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CLASSIFICATION AND VERSAL DEFORMATIONS OF L_∞ ALGEBRAS ON A 2|1-DIMENSIONAL SPACE

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(*communicated by James Stasheff*)

Abstract

This article explores $\mathbb{Z}_2$-graded L_∞ algebra structures on a 2|1-dimensional vector space. The reader should note that our convention on the parities is the opposite of the usual one, because we define our structures on the symmetric coalgebra of the parity reversion of a space, so our 2|1-dimensional L_∞ algebras correspond to the usual 1|2-dimensional algebras.

We give a complete classification of all structures with a nonzero degree 1 term. We also classify all degree 2 codifferentials, which is the same as a classification of all 1|2-dimensional $\mathbb{Z}_2$-graded Lie algebras. For each of these algebra structures, we calculate the cohomology and a miniversal deformation.

1. Introduction

L_∞ algebras - or strongly homotopy Lie algebras - were first described in [**21**] and have recently been the focus of much attention both in mathematics ([**22**], [**13**]) and in mathematical physics ([**15**], [**23**], [**1**], [**2**], [**20**], [**16**], [**11**]).

In the physics literature, one usually considers $\mathbb{Z}_2$-graded spaces. That's the case in our consideration also: throughout this paper, all spaces will be $\mathbb{Z}_2$-graded, and we will work in the parity reversed definition of the L_∞ structure. (See Section 2.)

In [**8**], all L_∞ algebras of dimension less than or equal to 2 were classified, in [**9**] miniversal deformations for all L_∞ structures on a space of dimension 0|3 (0 even and 3 odd dimension)- which correspond to ordinary Lie algebras - were constructed, and in [**10**], all L_∞ algebras of dimension 1|2 were classified.

The picture in the 2|1 dimensional case is more complicated than 1|2-dimensional algebras, because the space of n-cochains on a 1|2 dimensional space has dimension 6|6 for $n > 1$, while the space of n-cochains on a 2|1- dimensional space has dimension $3n+2|3n+1$, making it more difficult to classify the nonequivalent structures. Accordingly, we give a complete classification here of only those L_∞ algebras which

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are extensions of degree 1 coderivations, which are, as it turns out, equivalent to degree 1 coderivations, and those which are extensions of degree 2 coderivations, in other words, extensions of $\mathbb{Z}_2$-graded Lie algebras as L_∞ algebras. For each of these algebras we either construct a miniversal deformation or give the necessary steps for the construction.

2. Basic Definitions

2.1. L_∞ Algebras

We work in the framework of the parity reversion $W = \Pi V$ of the usual vector space V on which an L_∞ algebra structure is defined, because in the W framework, an L_∞ structure is simply an odd coderivation d of the symmetric coalgebra $S(W)$, satisfying $d^2 = 0$, in other words, it is an odd codifferential in the $\mathbb{Z}_2$-graded Lie algebra of coderivations of $S(W)$. As a consequence, when studying $\mathbb{Z}_2$-graded Lie algebra structures on V, the parity is reversed, so that a $2|1$-dimensional vector space W corresponds to a $1|2$-dimensional $\mathbb{Z}_2$-graded Lie structure on V. Moreover, the $\mathbb{Z}_2$-graded anti-symmetry of the Lie bracket on V becomes the $\mathbb{Z}_2$-graded symmetry of the associated coderivation d on $S(W)$.

A formal power series $d = d_1 + \cdots$, with $d_i \in L_i = \operatorname{Hom}(S^i(W), W)$ determines an element in $L = \operatorname{Hom}(S(W), W)$, which is naturally identified with $\operatorname{Coder}(S(W))$, the space of coderivations of the symmetric coalgebra $S(W)$. Thus L is a $\mathbb{Z}_2$-graded Lie algebra. An odd element d in L is called a *codifferential* if $[d, d] = 0$. We also say that d is an L_∞ structure on W.

If $g = g_1 + \cdots \in \operatorname{Hom}(S(W), W)$, and $g_1 : W \to W$ is invertible, then g determines a coalgebra automorphism of $S(W)$ in a natural way, which we will denote by the same letter g. Moreover, every coalgebra automorphism is determined in this manner. Two codifferentials d and d' are said to be *equivalent* if there is a coalgebra automorphism g such that $d' = g^*(d) = g^{-1}dg$.

An automorphism is said to be linear when it is determined by a linear map g_1. Two codifferentials are said to be *linearly equivalent* when there is a linear equivalence between them. If d and d' are codifferentials of a fixed degree N, then they are equivalent precisely when they are linearly equivalent. Thus we can restrict ourself to linear automorphisms when determining the equivalence classes of elements in L_N.

A detailed description of L_∞ algebras can be obtained in [**14, 15, 18, 19**]. The study of examples of L_∞ algebra structures in [**8, 9, 10**] may be useful to the reader, but we intend this article to be as self contained as possible.

Let us first establish some notation for the cochains. Suppose $W = \langle w_1, w_2, w_3\rangle$, with w_1 an odd element and w_2, w_3 even elements. If $I = \{i_1, i_2, i_3\}$ is a multi-index, with i_1 and i_2 either zero or one, let $w_I = w_1^{i_1} w_2^{i_2} w_3^{i_3}$. For simplicity, we will denote w_I by I. Then for $n \geqslant 1$,

$$(S^n(W))_e = \langle (0, p, n-p) | 0 \leqslant p \leqslant n\rangle, \qquad |(S^n(W))_e| = n+1$$
$$(S^n(W))_o = \langle (1, q, n-q-1) | 0 \leqslant q \leqslant n-1\rangle, \qquad |(S^n(W))_0| = n$$

If λ is a linear automorphism of $S(W)$, then in terms of the standard basis of

W, its restriction to W has matrix

$$\lambda = \begin{pmatrix} q & 0 & 0 \\ 0 & r & t \\ 0 & s & u \end{pmatrix} \tag{1}$$

where $q(ru - st) \neq 0$. We will sometimes express λ by the submatrix $\left(\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right)$. It is useful to note that for a linear automorphism

$$\lambda(w^I) = \lambda(w_1)^{i_1}\lambda(w_2)^{i_2}\lambda(w_3)^{i_3},$$

so that

$$\lambda(1, x, y) = \sum_{i=0}^{x}\sum_{j=0}^{y}(1, i+j, x+y-i-j)\binom{x}{i}\binom{y}{j} q r^i s^{x-i} t^j u^{y-j}. \tag{2}$$

Let $L_n := \mathrm{Hom}(S^n(W), W)$. Define

$$\varphi^I_j(w_J) = I!\delta^I_J w_j,$$

where $I! = i_1!i_2!i_3!$. If we let $|I| = i_1 + i_2 + i_3$, then $L_n = \langle \varphi^I_j, |I| = n\rangle$. If φ is odd, we denote it by the symbol ψ to make it easier to distinguish the even and odd elements. Then

$$(L_n)_e = \langle \varphi_1^{1,q,n-q-1}, \varphi_2^{0,p,n-p}, \varphi_3^{0,p,n-p} | 1 \leqslant q \leqslant n-1, 1 \leqslant p \leqslant n\rangle$$
$$(L_n)_o = \langle \psi_2^{1,q,n-q-1}, \psi_3^{1,q,n-q-1}, \psi_1^{0,p,n-p} | 1 \leqslant q \leqslant n-1, 1 \leqslant p \leqslant n\rangle,$$

so that $|L_n| = 3n + 2|3n + 1$.

2.2. Versal Deformations

The classical formal deformation theory for associative algebras was previously worked out in [**12**], and for Lie algebras in [**17**]. Versal deformation theory was first worked out in [**3, 4, 5**] and then extended to L_∞ algebras in [**7**].

An augmented local ring $\mathcal{A}$ with maximal ideal $\mathfrak{m}$ will be called an *infinitesimal base* if $\mathfrak{m}^2 = 0$, and a *formal base* if $\mathcal{A} = \overleftarrow{\lim}_n \mathcal{A}/\mathfrak{m}^n$. A deformation of an L_∞ algebra structure d on W with base given by a local ring $\mathcal{A}$ with augmentation $\epsilon : \mathcal{A} \to \mathfrak{K}$, where $\mathfrak{K}$ is the field over which W is defined, is an $\mathcal{A}$-L_∞ structure $\tilde{d}$ on $W\hat{\otimes}\mathcal{A}$ such that the morphism of $\mathcal{A}$-L_∞ algebras $\epsilon_* = 1\otimes\epsilon : L_{\mathcal{A}} = L\otimes\mathcal{A} \to L\otimes\mathfrak{K} = L$ satisfies $\epsilon_*(\tilde{d}) = d$. (Here $W\hat{\otimes}\mathcal{A}$ is an appropriate completion of $W\otimes\mathcal{A}$.) The deformation is called infinitesimal (formal) if $\mathcal{A}$ is an infinitesimal (formal) base.

In general, the cohomology $H(D)$ of d given by the operator $D : L \to L$ with $D(\varphi) = [\varphi, d]$ may not be finite dimensional. Moreover, if $\varphi \in L_n$, it may happen that $D(\varphi)$ in an infinite sum of terms, so that D does not act on the graded space $\bigoplus L_n$, but rather on the direct product L of these spaces. However, L has a natural filtration $L^n = \prod_{i=n}^{\infty} L_i$, which induces a filtration H^n on the cohomology, because D respects the filtration. Then $H(D)$ is of finite type if H^n/H^{n+1} is finite dimensional. Since this is always true when W is finite dimensional, the examples we study here will always be of finite type. A set δ_i will be called a *basis of the cohomology* if any element δ of the cohomology can be expressed uniquely as a formal sum $\delta = \delta_i a^i$. If we identify $H(D)$ with a subspace of the space of cocycles

$Z(D)$ and we choose a *basis* β_i of the coboundary space $B(D)$, then any element $\zeta \in Z(D)$ can be expressed uniquely as a sum $\zeta = \delta_i a^i + \beta_i b^i$.

For each δ_i, let u^i be a parameter of opposite parity. Then the infinitesimal deformation $d^1 = d + \delta_i u^i$, with base $\mathcal{A} = \mathfrak{K}[u_i]/(u_i u_j)$ is universal in the sense that if $\tilde{d}$ is any infinitesimal deformation with base $\mathcal{B}$, then there is a unique homomorphism $f : \mathcal{A} \to \mathcal{B}$, such that the morphism $f_* = 1 \otimes f : L_{\mathcal{A}} \to L_{\mathcal{B}}$ satisfies $f_*(\tilde{d}) \sim d$.

For formal deformations, there is no universal object in the sense above. A *versal deformation* is a deformation d^∞ with formal base $\mathcal{A}$ such that if $\tilde{d}$ is any formal deformation with base $\mathcal{B}$, then there is some morphism $f : \mathcal{A} \to \mathcal{B}$ such that $f_*(d^\infty) \sim \tilde{d}$. If f is unique whenever $\mathcal{B}$ is infinitesimal, then the versal deformation is called *miniversal*. In [**7**], we constructed a miniversal deformation for L_∞ algebras with finite type cohomology.

The method of construction is as follows. Define a coboundary operator D by $D(\varphi) = [\varphi, d]$. First, one constructs the universal infinitesimal deformation $d^1 = d + \delta_i u^i$, where δ_i is a graded basis of the cohomology $H(D)$ of d, or more correctly, a basis of a subspace of the cocycles which projects isomorphically to a basis in cohomology, and u^i is a parameter whose parity is opposite to δ_i. The infinitesimal assumption that the products of parameters are equal to zero gives the property that $[d^1, d^1] = 0$. Actually, we can express

$$[d^1, d^1] = (-1)^{\delta_j(\delta_i+1)}[\delta_i, \delta_j]u^i u^j = \delta_k a^k_{ij} u^i u^j + \beta_k b^k_{ij} u^i u^j,$$

where β_i is a basis of the coboundaries, because the bracket of d^1 with itself is a cocycle. Note that the right hand side is of degree 2 in the parameters, so it is zero up to order 1 in the parameters.

If we suppose that $D(\gamma_i) = -\frac{1}{2}\beta_i$, then by replacing d^1 with

$$d^2 = d^1 + \gamma_k b^k_{ij} u^i u^j,$$

one obtains

$$[d^2, d^2] = \delta_k a^k_{ij} u^i u^j + 2[\delta_l u^l, \gamma_k b^k_{ij} u^i u^j] + [\gamma_k b^k_{ij} u^i u^j, \gamma_l b^l_{ij} u^i u^j]$$

Thus we are able to get rid of terms of degree 2 in the coboundary terms β_i, but those which involve the cohomology terms δ_i can not be eliminated. This gives rise to a set of second order relations on the parameters. One continues this process, taking the bracket of the *n-th order deformation* d^n, adding some higher order terms to cancel coboundaries, obtaining higher order relations, which extend the second order relations.

