Homology, Homotopy and Applications, vol.4(2), 2002, pp.63–82

CROSSED EXTENSIONS OF ALGEBRAS AND HOCHSCHILD COHOMOLOGY

HANS-JOACHIM BAUES AND ELIAS GABRIEL MINIAN

(*communicated by Clas Löfwall*)

Abstract

We introduce the notion of crossed n-fold extensions of an algebra B by a bimodule M and prove that such extensions represent classes in the Hochschild cohomology of B with coefficients in M. Moreover we consider this way characteristic classes of chain (resp. cochain) algebras in Hochschild cohomology.

To Jan–Erik Roos on his sixty–fifth birthday

1. Introduction

Crossed modules over groups were introduced by J.H.C.Whitehead [**12**]. Mac Lane–Whitehead [**11**] observed that a crossed module over a group G with kernel a G-module M represents an element in the cohomology $H^3(G, M)$. This result was generalized by Huebschmann [**7**] by showing that crossed n-fold extensions over G by M represent elements in $H^{n+1}(G, M)$.

In this paper we prove similar results for the Hochschild cohomology $HH^{n+1}(B, M)$ of an algebra B with coefficients in a B-bimodule M. We show that *crossed modules over algebras* as introduced in [**2**] can be used to define crossed n-fold extensions of B by M which represent elements in $HH^{n+1}(B, M)$ for $n \geqslant 2$.

Our results are also available for graded algebras. In particular we show that each chain (resp. cochain) algebra C yields canonically a crossed module over the homology (resp. cohomology) algebra $B = HC$ and this crossed module represents a characteristic class $\langle C\rangle$ in the Hochschild cohomology of HC. The characteristic class $\langle C\rangle$ determines all triple Massey products which are secondary operations on HC determined by C. We can consider $\langle C\rangle$ as an analogue of the first k-invariant of a connected space X (in the Postnikov decomposition) which is an element in the cohomology of the fundamental group $G = \pi_1 X$. Berrick–Davydov [**6**] recently studied the class $\langle C\rangle$ without using crossed modules over algebras. We compute also the characteristic class $\langle A \otimes B\rangle$ of the tensor product of chain algebras A and B.

Received November 3, 2000, revised August 21, 2001; published on July 12, 2002.
2000 Mathematics Subject Classification: 18G99.
Key words and phrases: Hochschild cohomology, Massey products, crossed module.

2. Hochschild cohomology

Let k be a field. Classical Hochschild cohomology is defined for algebras and also for graded algebras over k. We consider here the graded and the non-graded case at the same time. In this paper an algebra B will mean an associative (graded) algebra with unit $k \to B$. A B-bimodule is a (graded) k-vector space V which is a left and a right B-module such that for $a, b \in B$ and $x \in V$ we have $(ax)b = a(xb)$. For example B can be considered as a B-bimodule via the multiplication in B. Given two (graded) k-vector spaces V and W we denote the tensor product $V \otimes_k W$ simply by $V \otimes W$.

Recall that the *Hochschild cohomology* of B with coefficients in a B-bimodule M is the family of extension groups

$$HH^*(B, M) = \mathrm{Ext}^*_{B-B}(B, M) \tag{2.1}$$

between the B-bimodules B and M.

One can associate to B the *bar complex* $B_*(B)$, where $B_n(B) = B^{\otimes(n+2)}$ with differential $d : B_n(B) \to B_{n-1}(B)$ given by

$$d(x_0 \otimes \ldots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i (x_0 \otimes \ldots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \ldots \otimes x_{n+1})$$

The bar complex is acyclic for any B. This follows from the existence of a homotopy h between the identity of $B_*(B)$ and the zero map. The homotopy $h : B_n(B) \to B_{n+1}(B)$ is defined by $h(x) = 1 \otimes x$.

The differential of the bar complex is B-bilinear. Thus we get the *standard* resolution of the B-bimodule B. Using this resolution one can identify the cohomology groups $HH^n(B, M)$ with the cohomology of the complex

$$F^n(B, M) = \mathrm{Hom}_{B-B}(B^{\otimes(n+2)}, M) = \mathrm{Hom}_k(B^{\otimes n}, M) \tag{2.2}$$

with differential $\delta : F^n(B, M) \to F^{n+1}(B, M)$ given by

$$\begin{aligned}(\delta f)(x_1 \otimes \ldots \otimes x_{n+1}) =& x_1 f(x_2 \otimes \ldots \otimes x_{n+1}) \\ &+ \left(\sum_{i=1}^{n} (-1)^i f(x_1 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x_{n+1}) \right) \\ &+ (-1)^{n+1} f(x_1 \otimes \ldots \otimes x_n) x_{n+1}.\end{aligned}$$

3. Crossed modules over algebras and HH^3

We recall from [**2**] the following definition of crossed modules over algebras.

Definition 3.1. A *Crossed module over a k-algebra* is a triple (V, A, ∂) where A is a (graded) k-algebra, V is a (graded) A-bimodule and $\partial : V \to A$ is a map of A-bimodules such that $(\partial v)w = v(\partial w)$ for $v, w \in V$. A map $(\alpha, \beta) : (V, A, \partial) \to (V', A', \partial')$ between crossed modules consists of a map $\alpha : V \to V'$ of k-vector spaces and a map $\beta : A \to A'$ of k-algebras such that $\partial' \alpha = \beta \partial$ and $\alpha(avb) = \beta(a)\alpha(v)\beta(b)$ for $a, b \in A$ and $v \in V$.

Given a crossed module $\partial : V \to A$, we consider $B = \operatorname{coker}(\partial)$ and $M = \ker(\partial)$ in the category of (graded) k-vector spaces. The algebra structure of A induces an algebra structure on B and the A-bimodule structure on V induces a B-bimodule structure on M given by $\pi(a)m = am$ and $m\pi(a) = ma$ where $\pi : A \to \operatorname{coker}(\partial)$ is the projection. This multiplication is well defined since $(a+\partial(v))m = am+\partial(v)m = am + v\partial(m) = am$. Hence a crossed module yields an exact sequence

$$0 \longrightarrow M \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{\pi} B \longrightarrow 0$$

in which all the maps are maps of A-bimodules. Here the A-bimodule structure on M and B is induced by the map π. We call $\partial : V \to A$ a crossed module over the k-algebra B with kernel M. Let $\operatorname{Cross}(B, M)$ be the category of such crossed modules. Morphisms are maps between crossed modules which induce the identity on M and B.

For a category C, let $\pi_0 C$ be the class of connected components in C. An object in $\pi_0 C$ is also termed a *connected class* of objects in C. In fact, $\pi_0 \operatorname{Cross}(B, M)$ is a set, as implied by the following result, which extends the well-known facts that $HH^1(B, M)$ is given by *derivations* and $HH^2(B, M)$ classifies the *singular algebra extensions of M by B* (cf. [**10**]). The proof of this result can also be found in [**9**].

Theorem 3.2. *There exists a bijection*

$$\psi : \pi_0 \operatorname{Cross}(B, M) \to HH^3(B, M).$$

Proof. We define $\psi : \pi_0 \operatorname{Cross}(B, M) \to HH^3(B, M)$ as follows. Given

$$\mathcal{E} = (\ 0 \longrightarrow M \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{\pi} B \longrightarrow 0\ )$$

choose k-linear sections $s : B \to A$, $\pi s = 1$ and $q : \operatorname{Im}(\partial) \to V$, $\partial q = 1$. For $x, y \in B$, we have $\pi(s(x)s(y) - s(xy)) = 0$ and then $s(x)s(y) - s(xy) \in \operatorname{Im}(\partial)$. Take $g(x, y) = q(s(x)s(y) - s(xy)) \in V$ and define

$$\theta_{\mathcal{E}}(x, y, z) = s(x)g(y, z) - g(xy, z) + g(x, yz) - g(x, y)s(z) \qquad (\S)$$

Since ∂ is a map of A-bimodules it follows that $\partial(\theta_{\mathcal{E}}(x, y, z)) = 0$ and therefore $\theta_{\mathcal{E}}(x, y, z) \in M = \ker(\partial)$. Thus we have defined a k-linear map $\theta_{\mathcal{E}} : B^{\otimes 3} \to M$ which is a cocycle with respect to the coboundary map δ in (2.2). In fact, one easily checks that $\delta(\theta_{\mathcal{E}}) = 0$. We define $\psi : \pi_0 \operatorname{Cross}(B, M) \to HH^3(B, M)$ by taking $\psi(\mathcal{E})$ to be the class of $\theta_{\mathcal{E}}$ in $HH^3(B, M)$.

One has to check that ψ is a well defined function from $\pi_0 \operatorname{Cross}(B, M)$ to $HH^3(B, M)$, i.e. the class of $\theta_{\mathcal{E}}$ in $HH^3(B, M)$ does not depend on the sections s and q. Moreover, if $\mathcal{E} \to \mathcal{E}'$ is a map in $\operatorname{Cross}(B, M)$, then $\theta_{\mathcal{E}} = \theta_{\mathcal{E}'}$ in $HH^3(B, M)$.

