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A HOMOTOPY LIE-RINEHART RESOLUTION AND CLASSICAL BRST COHOMOLOGY

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Abstract

We use an interlaced inductive procedure reminiscent of the integration process from traditional deformation theory to construct a homotopy Lie-Rinehart resolution for the Lie-Rinehart pair which arises as an exercise in Poisson reduction in the context of the BFV construction of classical BRST cohomology. We show that the associated homotopy Rinehart algebra and the BRST algebra are isomorphic as graded commutative algebras. In the irreducible case, the two have the same cohomology.

1. Introduction

In this paper, we utilize a strategy akin to the process of integration found in traditional deformation theory (see, for example, [**GS90**]) to construct a homotopy Lie-Rinehart resolution $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ for the Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ that appears in the BFV formulation of classical BRST cohomology. A Lie-Rinehart pair is a couple $(\mathbf{B}, s\mathfrak{g})$ which admits a structure analogous to that shared by the associative commutative algebra $C^\infty(M)$ of smooth functions and the Lie algebra $\Gamma(TM)$ of smooth vector fields on a smooth manifold M. The term Lie-Rinehart pair is not widely used. More often, $s\mathfrak{g}$ has been called a $(\mathbf{B}, k)$-Lie algebra, [**Rin63**] [**Pal61**] and [**Her72**]. More recently, Lie-Rinehart pairs have appeared as Lie algebroids (see [**dSW**]).

An associative algebra A is a *Poisson* algebra if it admits a Lie bracket $\{\ ,\ \}$ such that for any $a \in A$, the map $\{a,\ \}$ is a graded derivation with respect to the multiplication, i.e., $\{a, bc\} = \{a, b\}c \pm b\{a, c\}$. A multiplicative ideal $\mathcal{I}$ of a Poisson algebra A is *coisotropic* if it is closed under the Poisson bracket on A. Poisson reduction of a Poisson algebra A by a coisotropic ideal $\mathcal{I}$ is no more than the observation that while the quotient $A/\mathcal{I}$ is not a Poisson algebra (unless $\mathcal{I}$ is a Poisson ideal), the subset of $\mathcal{I}$-invariant classes in $A/\mathcal{I}$ is again a Poisson algebra. The $\mathcal{I}$-invariant classes comprise the zeroth cohomology of the Rinehart complex R for the Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$.

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Mathematicians and physicists (see, for exammple [**Sta88**], [**KS87**], [**FHST89**], [**HT92**], [**Kim92a**], [**Kim92b**], [**Sta92**], [**Kim93**] and [**Sta96**]) recognized the classical analogue of the Batalin, Fradkin and Vilkovisky (BFV) construction of the quantum BRST complex ([**BV77**], [**BF83**], [**BV83**] and [**BV85**]) as something new and interesting because BRST cohomology performs Poisson reduction without passing first to the quotient $A/\mathcal{I}$. The BFV construction of the classical BRST algebra (see, for example, [**Kim93**]) begins by replacing $A/\mathcal{I}$ with the Koszul-Tate resolution [**Tat57**] and adjoins formal (ghost) variables to the Koszul-Tate resolution. They then exploit a graded Poisson bracket to construct a differential. Under certain conditions, the BRST algebra $(\mathcal{A}, \mathcal{D})$ is a cohomological model for the Rinehart complex of $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$.

In general, constructing cohomological models for the Rinehart complex $(\mathbf{B}, s\mathfrak{g})$ by the traditional homological means of replacing both $\mathbf{B}$ and $s\mathfrak{g}$ with resolutions fails unless the Lie-Rinehart structure of the pair $(\mathbf{B}, s\mathfrak{g})$ is preserved. In [**Kje01**], we defined homotopy Lie-Rinehart pairs and the associated homotopy Rinehart algebra in the context of coalgebras. We defined homotopy Lie-Rinehart resolutions and presented conditions under which the associated homotopy Rinehart algebra is a cohomological model for the Rinehart algebra of the resolved Lie-Rinehart pair.

Summary

The coalgebra setting is used throughout this paper. First, we revisit the definitions of the Lie algebras of subordinate derivation sources, resting coderivations and shared Lie modules, which lie behind the coalgebra realizations of both homotopy and non-homotopy Lie algebras, Lie algebra modules, Lie-Rinehart pairs and Rinehart cohomology. We review homotopy Lie-Rinehart resolutions for Lie-Rinehart pairs and the conditions under which the homotopy Rinehart algebra for a homotopy Lie-Rinehart resolution is a model for the Rinehart cohomology complex for a Lie-Rinehart pair (§2). Next, using an interlaced inductive process, we construct a homotopy Lie-Rinehart resolution for the Lie-Rinehart pair in classical BRST algebra (§3). One surprising result is that the homotopy Rinehart complex associated with the homotopy Lie-Rinehart resolution is isomorphic to the BRST algebra (§4).

We have omitted all sign arguments from the proofs in this paper, as they are not generally instructive. All vector spaces, algebras and coalgebras in this paper are over a field k of characteristic zero. All tensor products are over k and all maps are at least k-linear or k-multilinear.