Either the process continues indefinitely, in which case the miniversal deformation is expressed as a formal power series in the parameters, or after a finite number of steps, the right hand side of the bracket is zero after applying the n-th order relations. In this case, the miniversal deformation is simply the n-th order deformation. In any case, we obtain a set of relations R_i on the parameters, one for each δ_i, and the algebra $A = \mathbb{C}[[u^i]]/(R_i)$ is called the base of the miniversal deformation. Examples of the construction of miniversal deformations can be found in [**6, 5, 8, 9**].

The miniversal deformation encodes the obstructions to extensions of deformations to formal deformations in terms of the relations on the base. In fact, the second order relations correspond to cocycles whose cohomology classes give the first obstruction to the extension of an infinitesimal deformation to a formal one. Moreover, the miniversal deformation contains all of the information necessary to answer the question of whether a deformation of order n extends to a higher order one.

3. Classification of Codifferentials

Let us compute the brackets of all odd cochains with each other.

$$
\begin{aligned}
[\psi_1^{0,p,n-p}, \psi_2^{1,q,m-q-1}] &= \varphi_1^{1,p+q-1,n-p+m-q-1} p + \varphi_2^{0,p+q,n-p+m-q-1} \\
[\psi_1^{0,p,n-p}, \psi_3^{1,q,m-q-1}] &= \varphi_1^{1,p+q,n-p+m-q-2}(n-p) + \varphi_3^{0,p+q,n-p+m-q-1} \\
[\psi_1^{0,p,n-p}, \psi_1^{0,q,m-q}] &= 0 \qquad [\psi_2^{1,p,n-p-1}, \psi_2^{1,q,m-q-1}] = 0 \\
[\psi_2^{1,p,n-p-1}, \psi_3^{1,q,m-q-1}] &= 0 \qquad [\psi_3^{1,p,n-p-1}, \psi_3^{1,q,m-q-1}] = 0
\end{aligned}
$$

Suppose that

$$
d = \sum_{p=0}^{n} \psi_1^{0,p,n-p} a_p + \sum_{q=0}^{n-1} \psi_2^{1,q,n-q} b_q + \psi_3^{1,q,n-q} c_q, \tag{3}
$$

where we sum over all odd codifferentials of degree n. Then using the above, we compute that

$$
\begin{aligned}
[d,d] = \sum_{p=0}^{n} \sum_{q=0}^{n-1} & \varphi_1^{1,p+q-1,2n-p-q-1} p a_p b_q + \varphi_2^{0,p+q,2n-p-q-1} a_p b_q \\
& + \varphi_1^{1,p+q,2n-p-q-2} (n-p) a_p c_q + \varphi_3^{0,p+q,2n-p-q-1} a_p c_q
\end{aligned} \tag{4}
$$

We claim that either all coefficients a_p must vanish or all coefficients b_q and c_q must vanish. For if p and q are the least indices for which a_p and b_q do not vanish, then there is only one term in the sum above of the form $\varphi_2^{0,p+q,2n-p-q-1}$, which would be a contradiction because its coefficient $a_p b_q$ must vanish.

As a consequence of this observation, we note that codifferentials of degree n fall into two distinct families, those of the *first kind*

$$
\sum_{q=0}^{n-1} \psi_2^{1,q,n-q} b_q + \psi_3^{1,q,n-q} c_q. \tag{5}
$$

and those of the *second kind*

$$
d = \sum_{p=0}^{n} \psi_1^{0,p,n-p} a_p. \tag{6}
$$

Moreover, any expression of either kind gives a codifferential. Thus we have determined all codifferentials of degree N. However, the process of classification requires that we determine the equivalence classes of codifferentials under the action of the automorphism group of the symmetric coalgebra, and we are a long way away from this classification at this stage.

Several things can be said in general. First, let us suppose that d is of the second kind. Then from the brackets computed already, we note that the odd d-cocycles are precisely the odd cochains of the second kind. The space of odd cocycles has dimension $n+1$, which means that the space of even d-coboundaries has dimension $2n$. Also, if φ is any even cocycle, then its bracket with d is an odd cocycle of the second kind. Precise computation of the cohomology depends on solving a linear system of equations whose coefficients depend on the coefficients in d.

Similarly, if d is of the first kind, then the odd d-cocycles are the ones of the first kind, and thus the dimension of the space of odd cocycles is $2n$, and the dimension of the space of even coboundaries is $n+1$. The bracket of any even cocycle with d is a cocycle of the first kind. The cohomology can be computed by solving a system of linear equations in coefficients depending on the coefficients of d.

Since there are $2n$ coefficients in a codifferential of the first kind, and $n+1$ coefficients in a codifferential of the second kind, there are potentially a lot of equivalence classes of codifferentials. The main strategy involved in classification is to reduce the number of independent variables to a manageable number.

It is useful to compute the brackets of even and odd cochains.

$$
\begin{aligned}
[\varphi_1^{1,p,m-p},\psi_1^{0,q,n-q}] =&\psi_1^{0,p+q,m+n-p-q}\\
[\varphi_2^{0,p,m-p},\psi_1^{0,q,n-q}] =&-\psi_1^{0,p+q-1,m+n-p-q}q\\
[\varphi_3^{0,p,m-p},\psi_1^{0,q,n-q}] =&-\psi_1^{0,p+q,m+n-p-q-1}(n-q)\\
[\varphi_1^{1,p,m-p},\psi_2^{1,q,n-q}] =&-\psi_2^{1,p+q,m+n-p-q}\\
[\varphi_2^{0,p,m-p},\psi_2^{1,q,n-q}] =&\psi_2^{1,p+q-1,m+n-p-q}(p-q)\\
[\varphi_3^{0,p,m-p},\psi_2^{1,q,n-q}] =&-\psi_2^{1,p+q,m+n-p-q-1}(n-q)+\psi_3^{1,p+q-1,m+n-p-q}p\\
[\varphi_1^{1,p,m-p},\psi_3^{1,q,n-q}] =&-\psi_3^{1,p+q,m+n-p-q}\\
[\varphi_2^{0,p,m-p},\psi_3^{1,q,n-q}] =&\psi_2^{1,p+q,m+n-p-q-1}(m-p)-\psi_3^{1,p+q-1,m+n-p-q}q\\
[\varphi_3^{0,p,m-p},\psi_3^{1,q,n-q}] =&\psi_3^{1,p+q,m+n-p-q-1}(m-p-(n-q))
\end{aligned}
$$

Notice that bracket of any even cochain with an odd cochain of a certain type is an odd cochain of the same type. This is very important in what follows, because this fact means that there is no mixing of types occurring in the cohomology of a codifferential of a fixed type.

Later on, in the computation of miniversal deformations, we will also need to know the brackets of even cochains with each other, so we include these calculations now.

$$
\begin{aligned}
[\varphi_1^{1,p,m-p},\varphi_1^{1,q,n-q}] =&0\\
[\varphi_2^{0,p,m-p},\varphi_1^{1,q,n-q}] =&-\varphi_1^{1,p+q-1,m+n-p-q}q\\
[\varphi_3^{0,p,m-p},\varphi_1^{1,q,n-q}] =&-\varphi_1^{1,p+q,m+n-p-q-1}(n-q)\\
[\varphi_2^{0,p,m-p},\varphi_2^{0,q,n-q}] =&\varphi_2^{0,p+q-1,m+n-p-q}(p-q)\\
[\varphi_3^{0,p,m-p},\varphi_2^{0,q,n-q}] =&-\varphi_2^{0,p+q,m+n-p-q-1}(n-q)+\varphi_3^{0,p+q-1,m+n-p-q}p\\
[\varphi_3^{0,p,m-p},\varphi_3^{0,q,n-q}] =&\varphi_3^{0,p+q,m+n-p-q-1}(m-p-(n-q))
\end{aligned}
$$

Let us call the degree of the leading term of a codifferential the *order* of that codifferential. We begin with a classification of all codifferentials of order 1.

4. Classification and miniversal deformations of codifferentials with $d_1 \neq 0$

Let us suppose that d is an odd, degree 1 codifferential of the first kind. Then $d = \psi_2^{1,0,0} a_1 + \psi_3^{1,0,0} a_2$ for some constants a_1 and a_2. To see that d is equivalent to $d' = \psi_2^{1,0,0}$, let t and u be such that $a_1 t + a_2 u \neq 0$. Suppose that $g = \begin{pmatrix} a_1 & t \\ a_2 & u \end{pmatrix}$. Then

$$dg(w_1) = d(w_1) = w_2 a_1 + w_3 a_2 = g(w_2) = gd'(w_1).$$

Since $dg(w_2) = gd'(w_2) = 0$ and $dg(w_3) = gd'(w_3) = 0$, it follows that d' and d are equivalent. Thus every codifferential of the first kind is equivalent to $\psi_2^{1,0,0}$.

Now let us study the cohomology of the codifferential $d = \psi_2^{1,0,0}$. We define the coboundary operator D by $D(\varphi) = [\varphi, d]$. Then computing brackets, we see that

$$\begin{aligned} D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,p-1,n-p} p, \quad & D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,p-1,n-p} p + \varphi_2^{0,p,n-p} \\ D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,p-1,n-p} p, \quad & D(\psi_2^{1,q,n-q-1}) &= 0 \\ D(\varphi_1^{1,q,n-q-1}) &= -\psi_2^{1,q,n-q-1}, \quad & D(\psi_3^{1,q,n-q-1}) &= 0 \end{aligned}$$

Note that for the n-cochains above, p ranges from 0 to n, while q ranges only from 0 to $n-1$. It is easy to see that $\psi_2^{1,q,n-q-1}$ and $\psi_3^{1,q,n-q-1}$ give a basis of the odd cocycles, and since both of these types are evidently coboundaries, of $\varphi_2^{0,p,n-p}$ and $\varphi_3^{0,p,n-p}$, resp., where $q = p-1$, all odd cocycles are coboundaries. Similarly, if we let $q = p-1$, then every even cocycle is a linear combination of elements of the form $\varphi_1^{1,p-1,n-p} p + \varphi_2^{0,p,n-p}$, and since these elements are coboundaries, it follows that all even cocycles are coboundaries. Thus the cohomology of d is zero.

If d is a codifferential of the second kind of degree 1, it is of the form $d = \psi_1^{0,1,0} a_1 + \psi_1^{0,0,1} a_2$. We show that it is equivalent to $d' = \psi_1^{0,1,0}$. For suppose that b_1 and b_2 are chosen so that $a_1 b_1 + a_2 b_2 = 1$. Then if g is given by $\begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix}$, we have

$$\begin{aligned} dg(w_2) &= d(w_2 b_1 + w_3 b_2) = w_1(a_1 b_1 + a_2 b_2) = w_1 = gd'(w_2) \\ dg(w_3) &= d(-w_2 a_2 + w_3 a_1) = -a_2 a_1 + a_1 a_2 = 0 = gd'(w_3) \end{aligned}$$

Now, we study the cohomology induced by $d = \psi_2^{1,0,0}$. Calculating coboundaries, we have

$$\begin{aligned} D(\varphi_2^{0,p,n-p}) &= -\,\psi_1^{0,p,n-p}, \qquad & D(\psi_1^{0,p,n-p}) &= 0 \\ D(\varphi_3^{0,p,n-p}) &= 0, & D(\psi_2^{1,q,n-q-1}) &= \varphi_1^{1,q,n-q-1} + \varphi_2^{0,q+1,n-q-1} \\ D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{1,q+1,n-q-1}, & D(\psi_3^{1,q,n-q-1}) &= \varphi_3^{0,q+1,n-q-1} \end{aligned}$$

It is not difficult to see from this table that the cohomology of this codifferential is also zero.

The picture for codifferentials of degree 1 is very simple. There are exactly two equivalence classes of codifferentials of order 1. Because the cohomology vanishes, there are no nontrivial deformations of the infinity algebra structures, so the miniversal deformation of the L_∞ algebras determined by degree 1 codifferentials coincides with the codifferentials.

Next, we will study codifferentials of degree two and the versal deformations of the L_∞ algebra structures determined by degree 2 (quadratic) codifferentials. Since quadratic codifferentials of the symmetric algebra precisely correspond to the $\mathbb{Z}_2$-graded Lie algebra structures on the parity reversion of our space W, what we are really doing here is giving a complete classification of $\mathbb{Z}_2$-graded Lie algebras on a $1|2$-dimensional vector space, and studying their versal deformations as L_∞ algebras. The versal deformations of these L_∞ algebras as superalgebras are given by considering only the deformations induced by quadratic cocycles, so are an easily identifiable part of the versal deformation we will study.