We show first that the class of $\theta_{\mathcal{E}}$ does not depend on the section s. Suppose $\overline{s} : B \to A$ is another section of π and let $\overline{\theta}_{\mathcal{E}}$ be the map defined using $\overline{s}$ instead of s. Since s and $\overline{s}$ are both sections of π there exists a linear map $h : B \to V$ with $s - \overline{s} = \partial h$. We have

$$\begin{aligned}(\theta_{\mathcal{E}} - \overline{\theta}_{\mathcal{E}})(x, y, z) = &h(x)(s(y)s(z) - s(yz)) - (s(x)s(y) - s(xy))h(z)\\ &+ \overline{s}(x)(g - \overline{g})(y, z) - (g - \overline{g})(xy, z)\\ &+ (g - \overline{g})(x, yz) - (g - \overline{g})(x, y)\overline{s}(z) \qquad (*)\end{aligned}$$

where $g(x,y) = q(s(x)s(y) - s(xy))$ and $\overline{g}(x,y) = q(\overline{s}(x)\overline{s}(y) - \overline{s}(xy))$. We define a map $b : B^{\otimes 2} \to V$ as follows.

$$b(x,y) = s(x)h(y) - h(xy) + h(x)s(y) - h(x)\partial h(y)$$

Since $\partial b = \partial(g - \overline{g})$ then $(g - \overline{g} - b)$ is a map from $B^{\otimes 2}$ to M. Moreover we can replace $(g - \overline{g})$ by b without changing the equality $(*)$ in $HH^3(B,M)$ since the difference is the coboundary $\delta(g - \overline{g} - b)$. After replacing $(g - \overline{g})$ by b, since $\partial : V \to A$ is a crossed module we obtain the following equality in $HH^3(B,M)$.

$$\begin{aligned}(*) \equiv &\partial h(x)s(y)h(z) - h(x)s(y)\partial h(z) - \partial h(x)h(yz) + h(x)\partial h(yz)\\ &+ \partial h(x)h(y)\overline{s}(z) - h(x)\partial h(y)\overline{s}(z) = 0\end{aligned}$$

That proves that the class of $\theta_{\mathcal{E}}$ does not depend on the section s.

Consider a map $\mathcal{E} \to \mathcal{E}'$ as follows.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & V \arrow[r, "\partial"] \arrow[d, "\alpha"'] & A \arrow[r, "\pi"] \arrow[d, "\beta"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "i'"] & V' \arrow[r, "\partial'"] & A' \arrow[r, "\pi'"] & B \arrow[r] & 0 \end{tikzcd}$$

Let $s : B \to A$ and $q : \mathrm{Im}(\partial) \to V$ be sections of π and ∂ and let $s' : B \to A'$ and $q' : \mathrm{Im}(\partial') \to V'$ be sections of π' and ∂'. Then

$$\begin{aligned}(\theta_{\mathcal{E}} - \theta_{\mathcal{E}'})(x,y,z) = &\alpha(s(x)q(s(y)s(z) - s(yz)) - q(s(xy)s(z) - s(xyz))\\ &+ q(s(x)s(yz) - s(xyz)) - q(s(x)s(y) - s(xy))s(z))\\ &- s'(x)q'(s'(y)s'(z) - s'(yz)) + q'(s'(xy)s'(z) - s'(xyz))\\ &- q'(s'(x)s'(yz) - s'(xyz)) + q'(s'(x)s'(y) - s'(xy))s'(z) \qquad (*)\end{aligned}$$

Since $\pi'\beta s = 1$ then βs is another section for π' and therefore we can now replace s' by βs and we obtain the following equality in $HH^3(B,M)$.

$$\begin{aligned}(*) \equiv &\beta s(x)((\alpha q - q'\beta)(s(y)s(z) - s(yz))) - (\alpha q - q'\beta)(s(xy)s(z).s(xyz))\\ &+ (\alpha q - q'\beta)(s(x)s(yz) - s(xyz)) - (\alpha q - q'\beta)(s(x)s(y) - s(xy))\beta s(z)\end{aligned}$$

Thus $(\theta_{\mathcal{E}} - \theta_{\mathcal{E}'})(x,y,z) = \delta\phi(x,y,z)$ for some $\phi : B^{\otimes 2} \to M$. This proves that $\theta_{\mathcal{E}} = \theta_{\mathcal{E}'}$ in $HH^3(B,M)$ and that the class of $\theta_{\mathcal{E}}$ in $HH^3(B,M)$ does not depend on the sections s and q. Therefore ψ is well defined.

The bijectivity of ψ follows from the following lemma.

□

Lemma 3.3. *Given $c \in HH^3(B,M)$ there exists a crossed module $\mathcal{E}_c \in \mathrm{Cross}(B,M)$ such that $\theta_{\mathcal{E}_c} = c$. Moreover if $\mathcal{E} \in \mathrm{Cross}(B,M)$ and $\theta_{\mathcal{E}} = c$ in $HH^3(B,M)$ there exists a map of crossed modules $\mathcal{E}_c \to \mathcal{E}$.*

Before we proceed with the proof, we show a construction which will be useful to prove the lemma.

Construction 3.4. *Free crossed module.* Given a (graded) k-algebra A, a (graded) k-vector space V and a k-linear map $d : V \to A$ (of degree 0) we obtain the *free* *crossed module* with basis (V, d) as follows. Define

$$A \otimes V \otimes A \otimes V \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A$$

by

$$d_2(a \otimes x \otimes b \otimes y \otimes c) = (a(dx)b \otimes y \otimes c) - (a \otimes x \otimes b(dy)c)$$
$$d_1(a \otimes x \otimes b) = a(dx)b$$

for $a, b, c \in A$ and $x, y \in V$. Since $d_1 d_2 = 0$ then d_1 induces

$$\partial : W = A \otimes V \otimes A/\mathrm{Im}(d_2) \to A.$$

It is easy to see that (W, A, ∂) is a crossed module which has the universal property of the free crossed module with basis (V, d).

Proof of Lemma 3.3. Let

$$T(B) = \bigoplus_{n \geqslant 0} B^{\otimes n}$$

be the tensor algebra generated by B as a k-vector space and let $\pi : T(B) \to B$ be the map of algebras given by $\pi(a_1 \otimes \ldots \otimes a_n) = a_1 \ldots a_n$. Let $V = B \otimes B$ and let $d : V \to T(B)$ be the linear map defined by

$$d(x \otimes y) = x \otimes y - xy$$

and consider $(W, T(B), \partial)$ the free crossed module with basis (V, d). The cokernel of this crossed module is the algebra B. Let N be the kernel of ∂.

Now consider the bar resolution $(B^{\otimes(n+2)}, d)$ and the following commutative diagram of vector spaces

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "h"] & W \arrow[r, "\partial"] \arrow[d, "h_1"] & T(B) \arrow[r] \arrow[d, "h_0"] & B \arrow[r] & 0 & \\
0 \arrow[r] & B^{\otimes 5}/\mathrm{Im}(d_4) \arrow[r, "d_3"] & B^{\otimes 4} \arrow[r, "d_2"] & B^{\otimes 3} \arrow[r, "d_1"] & B^{\otimes 2} \arrow[r, "\mu"] & B \arrow[r] & 0
\end{tikzcd}$$

Here the map $h_0 : T(B) \to B^{\otimes 3}$ is not a bimodule map but a derivation defined by $h_0(b) = 1 \otimes b \otimes 1$ for $b \in B$ and $h_0(xy) = \pi(x)h_0(y) + h_0(x)\pi(y)$ for $x, y \in T(B)$. The map $h_1 : W \to B^{\otimes 4}$ is the bimodule map defined by $h_1(x \otimes a \otimes b \otimes y) = \pi(x) \otimes a \otimes b \otimes \pi(y)$ for $x, y \in T(B)$ and $a, b \in B$. It is easy to see that $d_2 h_1 = h_0 \partial$. By restricting h_1 to N we obtain the map of B-bimodules $h : N \to B^{\otimes 5}/\mathrm{Im}(d_4)$.

An element $c \in HH^3(B, M)$ can be seen as a map $c : B^{\otimes 5}/\mathrm{Im}(d_4) \to M$ of B-bimodules. Composing with h we obtain the map $\tilde{c} = ch : N \to M$ of B-bimodules.

Consider the pushout of $T(B)$-bimodules

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\tilde{c}"'] \arrow[dr, phantom, "Push"] & W \arrow[d, "r"] \arrow[ddr, bend left, "\partial"] & \\ M \arrow[r] \arrow[drr, bend right, "0"'] & \overline{W} \arrow[dr, dotted, "\overline{\partial}"] & \\ & & T(B) \end{tikzcd} .$$

We show that $\mathcal{E}_c = (\ 0 \longrightarrow M \longrightarrow \overline{W} \xrightarrow{\overline{\partial}} T(B) \longrightarrow B \longrightarrow 0\ )$ is the desired crossed module.