Note that $s\mathfrak{g}$ is the suspension of the Lie algebra $\mathfrak{g}$, i.e., all elements of $\mathfrak{g}$ are assigned degree zero and hence, all elements of $s\mathfrak{g}$ have degree 1. Throughout this paper, we will identify all Lie algebras (and strongly homotopy Lie algebras) with their suspensions. At first, this identification will be explicit; later, when we work with $\mathcal{I}/\mathcal{I}^2$ and $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$, we will hide the suspension.

2. Necessary Background

Review of Chavalley-Eilenberg and Rinehart cohomologies

For any Lie algebra $s\mathfrak{g}$ and $s\mathfrak{g}$-module $\mathbf{B}$, an n-multilinear function $f_n : (s\mathfrak{g})^{\times n} \to \mathbf{B}$ is alternating if $f_n(sx_1, ..., sx_i, sx_{i+1}, ..., sx_n) = -f_n(sx_1, ..., sx_{i+1}, sx_i, ..., sx_n)$. The *Chevalley-Eilenberg* complex is the set of all alternating multilinear functions $Alt_k(s\mathfrak{g}, \mathbf{B})$, graded by n, and equipped with a degree $+1$ differential $\delta_{CE} : Alt_k^n(s\mathfrak{g}, \mathbf{B}) \to Alt_k^{n+1}(s\mathfrak{g}, \mathbf{B})$ given by

$$\delta_{\mathrm{CE}} f_n(sx_0, ..., sx_n) = \sum_{i=0}^{n} (-1)^i \omega(sx_i \otimes f_n(sx_0, ..., \widehat{sx_i}, ..., sx_n) - \sum_{i<j} (-1)^{i+j-1} f_n([sx_i, sx_j], sx_0, ..., \widehat{sx_i}, ..., \widehat{sx_j}, ..., sx_n), \tag{1}$$

where $\widehat{sx_k}$ indicates that sx_k should be omitted. The map ω is the $s\mathfrak{g}$-module action of $s\mathfrak{g}$ on $\mathbf{B}$. Any element $b \in \mathbf{B}$ is considered an element of $Alt_k^0(s\mathfrak{g}, \mathbf{B})$. The image of b under δ_{CE} is defined by setting $\delta_{CE} b(sx) = \omega(sx \otimes b)$. When $\mathbf{B}$ is an algebra, the Chevalley-Eilenberg complex is a differential graded commutative algebra. For f_n and g_m in $Alt_k(s\mathfrak{g}, \mathbf{B})$, the product $f_n \smile g_m$ is given by

$$(f_n \smile g_m)(sx_1, ..., sx_{n+m}) = \sum_{\substack{\sigma \\ (n,m) \\ \text{unshuffles}}} f_n(sx_{\sigma(1)}, ..., sx_{\sigma(n)}) g_m(sx_{\sigma(n+1)}, ..., sx_{\sigma(n+m)}).$$

An $(n, m)-$unshuffle is any permutation σ in the symmetric group Σ_{n+m} such that

$$\underbrace{\sigma(1) < \cdots < \sigma(n)}_{\text{first } \sigma \text{ hand}} \quad \text{and} \quad \underbrace{\sigma(n+1) < \cdots < \sigma(n+m),}_{\text{second } \sigma \text{ hand}}$$

where $\sigma(j)$ is the element of the set $\{1, ..., n+m\}$ moved to the j^{th} position under σ. The differential δ_{CE} acts as a derivation with respect to this multiplication. The cohomology of this complex with respect to δ_{CE} is the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in $\mathbf{B}$ [**CE48**].

Definition 2.1. [**Rin63**] *Let* $\mathbf{B}$ *is an algebra and* $s\mathfrak{g}$ *be a Lie algebra, both modules over an algebra* A *over a field* k *of characteristic zero and modules over each other. We denote the left* $\mathbf{B}$*-module action* μ *on* $s\mathfrak{g}$ *by* $\mu(a \otimes s\alpha) := as\alpha$. *Let* $\omega : s\mathfrak{g} \otimes \mathbf{B} \longrightarrow \mathbf{B}$ *(or, alternatively,* $\omega : s\mathfrak{g} \to \mathrm{Der}(\mathbf{B})$*) denote the* $s\mathfrak{g}$*-module action on* $\mathbf{B}$. *The pair* $(\mathbf{B}, s\mathfrak{g})$ *is a* Lie-Rinehart pair, *provided the* Lie-Rinehart relations (LRa) and (LRb) *are satisfied for all* $a, b \in \mathbf{B}$ *and* $sx, sy \in s\mathfrak{g}$:

LRa: $\omega(asx \otimes b) = a \cdot \omega(sx \otimes b)$, where $\cdot$ indicates the multiplication on $\mathbf{B}$.
LRb: $[sx, asy] = a[sx, sy] + \omega(sx \otimes a)sy$.