5. Codifferentials of Degree 2 of the First Kind

Let us suppose that $d = \psi_2^{1,1,0}x + \psi_3^{1,1,0}a + \psi_2^{1,0,1}b + \psi_3^{1,0,1}c$, and let us call the multi-index (x, a, b, c) the type of the codifferential. Let us say that a codifferential is of type (x, a, b, c) whenever it is equivalent to a codifferential of that type, so that the type of a codifferential is not unique. Our goal is to show that the equivalence classes of codifferentials reduce to only a few simple types. Let us first remark that if we express d as a matrix of the form $d = \begin{pmatrix} x & b \\ a & c \end{pmatrix}$, then if $d' = g^{-1}dg$, then its matrix is simply the product of the matrices expressing g^{-1}, d and g, multiplied by the scalar q, where $g(w_1) = w_1 q$.

If $x \neq 0$ then by applying a simple coalgebra automorphism, one can assume that it is equal to one. Similarly, if $a \neq 0$ one can assume it is also 1. Thus if both x and a are non zero, our codifferential is of type $(1, 1, b, c)$. We will show later that we can express codifferentials of this type in an even simpler form, but first we examine what possibilities have not been covered by our considerations.

If $x = 0$, but $c \neq 0$, then by interchanging the roles of w_2 and w_3 one can replace it with an equivalent one whose $\psi_2^{1,1,0}$ has nonzero coefficient. Similarly, if $x = 0$ or $a = 0$ and both b and c do not vanish, then by the same interchange, we can see that the codifferential has type $(1, 1, b, c)$ as well. This observation leads to the following possibilities,

If $x \neq 0$ but $a = 0$, then we can assume that either $b = 0$ or $c = 0$ (or both). This gives the possible types $(1, 0, 0, c)$ or $(1, 0, b, 0)$. If $b \neq 0$, then by a simple transformation, the type $(1, 0, b, 0)$ can be reduced to type $(1, 0, 1, 0)$.

The only other types which could arise have both the x and c coefficients vanishing, so they are of type $(0, a, b, 0)$. If both a and b do not vanish, they can be adjusted so we obtain type $(0, 1, 1, 0)$, and if one of the two vanishes but the other does not, we obtain type $(0, 1, 0, 0)$.

Actually, this myriad of types can be much reduced as we shall see shortly. Let us examine the type $(1, 1, b, c)$ and show that in most cases it can be reduced to type $(1,0,0,c)$.

Let d be of type $(1,1,b,c)$. Then if $d' = g^{-1}dg$, we compute

$$d' = \begin{pmatrix} \dfrac{q(ru + bus - rt - cst)}{ru - ts} & \dfrac{q(ut + bu^2 - t^2 - ctu)}{ru - ts} \\ -\dfrac{q(sr + bs^2 - r^2 - crs)}{ru - ts} & -\dfrac{q(ts + bus - rt - cru)}{ru - ts} \end{pmatrix} \tag{7}$$

Now, either r and u both do not vanish, or s and t both do not. Let us assume the former, and put $x = s/r$ and $y = t/u$. Substituting in the matrix for d', we obtain

$$d' = \begin{pmatrix} \dfrac{q(1 + bx - y - cxy)}{1 - xy} & \dfrac{q(y + b - y^2 - cy)\frac{u}{r}}{1 - xy} \\ -\dfrac{q(x + bx^2 - 1 - cx)\frac{r}{u}}{1 - xy} & -\dfrac{q(xy + bx - y - c)}{1 - xy} \end{pmatrix}$$

Our goal is to remove the off diagonal terms without violating the condition $xy \neq 1$. The terms vanish precisely when the equations $y + b - y^2 - cy$ and $x + bx^2 - 1 - cx$ are both equal to zero. When $b \neq 0$, these equations are quadratic in y and x respectively, with solutions

$$x_{\pm} = \frac{c - 1 \pm \sqrt{(1-c)^2 + 4b}}{2b} \qquad y_{\pm} = \frac{1 - c \pm \sqrt{(1-c)^2 + 4b}}{2}.$$

Oddly enough, we compute $x_+y_+ = x_-y_- = 1$, which is just what we want to avoid. On the other hand, $x_+y_- = 1$, if and only if $b = -\frac{(1-c)^2}{4}$. Assuming otherwise, we can eliminate the off diagonal terms, so that after applying a simple automorphism, we can reduce it to type $(1,0,0,c')$, where c' is given by some rational expression in b and c.

On the other hand, when $b = 0$, then the quadratic in x reduces to a linear expression, which is zero when $x = \frac{1}{1-c}$. Of course, x is not well defined if $c = 1$, so let us first assume otherwise. Now $y = 0$ is a solution of the quadratic equality for y, and substituting the expressions for x and y into the first and fourth terms yields that our codifferential is equivalent to one of type $(1,0,0,c)$ where the c in this expression is the same as the c occurring in the type $(1,1,0,c)$. In fact, it is also clear that even when $b \neq 0$, one can reduce any expression of type $(1,1,b,c)$ to the type $(1,1,0,c')$ by choosing $y = y_+$, and $x = 0$, except in the special case when $b = -\frac{(1-c)^2}{4}$.

Note that the case $b = 0$ and $c = 1$ is a special case of the equality $b = -\frac{(1-c)^2}{4}$, so all we have left is to consider the case where this equality holds. Then we certainly can set $y = y_+ = \frac{1-c}{2}$, and the condition $1 - xy \neq 0$ reduces to the inequality $\frac{c-1}{2}x + 1 \neq 0$. If we choose an arbitrary x so this inequality is satisfied, then it is easy to see that the first and the fourth coefficients of d' become, simply $q\frac{c+1}{2}$. This means that when $c \neq -1$, we can choose q to make the first and fourth coefficients of d' equal to 1, the third coefficient equal to 0, and by choosing r/u appropriately, the second coefficient equal to 1 as well. Thus we obtain an element of type $(1,1,0,1)$ unless $c = -1$. One can check that in this case, we obtain $b = -1$, so the element has type $(1,1,-1,-1)$, and also it is obvious from this argument that in this case d is equivalent to the codifferential $d=\psi_3^{1,1,0}$, so that in particular $\psi_3^{1,1,0}$ has type $(1,1,-1,-1)$. It is also easy to show that both types $(1,1,-1,-1)$ and $(1,1,0,1)$ can never be reduced to type $(1,0,0,c)$.

We now proceed to show that the other special types, $(1,0,1,0)$, $(0,1,1,0)$ also can be reduced to type $(1,0,0,c)$.

Type $(1,0,1,0)$ is the same as type $(1,0,0,0)$ and type $(1,1,0,0)$. To see this, apply the generic linear transformation g to produce d' as before, and we obtain

$$d'=\begin{pmatrix} \dfrac{q(r+s)u}{ru-ts} & \dfrac{q(t+u)u}{ru-ts} \\ -\dfrac{q(r+s)s}{ru-ts} & -\dfrac{q(t+u)s}{ru-ts} \end{pmatrix}$$

If we choose $q=1/2$, $s=t=r=-1$ and $u=1$, we obtain type $(1,1,0,0)$, and if instead we choose $q=r=u=1$, $s=0$, and $t=-1$, we obtain type $(1,0,0,0)$.

Type $(0,1,1,0)$ is the same as type $(1,0,0,-1)$ and type $(1,1,0,-1)$. To see this, apply the generic linear transformation g and we obtain

$$d'=\begin{pmatrix} \dfrac{q(su-rt)}{ru-ts} & \dfrac{q(u^2-t^2)}{ru-ts} \\ -\dfrac{q(s^2-r^2)}{ru-ts} & -\dfrac{q(su-rt)}{ru-ts} \end{pmatrix}$$

Choose $t=-1$ and $q=r=u=s=1$. Then this becomes $d'=\psi_2^{1,1,0}-\psi_3^{1,0,1}$ as desired. On the other hand, if $q=s=u=1$, $r=0$, and $t=-1$, then we obtain $d'=\psi_2^{1,1,0}+\psi_3^{1,1,0}-\psi_3^{1,0,1}$.

This completes the classification of types of codifferentials. We have one family $(1,0,0,c)$ and two special cases, $(1,1,0,1)$ and $(0,1,0,0)$ which cannot be reduced to elements of this family.

We show that an element of type $(1,0,0,c)$ is equivalent to one of type $(1,0,0,c')$ precisely when $c'=c^{\pm 1}$, so that the set of equivalence classes of codifferentials has a one parameter subfamily, parameterized by the unit disc in $\mathbb{C}$. To see this, apply the generic linear transformation and we obtain

$$d'=\begin{pmatrix} \dfrac{q(ru-cst)}{ru-ts} & -\dfrac{qtu(-1+c)}{ru-ts} \\ -\dfrac{qrs(-1+c)}{ru-ts} & \dfrac{q(-ts+rcu)}{ru-ts} \end{pmatrix}$$

It is interesting to note that when $c=1$, the middle two terms drop out and thus $\psi_2^{1,1,0}+\psi_3^{1,0,1}$ is not equivalent to any codifferential of type $(1,1,b,c)$. Otherwise, if $c\neq 0$, let $r=u=0$ and $s=t=1$ and $q=-1/c$ and we obtain type $(1,0,0,1/c)$. To see that this is the only other type that could occur, note that to cancel the middle terms, we must have either $r=u=0$ or $s=t=0$, so the claim is obvious.

For later purposes let us label the codifferentials representing the equivalence classes of degree 2 codifferentials of the first kind as follows.

$$\begin{aligned} d_* &= \psi_3^{1,1,0} \\ d_\sharp &= \psi_2^{1,1,0} + \psi_3^{1,1,0} + \psi_3^{1,0,1} \\ d_c &= \psi_2^{1,1,0} + \psi_3^{1,0,1} c \end{aligned}$$

6. Cohomology of Codifferentials of degree 2 of the First Kind

For a degree 2 codifferential d, with cohomology operator $D = [\bullet, d]$, the dimension of the cohomology is given by $h_n = z_n - b_{n-1}$.

6.1. Cohomology of $d_* = \psi_3^{1,1,0}$

The coboundaries of basic cochains for d_* are as follows:

$$\begin{aligned} D(\varphi_1^{1,q,n-q-1}) &= -\psi_3^{1,1+q,n-q-1} \\ D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,1+p,n-p-1}(n-p) - \psi_3^{1,p,n-p} \\ D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,1+p,n-p-1}(n-p) \\ D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,1+p,n-p-1}(n-p) + \varphi_3^{0,1+p,n-p} \\ D(\psi_2^{1,q,n-q-1}) &= D(\psi_3^{1,q,n-q-1}) = 0 \end{aligned}$$

From this table, we see that $\psi_2^{1,q,n-q-1}$, $\psi_3^{1,q,n-q-1}$ are cocycles for $q = 0 \dots n-1$, and $\varphi_3^{0,p,n-p} + \varphi_1^{1,p,n-p-1}(n-p)$ is a cocycle for $p = 0 \dots n$. Also, $\varphi_2^{0,n,0} + \varphi_3^{0,n-1,1}$ is a cocycle, so $z_n = n + 2|2n$, which means that $b_n = n + 1|2n$. It follows that $h_n = 2|2$ for $n > 1$ and $h_1 = 3|2$. Moreover,

$$\begin{aligned} H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_3^{0,1,0}, \varphi_3^{0,0,1} + \varphi_1^{1,0,0}, \varphi_2^{0,1,0} + \varphi_3^{0,0,1} \rangle \\ H^n &= \langle \psi_2^{1,0,n-1}, \psi_3^{1,0,n-1}, \varphi_2^{0,n,0} + \varphi_3^{0,n-1,1}, \varphi_3^{0,0,n} + \varphi_1^{1,0,n-1} n \rangle, \quad n > 1. \end{aligned}$$

6.2. Cohomology of $d_\sharp = \psi_2^{1,1,0} + \psi_3^{1,1,0} + \psi_3^{1,0,1}$

The coboundaries for $d_\sharp$ are given by

$$\begin{aligned} D(\varphi_1^{1,q,n-q-1}) &= -\psi_2^{1,1+q,n-q-1} - \psi_3^{1,1+q,n-q-1} - \psi_3^{1,q,n-q} \\ D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,1+p,n-p-1}(n-p) - \psi_2^{1,p,n-p}(n-1) - \psi_3^{1,p,n-p} \\ D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,1+p,n-p-1}(n-p) - \psi_3^{1,p,n-p}(n-1) \\ D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,p,n-p} n + \varphi_1^{1,1+p,n-p-1}(n-p) + \varphi_2^{0,1+p,n-p} \\ &\quad + \varphi_2^{0,p,n-p+1} + \varphi_3^{0,1+p,n-p} \\ D(\psi_2^{1,q,n-q-1}) &= D(\psi_3^{1,q,n-q-1}) = 0 \end{aligned}$$

We already know that $\psi_2^{1,q,n-q-1}$ and $\psi_3^{1,q,n-q-1}$ give a basis of the $2n$ odd cocycles. First note that $\varphi_3^{0,1,0}$ and $\varphi_2^{0,1,0} + \varphi_3^{0,0,1}$ are a basis of the even 1-cocycles. Thus $z_1 = h_1 = 2|2$ and $b_1 = 2|3$.