The free crossed module $\mathcal{F} = (\ 0 \longrightarrow N \longrightarrow W \xrightarrow{\partial} T(B) \longrightarrow B \longrightarrow 0\ )$ induces a cocycle $\theta_{\mathcal{F}} : B^{\otimes 5}/\mathrm{Im}(d_4) \to N$ as a map of B-bimodules by the formula (§) via the linear sections $s : B \to T(B)$ given by $s(b) = b$ and $q : \mathrm{Im}(\partial) = \ker(\pi) \to W$ defined by $q(x \otimes y - xy) = \overline{1 \otimes x \otimes y \otimes 1} \in W$. Now $\theta_{\mathcal{E}_c}$ can be computed from $\theta_{\mathcal{F}}$ since the map $\overline{q} = rq : \mathrm{Im}(\overline{\partial}) = \ker(\pi) \to \overline{W}$ is a section of $\overline{\partial}$, that is

$$\theta_{\mathcal{E}_c} = ch\theta_{\mathcal{F}} : B^{\otimes 5}/\mathrm{Im}(d_4) \to M.$$

Since $h\theta_{\mathcal{F}} = 1 : B^{\otimes 5}/\mathrm{Im}(d_4) \to B^{\otimes 5}/\mathrm{Im}(d_4)$ it follows that $\theta_{\mathcal{E}_c} = c$.

Suppose now we have a crossed module

$$\mathcal{E} = (\ 0 \longrightarrow M \longrightarrow V \xrightarrow{\alpha} A \xrightarrow{p} B \longrightarrow 0\ )$$

such that $\psi(\mathcal{E}) = c \in HH^3(B, M)$. That implies that for certain choice of linear sections $s : B \to A$ and $q : \mathrm{Im}(\alpha) \to V$ we have $\theta_{\mathcal{E}} = c$ where $\theta_{\mathcal{E}}$ is constructed by the formula (§). This induces a map of crossed modules

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "ch"'] & W \arrow[r, "\partial"] \arrow[d, "\overline{g}"'] & T(B) \arrow[r, "\pi"] \arrow[d, "\overline{s}"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & V \arrow[r] & A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where the map $\overline{s} : T(B) \to A$ is the map of algebras induced by s and the map $\overline{g} : W \to V$ is induced by the linear map $g : B^{\otimes 2} \to V,\ g(x, y) = q(s(x)s(y) - s(xy))$. By the universal property of the pushout

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\tilde{c}"'] \arrow[dr, phantom, "Push"] & W \arrow[d, "r"] \\ M \arrow[r] & \overline{W} \end{tikzcd} ,$$

there is a map $\mathcal{E}_c \to \mathcal{E}$ in $\mathrm{Cross}(B, M)$. □

Any differential graded k-algebra induces a crossed module as we can see in the following construction.

Construction 3.5. *The characteristic class of a cochain algebra.* Let C be a differential graded k-algebra with differential of degree $+1$, that is $C = \bigoplus_{i \geqslant 0} C^i$ with $C^i C^j \subseteq C^{i+j}$ and a differential $d : C \to C$ of degree $+1$ satisfying $d(xy) = (dx)y + (-1)^{|x|} x d(y)$ and $d^2 = 0$. Consider the graded k-vector spaces $V = \operatorname{coker}(d)[1]$ and $A = \ker(d)$. Here we define for a graded vector space W the *shifted* graded vector space $W[1]$ by

$$(W[1])^{n+1} = W^n$$

The elements in $(W[1])^{n+1}$ are denoted by $s(w)$, where $w \in W^n$. Hence for the cokernel of the differential $W = \operatorname{coker}(d)$ we obtain the shifted object $V = \operatorname{coker}(d)[1]$. We denote by $s(\overline{x}) \in \operatorname{coker}(d)[1]$ the element corresponding to $x \in C$ via the projection $C \twoheadrightarrow \operatorname{coker}(d)$. Then d induces a map of graded k-vector spaces

$$\partial : V = \operatorname{coker}(d)[1] \to A = \ker(d)$$

carrying $s(\overline{x})$ to dx. The multiplication in C induces a structure of k-algebra on A. Moreover it induces a structure of A-bimodule on V by setting

$$a * s(\overline{x}) = (-1)^{|a|} s(\overline{ax})$$
$$s(\overline{x}) * b = s(\overline{xb})$$

In fact, for $y = dz$ and $a \in A$ we have $(-1)^{|a|} ay = d(az)$ and therefore the multiplication is well defined. We now check that $\partial : V \to A$ is a crossed module. Given $a \in A$ and $s(\overline{x}) \in V$ we have

$$\partial(a * s(\overline{x})) = (-1)^{|a|} \partial(s(\overline{ax})) = (-1)^{|a|} d(ax) = ad(x) = a\partial(s(\overline{x}))$$

In the same way one can check that $\partial(s(\overline{x}) * a) = \partial(s(\overline{x}))a$. Given now $s(\overline{x}), s(\overline{y}) \in V$ we have

$$\partial(s(\overline{x})) * s(\overline{y}) = (dx) * s(\overline{y}) = (-1)^{|x|+1} s(\overline{(dx)y}) = s(\overline{x(dy)}) = s(\overline{x}) * (dy) =$$

$$s(\overline{x}) * \partial(s(\overline{y})).$$

Thus the DG-algebra C induces a crossed module (V, A, ∂), the cokernel of which is the algebra $H^*(C)$ and the kernel is the $H^*(C)$-bimodule whose underlying k-vector space is $H^*(C)[1]$ and where the left multiplication is twisted, i.e. $x * y = (-1)^{|x|} xy$ and the right multiplication is the ordinary one. We denote this $H^*(C)$-bimodule by $\overline{H^*(C)}[1]$. The crossed module $\partial : V \to A$ represents by 3.2 an element $\langle C \rangle \in HH^3(H^*(C), \overline{H^*(C)}[1])$ which is termed the *characteristic class* of the cochain algebra C (compare with [**6**]).

Construction 3.6. *The characteristic class of a chain algebra.* For a chain algebra $C = \{C_i,\ i \geqslant 0\}$ concentrated in non-negative degrees with differential $d : C \to C$ of degree -1 satisfying $d(xy) = (dx)y + (-1)^{|x|} x d(y)$ and $d^2 = 0$ we proceed in the same way as in 3.5. We consider the graded vector spaces $V = \operatorname{coker}(d)_{\geqslant 1}[-1]$ and $A = \ker(d)$. Here $\operatorname{coker}(d)_{\geqslant 1}[-1]$ denotes the shifted graded vector space from $\operatorname{coker}(d)_{\geqslant 1}$ similarly as above. The differential d induces as before a crossed module

$$\partial : V = \operatorname{coker}(d)_{\geqslant 1}[-1] \to A = \ker(d)$$

the cokernel of which is the algebra $H_*(C)$ and the kernel is the $H_*(C)$-bimodule whose underlying vector space is $H_{\geqslant 1}(C)[-1]$ and where the left multiplication is twisted and the right multiplication is the ordinary one. We denote this bimodule by $\overline{H_{\geqslant 1}(C)}[-1]$. The crossed module represents by 3.2 an element $\langle C\rangle \in HH^3(H_*(C), \overline{H_{\geqslant 1}(C)}[-1])$ which is termed the characteristic class of the chain algebra C.

Defination and Remark 3.7. *Massey triple products of crossed modules.* Let

$$\mathcal{E} = (\ 0 \longrightarrow M \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{\pi} B \longrightarrow 0\ )$$

be a crossed module over B with kernel M. Given $a, b, c \in B$ with $ab = bc = 0$, we define the *Massey triple product* $\langle a, b, c\rangle \in M/(aM + Mc)$ as follows. Let $s : B \to A$ be a k-linear section of π, i.e. $\pi s = 1$ and let $q : \mathrm{Im}(\partial) \to V$ be a k-linear section of ∂. Since $ab = 0$ then $s(a)s(b) \in \ker(\pi)$ and we can take $q(s(a)s(b)) \in V$. In the same way, since $bc = 0$, we consider $q(s(b)s(c)) \in V$. Now consider the element $\{a, b, c\} = s(a)q(s(b)s(c)) - q(s(a)s(b))s(c) \in V$. Since $\partial(\{a, b, c\}) = 0$ this element is in fact in M and we define

$$\langle a, b, c\rangle = \overline{\{a, b, c\}} \in M/(aM + Mc),$$

where $\overline{\{a, b, c\}}$ denotes the class of $\{a, b, c\}$ in the quotient. One can check that $\langle a, b, c\rangle$ does not depend on the choice of s and q. Moreover it depends only on the class of $\mathcal{E}$ in $\pi_0\,\mathrm{Cross}(B, M)$ and the elements a, b and c. In fact $\langle a, b, c\rangle$ can be computed from $HH^3(B, M)$ by taking

$$\langle a, b, c\rangle = \overline{\theta_{\mathcal{E}}(a, b, c)}$$

where $\theta_{\mathcal{E}}$ is any cocycle representing the class of $\psi(\mathcal{E}) \in HH^3(B, M)$. Note that for any DG-algebra C and any Massey triple $x, y, z \in H^*(C)$ the Massey product defined here in terms of ∂ in 3.5 coincides with the classical one.