Suppose $(\mathbf{B}, s\mathfrak{g})$ is a Lie-Rinehart pair and the alternating function f_n is $\mathbf{B}$-multilinear, i.e., $f_n(a_1sx_1, ..., a_nsx_n) = a_1 \cdots a_n f_n(sx_1, ..., sx_n)$. Because the Lie action map ω maps $s\mathfrak{g}$ into the derivations of $\mathbf{B}$ and as a result of the Lie-Rinehart relations, the image of f_n under the Chevalley-Eilenberg differential δ_{CE} is again

B-multilinear, despite the fact that the bracket is not **B**-multilinear. The *Rinehart* algebra $R = Alt_{\mathbf{B}}(s\mathfrak{g}, \mathbf{B})$ with differential $\delta_R = \delta_{CE}$ is a subcomplex of the Chevalley-Eilenberg algebra and the cohomology with respect to δ_R is the *Rinehart cohomology* of $s\mathfrak{g}$ with coefficients in **B** [**Rin63**].

Coalgebras and Subcoalgebras

A *graded coassociative coalgebra* is a pair $(\mathbf{C}, \Delta)$, where $\mathbf{C}$ is a graded module over k together with a 0-degree coassociative comultiplication $\Delta : \mathbf{C} \to \mathbf{C} \otimes \mathbf{C}$. A function f with degree $|f| = r$ is a *coderivation* on $\mathbf{C}$ if $(f \otimes 1 + 1 \otimes f)\Delta = \Delta f$. The set of all coderivations on a coalgebra C, denoted $\mathrm{Coder}(C)$, is a graded Lie algebra under the graded commutator bracket, that is to say, $[f, g] = fg - (-1)^{|f||g|} gf$ for all f and $g \in \mathrm{Coder}(C)$, where $|f|$ and $|g|$ are the degrees of f and g.

We will work with the tensor coalgebra $\mathrm{T}^c(sV) = \bigotimes(sV)$, where sV is the suspension a graded module V over a k-algebra A, where k is a field of characteristic 0. We will let $sv_{[1 \text{ to } n]}$ denote the element $sv_1 \otimes \cdots \otimes sv_n \in (sV)^{\otimes n}$. The *internal graded* action $\rho_n^\wedge$ of the symmetric group Σ_n on $(sV)^{\otimes n}$ is given by $\sigma \cdot (sv_{[1 \text{ to } n]}) = \varkappa_{\mathbf{id}}(\sigma) sv_{\sigma[1 \text{ to } n]}$ for all n, $\sigma \in \Sigma_n$ and $sv_{[1 \text{ to } n]} \in (sV)^{\otimes n}$. The symbol $sv_{\sigma[1 \text{ to } n]}$ is shorthand for $sv_{\sigma(1)} \otimes \cdots \otimes sv_{\sigma(n)}$, where $\sigma(i)$ is the element of the ordered set $\{1, ..., k\}$ which moves to the i^{th} position under σ. The factor $\varkappa_{\mathbf{id}}(\sigma)$ is the sign produced by rearranging the sv_i's into the σ order, following the Koszul sign convention, which states that exchanging two objects of homogeneous degrees p and q (whether elements or maps) introduces a factor of $(-1)^{pq}$. The action $\rho^\wedge = \{\rho_n^\wedge\}_{n=1}^\infty$ of $\{\Sigma_n\}_{n=1}^\infty$ on $\mathrm{T}^c(sV)$ is defined in the obvious way. The $\rho^\wedge$-invariant subcoalgebra is well-known as the graded commutative coalgebra $\bigwedge(sV)$ and is generated by elements of the form

$$\sum_{\sigma \in \Sigma_n} \varkappa_{\mathbf{id}}(\sigma) sv_{\sigma[1 \text{ to } n]},$$

which we will denote by $sv^\wedge_{[1 \text{ to } n]}$. The coassociative comultiplication on $\bigwedge(sV)$ is given by

$$\Delta(sv^\wedge_{[1 \text{ to } n]}) = \sum_{j=0}^{n} \sum_{\substack{\rho \\ (j, n-j)}} \varkappa_{\mathbf{id}}(\rho) \left(sv^\wedge_{\rho[1 \text{ to } j]} \right) \otimes \left(sv^\wedge_{\rho[j+1 \text{ to } n]} \right).$$

where it is understood that the second sum is over all $(j, n-j)$-unshuffles.

Shared Lie modules, Subordinate and resting coderivations

Three algebraic objects are vital for what follows. The graded Lie algebra $\mathrm{Coder}_W^W(\bigwedge(sV))$ is generated by the set of coderivations l_n on $\bigwedge(sV)$, each of which *rests* on a specific *subordinate* coderivation m_n in the Lie algebra $\mathrm{Coder}_W^W(\bigwedge(sV) \otimes W)$. For a graded commutative algebra W, the Lie algebra $\mathrm{Coder}_W^W(\bigwedge(sV) \otimes W)$ is generated by the set of all coderivations m_n, each of which is both *subordinate* to a specific *resting* coderivation l_n and a W*-derivation source*. Finally, the graded commutative algebra $\mathrm{Hom}_W(\bigwedge(sV), \bigwedge(sV) \otimes W)$ is generated by the set of all W-linear maps f_n from $\bigwedge(sV)$ into $\bigwedge(sV) \otimes W$. This algebra admits a *shared* Lie module structure over both $\mathrm{Coder}_W^W(\bigwedge(sV) \otimes W)$ and $\mathrm{Coder}_W^W(\bigwedge(sV))$. We review what the terms *subordinate coderivation*, W-*derivation source*, *resting coderivation* and *shared Lie module* mean. For a more detailed development, see [**Kje01**].