For $n > 1$ it is easy to see that images of the cochains of the form $\varphi_3^{0,p,n-p}$ are a basis of the $n+1$-dimensional subspace of cochains of the form $\psi_3^{1,p,n-p}$. If we consider the subspace X spanned by elements of the form $\psi_2^{1,1+q,n-q-1}$ and $\psi_3^{1,p,n-p}$, then D maps the $(2n+1)$-dimensional space spanned by elements of the form $\varphi_1^{1,q,n-q-1}$ and $\varphi_3^{0,p,n-p}$ bijectively onto X.

If $p > 0$, it is clear that the image of $\varphi_2^{0,p,n-p}$ lies in X. Thus we obtain a cocycle as a sum of the element $\varphi_2^{0,p,n-p}$ and a unique linear combination of the elements $\varphi_3^{0,p,n-p}$ and $\varphi_1^{1,q,n-q-1}$. Thus there are n independent cocycles generated by these elements.

For $p = 0$, the image of $\varphi_2^{0,0,n}$ does not lie in X, so it cannot contribute to any cocycle. Thus we see that there are exactly n independent even cocycles. Thus $z_n = n|2n$ and $b_n = n+1|2n+2$.

This means that $h_n = z_n - b_{n-1} = 0$ if $n > 2$. Furthermore, $h_2 = 2|4 - 2|3 = 0|1$. It is easy to see that $\psi_2^{1,0,1}$ can be taken as the basis for H^2. Thus we have

$$\begin{aligned}
H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_3^{0,1,0}, \varphi_2^{0,1,0} + \varphi_3^{0,0,1} \rangle \\
H^2 &= \langle \psi_2^{1,0,1} \rangle \\
H^n &= 0, \qquad \text{if } n > 2
\end{aligned}$$

We will discuss the versal deformation of this codifferential in the next section.

6.3. Cohomology of $d_c = \psi_2^{1,1,0} + \psi_3^{1,0,1}c$

Since d_c is equivalent to $d_{1/c}$, we can assume that c lies in the unit circle. Thus we will assume that $|c| \leqslant 1$ in the following. The coboundaries are given by

$$\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= -\psi_2^{1,1+q,n-q-1} - \psi_3^{1,q,n-q}c \\
D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,p,n-p}(p-1+c(n-p)) \\
D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,p,n-p}(p+c(n-p-1)) \\
D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,p,n-p}(p+c(n-p)) + \varphi_2^{0,1+p,n-p} + \varphi_3^{0,p,n-p+1}c \\
D(\psi_2^{1,q,n-q-1}) &= 0 \\
D(\psi_3^{1,q,n-q-1}) &= 0
\end{aligned}$$

Let $Q_p = p + c(n-p-1)$. When $Q_p \neq 0$, then

$$\xi_p = \varphi_2^{0,p+1,n-p-1} + \varphi_3^{0,p,n-p}c + \varphi_1^{1,p,n-p-1}Q_p, \quad p = 0 \dots n-1,$$

give n independent even cocycles, which are obviously coboundaries. In most cases, the ξ_p give a basis of the even cocycles. However, when $Q_p = 0$ we get an additional even cocycle $\varphi_3^{0,p,n-p}$ which is never a coboundary, and $\psi_3^{1,p,n-p}$ is no longer a coboundary. When this happens the even part of h_n increases by one, and the odd part of h_{n+1} also increases by one. Note that if $n > 1$, for most values of c it never happens that $Q_p = 0$. In fact, if c is not a nonpositive rational number, then Q_p is never zero when $n > 1$.

There is another source of possible even cocycles, given by the terms $\varphi_2^{0,0,n}$, which is a cocycle if $nc = 1$, and $\varphi_3^{0,n,0}$, which is a cocycle if $n = c$. When this happens, the even part of z_n increases by 1, so the odd part of b_n decreases by 1. Thus we see again that the even part of h_n and the odd part of h_{n+1} both increase by 1. Moreover, if $nc = 1$, then we need to add $\psi_2^{1,0,n}$ to the basis of H^{n+1} and if $n = c$, we need to add $\psi_3^{1,n,0}$ to the basis. If c or its reciprocal is not a positive integer, then neither of these two cases hold.

6.3.1. Cohomology for generic values of c

Let us say that c is generic if it is not a nonpositive rational number, nor is it or its reciprocal a positive integer. If c is generic, and $n > 1$, then ξ_p are the only even cocycles, so that $z_n = n|2n$ and thus we have $b_n = n + 1|2n + 2$. It follows that $h_n = 0|0$ for $n > 2$. For $n = 1$, we always have $Q_0 = 0$, so we have two even cocycles, $\varphi_2^{0,1,0}$ and $\varphi_3^{0,0,1}$, which along with the two odd cocycles $\psi_2^{1,0,0}$ and $\psi_3^{1,0,0}$ generically form a basis for H^1. Thus $z_1 = h_1 = 2|2$ in the generic case.

Also, in the generic case, we obtain $b_1 = 2|3$. Since $z_2 = 2|4$, it follows that $h_2 = 0|1$. In fact, $\varphi_3^{1,0,1}$ can be taken as a basis for H^2. What this says is that you can deform d_c in the direction of the family.

Thus we conclude that for generic values of c we have

$$
\begin{aligned}
H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle \\
H^2 &= \langle \psi_3^{1,0,1} \rangle \\
H^n &= 0, \qquad \text{if } n > 2
\end{aligned}
$$

Note that for generic values of c, the picture is the same as for $d_\sharp$. The pattern for versal deformations will be seen to be similar to $d_\sharp$ as well.

6.3.2. Cohomology for the special value $c = 1$

In this case both $nc = 1$ and $n = c$ hold for $n = 1$. Thus we obtain two additional 1-cohomology classes, given by $\varphi_2^{0,0,1}$ and $\varphi_3^{0,1,0}$ and $h_1 = z_1 = 4|2$. Thus $b_1 = 2|1$, so $h_2 = z_2 - b_1 = 2|4 - 2|1 = 0|3$. Thus for $c = 1$ we have

$$
\begin{aligned}
H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1}, \varphi_2^{0,0,1}, \varphi_3^{0,1,0} \rangle \\
H^2 &= \langle \psi_3^{1,0,1}, \psi_2^{1,0,1}, \psi_3^{1,1,0} \rangle \\
H^n &= 0, \qquad \text{if } n > 2
\end{aligned}
$$

This suggests that somehow there are additional directions in which the codifferential can be deformed, and we will comment on this later.

6.3.3. Cohomology when $1/c \neq 1$ is a positive integer

Let $m = 1/c$. Then we have

$$
\begin{aligned}
H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle \\
H^2 &= \langle \psi_3^{1,0,1} \rangle \\
H^m &= \langle \varphi_2^{0,0,m} \rangle \\
H^{m+1} &= \langle \psi_2^{1,0,m} \rangle \\
H^n &= 0, \qquad \text{otherwise}
\end{aligned}
$$

except when $c = 1/2$, in which case, since $m = 2$, $H^2 = \langle \psi_3^{1,0,1}, \varphi_2^{0,0,2} \rangle$.

In this casethe versal deformation picture is more complicated.

6.3.4. Cohomology when $c = 0$

This case is special because $Q_0 = 0$ for all n. Thus we always have the even cohomology class $\varphi_3^{0,0,n}$, and the odd cohomology class $\psi_3^{1,0,n-1}$. Since Q_0 is zero when $n = 1$ in all cases, H^1 is not changed from the generic pattern. Thus

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle$$
$$H^n = \langle \psi_3^{1,0,n-1}, \varphi_3^{0,0,n} \rangle, \qquad \text{if } n > 1$$

For this special case, the versal deformation picture is more involved.

6.3.5. Cohomology when c is a negative rational number

Let us rewrite the equality $Q_p = 0$ in the form $p = \frac{(n-1)c}{c-1}$ When $-1 \leqslant c < 0$ is rational, note that $0 < c/(c-1) \leqslant \frac{1}{2}$, so that $Q = 0$ has an integral solution for p with $0 < p < n-1$ for infinitely many values of n. In fact, suppose that $\frac{c}{c-1} = \frac{r}{s}$, expressed as a fraction in lowest terms. Then $\frac{(n-1)c}{c-1}$ is a positive integer precisely when $n = ks+1$ for a positive integer k, in which case we have $kr = \frac{(n-1)c}{c-1}$, and $0 < kr < n-1$. From this, we can calculate the table of cohomology of d_c as follows.

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle$$
$$H^2 = \langle \psi_3^{1,0,1} \rangle$$
$$H^{ks+1} = \langle \varphi_3^{0,kr,k(s-r)+1} \rangle$$
$$H^{ks+2} = \langle \psi_3^{1,kr,k(s-r)+1} \rangle$$
$$H^n = 0, \qquad \text{otherwise}$$

6.3.6. The Moduli Space of Codifferentials of the First Kind

Let us consider now the deformations of these various types of codifferentials of degree 2 only as graded Lie algebras, *i.e.*, consider only H^2. Consider the following table of codifferentials and bases of the odd part of the second cohomology group:

Type	$(0,1,0,0)$	$(1,1,0,1)$	$(1,0,0,1)$	$(1,0,0,c)$
d	d_*	$d_\sharp$	d_1	d_c
$(H^2)_o$	$\psi_2^{1,0,1}, \psi_3^{1,0,1}$	$\psi_2^{1,0,1}$	$\psi_2^{1,0,1}, \psi_3^{1,0,1}, \psi_3^{1,1,0}$	$\psi_3^{1,0,1}$

There are three special cases, and the generic pattern. Note that even though the dimension of H^2 is not generic for d_0, the extra dimension is even, so does not contribute to the deformations over $\mathbb{C}$.

Clearly, there is only one family of codifferentials, so what is going on with the extra degrees of freedom in the cohomology? To understand this better, let us examine the moduli space of codifferentials of degree 2 in some more detail. Here we use the term moduli space in the following sense. The space of all codifferentials of degree 2 is a variety in a 4 dimensional complex space, preserved under the action of the group of linear automorphisms of the symmetric coalgebra. A quotient space of a variety by such a group action is called a moduli space. The structure of such moduli spaces can be very strange, from a topological point of view.

Let us parameterize our moduli space by types, and note that since type $(1, 0, 0, c)$ is the same as type $(1, 0, 0, 1/c)$, it is natural to think of the moduli space as the unit disc in $\mathbb{C}$, with an identification of the upper semicircle with the bottom. Then every point except 1 and -1 have neighborhoods which are discs, but 1 and -1 are orbifold points of degree 2. Of course, we are really describing the action of the group generated by the transformation $\{z \to 1/z\}$ on the Riemann sphere, and identifying our standard points of the moduli space with the resulting images.

We should like to have some notion of neighborhood of a point in our moduli space, and the natural notion is to consider two elements of the moduli space to be close if they have inverse images which are close in the space of codifferentials. Of course, since any codifferential is equivalent to any multiple of itself, this would make all codifferentials close, so we have to be a bit more careful in our definition.

Consider the standard representatives of the equivalence classes of codifferentials, which are either $(1, 0, 0, c)$, $(1, 1, 0, 1)$ and $(0, 1, 0, 0)$. Let P and Q be equivalence classes. Let us say that Q is ϵ close to P if Q is among the types which occur by adding coordinates to the standard representation of P of absolute value no larger than ϵ. Then P is said to be infinitesimally close to Q if Q is epsilon close to P for all positive values of ϵ.

For most of our points, the notion of neighborhood we have just described yields no surprises. For any standard point P of type $(1, 0, 0, c)$ with $c \neq 1$, ϵ neighborhoods of P for small values of ϵ correspond to standard points $(1, 0, 0, c')$ with c' close to c.