Remark 3.8. *Connection with Baues–Wirsching cohomology of categories.* Given a monoid C one can consider C as a category $\mathcal{C}$ with one object $*$. Let $M : \mathcal{C} \times \mathcal{C}^{op} \to \mathrm{Vect}_k$ be a functor, where Vect_k denotes the category of k-vector spaces. Then the $\mathcal{C}$-bimodule M induces a *natural system* on $\mathcal{C}$ also denoted by M (see [**5**]) and we have the Baues–Wirsching cohomology groups of $\mathcal{C}$ with coefficients in M denoted by $H^n(\mathcal{C}, M)$. On the other hand one can consider the k-algebra $k[C]$ and the $k[C]$-bimodule iM induced by the $\mathcal{C}$-bimodule M. It is easy to see that $HH^n(k[C], iM) = H^n(\mathcal{C}, M)$. For $n = 3$ this isomorphism induces a bijection

$$\pi_0\mathrm{Track}(\mathcal{C}, M) = \pi_0\,\mathrm{Cross}(k[C], iM).$$

Here $\mathrm{Track}(\mathcal{C}, M)$ denotes the category of *track extensions* over $\mathcal{C}$ with kernel M (cf. [**3**],[**4**]).

In the last section of this paper we define the $\odot$-product of crossed modules in order to compute the characteristic class of a tensor product of differential algebras.

4. crossed n-fold extensions and main result

We introduce in this section the groups $\mathrm{Opext}^n(B,M)$ of *crossed n-fold extensions* of a k-algebra B by a B-bimodule M, $n \geqslant 2$. These extensions are analogous to crossed extensions of groups (cf. [**7**]). Our result 4.3 shows that the connected classes of such extensions represent cohomology classes in $HH^{n+1}(B,M)$.

Definition 4.1. Let B be a k-algebra and M a B-bimodule. For $n \geqslant 2$, a *crossed n-fold extension of B by M* is an exact sequence

$$0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} A \xrightarrow{\pi} B \to 0$$

of k-vector spaces with the following properties.

1. (M_1, A, ∂_1) is a crossed module with cokernel B,
2. M_i is a B-bimodule for $1 < i \leqslant n-1$ and ∂_i and f are maps of B-bimodules.

Note that the map ∂_1 is a map of A-bimodules since (M_1, A, ∂_1) is a crossed module and it makes sense to require ∂_2 to be a map of B-bimodules since the kernel of ∂_1 is naturally a B-bimodule.

Definition 4.2. Given a crossed n-fold extension of B by M

$$\mathcal{E} = (\ 0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} A \xrightarrow{\pi} B \to 0\ )$$

and a crossed n-fold extension of B by M'

$$\mathcal{E}' = (\ 0 \to M' \xrightarrow{f'} M'_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} M'_1 \xrightarrow{\partial'_1} A' \xrightarrow{\pi'} B \to 0\ )$$

a map from $\mathcal{E}$ to $\mathcal{E}'$ is a sequence $(\alpha, \delta_{n-1}, \ldots, \delta_1, \beta)$ such that $\alpha : M \to M'$ and $\delta_i : M_i \to M'_i$ are morphisms of B-bimodules for $i \geqslant 2$, $(\delta_1, \beta) : (M_1, A, \partial_1) \to (M'_1, A', \partial'_1)$ is a map of crossed modules which induces the identity on B and the whole diagram commutes.

Let $\mathcal{E}^n(B,M)$ be the following category. The objects are the crossed n-fold extensions of B by M and the morphisms are the maps between such extensions that induce the identity on M. We denote $\mathrm{Opext}^n(B,M) = \pi_0\mathcal{E}^n(B,M)$. Of course $\mathrm{Opext}^2(B,M)$ coincides with $\pi_0\,\mathrm{Cross}(B,M)$.

We will exhibit a natural structure of Abelian group on $\mathrm{Opext}^n(B,M)$ and prove the main result of this section.

Theorem 4.3. *There exists an isomorphism of Abelian groups*

$$\mathit{Opext}^n(B,M) = HH^{n+1}(B,M),\ \ n \geqslant 2.$$

Definition 4.4. For $n \geqslant 3$ we define the element $0 \in \mathrm{Opext}^n(B,M)$ as the class of the extension

$$0 \longrightarrow M = M \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow B = B \longrightarrow 0\ .$$

Remark 4.5. If B is a projective algebra or M is injective as a B-bimodule, then $\mathrm{Opext}^n(B, M) = 0$. In general, if

$$\mathcal{E} = (\ 0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} A \xrightarrow{\pi} B \to 0\ )$$

and there is a map $g : M_{n-1} \to M$ such that $gf = 1_M$, then $\mathcal{E} = 0$ in $\mathrm{Opext}^n(B, M)$, $n \geqslant 3$.

Proposition 4.6. *Given $\mathcal{E} \in$ Opext$^n(B, M)$ and a map $\alpha : M \to M'$ of B-bimodules, there exists an extension $\alpha\mathcal{E} \in$ Opext$^n(B, M')$ and a morphism of the form $(\alpha, \delta_{n-1}, \dots, \beta)$ from $\mathcal{E}$ to $\alpha\mathcal{E}$. Moreover, $\alpha\mathcal{E}$ is unique in Opext$^n(B, M')$ with this property.*

Proof. Let $\mathcal{E} = (\ 0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} A \xrightarrow{\pi} B \to 0\ )$. Consider the following pushout of B-bimodules

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "Push"] & M_{n-1} \arrow[d, "i"] \arrow[ddr, bend left] & \\
M' \arrow[r] \arrow[drr, bend right, "0"] & \overline{M_{n-1}} \arrow[dr, dotted] & \\
 & & M_{n-2}
\end{tikzcd}$$

Take $\alpha\mathcal{E} = (\ 0 \to M' \to \overline{M_{n-1}} \to \cdots\cdots \to M_1 \xrightarrow{\partial_1} A \to B \to 0\ )$ in $\mathrm{Opext}^n(B, M')$ and the morphism $(\alpha, i, 1, \dots, 1) : \mathcal{E} \to \alpha\mathcal{E}$.

Given $\mathcal{E}' \in \mathrm{Opext}^n(B, M')$ and a morphism of the form $(\alpha, \delta_{n-1}, \dots, \beta) : \mathcal{E} \to \mathcal{E}'$, by properties of the pushout we find a map $(1, j, \delta_{n-2}, \dots, \beta) : \alpha\mathcal{E} \to \mathcal{E}'$ and therefore $\alpha\mathcal{E} = \mathcal{E}' \in \mathrm{Opext}^n(B, M')$.

□

Defination and Remark 4.7. By 4.6, a morphism of B-bimodules $\alpha : M \to M'$ induces a well defined function

$$\alpha_* : \mathrm{Opext}^n(B, M) \to \mathrm{Opext}^n(B, M')$$

by $\alpha_*(\mathcal{E}) = \alpha\mathcal{E}$.

Lemma 4.8. *If $\mathcal{E} = (0 \longrightarrow M \xrightarrow{f} M_{n-1} \longrightarrow \dots) \in$ Opext$^n(B, M)$, then $f\mathcal{E} = 0 \in$ Opext$^n(B, M_{n-1})$.*

Proof. Consider the morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f"'] & M_{n-1} \arrow[r, "g"] \arrow[d, "(1,g)"] & M_{n-2} \arrow[r] \arrow[d, equal, "Identity"] & \cdots \\
0 \arrow[r] & M_{n-1} \arrow[r, "(1,0)"] & M_{n-1} \oplus M_{n-2} \arrow[r, "p_2"] & M_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

By definition the row in the bottom corresponds to $f\mathcal{E}$, therefore by 4.5 $f\mathcal{E} = 0$. □

Definition 4.9. Given two crossed n-fold extensions of B

$$\mathcal{E} = (\ 0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} A \xrightarrow{\pi} B \to 0\ )$$

and

$$\mathcal{E}' = (\ 0 \to M' \xrightarrow{f'} M'_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} M'_1 \xrightarrow{\partial'_1} A' \xrightarrow{\pi'} B \to 0\ )$$

the *sum of* $\mathcal{E}$ *and* $\mathcal{E}'$ *over* B is denoted by $\mathcal{E} \oplus_B \mathcal{E}'$ and corresponds to the following crossed n-fold extension

$$0 \longrightarrow M \oplus M' \longrightarrow M_{n-1} \oplus M'_{n-1} \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow M_1 \oplus M'_1 \xrightarrow{(\partial_1, \partial'_1)} A \times_B A' \xrightarrow{q} B \longrightarrow 0\,.$$

Here the algebra $A \times_B A'$ is defined as follows. The elements of it are the pairs (a, a') with $a \in A$ and $a' \in A'$ such that $\pi a = \pi' a'$, addition and multiplication is defined coordinatewise. The map $q : A \times_B A' \to B$ is the map $q(a, a') = \pi(a) = \pi'(a')$. The action of $A \times_B A'$ on $M_1 \oplus M'_1$ is also defined coordinatewise. It is easy to check that this defines a crossed module $(M_1 \oplus M'_1, A \times_B A', (\partial_1, \partial'_1))$.

