l[l^k]) & \longrightarrow & \text{Hom}(g_{u_2}; A_l[l^k]),
 \end{array} \tag{2.9}$$

where u_2 is the place in F_{l^k} below w , and F_{l^k, S_l} denotes the maximal extension of F_{l^k} contained in \bar{F} and unramified outside of S_l . The image of \hat{P}_j by the composition of left vertical maps is the homomorphism $\phi_j : (G_{F_{l^k}, S_l})^{ab} \longrightarrow A_l[l^k]$ induced by the Kummer map $\phi_{\hat{P}_j}^{(k)}$. Let $\tilde{\gamma}$ denote the extension to F_{l^k, S_l} of the element γ selected in Step 3. By the choice of σ in Step 2 we obtain that $\phi_j(\tilde{\gamma})$ has order $\geq l^M$. The lower horizontal map in diagram (2.9) sends ϕ_j to a homomorphism whose value at the Frobenius element of g_{u_2} has order at least l^M by the construction of γ and the choice of v in step 3. So the point \hat{P}_j maps to a homomorphism of order at least l^M in the lower right corner of the diagram (2.9) \square

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G. Banaszak banaszak@amu.edu.pl

Mathematics Department
Adam Mickiewicz University
Poland
Poznań
ul.Umułtowska 87

W. Gajda gajda@amu.edu.pl

Mathematics Department
Adam Mickiewicz University
Poland
Poznań
ul.Umułtowska 87

P. Krason krason@sus.univ.szczecin.pl

Mathematics Department
Szczecin University
Poland
Szczecin
ul.Wielkopolska 15