However, for $c = 1$, things are quite different. As we have described before, the one parameter family $(1, 1, b, c)$ with $b = -\frac{1}{4}(c-1)^2$, contains types $(1, 0, 0, 1)$ and $(0, 1, 0, 0)$ for two special values of c, but gives type $(1, 1, 0, 1)$ otherwise. It follows that d_1 and d_* are infinitesimally close to $d_\sharp$. One can check that for ϵ small enough, a neighborhood of d_1 contains only this extra point, along with the points one would usually expect. It is hard to reconcile the fact that for the codifferential d_1, the dimension of the cohomology is 3. One might expect one extra dimension for the deformation in the $d_\sharp$ direction, but two extra dimensions are obtained instead.

Recall that type $(1, 1, 0, a)$ is the same as type $(1, 0, 0, a)$ when $a \neq 1$. This means that a neighborhood of $d_\sharp$ looks just like a neighborhood of d_1) (minus point d_1). Note that although d_1 is infinitesimally close to $d_\sharp$, the converse is not true. Notice that the dimension of the cohomology for $d_\sharp$ is just 1, corresponding to the fact that any small deformation of this codifferential just gives an ordinary element in the main family. Note that the topology of the moduli space in not Hausdorff.

Finally, consider type $(0, 1, 0, 0)$. Note that $(0, 1, \epsilon, 0)$ is the same as $(1, 0, 0, -1)$, so d_* is infinitesimally close to d_{-1}. It is easy to see that type $(0, 1, \epsilon_1, \epsilon_2, 0)$ is the same as type $(1, 1, \epsilon_1/\epsilon_2^2, 0)$, if $\epsilon_2 \neq 0$. One also sees that type $(1, 0, 0, c)$ is the same as type $(1, 1, -\frac{c}{(c+1)^2}, 0)$, if $c \neq \pm 1$. When $c = 1$, we obtain type $(1, 1, -\frac{1}{4}, 0)$ which is the same as type $(1, 1, 0, 1)$. Thus d_* is infinitesimally close to every element of the moduli space except d_1. Note that the cohomology has odd dimension 2, and the type $(0, 1, \epsilon_1, \epsilon_2)$ corresponds to adding a small cocycle to d_*.

7. Miniversal deformations of degree 2 codifferentials of the first kind

The most important part of the construction of a miniversal deformation is the computation of the relations on its base, because they determine the answer to the classical question "Given an infinitesimal deformation, when does it extend to a formal deformation?". Sometimes it is possible to calculate the relations on the base of a miniversal deformation, without explicitly computing the miniversal deformation. In most of the examples here, we give explicit computations of the versal deformation, but for the structure d_*, even the computation of the second order deformation is quite involved, so a general formula would be difficult to develop.

7.1. A miniversal deformation of $d_\sharp$

We study this one first because it is very simple in comparison. The universal infinitesimal deformation is given by

$$d^1 = d_\sharp + \psi_2^{1,0,0} t_1 + \psi_3^{1,0,0} t_2 + \varphi_3^{0,1,0}\theta_1 + (\varphi_2^{0,1,0} + \varphi_3^{0,0,1})\theta_2 + \psi_2^{1,0,1} t_3.$$

To compute the versal deformation, we first compute

$$\tfrac{1}{2}[d^1, d^1] = -\,\psi_3^{1,0,0}(t_1\theta_1 + t_2\theta_2) - \psi_2^{1,0,0} t_1\theta_2 + (\psi_2^{1,1,0} - \psi_3^{1,0,1}) t_3\theta_1.$$

The first two terms are cohomology classes, so give rise to the second order relations

$$t_1\theta_1 + t_2\theta_2 = 0, \qquad t_1\theta_2 = 0.$$

The third term is a coboundary, in fact $D(\varphi_2^{0,0,1}) = \psi_2^{1,1,0} - \psi_3^{1,0,1}$. Thus the second order deformation of $d_\sharp$ is given by

$$d^2 = d^1 - \varphi_2^{0,0,1} t_2\theta_1.$$

Continuing the process, we obtain

$$\tfrac{1}{2}[d^2, d^2] = \psi_2^{1,0,0} t_2 t_3\theta_1.$$

As a consequence of the fact that no coboundary terms appear in this bracket, we see that d^2 is a miniversal deformation of $d_\sharp$, and the relations become

$$t_1\theta_1 + t_2\theta_2 = 0, \qquad -t_1\theta_2 + t_2 t_3\theta_1 = 0.$$

The base $\mathcal{A}$ of the versal deformation is thus

$$\mathcal{A} = \mathfrak{K}[[t_1, t_2, t_3, \theta_1, \theta_2]]/(t_1\theta_1 + t_2\theta_2, -t_1\theta_2 + t_2 t_3\theta_1).$$

7.2. A miniversal deformation of d_c for generic values of c

The universal infinitesimal deformation is given by

$$d^1 = \psi_2^{1,1,0} + \psi_3^{1,0,1} c + \psi_2^{1,0,0} t^1 + \psi_3^{1,0,0} t^2 + \varphi_2^{0,1,0}\theta^1 + \varphi_3^{0,0,1}\theta^2 + \psi_3^{1,0,1} t^3.$$

Then

$$\tfrac{1}{2}[d^1, d^1] = -\psi_2^{1,0,0} t^1\theta^1 - \psi_3^{1,0,0} t^2\theta^2.$$

Since both of the terms in the bracket are cohomology classes, d^1 is already a miniversal deformation of d_c, and the relations on the base are simply

$$t^1\theta^1 = 0, \qquad t^2\theta^2 = 0$$

7.3. A miniversal deformation of d_1

The universal infinitesimal deformation is given by

$$\begin{aligned} d^1 =& \psi_2^{1,1,0} + \psi_3^{1,0,1} + \psi_2^{1,0,0}t^1 + \psi_3^{1,0,0}t^2 + \varphi_2^{0,1,0}\theta^1 + \varphi_3^{0,0,1}\theta^2 + \varphi_2^{0,0,1}\theta^3 \\ &+ \varphi_3^{0,1,0}\theta^4 + \psi_3^{1,0,1}t^3 + \psi_2^{1,0,1}t^4 + \psi_3^{1,1,0}t^5. \end{aligned}$$

Then

$$\begin{aligned} \tfrac{1}{2}[d^1, d^1] =& -\psi_2^{1,0,0}(t^1\theta^1 + t^2\theta^3) - \psi_3^{1,0,0}(t^2\theta^2 + t^1\theta^4) \\ &- \psi_2^{1,0,1}(t^3\theta^3 + t^4\theta^1 - t^4\theta^2) + \psi_3^{1,1,0}(t^3\theta^4 + t^5\theta^1 - t^5\theta^2) \\ &+ (\psi_2^{1,1,0} - \psi_3^{1,0,1})(t^4\theta^4 - t^5\theta^3). \end{aligned}$$

This is a rather interesting situation, because

$$(\psi_2^{1,1,0} - \psi_3^{1,0,1}) = D(\varphi_1^{1,0,0}) + 2\psi_3^{1,0,1},$$

so it is a sum of a coboundary term and a cohomology class. Therefore, a second order relation is $t^4\theta^4 - t^5\theta^3 = 0$, and after taking the second order relations into account, one obtains that the bracket vanishes identically. Thus, the versal deformation of d_1 is still given by the universal infinitesimal deformation, as in the generic case, but with the relations

$$\begin{aligned} t^1\theta^1 + t^2\theta^3 = t^2\theta^2 + t^1\theta^4 = t^4\theta^4 - t^5\theta^3 = 0 \\ t^3\theta^3 + t^4\theta^1 - t^4\theta^2 = t^3\theta^4 + t^5\theta^1 - t^5\theta^2 = 0 \end{aligned}$$

7.4. Deformations of d_c when $m = 1/c \neq 1$ is a positive integer

The universal infinitesimal deformation is given by

$$\begin{aligned} d^1 =& \psi_2^{1,1,0} + \psi_3^{1,0,1}c + \psi_2^{1,0,0}t^1 + \psi_3^{1,0,0}t^2 + \varphi_2^{0,1,0}\theta^1 + \varphi_3^{0,0,1}\theta^2 + \psi_3^{1,0,1}t^3 \\ &+ \varphi_2^{0,0,m}\theta^3 + \psi_2^{1,0,m}t^4. \end{aligned}$$

Then

$$\begin{aligned} \tfrac{1}{2}[d^1, d^1] =& -\psi_2^{1,0,0}t^1\theta^1 - \psi_3^{1,0,0}t^2\theta^2 + \varphi_2^{0,0,m}(\theta^1\theta^3 - m\theta^2\theta^3) \\ &- \psi_2^{1,0,m}(mt^3\theta^3 + t^4\theta^1 - mt^4\theta^2) - \psi_2^{1,0,m-1}mt^2\theta^3. \end{aligned}$$

The only coboundary term appearing is $-\psi_2^{1,0,m-1} = D(\varphi_2^{0,0,m-1}m)$, so the second order relations are

$$t^1\theta^1 = t^2\theta^2 = \theta^1\theta^3 - m\theta^2\theta^3 = mt^3\theta^3 + t^4\theta^1 - mt^4\theta^2 = 0,$$

and the second order deformation is given by

$$d^2 = d^1 - \varphi_2^{0,0,m-1} m^2 t^2 \theta^3.$$

We compute

$$\begin{aligned}\tfrac{1}{2}[d^2, d^2] = &- \psi_2^{1,0,0} t^1\theta^1 - \psi_3^{1,0,0} t^2\theta^2 + \varphi_2^{0,0,m}(\theta^1\theta^3 - m\theta^2\theta^3)\\ &- \psi_2^{1,0,m}(mt^3\theta^3 + t^4\theta^1 - mt^4\theta^2) + \psi_2^{1,0,m-1}(m-1)m^2t^2t^3\theta^3\\ &+ \psi_2^{1,0,m-2}(m-1)m^2(t^2)^2\theta^3\\ &+ \varphi_2^{0,0,m-1}(-\theta^1 + (m-1)\theta^2)m^2t^2\theta^3.\end{aligned}$$

Notice that $\varphi_2^{0,0,m-1}$ is not even a cocycle, so at first this may seem to be a problem, because we know the right hand side is a cocycle, and all the other terms are cocycles. The answer is that by using the second order relations we can see that the coefficient of this term is zero.

Second, note that the original coboundary term $\psi_2^{1,0,m-1}$ reappears, as well as the new coboundary term $\psi_2^{1,0,m-2}$. In fact, it is not hard to guess that as we take higher order deformations of d_c, eventually every coboundary of the form $\psi_2^{1,0,k}$ with $k \leqslant m-1$ will appear. It is not hard to guess that the miniversal deformation d^∞ is given by

$$d^\infty = d^1 + \sum_{k=0}^{m-1} \varphi_2^{1,0,k} x^k,$$

where x^k is a power series in t^3, multiplied by $(t^2)^{m-k}\theta^3$. To see this, suppose that d^∞ has the form above, and compute

$$\begin{aligned}\tfrac{1}{2}[d^\infty, d^\infty] = &- \psi_2^{1,0,0} t^1\theta^1 - \psi_3^{1,0,0} t^2\theta^2 + \varphi_2^{0,0,m}(\theta^1\theta^3 - m\theta^2\theta^3)\\ &- \psi_2^{1,0,m}(mt^3\theta^3 + t^4\theta^1 - mt^4\theta^2) + \sum_{k=1}^{m-1} \varphi_2^{1,0,k} x^k(-\theta^1 + k\theta^2)\\ &+ \psi_2^{1,0,m-1}(-mt^2\theta^3 + x^{m-1}(\tfrac{1}{m} - (m-1)t^3)\\ &+ \sum_{k=1}^{m-2} \psi_2^{1,0,k}((\tfrac{m-k}{m} - kt^3)x^k - (k+1)x^{k+1}).\end{aligned}$$

From this one obtains that

$$x^{m-1} = \frac{m^2t^2\theta^3}{1 - m(m-1)t^3} \qquad x^k = \frac{(k+1)mt^2x^{k+1}}{m-k-mkt^3}, \quad k < m-1,$$

The second order relations remain unmodified, and using them, we obtain that the term involving $\varphi_2^{1,0,k}$ is zero. Thus we obtain a closed form expression for the miniversal deformation, even though it cannot be obtained as a finite order deformation.