Definition 4.10. Given $\mathcal{E}, \mathcal{E}' \in \operatorname{Opext}^n(B, M)$ with $n \geqslant 3$, we define the *Baer Sum* $\mathcal{E} + \mathcal{E}' \in \operatorname{Opext}^n(B, M)$ as follows.

$$\mathcal{E} + \mathcal{E}' = \nabla_M(\mathcal{E} \oplus_B \mathcal{E}')$$

where $\nabla_M : M \oplus M \to M$ is the codiagonal.

Theorem 4.11. *For $n \geqslant 3$ the set $\operatorname{Opext}^n(B, M)$ equipped with the Baer sum is an abelian group with the zero element defined as in 4.4. The inverse of an extension*

$$\mathcal{E} = (\ 0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{g} \cdots\cdots \to M_1 \xrightarrow{\partial_1} A \to B \to 0\ )$$

is the extension

$$(-1_M)\mathcal{E} = (\ 0 \to M \xrightarrow{-f} M_{n-1} \xrightarrow{g} \cdots\cdots \to M_1 \xrightarrow{\partial_1} A \to B \to 0\ )$$

Moreover, the maps $\alpha_* : \operatorname{Opext}^n(B, M) \to \operatorname{Opext}^n(B, M')$ are morphisms of groups.

Proof. Follows the classical one (cf. [**10**]). One has to check that

1. $(\alpha + \beta)\mathcal{E} = \alpha\mathcal{E} + \beta\mathcal{E}$
2. $\alpha(\mathcal{E} + \mathcal{E}') = \alpha\mathcal{E} + \alpha\mathcal{E}'$

□

The Baer sum in $\operatorname{Opext}^2(B, M)$ is defined in a slightly different way. Recall that the elements in $\operatorname{Opext}^2(B, M)$ are classes of crossed modules with cokernel B and kernel M. The class of $0 \in \operatorname{Opext}^2(B, M)$ is the class of the extension

$$0 \longrightarrow M = M \xrightarrow{0} B = B \longrightarrow 0$$

Now given

$$\mathcal{E} = (\ 0 \longrightarrow M \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{\pi} B \longrightarrow 0\ )$$

and

$$\mathcal{E}' = (\ 0 \longrightarrow M \xrightarrow{i'} V' \xrightarrow{\partial'} A' \xrightarrow{\pi'} B \longrightarrow 0\ ),$$

the Baer sum $\mathcal{E} + \mathcal{E}'$ is the class of the extension

$$\mathcal{E} + \mathcal{E}' = (\ 0 \longrightarrow M \xrightarrow{j} V + V' \xrightarrow{\tilde{\partial}} A \times_B A' \xrightarrow{q} B \longrightarrow 0\ )$$

where $q : A \times_B A' \to B$ is defined as in 4.9 and $V + V'$ is the pushout of k-vector spaces

$$\begin{tikzcd}
M \oplus M \arrow[r, "i \oplus i'"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "Push"] & V \oplus V' \arrow[d, "r"] \arrow[ddr, bend left, "(\partial, \partial')"] & \\
M \arrow[r, "j"] \arrow[drr, bend right, "0"'] & V + V' \arrow[dr, dotted, "\tilde{\partial}"] & \\
 & & A \times_B A'
\end{tikzcd}.$$

The structure of $(A \times_B A')$-bimodule on $V + V'$ is induced by the structure on $V \oplus V'$ (coordinatewise) via the quotient map $r : V \oplus V' \to V + V'$ by $(a, a')r(v, v') = r(av, a'v')$ and $r(v, v')(a, a') = r(va, v'a')$. Note that the multiplication is well defined since $(a, a') \in A \times_B A'$ and therefore $\pi(a) = \pi'(a')$. It is easy to check that $\tilde{\partial} : V + V' \to A \times_B A'$ is a crossed module.

Remark 4.12. With this structure of abelian group in $\mathrm{Opext}^2(B, M)$ the bijection

$$\psi : \mathrm{Opext}^2(B, M) \to HH^3(B, M)$$

of 3.2 is an isomorphism of groups.

Definition 4.13. Given a short exact sequence of B-bimodules

$$0 \longrightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \longrightarrow 0$$

we define a connecting homomorphism ($n \geqslant 2$)

$$\delta : \mathrm{Opext}^n(B, M'') \to \mathrm{Opext}^{n+1}(B, M)$$

as follows. Given an extension $\mathcal{E} = (\ 0 \longrightarrow M'' \xrightarrow{f} M_{n-1} \longrightarrow \cdots\ )$, take $\delta(\mathcal{E})$ to be the class of the extension $(\ 0 \longrightarrow M \xrightarrow{\alpha} M' \xrightarrow{f\beta} M_{n-1} \longrightarrow \cdots\ )$.

Note that δ is a well defined homomorphism for all $n \geqslant 2$.

Theorem 4.14. *A short exact sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \longrightarrow 0$$

of B-bimodules induces a long exact sequence of abelian groups ($n \geqslant 2$)

$$Opext^n(B,M) \xrightarrow{\alpha_*} Opext^n(B,M') \xrightarrow{\beta_*} Opext^n(B,M'') \xrightarrow{\delta}$$

$$Opext^{n+1}(B,M) \longrightarrow \cdots$$

Proof. To prove exactness at $\mathrm{Opext}^n(B,M')$ with $n \geqslant 3$ note first that $\beta_*\alpha_* = (\beta\alpha)_* = 0$. Now let

$$\mathcal{E} = (\ 0 \to M' \xrightarrow{f} M_{n-1} \xrightarrow{g} M_{n-2} \to \cdots \to M_1 \xrightarrow{\partial} A \to B \to 0\ ) \in \mathrm{Opext}^n(B,M')$$

and $\beta\mathcal{E} = 0$. We suppose first that there is a map $\beta\mathcal{E} \to 0$, i.e.

$$\beta\mathcal{E} = (\ 0 \to M'' \xrightarrow{h} \overline{M_{n-1}} \xrightarrow{g'} M_{n-2} \to \cdots \to M_1 \xrightarrow{\partial} A \to B \to 0\ )$$

and there is a map $r : \overline{M_{n-1}} \to M''$ such that $rh = 1$. The following diagram shows that $\mathcal{E} = \alpha\overline{\mathcal{E}}$.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f\alpha"] \arrow[d, "\alpha"'] & \ker rt \arrow[r, "g"] \arrow[d, "i"'] & M_{n-2} \arrow[r] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "Id"] & \cdots \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\beta"'] & M_{n-1} \arrow[r, "g"] \arrow[d, "t"] & M_{n-2} \arrow[r] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "Id"] & \cdots \\
0 \arrow[r] & M'' \arrow[r, "h"] & \overline{M_{n-1}} \arrow[r, "g'"] & M_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose now that there is a map $0 \to \beta\mathcal{E}$. In this case it is easy to see that $\mathcal{E} = 0$. The general case follows combining these both cases. Suppose for example there exists an extension $\tilde{\mathcal{E}} = (\ 0 \longrightarrow M'' \xrightarrow{l} \tilde{M}_{n-1} \longrightarrow \tilde{M}_{n-2} \longrightarrow \cdots\ ) \in \mathrm{Opext}^n(B,M'')$ and maps $\tilde{\mathcal{E}} \to \beta\mathcal{E}$ and $\tilde{\mathcal{E}} \to 0$. In this case we construct the extension $\overline{\mathcal{E}}$ with $\alpha\overline{\mathcal{E}} = \mathcal{E}$ as follows. There exists a retraction $r : \tilde{M}_{n-1} \to M''$ such that $rl = 1$. Consider the pushout of B-bimodules

$$\begin{tikzcd}
\tilde{M}_{n-1} \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "Push"] & \overline{M}_{n-1} \arrow[d, "\overline{r}"] \\
M'' \arrow[r] & \overline{M''}
\end{tikzcd}$$

and take $\overline{\mathcal{E}} = (\ 0 \longrightarrow M \xrightarrow{f\alpha} \mathrm{Ker}\ \overline{r}t \xrightarrow{g} M_{n-2} \longrightarrow \cdots\ )$.

For $n = 2$ exactness at $\mathrm{Opext}^2(B,M')$ follows from 3.2.