We extend any map $l_n : (sV)^{\wedge n} \to sV$ to a coderivation on the coalgebra $\bigwedge(sV)$ by setting

$$l_n(sv^{\wedge}_{[1 \text{ to } k]}) = \sum_{\substack{\rho \\ (n,k-n)}} \kappa_{\mathbf{id}}(\rho) l_n(sv^{\wedge}_{\rho[1 \text{ to } n]}) \wedge sv^{\wedge}_{\rho[n+1 \text{ to } k]}. \tag{2}$$

Definition 2.2. *A map m_n on $\bigwedge(sV) \otimes W$ is an l_n-*subordinate coderivation *if the degree of m_n and l_n agree and m_n, as the extension of a map $m_n : (sV)^{\wedge n-1} \otimes W \to W$ as a coderivation on $\bigwedge(sV) \otimes W$, is defined on $(sV)^k \otimes W$ by setting $m_n(sv^{\wedge}_{[1 \text{ to } k]} \otimes w) = l_n(sv^{\wedge}_{\rho[1 \text{ to } k]}) \otimes w \;\; +$*

$$\sum_{\substack{\sigma \\ (k-n+1,n-1)}} \kappa_{\mathbf{id}}(\sigma)\, \kappa_{\mathbf{id}}(m_n; sv^{\wedge}_{\sigma[1 \text{ to } k-n+1]}) \cdot sv^{\wedge}_{\sigma[1 \text{ to } k-n+1]} \otimes m_n(sv^{\wedge}_{\sigma[k-n+2 \text{ to } k]} \otimes w). \tag{3}$$

when $k \geqslant n-1$ and setting $m_n = 0$ when $k < n-1$.

The set of all subordinate coderivations $\mathrm{Coder}(\bigwedge(sV) \otimes W)$ forms a graded Lie algebra under the graded commutator bracket. The bracket respects subordination, that is to say, if m_i and m_j are l_i and l_j-subordinate coderivations respectively, then $[m_i, m_j]$ is a coderivation subordinate to $[l_i, l_j]$.

Definition 2.3. *An l_n-subordinate map m_n is a W-*derivation source *if for every $sv^{\wedge}_{[1 \text{ to } n-1]}$ in $(sV)^{\wedge n-1}$, the map $m_n(sv^{\wedge}_{[1 \text{ to } n-1]} \otimes (\quad)) : W \to W$ is a graded derivation on W.*

The subset of all W-derivation sources in $\mathrm{Coder}(\bigwedge(sV) \otimes W)$ forms a graded Lie subalgebra denoted by $\mathrm{Coder}^W(\bigwedge(sV) \otimes W)$. The subset of all coderivations $l_n \in \mathrm{Coder}(\bigwedge(sV))$ which admit a subordinate W-derivation source m_n forms a graded Lie subalgebra which we will denote by $\mathrm{Coder}^W(\bigwedge(sV))$.

We can extend any map $f_n : (sV)^{\wedge n} \to W$ to function from $\bigwedge(sV)$ into $\bigwedge(sV) \otimes W$ by setting $f_n = 0$ for $k < n$ and, for $k \geqslant n$, setting $f_n(sv^{\wedge}_{[1 \text{ to } k]}) =$

$$\sum_{\substack{\sigma \\ (k-n,n)}} \kappa_{\mathbf{id}}(\sigma)\, \kappa_{\mathbf{id}}(f_n; sv^{\wedge}_{\sigma[1 \text{ to } k-n]}) sv^{\wedge}_{\sigma[1 \text{ to } k-n]} \otimes f_n(sv^{\wedge}_{\sigma[k-n+1 \text{ to } k]}). \tag{4}$$

The set of all such maps is denoted by $\mathrm{Hom}(\bigwedge(sV), \bigwedge(sV) \otimes W)$. The algebra structure on W extends to a graded commutative algebra structure on $\bigwedge(sV) \otimes W$, which provides $\mathrm{Hom}(\bigwedge(sV), \bigwedge(sV) \otimes W)$ with a cup product. If f_n and g_s are maps in $\mathrm{Hom}(\bigwedge(sV), \bigwedge(sV) \otimes W)$. Then $f_n \smile g_s$ is given by setting $(f_n \smile g_s)(sv^{\wedge}_{[1 \text{ to } n+s]}) =$

$$\sum_{\substack{\rho \\ (n,s)}} \kappa_{\mathbf{id}}(\rho)\, \kappa_{\mathbf{id}}(g_s; sv^{\wedge}_{\rho[1 \text{ to } n]}) f_n(sv^{\wedge}_{\rho[1 \text{ to } n]}) \cdot g_s(sv^{\wedge}_{\rho[n+1 \text{ to } n+s]})$$

on $(sV)^{\wedge n+s}$, where $\cdot$ represents the the multiplication on W (henceforth, the $\cdot$ will be suppressed). The map $f_n \smile g_s$ is then extended to all of $\bigwedge(sV)$ as in equation (4).