7.5. A miniversal deformation of d_0

Since there are an infinite number of cohomology classes, let us give them some simple labels. Let

$$\xi = \psi_2^{1,0,0}, \quad \sigma = \varphi_2^{0,1,0} \quad \psi_k = \psi_3^{1,0,k-1}, \quad \phi_k = \varphi_3^{0,0,k} \quad k \geqslant 1$$

The universal infinitesimal deformation is given by

$$d^1 = \xi s + \sigma\eta + \psi_k t^k + \phi_k \theta^k,$$

where s, t^k are even and η, θ^k are odd parameters. Then

$$\tfrac{1}{2}[d^1, d^1] = -\xi s\eta + \psi_{k+l-1}(k-l-1)t^k\theta^l + \tfrac{1}{2}\phi_{k+l-1}(k-l)\theta^k\theta^l.$$

Thus $[d^1, d^1]$ has no coboundary terms, so the universal infinitesimal deformation is miniversal, and the relations are given by

$$s\eta = 0$$
$$\sum_{k+l=n+1}(k-l-1)t^k\theta^l = \sum_{k+l=n+1}(k-l)\theta^k\theta^l = 0, \qquad n = 1\cdots$$

7.6. A miniversal deformation of d_c when c is a negative rational number

Decomposing $\frac{c}{c-1} = \frac{r}{s}$ as before, we obtain an infinite number of cohomology classes, which we label as follows.

$$\begin{aligned} \xi_1 &= \psi_2^{1,0,0} & \xi_2 &= \psi_3^{1,0,0} & \sigma_1 &= \varphi_2^{0,1,0} \\ \psi_k &= \psi_3^{1,kr,k(s-r)+1} & \phi_k &= \varphi_3^{0,kr,k(s-r)+1} \quad k \geqslant 0 \end{aligned}$$

The universal infinitesimal deformation is given by

$$d^1 = \xi_1 s^1 + \xi_2 s^2 + \sigma_1\eta^1 + \psi_k t^k + \phi_k\theta^k,$$

where s^i, t^i are even and η^1, θ^k are odd parameters. The bracket calculations needed to compute $[d^1, d^1]$ are

$$\begin{aligned} [\xi_1, \sigma_1] &= -\xi_1 & [\xi_1, \phi_k] &= -\psi_3^{1,kr-1,k(s-r)+1}kr \\ [\xi_2, \sigma_1] &= 0 & [\xi_2, \phi_k] &= -\psi_3^{1,kr,k(s-r)}(k(s-r)+1) \\ [\psi_k, \sigma_1] &= \psi_k kr & [\psi_k, \phi_l] &= \psi_{k+l}(k-l)(s-r) \\ [\sigma_1, \phi_k] &= -\phi_k kr & [\phi_k, \phi_l] &= \phi_{k+l}(k-l)(s-r) \end{aligned}$$

The coboundary terms appearing above are

$$\begin{aligned} -\psi_3^{1,kr-1,k(s-r)+1}kr &= D\left(\varphi_3^{0,kr-1,k(s-r)+1}kr\right), \quad k > 0 \\ -\psi_3^{1,kr,k(s-r)}(k(s-r)+1) &= D\left(\varphi_3^{0,kr,k(s-r)}\tfrac{(k(s-r)+1)(s-r)}{r}\right), \quad k > 0. \end{aligned}$$

The second order relations are

$$s^1\eta^1 = 0, \qquad s^2\theta^0 = 0$$
$$nrt^n\eta^1 + (s-r)\sum_{k+l=n} t^k\theta^l(k-l) = 0, \quad n = 0\cdots$$
$$-2nr\eta^1\theta^n + (s-r)\sum_{k+l=n}\theta^k\theta^l(k-l) = 0, \quad n = 0\cdots$$

The factor 2 appearing in the first summand of the second relation occurs because the terms $[\phi_k, \phi_l]$ appear only once in the bracket $[d^1, d^1]$ because they are like terms. (Note that all the other like terms have zero self-brackets.) The second order deformation is given by

$$d^2 = d^1 + \gamma_k s^1\theta^k + \epsilon_k s^2\theta^k,$$

where $\gamma_k = \varphi_3^{0,kr-1,k(s-r)+1}kr$ and $\epsilon_k = \varphi_3^{0,kr,k(s-r)}\frac{(k(s-r)+1)(s-r)}{r}$ for $k \geqslant 1$. In order to compute $[d^2, d^2]$ we need to calculate the following brackets.

$$\begin{aligned}
&[\xi_1, \gamma_k] = -\psi_3^{1,kr-2,k(s-r)+1}kr(kr-1)\\
&[\xi_2, \gamma_k] = -\psi_3^{1,kr-1,k(s-r)}kr(k(s-r)+1)\\
&[\sigma_1, \gamma_k] = -\varphi_3^{0,kr-1,k(s-r)+1}kr(kr-1)\\
&[\psi_k, \gamma_l] = -\psi_3^{1,(k+l)r-1,(k+l)(s-r)+1}lr(l-k)(s-r)\\
&[\phi_k, \gamma_l] = -\varphi_3^{0,(k+l)r-1,(k+l)(s-r)+1}lr(l-k)(s-r)\\
&[\gamma_k, \gamma_l] = \varphi_3^{0,(k+l)r-2,(k+l)(s-r)+1}r^2lk(l-k)(s-r)\\
&[\xi_1, \epsilon_k] = -\psi_3^{1,kr-1,k(s-r)}k(s-r)(k(s-r)+1)\\
&[\xi_2, \epsilon_k] = -\psi_3^{1,kr,k(s-r)-1}\tfrac{k(s-r)^2(k(s-r)+1)}{r}\\
&[\sigma_1, \epsilon_k] = -\varphi_3^{0,kr,k(s-r)}k(s-r)(k(s-r)+1)\\
&[\psi_k, \epsilon_l] = -\psi_3^{1,(k+l)r,(k+l)(s-r)}\tfrac{(l(s-r)+1)(s-r)((l-k)(s-r)-1)}{r}\\
&[\phi_k, \epsilon_l] = -\varphi_3^{0,(k+l)r,(k+l)(s-r)}\tfrac{(l(s-r)+1)(s-r)((l-k)(s-r)-1)}{r}\\
&[\epsilon_k, \epsilon_l] = \varphi_3^{0,(k+l)r,(k+l)(s-r)-1}\tfrac{(s-r)^3(k(s-r)+1)(l(s-r)+1)(l-k)}{r^2}\\
&[\gamma_k, \epsilon_l] = \varphi_3^{0,(k+l)r-1,(k+l)(s-r)}k(s-r)(l(s-r)+1)((l-k)(s-r)-1)
\end{aligned}$$

Note that all the odd terms appearing above are automatically cocycles, but none of the even terms are. Therefore, the sum of the even terms must be zero up to third order, using only the third order relations. Since none of the terms appearing above are cohomology classes, the third order relations are the same as the second order ones. Let us examine the even terms to see what is going on.

The term in $[\sigma_1, \gamma_k]$ will be multiplied by the parameters η^1 and $s^1\theta^k$, so it vanishes by the first relation $\eta^1 s^1 = 0$. To see why the terms arising in $[\phi_k, \gamma_l]$ cancel, note that we need to sum the terms with $k + l = n$. Adding the coefficients from the term $[\phi_k, \gamma_l]$ with that from $[\gamma_k, \phi_l]$, we get the coefficient $nr(l-k)(s-r)s^1\theta^k\theta^l$.

Thus

$$\sum_{k+l=n} lr(l-k)(s-r)s^1\theta^k\theta^l = \frac{1}{2}\sum_{k+l=n} nr(l-k)(s-r)s^1\theta^k\theta^l.$$

If you multiply the third relation by s^1, the first term drops out by the first relation, so it follows that the sum above is zero.

The terms $[\gamma_k, \gamma_l]$, $[\epsilon_k, \epsilon_l]$ and $[\gamma_k, \epsilon_l]$ are multiplied by four parameters, so are automatically zero up to third order. Thus we are left with considering the terms $[\sigma_1, \epsilon_k]$ and $[\phi_k, \epsilon_l]$. Note that these terms $[\sigma_1, \epsilon_n]$ and $[\phi_k, \epsilon_l]$ involve the same cochain when $k + l = n$. Consider the coefficient arising from $[\phi_l, \epsilon_k]$, which is $\frac{(k(s-r)+1)((l-k)(s-r)-1)s^2}{r}\theta^l\theta^k$. When we add this to $[\phi_k, \epsilon_l]$ we obtain the coefficient

$$-(k-l)(s-r)(n(s-r)+1)\theta^k\theta^l.$$

After multiplying this coefficient by $1/2$, summing, and adding the coefficient from $[\sigma, \epsilon_k]$, we obtain exactly zero by the third relation.

While all these cancellations seem to appear miraculously, it really is not necessary to carry out this verification. By the results in [**7**], the construction of the miniversal deformation which we are engaging in is guaranteed to work, so that any terms which arise in the brackets which are not cocycles must cancel. Therefore, we can ignore these terms and concentrate on the coboundary terms above.

Of the five types of odd coboundary terms appearing in $[d^2, d^2]$, there are three new types: $\psi_3^{1,kr-2,k(s-r)+1}$, $\psi_3^{1,kr-1,k(s-r)}$, and $\psi_3^{1,kr,k(s-r)-1}$, while the other two are coboundaries of γ and ϵ terms. The parameters for the new types are $(s^1)^2\theta^k$, $s^1s^2\theta^k$ and $(s^2)^2\theta^k$, and the powers of s^1 and s^2 which occur can be read by looking at how much the middle and third upper indices have decreased from the values kr and $k(s-r)+1$ in the cohomology class $\psi_3^{1,kr,k(s-r)+1}$. Notice that this observation is also true for γ_k and ϵ_k as well. Putting this information together, we now construct the miniversal deformation.

Let

$$\beta_{k,x,y} = \varphi_3^{0,kr-x,k(s-r)+1-y}, \quad 0 \leqslant x \leqslant kr, \quad 0 \leqslant y \leqslant k(s-r)+1.$$

Note that if $x \geqslant r$, then for $x' = x - r$, $k' = k + 1$ and $y' = y + s - r$, we have $\beta_{k,x,y} = \beta_{k+1,x',y'}$, so we only need to consider the case $0 \leqslant x < r$. Also, note that $\phi_k = \beta_{k,0,0}$. We claim that the miniversal deformation d^∞ of d_c is given by

$$d^\infty = \xi_1 s^1 + \xi_2 s^2 + \psi_k t^k + \sigma_1\eta^1 + \phi_0\theta^0 + \beta_{k,x,y}u^{k,x,y}\theta^k,$$

where we restrict ourselves to the case $0 \leqslant x < r$, where $u^{k,x,y}$ is a power series in the parameters, which we will determine by a recursive process. Since $\beta_{k,0,0} = \phi_k$, we know that $u^{k,0,0} = \theta^k$.

Consider the brackets

$$\begin{aligned}
[\xi_1, \beta_{k,x,y}] =& -\psi_3^{1,kr-x-1,k(s-r)+1-y}(kr-x)\\
[\xi_2, \beta_{k,x,y}] =& -\psi_3^{1,kr-x,k(s-r)-y}(k(s-r)+1-y) =\\
[\psi_k, \beta_{l,x,y}] =& \psi_3^{1,(k+l)r-x,(k+l)(s-r)+1-y}((k-l)(s-r)+y)\\
[\sigma_1, \beta_{k,x,y}] =& -\beta_{k,x,y}(kr-x)\\
[\phi_0, \beta_{k,x,y}] =& -\beta_{k,x,y}(k(s-r)-y)\\
[\beta_{k,x,y}, \beta_{l,u,v}] =& \beta_{k+l,x+u,y+v}((k-l)(s-r)-(y-v)).
\end{aligned}$$

Now

$$\begin{aligned}
-\psi_3^{1,kr-x-1,k(s-r)+1-y} =& D(\beta_{k,x+1,y})\tfrac{s-r}{(x+1)(s-r)-ry}\\
-\psi_3^{1,kr-x,k(s-r)-y} =& D(\beta_{k,x,y+1})\tfrac{s-r}{x(s-r)-r(y+1)}\\
-\psi_3^{1,(k+l)r-x,(k+l)(s-r)+1-y} =& D(\beta_{k+l,x,y})\tfrac{s-r}{x(s-r)-ry}
\end{aligned}$$

except, of course, when the denominators on the right hand side vanish. Since r and s are relatively prime, $(x+1)(s-r)-ry=0$ only when $x+1=r$ and $y=s-r$, in which case $\psi_3^{1,kr-x-1,k(s-r)+1-y} = \psi_{k-1}$. Since $x<r$, the denominator of the second fraction never vanishes. The denominator of the third fraction only vanishes when $x=y=0$, in which case, since $\beta_{l,0,0}=\varphi_l$, the bracket is just $\psi_{k+l}(k-l)(s-r)$ as computed earlier. From this, we compute

$$\begin{aligned}
\tfrac{1}{2}[d^\infty, d^\infty] =& -\xi_1 s^1\eta^1 - \xi_2 s^2\theta^0 + \psi_k krt^k\eta^1 + \psi_k k(s-r)t^k\theta^0\\
&- D(\beta_{k,x,y}u^{k,x,y}) + D(\beta_{k,x+1,y})\left(\tfrac{(kr-x)(s-r)}{(x+1)(s-r)-ry}\right)s^1u^{k,x,y}\\
&+ D(\beta_{k,x,y+1})\left(\tfrac{(k(s-r)+1-y)(s-r)}{x(s-r)-r(y+1)}\right)s^2u^{k,x,y}\\
&- D(\beta_{k+l,x,y})\left(\tfrac{((k-l)(s-r)+y)(s-r)}{x(s-r)-ry}\right)t^ku^{l,x,y}\\
&- \beta_{k,x,y}(kr-x)\eta^1u^{k,x,y} - \beta_{k,x,y}(k(s-r)-y)\theta^0u^{k,x,y}\\
&+ \tfrac{1}{2}\beta_{k+l,x+u,y+v}((k-l)(s-r)-(y-v)))u^{k,x,y}u^{l,u,v},
\end{aligned}$$

except for the cases when the denominators above vanish, *i.e.*when $x=r-1$ and $y=s-r$, then the first coboundary term above is replaced by $-\psi_{k+1}((k-1)r+1)s^2u^{k,r-1,s-r}$, and when $x=y=0$ the second coboundary term is replaced by $\psi_{k+l}(k-l)(s-r)t^k\theta^l$.