To prove exactness at $\mathrm{Opext}^{n+1}(B,M)$ for $n \geqslant 2$ note first that $\delta(\mathcal{E})$ has the form

$$\delta(\mathcal{E}) = (\ 0 \to M \xrightarrow{\alpha} M' \xrightarrow{f\beta} M_{n-1} \to \cdots \to M_1 \xrightarrow{\partial} A \to B \to 0\ )$$

and therefore $\alpha\delta(\mathcal{E}) = 0$ by 4.8. Now let

$$\mathcal{E} = (\ 0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{g} M_{n-2} \to \cdots \to M_1 \xrightarrow{\partial} A \to B \to 0\ ) \in \mathrm{Opext}^n(B, M)$$

with $\alpha\mathcal{E} = 0$. Applying the same argument as above, we can suppose that there is a map $\alpha\mathcal{E} \to 0$, i.e.

$$\alpha\mathcal{E} = (\ 0 \to M' \xrightarrow{l} \overline{M_{n-1}} \xrightarrow{g'} M_{n-2} \to \cdots \to M_1 \xrightarrow{\partial} A \to B \to 0\ )$$

and there is a map $t : \overline{M_{n-1}} \to M'$ such that $tl = 1$.

Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, equal] & M_{n-1} \arrow[r, "g"] \arrow[d, "tr"] \arrow[dr, phantom, "Push"] & M_{n-2} \arrow[r] \arrow[d] & M_{n-3} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "Id"] & \cdots \\
0 \arrow[r] & M \arrow[r, "\alpha"] & M' \arrow[r, "j"'] & \overline{M_{n-2}} \arrow[r] & M_{n-3} \arrow[r] & \cdots
\end{tikzcd}$$

The map j can be factored $j = h\beta$ for some $h : M'' \to \overline{M_{n-2}}$ and therefore $\mathcal{E} = \delta(\mathcal{E}')$ with

$$\mathcal{E}' = (\ 0 \longrightarrow M'' \xrightarrow{h} \overline{M_{n-2}} \longrightarrow M_{n-3} \longrightarrow \cdots\ ).$$

To prove exactness at $\mathrm{Opext}^n(B, M'')$ for $n \geqslant 2$ consider the following diagrams. The first row of the first diagram corresponds to $\mathcal{E} \in \mathrm{Opext}^n(B, M')$ and the second row corresponds to $\beta\mathcal{E} \in \mathrm{Opext}^n(B, M'')$.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\beta"'] & M_{n-1} \arrow[r] \arrow[d] & M_{n-2} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "Id"] & \cdots \\
0 \arrow[r] & M'' \arrow[r, "h"] & \overline{M_{n-1}} \arrow[r] & M_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "{(1_M,0)}"] \arrow[d, equal] & M \oplus M' \arrow[r, "0+f"] \arrow[d, "\alpha+1_{M'}"] & M_{n-1} \arrow[r] \arrow[d] & M_{n-2} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "Id"] & \cdots \\
0 \arrow[r] & M \arrow[r, "\alpha"] & M' \arrow[r, "h\beta"] & \overline{M_{n-1}} \arrow[r] & M_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

□

Proof of 4.3. The result is true for $n = 2$ by 3.2. For $n \geqslant 3$ we use theorem 4.14. Since the category of B-bimodules has enough injectives, we can find a short exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \longrightarrow 0$$

with M' injective. By 4.14 and 4.5 we have

$$\mathrm{Opext}^{n+1}(B, M) = \mathrm{Opext}^n(B, M'').$$

On the other hand we have $HH^{n+2}(B,M) = HH^{n+1}(B,M'')$ by the long exact sequence of cohomology. Hence the result follows by induction from 3.2. $\square$

Remark 4.15. Theorem 4.3 is the analogue of a corresponding result for the cohomology of groups. In fact, using crossed modules in the category of groups as introduced by J.H.C.Whitehead [**12**] one can consider crossed extensions of groups which represent elements in the cohomology of groups (cf. Huebschmann [**7**]).

5. The characteristic class of a tensor product of differential algebras

In this section we define the $\odot$-product of crossed modules. The definition of $\partial_1 \odot \partial_2$ is used below for the computation of the characteristic class of a tensor product of differential algebras (see 3.5 above).

Definition 5.1. Let $\partial_1 : V_1 \to A_1$ and $\partial_2 : V_2 \to A_2$ be crossed modules. Consider the diagram of (graded) vector spaces

$$V_1 \otimes V_2 \xrightarrow{d_2} (V_1 \otimes A_2) \oplus (A_1 \otimes V_2) \xrightarrow{d_1} (A_1 \otimes A_2) \tag{*}$$

where d_1 and d_2 are defined as follows.

$$\begin{aligned} d_2(v_1 \otimes v_2) &= \partial_1 v_1 \otimes v_2 - v_1 \otimes \partial_2 v_2 \\ d_1(v_1 \otimes a_2) &= \partial_1 v_1 \otimes a_2 \\ d_1(a_1 \otimes v_2) &= a_1 \otimes \partial_2 v_2 \end{aligned}$$

Since $d_1 d_2 = 0$ we obtain a map ∂ induced by d_1:

$$\partial : W = \frac{(V_1 \otimes A_2) \oplus (A_1 \otimes V_2)}{\mathrm{Im}(d_2)} \to A_1 \otimes A_2$$

Note that the diagram (*) is in fact a diagram of $(A_1 \otimes A_2)$-bimodules. Here the $(A_1 \otimes A_2)$-bimodule structure on $V_1 \otimes V_2$ is given by

$$\begin{aligned} (a_1 \otimes a_2)(v_1 \otimes v_2) &= (-1)^{|a_2||v_1|}(a_1 v_1 \otimes a_2 v_2) \\ (v_1 \otimes v_2)(a_1 \otimes a_2) &= (-1)^{|v_2||a_1|}(v_1 a_1 \otimes v_2 a_2) \end{aligned}$$

Thus the map $\partial : W \to A_1 \otimes A_2$ is a map of $(A_1 \otimes A_2)$-bimodules. We show now that ∂ is a crossed module. Given $w, w' \in W$ we have to check that $\partial(w)w' = w\partial(w')$. For $v_1, v_1' \in V_1, a_2, a_2' \in A_2$ we have

$$\partial(\overline{v_1 \otimes a_2})(\overline{v_1' \otimes a_2'}) = (\partial_1 v_1 \otimes a_2)(\overline{v_1' \otimes a_2'})$$

$$= (-1)^{|a_2||v_1'|}(\overline{(\partial_1 v_1)v_1' \otimes a_2 a_2'}) = (\overline{v_1 \otimes a_2})\partial(\overline{v_1' \otimes a_2'})$$

We have similar equation for $\partial(\overline{a_1 \otimes v_2})(\overline{a_1' \otimes v_2'})$. Now for $(\overline{v_1 \otimes a_2})$ and $(\overline{a_1 \otimes v_2})$ we have

$$\begin{aligned} \partial(\overline{v_1 \otimes a_2})(\overline{a_1 \otimes v_2}) &= (\partial_1 v_1 \otimes a_2)(\overline{a_1 \otimes v_2}) = (-1)^{|a_2||a_1|}(\overline{\partial_1(v_1 a_1) \otimes a_2 v_2}) \\ &= (-1)^{|a_2||a_1|}(\overline{v_1 a_1 \otimes \partial_2(a_2 v_2)}) = (\overline{v_1 \otimes a_2})\partial(\overline{a_1 \otimes v_2}) \end{aligned}$$

Thus $\partial : W \to A_1 \otimes A_2$ is a crossed module termed the $\odot$-*product of* ∂_1 and ∂_2 and is denoted by $\partial_1 \odot \partial_2$.

Notation. Given a crossed module $\partial : V \to A$ we denote the cokernel of ∂ by $\pi_0(\partial)$ and the kernel by $\pi_1(\partial)$.

Proposition 5.2. *The $\odot$-product of two crossed modules ∂_1 and ∂_2 satisfies*

$$\pi_0(\partial_1 \odot \partial_2) = \pi_0(\partial_1) \otimes \pi_0(\partial_2) \tag{1}$$

$$\pi_1(\partial_1 \odot \partial_2) = (\pi_0(\partial_1) \otimes \pi_1(\partial_2)) \oplus (\pi_1(\partial_1) \otimes \pi_0(\partial_2)) \tag{2}$$

Proof. To prove (1) consider the map

$$\psi : \pi_0(\partial_1 \odot \partial_2) \to \pi_0(\partial_1) \otimes \pi_0(\partial_2)$$

defined by $\psi(\overline{a_1 \otimes a_2}) = \overline{a_1} \otimes \overline{a_2}$. It is easy to check that this map is well defined and is and isomorphism.

To prove (2) consider the map

$$\phi : (\pi_0(\partial_1) \otimes \pi_1(\partial_2)) \oplus (\pi_1(\partial_1) \otimes \pi_0(\partial_2)) \to \pi_1(\partial_1 \odot \partial_2)$$

given by $\phi(\overline{a_1} \otimes v_2) = \overline{a_1 \otimes v_2}$ and $\phi(v_1 \otimes \overline{a_2}) = \overline{v_1 \otimes a_2}$.