If the graded module sV is a module over the graded commutative algebra W, then $\bigwedge(sV)$ is a module over the tensor algebra $\mathrm{T}W$. We will follow our general convention and denote $w_1sv_1 \wedge \cdots \wedge w_nsv_n$ by $wsv^{\wedge}_{[1 \text{ to } n]}$. The *unraveling map*

$$\begin{aligned} \mathrm{U} : \bigwedge(sV) &\to W \otimes \bigwedge(sV) \\ wsv_{[1 \text{ to } n]} &\mapsto \mathrm{u}([1 \text{ to } n]) w^{\cdot}_{[1 \text{ to } n]} \otimes sv_{[1 \text{ to } n]}, \end{aligned}$$

where $\mathrm{u}([1 \text{ to } n])$ is the sign produced when moving the w_i's past the sv_j's. The map U respects the coalgebra structure of $\bigwedge(sV)$.

Definition 2.4. *A map* $f_n : (sV)^{\wedge n} \to W$ *is* W-linear *if*

$$f_n(wsv^{\wedge}_{[1 \text{ to } n]}) = \mathrm{u}([1 \text{ to } n])\ \kappa_{\mathbf{id}}(f_n; w^{\cdot}_{[1 \text{ to } n]}) w^{\cdot}_{[1 \text{ to } n]} f_n(sv^{\wedge}_{[1 \text{ to } n]}).$$

Likewise, a map $m_n : (sV)^{\wedge n-1} \otimes W \to W$ *is* W-linear *if*
$m_n(wsv^{\wedge}_{[1 \text{ to } n-1]} \otimes w) =$

$$\mathrm{u}([1 \text{ to } n-1])\ \kappa_{\mathbf{id}}(m_n; w^{\cdot}_{[1 \text{ to } n-1]}) w^{\cdot}_{[1 \text{ to } n-1]} m_n(sv^{\wedge}_{[1 \text{ to } n-1]} \otimes w).$$

For a map $m_n : (sV)^{\wedge n-1} \otimes W \to W$, we define the V*-extension* map $m^e_n : (sV)^{\wedge n} \to V$ by setting

$$m^e_n(wsv^{\wedge}_{[1 \text{ to } n]}) = \sum_{\substack{\rho \\ (n-1,1)}} \kappa_{\mathbf{id}}(\rho) m_n(wsv^{\wedge}_{\rho[1 \text{ to } n-1]} \otimes w_{\rho(n)}) sv_{\rho(n)}.$$

Definition 2.5. *Given a* W*-linear,* W*-derivation source* $m_n \in \mathrm{Coder}^W(\bigwedge(sV) \otimes W)$ *which is subordinate to* $l_n \in \mathrm{Coder}^W(\bigwedge(sV))$*, the map* l_n *is said to* rest *on* m_n *if* $l_n(wsv^{\wedge}_{[1 \text{ to } n]}) =$

$$\mathrm{u}([1 \text{ to } n])\ \kappa_{\mathbf{id}}(l_n; w^{\cdot}_{[1 \text{ to } n]}) w^{\cdot}_{[1 \text{ to } n]} l_n(sv^{\wedge}_{[1 \text{ to } n]}) + m^e_n(wsv^{\wedge}_{[1 \text{ to } n]}).$$

The subset of $\mathrm{Coder}^W(\bigwedge(sV))$ of all coderivations l_p which rest on W-linear W-derivation sources m_p forms $\mathrm{Coder}^W_W(\bigwedge(sV))$. Similarly, the subset of $\mathrm{Coder}^W(\bigwedge(sV) \otimes W)$ consisting of all W-linear W-derivation sources m_p subordinate to maps l_p, which in turn rest on m_p, forms $\mathrm{Coder}^W_W(\bigwedge(sV) \otimes W)$.

Suppose $m_p \in \mathrm{Coder}^W_W(\bigwedge(sV) \otimes W)$ is subordinate to $l_p \in \mathrm{Coder}(\bigwedge(sV))$. Remarkably, the map

$$\langle m_p, \ \ \rangle : \mathrm{Hom}_W(\bigwedge(sV), \bigwedge(sV) \otimes W) \to \mathrm{Hom}_W(\bigwedge(sV), \bigwedge(sV) \otimes W),$$

given by setting

$$\langle m_p, f_n \rangle = m_p f_n - \kappa_{\mathbf{id}}(m_p; f_n) f_n l_p$$

on $(sV)^{\wedge n+p-1}$ and extending $\langle m_p, f_n \rangle$ to all of $\bigwedge(sV)$ as in equation (4) is well-defined [**Kje01**]. This map exposes the *shared* Lie module structure on $\mathrm{Hom}_W(\bigwedge(sV), \bigwedge(sV) \otimes W)$ [**Kje01**]. In a sense, it is simultaneously a Lie-module over *both* $\mathrm{Coder}^W_W(\bigwedge(sV) \otimes W)$ and $\mathrm{Coder}^W_W(\bigwedge(sV))$ in that $\langle [m_i, m_j], \ \ \rangle = \langle m_i, \langle m_j, \ \ \rangle\rangle - \kappa_{\mathbf{id}}(m_i; m_j) \langle m_j, \langle m_i, \ \ \rangle\rangle$. The map $\langle m_p, \ \ \rangle$ acts as a derivation with respect to the cup product.