From the equation above, we can easily determine the relations on the base of the miniversal deformation. In fact, only one of the relations is modified from the second order relations. The $\beta_{k,0,0}$ terms are cohomology classes, so they give rise to relations, and the other cohomology classes are immediately identifiable. We obtain the relations

$$s^1\eta^1 = 0, \qquad s^2\theta^0 = 0$$

$$nrt^n\eta^1 - (nr+1)s^1u^{n+1,r-1,s-r} + (s-r)\sum_{k+l=n} t^k\theta^l(k-l) = 0, n = 0\cdots$$

$$-2nr\eta^1\theta^n + (s-r)\sum_{k+l=n}\theta^k\theta^l(k-l) = 0, \quad n = 0\cdots$$

Let us determine the coefficients $u^{k,x,y}$. We already know $u^{n,0,0}$. The coefficients of each coboundary term must vanish. First, when $x = 0$, by summing up all the terms involving $D(\beta_{n,0,y+1}$ we obtain

$$\begin{aligned}0 =&(n(s-r)+1-y)(s-r)s^2u^{n,0,y}\\ &-\sum_{k=1}^{n-1}((2k-n)(s-r)+y+1)(s-r)t^ku^{n-k,0,y+1}\\ &+(r(y+1)-(n(s-r)+y+1)t^0)u^{n,0,y+1},\end{aligned}$$

which determines $u^{n,0,y+1}$ in terms of $u^{n,0,y}$ and $u^{k,0,y+1}$ for $k < n$. Note that this solution is a power series in t^0, and is polynomial in the parameters s^2, t^k and θ^k for $0 < k < n$. Secondly, summing up all coefficients involving $D(\beta_{n,x+1,y})$ we obtain that

$$\begin{aligned}0 =&(nr-x-1)s^1u^{n,x,y} + (n(s-r)-y)s^2u^{n,x+1,y-1}\\ &-\sum_{k=1}^{n-1}((k-l)(s-r)+y)t^ku^{l,x+1,y}\\ &-\left(\tfrac{(x+1)(s-r)-ry}{s-r} + (n(s-r)+y)t^0\right)u^{n,x+1,y},\end{aligned}$$

which determines $u^{n,x+1,y}$ in terms of $u^{n,x,y}$, $u^{n,x+1,y-1}$ and the values of $u^{l,x+1,y}$ for $l < n$. Note that we obtain the solution for $u^{n,x+1,y}$ as a power series in t^0, and that it is polynomial in the parameters s^1, s^2, t^k, and θ^k, again for $1 < k < n$.

We have provided the details of the recursive construction here in order to illustrate the method involved. For subsequent examples we will not provide such complete details.

7.7. A miniversal deformation of d_*

Since there are a lot of cohomology classes, let us make some abbreviations for the cohomology classes. Let

$$\psi_k = \psi_2^{1,0,k-1} \qquad \xi_k = \psi_3^{1,0,k-1}$$

be the odd cohomology classes of degree k,

$$\phi_k = \varphi_3^{0,0,k} + \varphi_1^{1,0,k-1}k \qquad \sigma_k = \varphi_2^{0,k,0} + \varphi_3^{0,k-1,1}$$

be the even cohomology classes of degree k, and $\tau = \varphi_3^{0,1,0}$ be the extra even cohomology class in L_1. Then the universal infinitesimal deformation of d_* is given by

$$d^1 = d_* + \psi_k s^k + \xi_k t^k + \tau\zeta + \phi_k\theta^k + \sigma_k\eta^k,$$

where s^k and t^k are even parameters, and ζ, θ^k and η^k are odd parameters. Computing the brackets necessary to compute $[d^1, d^1]$, and comparing them to the coboundary calculations, we obtain

$$\begin{aligned}
[\psi_k, \tau] &= \left(\psi_2^{1,1,k-2}(k-1) - \psi_3^{1,0,k-1}\right) = \begin{cases} D(\varphi_2^{0,0,k-1}) & k > 1 \\ -\xi_1 & k = 1 \end{cases} \\
[\psi_k, \phi_l] &= \psi_{k+l-1}(k+l-1) \\
[\psi_k, \sigma_l] &= -\psi_2^{1,l-1,k-1}(l+1-k) - \psi_3^{1,l-2,k}(l-1) \\
&= \begin{cases} \psi_k(k-2) & l = 1 \\ D\left(-\varphi_2^{0,0,k}\left(\frac{3-k}{k}\right)\right) + \psi_{k+1}\frac{3-2k}{k} & l = 2 \\ D\left(-\varphi_3^{0,l-3,k+1}\left(\frac{kl-2k+l+1}{k(k+1)}\right) - \varphi_2^{0,l-2,k}\left(\frac{l+1-k}{k}\right)\right) & \text{otherwise} \end{cases} \\
[\xi_k, \tau] &= \psi_3^{1,1,k-2}(k-1) = \begin{cases} D(\varphi_3^{0,0,k-1}) & k > 1 \\ 0 & k = 1 \end{cases} \\
[\xi_k, \phi_l] &= \xi_{k+l-1}(k-1) \\
[\xi_k, \sigma_l] &= \psi_3^{1,l-1,k-1}(k-2) = \begin{cases} D(\varphi_3^{0,l-2,k})\frac{k-2}{k} & l > 1 \\ \xi_k(k-2) & l = 1 \end{cases} \\
[\tau, \phi_k] &= -(\varphi_1^{1,1,k-2}(k-1) + \varphi_3^{0,1,k-1}))k = \begin{cases} D(-\psi_1^{0,0,k-1})k & k > 1 \\ -\tau & k = 1 \end{cases} \\
[\tau, \sigma_k] &= 0 \\
[\phi_k, \phi_l] &= \phi_{k+l-1}(k-l) \\
[\phi_k, \sigma_l] &= \left(\varphi_1^{1,l-1,k-1}k + \varphi_3^{0,l-1,k}\right)(k-1) = \begin{cases} D(\psi_1^{0,l-2,k})(k-1) & l > 1 \\ \phi_k(k-1) & l = 1 \end{cases} \\
[\sigma_k, \sigma_l] &= \sigma_{k+l-1}(k-l)
\end{aligned}$$

The second order deformation is given by

$$\begin{aligned}
d^2 = &d^1 - \varphi_2^{0,0,k-1}s^k\zeta + \left(-\varphi_3^{0,l-3,k+1}\left(\frac{kl-2k+l+1}{k(k+1)}\right) - \varphi_2^{0,l-2,k}\left(\frac{l+1-k}{k}\right)\right)s^k\eta^l \\
&- \varphi_3^{0,0,k-1}t^k\zeta + \varphi_3^{0,l-2,k}\frac{2-k}{k}t^k\eta^l + \varphi_1^{0,0,k-1}\zeta\theta^k + \psi_1^{0,l-2,k}(1-k)\theta^k\eta^l.
\end{aligned}$$

The second order relations are

$$\begin{aligned}
&s^1\zeta + t^1\eta^1 = 0\\
&\zeta\theta^1 = 0\\
&-s^1\eta^1 + s^1\theta^1 = 0\\
&(n-2)s^n\eta^1 + \left(\tfrac{5-2n}{n-1}\right)t^{n-1}\eta^2 + n\sum_{k+l=n+1} s^k\theta^l = 0, \qquad n>1\\
&(n-2)t^n\eta^1 + \sum_{k+l=n+1}(k-1)t^k\theta^l = 0, \qquad n>1\\
&(n-1)\theta^n\eta^1 + \sum_{k+l=n+1}(k-l)\theta^k\theta^l = 0, n\geqslant 1\\
&\sum_{k+l=n}(k-l)\eta^k\eta^l = 0
\end{aligned}$$

In order to determine $[d^2, d^2]$, it is necessary to compute 51 additional brackets. Many of the brackets give coboundary terms, so it is clear that d^2 is far from the end of the story. The calculations do not seem especially illuminating, so we felt that it would not be useful to provide them here. It is not that surprising that the miniversal deformation of d_* is a complicated object, given the complexity of its cohomology.

8. Codifferentials of degree 2 of the second kind

A codifferential of degree 2 of the second kind is of the form

$$d = \psi_1^{0,2,0}a + \psi_1^{0,1,1}b + \psi_1^{0,0,2}c,$$

which we will say is of type (a, b, c). If either a or b is nonzero, then it is clearly equivalent to a codifferential of type $(1, b', c')$, for some b' and c'. Note that the only type which cannot be reduced in this way is type $(0, b, 0)$, which is clearly also of type $(0, 1, 0)$. However, let us examine type $(0, 1, 0)$ to see what it is equivalent to. Applying a standard linear automorphism, we obtain that $d = \psi_1^{0,1,1}$ is equivalent to any codifferential of the form

$$d' = \psi_1^{0,2,0}\frac{2rs}{q} + \psi_1^{0,1,1}\frac{ru+ts}{q} + \psi_1^{0,0,2}\frac{2tu}{q}, \qquad ru - ts \neq 0, q \neq 0.$$

If we set $q = 2rs$, $x = u/s$ $y = s/r$, then we obtain type $(1, b, c)$ where $2b = x + y$, $c = xy$, and we must avoid the condition $xy = 1$. But this occurs exactly when $b^2 = c$. Thus type $(0, 1, 0)$ is equivalent to type $(1, b, c)$ whenever $b^2 \neq c$. It is also clear that type $(0, 1, 0)$ is not equivalent to type $(1, 0, 0)$.

Let us next study type $(1, 0, 0)$. Applying a linear automorphism, we see that $\psi_1^{0,2,0}$ is equivalent to codifferentials of the form

$$d' = \psi_1^{0,2,0}\frac{r^2}{q} + \psi_1^{0,1,1}\frac{rt}{q} + \psi_1^{0,0,2}\frac{t^2}{q}.$$

If we set $r^2 = q$, then we see that this d is equivalent to any codifferential of the form $(1, b, b^2)$, exactly the types not covered by the first case.

Thus there are only two types of codifferentials, represented by $\psi_1^{0,2,0}$ and $\psi_1^{0,1,1}$. Let us study the second type first.

8.1. Type $(0, 1, 0)$

Let $D(\varphi) = [\varphi, \psi_1^{0,1,1}]$. Then we obtain the following table of coboundaries.

$$\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{0,1+q,n-q} \\
D(\varphi_2^{0,p,n-p}) &= -\psi_1^{0,p,n-p+1} \\
D(\varphi_3^{0,p,n-p}) &= -\psi_1^{0,p+1,n-p} \\
D(\psi_1^{0,p,n-p}) &= 0 \\
D(\psi_2^{1,q,n-q-1}) &= \varphi_1^{1,q,n-q} + \varphi_2^{0,q+1,n-q} \\
D(\psi_3^{1,q,n-q-1}) &= \varphi_1^{1,q+1,n-q-1} + \varphi_3^{0,q+1,n-q}
\end{aligned}$$

Then we have $n+1$ odd cocycles of the form $\psi_1^{0,p,n-p}$, and $2n$ even cocycles of the form $\varphi_1^{1,q,n-q} + \varphi_3^{0,q,n-q-1}$ and $\varphi_1^{1,q,n-q-1} + \varphi_2^{0,q+1,n-q-1}$ This means $z_n = 2n|n+1$, so $b_n = 2n|n+2$, and $h_n = 2|0$ if $n > 1$. We have

$$\begin{aligned}
H^1 &= \langle \psi_1^{0,0,1}, \psi_1^{0,1,0}, \varphi_1^{1,0,0} + \varphi_3^{0,0,1}, \varphi_1^{1,0,0} + \varphi_2^{0,1,0} \rangle \\
H^n &= \langle \varphi_1^{1,0,n-1} + \varphi_3^{0,0,n}, \varphi_1^{1,n-1,0} + \varphi_2^{0,n,0} \rangle, \qquad \text{if } n > 1
\end{aligned}$$

and all cohomology for $n > 1$ is even. Thus we don't obtain any deformations inside this L_∞ algebra.