We show that ϕ is well defined. For $a_1 = \partial_1 v_1$ and $v_2 \in \ker(\partial_2)$ we have

$$\overline{\partial_1 v_1 \otimes v_2} = \overline{v_1 \otimes \partial_2 v_2} = 0$$

The same procedure for $a_2 = \partial_2 v_2$ and $v_1 \in \ker(\partial_1)$. Moreover $\partial(\overline{a_1 \otimes v_2}) = 0$ if $v_2 \in \ker(\partial_2)$ and $\partial(\overline{v_1 \otimes a_2}) = 0$ for $v_1 \in \ker(\partial_1)$.

To prove that ϕ is and isomorphism consider a k-linear section of ∂_1, $q_1 : \mathrm{Im}(\partial_1) \to V_1$. Suppose $\overline{v_1 \otimes a_2} \in W$ and $\partial_1 v_1 \otimes a_2 = 0$. Then $(q_1 \partial_1 v_1 \otimes a_2) = 0$ and therefore $v_1 \otimes a_2 = (v_1 - q_1 \partial_1 v_1) \otimes a_2$ and $v_1 - q_1 \partial_1 v_1 \in \ker(\partial_1)$. The same procedure for $\overline{a_1 \otimes v_2}$. This implies that ϕ is an isomorphism. $\square$

Proposition 5.3. *Let ∂_1 and ∂_2 be crossed modules with cokernel B_i and kernel M_i, $i = 1, 2$. Then the class*

$$\langle \partial_1 \odot \partial_2 \rangle \in \pi_0 \, \mathrm{Cross}(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2)) =$$

$$HH^3(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2))$$

depends only on the classes $\langle \partial_1 \rangle \in HH^3(B_1, M_1)$ and $\langle \partial_2 \rangle \in HH^3(B_2, M_2)$. Moreover one obtains a group homomorphism

$$\Gamma : HH^3(B_1, M_1) \oplus HH^3(B_2, M_2) \to HH^3(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2))$$

defined by $\Gamma(\langle \partial_1 \rangle, \langle \partial_2 \rangle) = \langle \partial_1 \odot \partial_2 \rangle$.

Proof. To check that $\langle \partial_1 \odot \partial_2 \rangle$ depends only on the class of ∂_1 and ∂_2 consider a map $\alpha : \partial_1 \to \partial_1'$ in $\mathrm{Cross}(B_1, M_1)$

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, equal] & V_1 \arrow[r, "\partial_1"] \arrow[d, "\alpha_1"'] & A_1 \arrow[r] \arrow[d, "\alpha_0"] & B_1 \arrow[d, equal] \\
M_1 \arrow[r] & V_1' \arrow[r, "\partial_1'"] & A_1' \arrow[r] & B_1
\end{tikzcd}$$

Then α induces a map

$$\alpha \odot 1 : \partial_1 \odot \partial_2 \to \partial_1' \odot \partial_2$$

given by $(\alpha \odot 1)_0 : A_1 \otimes A_2 \to A_1' \otimes A_2$, $(\alpha \odot 1)_0(a_1 \otimes a_2) = \alpha_0(a_1) \otimes a_2$ and $(\alpha \odot 1)_1 : (V_1 \otimes A_2) \oplus (A_1 \otimes V_2) \to (V_1' \otimes A_2) \oplus (A_1' \otimes V_2)$ defined by

$$(\alpha \odot 1)_1(v_1 \otimes a_2) = \alpha_1(v_1) \otimes a_2$$
$$(\alpha \odot 1)_1(a_1 \otimes v_2) = \alpha_0(a_1) \otimes v_2$$

It is easy to check that $(\alpha \odot 1)$ is a well defined map in $\mathrm{Cross}(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2))$ from $\partial_1 \odot \partial_2$ to $\partial_1' \odot \partial_2$. The same argument applies to $\beta : \partial_2 \to \partial_2'$. That proves the first part of the proposition.

To prove that Γ is a well defined homomorphism one has to check that

$$\langle (\partial_1 + \partial_1') \odot 0 \rangle = \langle \partial_1 \odot 0 \rangle + \langle \partial_1' \odot 0 \rangle \qquad (*)$$

and the same for $\langle 0 \odot (\partial_2 + \partial_2') \rangle$. The sum $(\partial_1 + \partial_1') \in HH^3(B_1, M_1)$ and the element $0 \in HH^3(B_2, M_2)$ are defined explicitly in section 4 below (Baer Sum in $\mathrm{Opext}^2(B, M)$).

It is easy to check that $(*)$ holds. In fact the class $\langle (\partial_1 + \partial_1') \odot 0 \rangle \in HH^3(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2))$ corresponds to the class of the crossed module

$$\partial : ((V_1 + V_1') \otimes B_2) \oplus (B_1 \otimes M_2) \to (A_1 \times_{B_1} A_1') \otimes B_2$$

with $\partial((v_1 + v_1') \otimes b_2) = (\partial_1 v_1, \partial_1' v_1') \otimes b_2$ and $\partial(b_1 \otimes m_2) = 0$. Here $(V_1 + V_1')$ and $(A_1 \times_{B_1} A_1')$ are defined as in section 4 below. $\square$

We can describe the $\odot$-product in terms of classical cohomology products

$$HH^n(B_1, M_1) \otimes HH^m(B_2, M_2) \to HH^{n+m}(B_1 \otimes B_2, M_1 \otimes M_2)$$

(cf. [**10**],Chapter X). Given $f \in HH^3(B_1, M_1)$ we denote by $f \otimes 1_{B_2} \in HH^3(B_1 \otimes B_2, M_1 \otimes B_2)$ the tensor product of f with $1_{B_2} \in HH^0(B_2, B_2)$ given by the map

$$HH^3(B_1, M_1) \otimes HH^0(B_2, B_2) \to HH^3(B_1 \otimes B_2, M_1 \otimes B_2)$$

In similar way we define for an element $g \in HH^3(B_2, M_2)$ the element $1_{B_1} \otimes g \in HH^3(B_1 \otimes B_2, B_1 \otimes M_2)$.

Proposition 5.4. *Let ∂_1 and ∂_2 be crossed modules with cokernel B_i and kernel M_i, $i = 1, 2$. There is an equivalence of crossed modules*

$$\partial_1 \odot \partial_2 = i_1(1_{B_1} \otimes \partial_2) + i_2(\partial_1 \otimes 1_{B_2})$$

where

$$i_1 : HH^3(B_1 \otimes B_2, B_1 \otimes M_2) \to HH^3(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2))$$

and

$$i_2 : HH^3(B_1 \otimes B_2, M_1 \otimes B_2) \to HH^3(B_1 \otimes B_2, (B_1 \otimes M_2) \oplus (M_1 \otimes B_2))$$

are induced by the inclusions $i_1 : B_1 \otimes M_2 \to (B_1 \otimes M_2) \oplus (M_1 \otimes B_2)$ and $i_2 : M_1 \otimes B_2 \to (B_1 \otimes M_2) \oplus (M_1 \otimes B_2)$.

Proof. The crossed module $i_1(1_{B_1} \otimes \partial_2)$ corresponds by definition to the crossed module

$$p_1 : (B_1 \otimes V_2) \oplus (M_1 \otimes B_2) \to B_1 \otimes A_2$$

with $p_1(b_1 \otimes v_2) = b_1 \otimes \partial_2 v_2$ and $p_1(m_1 \otimes b_2) = 0$. The crossed module $i_2(\partial_1 \otimes 1_{B_2})$ corresponds to

$$p_2 : (B_1 \otimes M_2) \oplus (V_1 \otimes B_2) \to A_1 \otimes B_2$$

with $p_2(v_1 \otimes b_2) = \partial_1 v_1 \otimes b_2$ and $p_2(b_1 \otimes m_2) = 0$.

By definition of Baer Sum it is easy to check that $i_1(1_{B_1} \otimes \partial_2) + i_2(\partial_1 \otimes 1_{B_2})$ is isomorphic to the crossed module $\partial_1 \odot \partial_2$. □

Now let A and B be DG-algebras with differentials d_A and d_B of degree -1. Consider the tensor product $A \otimes B$ which is a DG-algebra with differential defined as follows.

$$d_{A\otimes B}(x_i \otimes y_j) = d_A x_i \otimes y_j + (-1)^i x_i \otimes d_B y_j$$

for $x_i \in A_i$ and $y_j \in B_j$.

Theorem 5.5. *The characteristic class* $\langle A\otimes B\rangle \in HH^3(H_*(A\otimes B), \overline{H_*(A \otimes B)}[-1])$ *can be computed as*

$$\langle A \otimes B\rangle = (\phi_1)_*(1 \otimes \langle B\rangle) + (\phi_2)_*(\langle A\rangle \otimes 1)$$

where

$$\phi_1 : H_*(A) \otimes \overline{H_*(B)}[-1] \to \overline{H_*(A \otimes B)}[-1]$$

$$\phi_2 : \overline{H_*(A)}[-1] \otimes H_*(B) \to \overline{H_*(A \otimes B)}[-1]$$

are defined by $\phi_1(a \otimes s^{-1}(b)) = (-1)^{|a|} s^{-1}(a \otimes b)$ *and* $\phi_2(s^{-1}(a) \otimes b) = s^{-1}(a \otimes b)$.