Homotopy (and non-homotopy) Lie algebras and their modules

Strongly homotopy Lie algebras (shLie algebras) first appeared implicitly in [**Sul77**] and explicitly in [**SS**] in the context of deformation theory. A concise introduction to shLie algebras is found in [**LM95**]. The definitions below were lifted directly from [**LM95**] and [**LS93**] and then modified in [**Kje01**] to fit the language of this paper. The machinery used in deformation theory (see, for example, [**GS90**]) will prove useful in the process of constructing an shLie structure and an shLie module structure on the resolution of the Lie-Rinehart pair in §3, although we will not use the language in the traditional way.

Definition 2.6. *An* $\mathcal{L}(m)$-structure *on a graded module* sL *consists of a system of coderivations* $\{l_k : \bigwedge(sL) \to \bigwedge(sL) \;:\; 1 \leqslant k \leqslant m \leqslant \infty, k \neq \infty\}$ *which are extensions of maps* $l_k : (sL)^{\wedge k} \to sL$*. Each* l_k *has degree* -1*. Moreover, the following generalized form of the Jacobi identity is satisfied for* $n \leqslant m$*:*

$$\mathcal{J}_{\mathrm{ID}\,n} = \frac{1}{2} \sum_{\substack{i+j=n+1 \\ i,j\geqslant 1}} [l_j, l_i] = 0 \tag{5}$$

on $\bigwedge(sL)$*. If* L *admits an* $\mathcal{L}(\infty)$-structure*, then* L *is a* strongly homotopy Lie algebra (an shLie algebra).

For this paper, we will view a standard Lie algebra $s\mathfrak{g}$ as a degenerate shLie algebra $\bigwedge(s\mathfrak{g})$ where all coderivations $\widehat{l_i}$ are zero except $\widehat{l_2}$. We decorate the bracket $\widehat{l_2}$ on $s\mathfrak{g}$ to distinguish it from l_2 on the shLie algebra which resolves $\bigwedge(s\mathfrak{g})$.

The following maps will be useful in §3, where we will use a deformation theoretic approach to construct an shLie algebra together with an shLie module.

Definition 2.7. *For* $n > 1$*, the* n^{th} Jacobi coderivation $\mathcal{J}_n : \bigwedge(sL) \to \bigwedge(sL)$ *is given by*

$$\mathcal{J}_n = -\frac{1}{2} \sum_{\substack{i+j=n+1 \\ i,j>1}} [l_j, l_i].$$

For an $\mathcal{L}(m)$-algebra, $\mathcal{J}_n = [l_1, l_n]$ for every $2 < n < m$. Since $[l_1, l_n] = l_1 l_n + (-1)^{n-1} l_n l_1$ and l_1 is a differential on L, it will be useful to have a name for the following map even though it is not a coderivation on $\bigwedge(sL)$.

Definition 2.8. *For* $n > 1$*, the* n^{th} Jacobi obstruction map $\mathcal{J}_{\mathrm{OBST}\,n} : (sL)^{\wedge n} \to sL$ *is given by*

$$\mathcal{J}_{\mathrm{OBST}\,n} = \mathcal{J}_n - l_n l_1 = -\sum_{\substack{i+j=n+1 \\ i>0\;,\; j>1}} l_j l_i$$

In the language of deformation theory (see, among others, [**Ger63**], [**Ger64**], [**Ger66**], [**Ger68**], [**GS90**], [**GS88**], [**FGV95**] and [**FGV**]), an $\mathcal{L}(m)$-structure on L may be extended to an $\mathcal{L}(m+1)$-structure if there is a linear map $l_{m+1} : L^{\wedge m+1} \to L$ such that $\mathcal{J}_{\mathrm{ID}\,m+1} = 0$ or, equivalently, such that $\frac{1}{2}[l_1, l_{m+1}] = \mathcal{J}_{m+1}$. What we are deforming here is a bit obscure and less important than the process of extending an $\mathcal{L}(m)$-structure. The graded Lie algebra Coder$(\bigwedge L)$, together with the differential given by $[l_1,\ \ ]$, governs the extension theory of $\mathcal{L}(m)$-structures on $\bigwedge L$. If we were viewing the construction of an $\mathcal{L}(\infty)$-structure strictly as a deformation theory problem rather than simply borrowing the machinery, the map $\mathcal{J}_n$ would be called the n^{th} obstruction. However, in the context of our construction in §3, the use of the map $\mathcal{J}_{\text{OBST}n}$ is more consistent with the common notion of an obstruction. The following lemmas concerning $\mathcal{J}_n$ and $\mathcal{J}_{\text{OBST}n}$ will also be useful in §3.