8.2. Type $(1, 0, 0)$

Let $D(\varphi) = [\varphi, \psi_1^{0,2,0}]$. Then we obtain the following table of coboundaries.

$$\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{0,2+q,n-q-1} \\
D(\varphi_2^{0,p,n-p}) &= -2\psi_1^{0,p+1,n-p} \\
D(\varphi_3^{0,p,n-p}) &= 0 \\
D(\psi_1^{0,p,n-p}) &= 0 \\
D(\psi_2^{1,q,n-q-1}) &= 2\varphi_1^{1,q+1,n-q-1} + \varphi_2^{0,q+2,n-q-1} \\
D(\psi_3^{1,q,n-q-1}) &= \varphi_3^{0,q+2,n-q-1}
\end{aligned}$$

Besides the obvious $n+1$ odd cocycles $\psi_1^{0,p.n-p}$ and $n+1$ even ones $\varphi_3^{0,p,n-p}$, we also have n more even cocycles

$$2\varphi_1^{1,q,n-q-1} + \varphi_2^{0,q+1,n-q-1},$$

so $z_n = 2n+1|n+1$ and $h_n = 3|1$, if $n>1$. In fact, it is easily seen that

$$\begin{aligned} H^1 &= \langle \psi_1^{0,0,1}, \psi_1^{0,1,0}, \varphi_3^{0,0,1}, \varphi_3^{0,1,0}, 2\varphi_1^{1,0,0} + \varphi_2^{0,1,0} \rangle \\ H^n &= \langle \psi_1^{0,0,n}, \varphi_3^{0,0,n}, \varphi_3^{0,1,n-1}, 2\varphi_1^{1,0,n-1} + \varphi_2^{0,1,n-1}, \rangle, \quad \text{if } n>1 \end{aligned}$$

Let us think about the moduli space of two points given by these codifferentials. Note that type $(1,0,\epsilon)$ is the same as type $(0,1,0)$ for any nonzero value of ϵ. Thus type $(1,0,0)$ is infinitesimally close to $(0,1,0)$, but not the other way around. It is not surprising, therefore to see that type $(1,0,0)$ has a nontrivial deformation as a Lie algebra.

9. Miniversal deformations of degree 2 codifferentials of the second kind

9.1. Miniversal deformations of Type (0,1,0)

Let $d = \psi_1^{0,1,1}$, and let us label the cohomology classes as follows

$$\psi_1 = \psi_1^{0,0,1}, \quad \psi_2 = \psi_1^{0,1,0}, \ \phi_n = \psi_1^{1,0,n-1} + \psi_3^{0,0,n}, \quad \sigma_n = \varphi_1^{1,n-1,0} + \varphi_2^{0,n,0}$$

The universal infinitesimal deformation is given by

$$d^1 = \psi_1^{0,1,1} + \psi_1 s^1 + \psi_2 s^2 + \phi_n \theta^n + \sigma_n \eta^n,$$

where s^i are even parameters and θ^n and η^n are odd parameters. The brackets we need to compute are

$$\begin{aligned} [\psi_1, \phi_k] =& [\psi_2, \sigma_k] = [\phi_k, \sigma_l] = 0 \\ [\psi_1, \sigma_k] =& -\psi_1^{0,k-1,1} = \begin{cases} D(\varphi_2^{0,k-1,0}) & k>1 \\ -\psi_1 & k=1 \end{cases} \\ [\psi_2, \phi_k] =& -\psi_1^{0,1,k-1} = \begin{cases} D(\varphi_2^{0,1,k-2}) & k>1 \\ -\psi_2 & k=1 \end{cases} \\ [\phi_k, \phi_l] =& \phi_{k+l-1}(k-l), \quad [\sigma_k, \sigma_l] = \sigma_{k+l-1}(k-l) \end{aligned}$$

Thus the second order relations are

$$\begin{aligned} s^1\eta^1 &= s^2\theta^1 = 0 \\ \sum_{k+l=n+1} (k-l)\theta^k\theta^l &= 0, \quad n \geqslant 1 \\ \sum_{k+l=n+1} (k-l)\eta^k\eta^l &= 0, \quad n \geqslant 1 \end{aligned}$$

Let $\gamma_k = -\varphi_2^{0,k-1,0}$ and $\epsilon_k = -\varphi_2^{0,1,k-2}$. Then the second order deformation is

$$d^2 = d^1 + \gamma_k s^1 \eta^k + \epsilon_k s^2 \theta^k, k>1.$$

The brackets needed to compute $[d^2, d^2]$ are

$$\begin{aligned}
[\psi_1,\gamma_k] =&[\phi_k,\gamma_k] = [\psi_1,\epsilon_k] = [\epsilon_k,\epsilon_l] = 0\\
[\psi_2,\gamma_k] =&-\psi_1^{0,k-1,0} \qquad\qquad [\psi_2,\epsilon_k] = -\psi_1^{0,1,k-1}\\
[\gamma_k,\gamma_l] =&\varphi_2^{0,k+l-3,0}(k-l) \qquad [\phi_k,\epsilon_l] = \varphi_2^{0,1,k+l-2}(l-1)\\
[\sigma_k,\gamma_l] =&\varphi_1^{1,k+l-3,0}(1-k) + \varphi_2^{0,k+l-2,0}(l-k-1)\\
[\sigma_k,\epsilon_l] =&\varphi_1^{1,k-1,l-1}(1-k) + \varphi_2^{0,k,l-1}(1-k)
\end{aligned}$$

With the exception of the terms $[\psi_2, \gamma_2]$ and $[\sigma_k, \gamma_l]$, the terms above do not involve any cohomology classes. Most of the terms vanish, after taking into account the third order relations, which are the same as the second order relations except that the relation $s^2\theta^1 = 0$ becomes $s^2\theta^1 + s^1s^2\eta^1 = 0$. The modification of the second order relations by addition of higher order terms is a common pattern that occurs in the construction of miniversal deformations, as was illustrated by several examples in [**8**].

The coboundary terms which do not vanish are

$$\begin{aligned}
&-\psi_1^{0,k-1,0} = D\left(\varphi_3^{0,k-2,0}\right), k > 2\\
&-\psi_1^{0,1,k-1} = D\left(\varphi_3^{0,0,k-1}\right)\\
&\varphi_1^{1,k-1,l-1}(1-k) + \varphi_2^{0,k,l-1}(1-k) = D\left(\psi_2^{1,k-1,l-2}(1-k)\right).
\end{aligned}$$

Thus we have

$$d^3 = d^2 - \varphi_3^{0,k-2,0}s^1s^2\eta^k - \varphi_3^{0,0,k-1}(s^2)^2\theta^k - \psi_2^{1,k-1,l-2}(1-k)s^2\eta^k\theta^l.$$

The next step would involve calculating brackets of these three new terms with all the terms introduced in d^1 and d^2, as well as with each other. We did not see any simple pattern to the construction, but the reader may see something which eluded us. However, in the next example, we will give some more detail, showing how the process can be reduced to a recursion.

9.2. Miniversal deformations of Type (1,0,0)

Let $d = \psi_1^{0,2,0}$, and let us label the cohomology classes as follows

$$\begin{gathered}
\xi = \psi_1^{0,1,0}\\
\psi_n = \psi_1^{0,0,n}, \quad \phi_n = \varphi_3^{0,0,n}, n > 0\\
\sigma_n = \varphi_3^{0,1,n-1}, \quad \tau_n = 2\varphi_1^{1,0,n-1} + \varphi_2^{0,1,n-1}, \quad n > 0
\end{gathered}$$

The universal infinitesimal deformation is given by

$$d^1 = \psi_1^{0,2,0} + \xi s^1 + \psi_n t^n + \phi_n\theta^n + \sigma_n\eta^n + \tau_n\zeta^n,$$

where s^1 and t^n are even parameters and θ^n, η^n and ζ^n are odd parameters. The brackets we need to compute are

$$\begin{aligned}
[\xi,\phi_k] =&[\xi,\sigma_k] = [\tau_k,\tau_l] = 0\\
[\xi,\tau_k] =&-\psi_1^{0,1,k-1}, \qquad [\psi_k,\phi_l] = \psi_{k+l-1}k\\
[\psi_k,\sigma_l] =&\psi_1^{0,1,k+l-2}k, \qquad [\psi_k,\tau_l] = -2\psi_{k+l-1}\\
[\phi_k,\phi_l] =&\phi_{k+l-1}(k-l) \quad [\phi_k,\sigma_l] = \sigma_{k+l-1}(k-l+1)\\
[\phi_k,\tau_l] =&\tau_{k+l-1}(1-l), \quad [\sigma_k,\sigma_l] = \varphi_3^{0,2,k+l-3}(k-l)\\
[\sigma_k,\tau_l] =&2\varphi_1^{1,1,k+l-3}(1-l) + \varphi_2^{0,2,k+l-3}(1-l) + \varphi_3^{0,1,k+l-2}
\end{aligned}$$

Note that there are a few special cases:

$$[\xi,\tau_1] = -\xi, \quad [\psi_1,\sigma_1] = \xi, \quad [\sigma_1,\tau_1] = \sigma_1$$

Thus the second order relations are

$$\begin{gathered}
-s^1\zeta^1 + t^1\eta^1 = \theta^1\eta^1 + \eta^1\zeta^1 = 0\\
\sum_{k+l=n+1}(k-l-1)\theta^k\eta^l + \eta\zeta = 0, \quad n = 2\cdots\\
\sum_{k+l=n+1}(k-l)\theta^k\theta^l = 0, n = 1\cdots\\
\sum_{k+l=n+1}(1-l)\theta^k\zeta^l = \sum_{k+l=n+1}kt^k(\theta^l - 2\eta^l) = 0, n = 1\cdots
\end{gathered}$$

In order to set up the recursion relations, let

$$\gamma_{k,l} = \varphi_2^{0,k,l}, \quad \alpha_{k,l} = \psi_2^{1,k,l}, \quad \beta_{k,l} = \psi_3^{1,k,l}$$

The coboundary terms arising in $[d^1, d^1]$ can all be expressed in terms of coboundaries of these cochains.

$$\begin{aligned}
[\xi,\tau_k] &= -\tfrac{1}{2}D(\gamma_{0,k-1}), \quad k > 1\\
[\psi_k,\sigma_l] &= \tfrac{1}{2}D(\gamma_{0,k+l-2}), \quad k+l > 2\\
[\sigma_k,\sigma_l] &= D(\beta_{0,k+l-3})(k-l), \quad k+l > 2\\
[\sigma_k,\tau_l] &= D(\alpha_{0,k+l-3}(1-l)) + \sigma_{k+l-1}
\end{aligned}$$

It is also easy to express the brackets of the γ, β and α cochains with each other and all of the cohomology classes. The resulting cochains can either be expressed in terms of these cochains and cocycles, which makes it possible to find the recursion relations. We do not include the details of these calculations here, due to limitations of space.

10. Conclusions

In this paper, we have explored the construction of all linear and quadratic L_∞ structures on a $2|1$-dimensional space. We also provided some information about L_∞ structures in general on this space. We showed that there are two kinds of codifferentials, those of the first and second kinds , and that any codifferential is either a sum of cochains of the first kind, or a sum of cochains of the second kind, and every such sum is a codifferential. In this sense, we have described all L_∞ structures on a $2|1$ dimensional space. However, because we did not address equivalence in general, we are a long way from classification of the L_∞ structures.

For linear and quadratic L_∞ structures, we did give a complete classification, as well as classifying all L_∞ structures with a leading linear term (they are equivalent to the structure given by the leading term). For L_∞ structures with a leading quadratic term, it is necessary to classify the extensions of the quadratic codifferentials, which we leave to a separate paper. We have not studied the problem of classifying L_∞ structures with a leading term of degree 3 or higher, or even classified the structures of a fixed degree larger than 2.

In addition, we have shown how to construct a miniversal deformation for each of the equivalence classes of degree 2 codifferentials, providing complete details when the cohomology is not too complicated, and indicating some methods of computation in general. The main goal of this paper is not to provide an exhaustive description of a miniversal deformation, but rather to give the reader an idea of the process involved in its computation.

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