Proof. Let ∂_A, ∂_B and $\partial_{A\otimes B}$ be the crossed modules induced by A, B and $A \otimes B$. There exists a morphism of crossed modules $\Upsilon : \partial_A \odot \partial_B \to \partial_{A\otimes B}$ defined as follows.

$$\begin{tikzcd}
\dfrac{(\mathrm{coker}(d_A)[-1] \otimes \ker(d_B)) \oplus (\ker(d_A) \otimes \mathrm{coker}(d_B)[-1])}{\mathrm{Im}(d_2)} \arrow[r, "\partial"] \arrow[d, "\Upsilon_1"'] & \ker(d_A) \otimes \ker(d_B) \arrow[d, "\Upsilon_0"] \\
\mathrm{coker}(d_{A\otimes B})[-1] \arrow[r, "\partial_{A\otimes B}"] & \ker(d_{A\otimes B})
\end{tikzcd}$$

The top row in the diagram corresponds to the $\odot$-product $\partial_A \odot \partial_B$ and the bottom row corresponds to the crossed module $\partial_{A\otimes B}$. The map $\Upsilon_0 : \ker(d_A) \otimes \ker(d_B) \to \ker(d_{A\otimes B})$ is defined by

$$\Upsilon_0(x_i \otimes y_j) = x_i \otimes y_j \qquad x_i \in (\ker(d_A))_i,\ y_j \in (\ker(d_B))_j$$

The map Υ_1 is defined as follows. For $s^{-1}(\overline{x_i}) \in (\mathrm{coker}(d_A)[-1])_i$ and $y_j \in (\ker(d_B))_j$ we define $\Upsilon_1(s^{-1}(\overline{x_i}) \otimes y_j)$ to be the element

$$s^{-1}(\overline{x_i \otimes y_j}) \in (\mathrm{coker}(d_{A\otimes B})[-1])_{i+j}.$$

For $x_i \in (\ker(d_A))_i$ and $s^{-1}(\overline{y_j}) \in (\operatorname{coker}(d_B)[-1])_j$ we define

$$\Upsilon_1(x_i \otimes s^{-1}(\overline{y_j})) = (-1)^i s^{-1}(\overline{x_i \otimes y_j}) \in (\operatorname{coker}(d_{A\otimes B})[-1])_{i+j}.$$

We check that the map Υ_1 is well defined. Suppose $\overline{x_i} = \overline{0} \in (\operatorname{coker}(d_A))_i$ i.e. $x_i = d_A a_{i+1}$ for some $a_{i+1} \in A_{i+1}$. Then

$$\Upsilon_1(s^{-1}(\overline{x_i}) \otimes y_j) = s^{-1}(\overline{d_A a_{i+1} \otimes y_j}) = s^{-1}(\overline{d_{A\otimes B}(a_{i+1} \otimes y_j)}) = 0$$

The same argument for $\overline{y_j} = \overline{0} \in (\operatorname{coker}(d_B))_j$. For $z = (d_A x_i \otimes y_j - x_i \otimes d_B y_j) \in \operatorname{Im}(d_2)$ we have $\Upsilon_1(z) = d_{A\otimes B}((-1)^i x_i \otimes y_j)$. Thus Υ_1 is well defined.

It is easy to check that the diagram above is a morphism of crossed modules. Moreover $\Upsilon : \partial_A \odot \partial_B \to \partial_{A\otimes B}$ induces an isomorphism

$$\Upsilon_* : \pi_0(\partial_A \odot \partial_B) = \pi_0(\partial_A) \otimes \pi_0(\partial_B) = H_*(A) \otimes H_*(B) \to \pi_0(\partial_{A\otimes B}) = H_*(A \otimes B)$$

and an epimorphism

$$\Upsilon_* : \pi_1(\partial_A \odot \partial_B) = (\overline{H_*(A)}[-1] \otimes H_*(B)) \oplus (H_*(A) \otimes \overline{H_*(B)}[-1]) \to \pi_1(\partial_{A\otimes B}) = \overline{H_*(A \otimes B)}[-1]$$

The crossed module $\partial_A \odot \partial_B$ induces an element $\langle A\rangle \odot \langle B\rangle = \langle \partial_A \odot \partial_B\rangle \in HH^3(H_*(A \otimes B), \pi_1(\partial_A \odot \partial_B))$ which is mapped by Υ to the characteristic class $\langle A \otimes B\rangle \in HH^3(H_*(A \otimes B), \overline{H_*(A \otimes B)}[-1])$ of the chain algebra $A \otimes B$, i.e.

$$\langle A \otimes B\rangle = \Upsilon_*(\langle A\rangle \odot \langle B\rangle)$$

where the homomorphism

$$\Upsilon_* : HH^3(H_*(A \otimes B), \pi_1(\partial_A \odot \partial_B)) \to HH^3(H_*(A \otimes B), \overline{H_*(A \otimes B)}[-1])$$

is the homomorphism induced by the map $\Upsilon_* : \pi_1(\partial_A \odot \partial_B) \to \pi_1(\partial_{A\otimes B})$.

By 5.4 we have $\langle A\rangle \odot \langle B\rangle = i_1(1 \otimes \langle B\rangle) + i_2(\langle A\rangle \otimes 1)$ and therefore

$$\langle A \otimes B\rangle = (\phi_1)_*(1 \otimes \langle B\rangle) + (\phi_2)_*(\langle A\rangle \otimes 1)$$

with $(\phi_1)_* = (\Upsilon i_1)_*$ and $(\phi_2)_* = (\Upsilon i_2)_*$. □

For cochain algebras one can prove the following analogous result.

Theorem 5.6. *Let A and B be DG-algebras with differentials of degree 1. Then the characteristic class $\langle A \otimes B\rangle \in HH^3(H^*(A \otimes B), \overline{H^*(A \otimes B)}[1])$ of the tensor algebra $A \otimes B$ can be computed from $\langle A\rangle$ and $\langle B\rangle$ as*

$$\langle A \otimes B\rangle = (\phi_1)_*(1 \otimes \langle B\rangle) + (\phi_2)_*(\langle A\rangle \otimes 1)$$

where

$$\phi_1 : H^*(A) \otimes \overline{H^*(B)}[1] \to \overline{H^*(A \otimes B)}[1] \quad \phi_2 : \overline{H^*(A)}[1] \otimes H^*(B) \to \overline{H^*(A \otimes B)}[1]$$

are the maps defined by $\phi_1(a \otimes s(b)) = (-1)^{|a|} s(a \otimes b)$ and $\phi_2(s(a) \otimes b) = s(a \otimes b)$.

References

[1] H.-J. Baues, *Combinatorial homotopy and 4-dimensional complexes*, De Gruyter Expositions in Mathematics, 1991.

[2] H.-J. Baues, *Secondary cohomology and the Steenrod square*, Preprint.

[3] H.-J. Baues, W.Dreckmann, *The cohomology of homotopy categories and the general linear group*, K-Theory, **3**(1989), 307–338.

[4] H.-J. Baues, E.G. Minian, *Track extensions of categories and cohomology*, Preprint.

[5] H.-J. Baues, G. Wirsching, *Cohomology of small categories*, J.Pure Appl. Algebra, **38**(1985), 187–211.

[6] A.J.Berrick, A.A.Davydov, *Splitting of Gysin extensions*, Preprint.

[7] J. Huebschmann, *Crossed n-fold extensions of groups and cohomology*, Comment. Math. Helvetici, **55**(1980), 302–314.

[8] M. Jibladze and T. Pirashvili, *Cohomology of algebraic theories*, Journal of Algebra, **137**(1991), 253–296.

[9] J.-L. Loday, *Cyclic Homology*, Springer-Verlag, 1992.

[10] S. Mac Lane, *Homology*, Springer-Verlag, 1963.

[11] S. Mac Lane and J.H.C. Whitehead, *On the 3-type of a complex*, Proc. N.A.S., **36**(1950), 41–48.

[12] J.H.C. Whitehead, *Combinatorial homotopy* II, Bull. Amer. Math. Soc., **55**(1949), 453–496.

This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/
or by anonymous ftp at
ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2002/n2a3/v4n2a3.(dvi,ps,pdf)

Hans-Joachim Baues `baues@mpim-bonn.mpg.de`

Max-Planck-Institut für Mathematik,
Vivatsgasse 7, 53111 Bonn,
Germany

Elias Gabriel Minian `gminian@mpim-bonn.mpg.de`

Max-Planck-Institut für Mathematik,
Vivatsgasse 7, 53111 Bonn,
Germany