Lemma 2.9. *The coderivation $\mathcal{J}_n$ is a cocycle, i.e., $[l_1, \mathcal{J}_n] = 0$.*

Proof of Lemma 2.9. We will show that $[l_1, -4\mathcal{J}_n] = 0$.

$$[l_1, -4\mathcal{J}_n] = \left[l_1, \sum_{\substack{i+j=n+1 \\ i,j\geqslant 1}} 2[l_j, l_i] \right] = \left[l_1, -\sum_{\substack{i+j=n+1 \\ i,j\geqslant 1}} ([l_j, l_i] + [l_i, l_j]) \right]$$
$$= \sum_{\substack{i+j=n+1 \\ i,j\geqslant 1}} ([l_1, [l_j, l_i]] + [l_1, [l_i, l_j]]) .$$

Since $[l_1, [l_j, l_i]] = [[l_1, l_j], l_i] - [[l_1, l_i], l_j]$, it follows that

$$\begin{aligned} &[l_1, [l_j, l_i]] + [l_1, [l_i, l_j]] \\ &\qquad = [[l_1, l_j], l_i]] - [[l_1, l_i], l_j]] + [[l_1, l_i], l_j]] - [[l_1, l_j], l_i]] \\ &\qquad = 0. \end{aligned}$$

Therefore, $[l_1, \mathcal{J}_n] = -\frac{1}{4}[l_1, -4\mathcal{J}_n] = 0$. □

Lemma 2.10. *If sL admits an $\mathcal{L}(n)$-structure, then $l_1 \mathcal{J}_{\text{OBST}n} = 0$ on $(sL)^{\wedge n}$.*

Proof of Lemma 2.10. The map $\mathcal{J}_{\text{OBST}n} = -\sum_{\substack{i+j=n+1 \\ i>0,j>1}} (-1)^{i(j-1)} l_j l_i$, so

$$l_1 \mathcal{J}_{\text{OBST}n} = l_1 \left(-\sum_{\substack{i+j=n+1 \\ i>0,j>1}} l_j l_i \right) = -\sum_{\substack{i+j=n+1 \\ i>0,j>1}} l_1(l_j l_i)$$
$$= -\sum_{\substack{i+j=n+1 \\ i>0,j>1}} (l_1 l_j) l_i.$$

We substitute $l_1 l_j$ with $-\sum_{\substack{s+r=j+1 \\ s>0,r>1}} l_r l_s$ and reassociate so that

$$\sum_{\substack{i+j=n+1 \\ i>0,j>1}} \left(\sum_{\substack{s+r=j+1 \\ s>0,r>1}} l_r l_s \right) l_i = \sum_{\substack{i+s+r=n+2 \\ i,s>0,r>1}} l_r(l_s l_i).$$

If we let $i+s=t+1$, the sum above becomes

$$\sum_{\substack{t+r=n+1\\ t>0,r>1}} l_r \left(\sum_{\substack{i+s=t+1\\ i,s>0}} l_s l_i \right) = \sum_{\substack{t+r=n+1\\ t>0,r>1}} l_r \mathcal{J}_{\mathrm{ID}\,t}.$$

Since $\mathcal{J}_{\mathrm{ID}\,t}=0$ on $(sL)^{\wedge t}$ for all $1\leqslant t\leqslant n-1$, the final sum above is zero. □

Definition 2.11. [**LM95**] *Let* (sL,l_i) *be an* $\mathcal{L}(p)$*-algebra* $(0<p<\infty)$ *and let* M *be a differential graded module with differential* m_1*. Then a* left $\mathcal{L}(k)$-module structure over sL *on* M *(for* $k\leqslant p$*) is a collection of coderivations* $\{m_n : \bigwedge(sL)\otimes M \to \bigwedge(sL)\otimes M \ : \ 1\leqslant n\leqslant k, n\neq\infty\}$*. Each* m_n *is* l_n*-subordinate and the extension of a map* $m_n : L^{\wedge n-1}\otimes M\to M$ *such that the* n^{th} action identity map

$$\mathcal{ACT}_{\mathrm{ID}\,n} \quad = \quad \frac{1}{2}\sum_{\substack{i+j=n+1\\ i,j\geqslant 1}} [m_j,m_i] \quad = \quad 0$$

on $\bigwedge(sL)\otimes M$*. The differential graded module* M *is a* strongly homotopy Lie module over sL (or an sL-shLie module) *if* sL *admits an* $\mathcal{L}(\infty)$*-structure and* M *is a module with respect to that* $\mathcal{L}(\infty)$*-structure.*

Definition 2.11 implies that the differential m_1 on M must be l_1-subordinate. It is simple to verify that m_1 is a differential on $\bigwedge L\otimes M$.

When M is a differential graded commutative algebra, we will insist the maps m_i be M-derivation sources, that is to say, the map $m_i(v_{[1 \text{ to } i-1]}\otimes(\quad)) : M\to M$ is in $\mathrm{Der}(M)$ for every $v^{\wedge}_{[1 \text{ to } i-1]}\in L^{\wedge i-1}$.

In this paper a module $\mathbf{B}$ over a Lie algebra $\mathfrak{g}$ is a degenerate shLie module $\bigwedge(s\mathfrak{g})\otimes\mathbf{B}$ over the shLie algebra $\bigwedge(s\mathfrak{g})$, where all the coderivations m_i are zero except for $\widehat{m_2}$ (again, we decorate this particular map).

Below, we define the maps $\mathcal{ACT}_n$ and $\mathcal{ACT}_{\mathrm{OBST}\,n}$, the analogs of $\mathcal{J}_n$ and $\mathcal{J}_{\mathrm{OBST}\,n}$.

Definition 2.12. *For* $n>1$*, the* n^{th} action map $\mathcal{ACT}_n : \bigwedge(sL)\otimes M\to\bigwedge(sL)\otimes M$ *is given by*

$$\mathcal{ACT}_n \quad = \quad -\frac{1}{2}\sum_{\substack{i+j=n+1\\ i,j>1}} [m_j,m_i].$$

Definition 2.13. *For* $n>1$*, the* n^{th} action obstruction map $\mathcal{ACT}_{\mathrm{OBST}\,n}$: $(sL)^{\wedge n-1}\otimes M\to M$ *is given by*

$$\mathcal{ACT}_{\mathrm{OBST}\,n} = \mathcal{ACT}_n - m_n m_1 = -\sum_{\substack{i+j=n+1\\ i>0,j>1}} m_j m_i.$$

If sL is an $\mathcal{L}(p)$-algebra and M is an $\mathcal{L}(n)$-module over sL with $n<p-1$ (or $n<\infty$ if $p=\infty$), we can extend the $\mathcal{L}(n)$-module structure to an $\mathcal{L}(n+1)$-module structure if an l_{n+1}-subordinate M-derivation source $m_{n+1} : (sL)^{\wedge n}\otimes M\to M$ can be found such that $\mathcal{ACT}_{\mathrm{ID}\,n+1}=0$. Again, what exactly we are deforming here is less important than the language and machinery of deformation theory.

The following lemmas and their proofs mimic lemmas 2.9 and 2.10 and their proofs.

LEMMA 2.14. *The coderivation $\mathcal{A}CT_n$ is a cocycle, i.e.,* $[m_1, \mathcal{A}CT_n] = 0$.

LEMMA 2.15. *If L admits an $\mathcal{L}(p)$-structure and M is an L-$\mathcal{L}(n)$-module for $n < p-1$ (or $n < \infty$ if $p = \infty$), then $m_1 \mathcal{A}CT_{\mathrm{OBST}\,n} = 0$ on $L^{\wedge n-1} \otimes M$.*

PROPOSITION 2.16. [**LS93**] *If (sL, l_i) is an shLie algebra, the map $\mathrm{D}_{sL} = \sum_{i=1}^{\infty} l_i$ is a differential on $\bigwedge(sL)$, i.e. it is a map of degree -1 such that $\mathrm{D}_{sL} \circ \mathrm{D}_{sL} = 0$.*

Proof of Proposition 2.16. A proof can be found in [**LS93**]. Here the proof is essentially the same but takes advantage of the bracket of coderivations:

$$\mathrm{D}_{sL} \circ \mathrm{D}_{sL} = \left(\sum_{j=1}^{\infty} l_j\right) \circ \left(\sum_{i=1}^{\infty} l_i\right) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} l_j l_i = \sum_{n=1}^{\infty} \sum_{\substack{i+j=n+1 \\ i,j \geqslant 1}} l_j l_i.$$

Using equation in definition 2.6, we see that

$$\sum_{n=1}^{\infty} \sum_{\substack{i+j=n+1 \\ i,j \geqslant 1}} l_j l_i = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\substack{i+j=n+1 \\ i,j \geqslant 1}} [l_j, l_i] = \frac{1}{2} \sum_{n=1}^{\infty} \mathcal{J}_{\mathrm{ID}\,n} = 0.$$

Note also that D_{sL} is a degree -1 coderivation, so $\mathrm{D}_{sL} \circ \mathrm{D}_{sL} = \frac{1}{2}[\mathrm{D}_{sL}, \mathrm{D}_{sL}]$ and the graded skew-commutativity of the bracket ensures that $[\mathrm{D}_{sL}, \mathrm{D}_{sL}] = 0$. □

For the Lie algebra $\bigwedge(s\mathfrak{g})$, the differential $\mathrm{D}_{s\mathfrak{g}} = \widehat{l_2}$.

PROPOSITION 2.17. *The map $\mathrm{D}_M = \sum_{i=1}^{\infty} m_i$ is a differential on $\bigwedge(sL) \otimes M$, i.e., $\mathrm{D}_M \circ \mathrm{D}_M = 0$.*

Proof of Proposition 2.17. Isomorphic to the proof of proposition 2.16. Notice that D_M is D_{sL}-subordinate and that $\mathrm{D}_M \circ \mathrm{D}_M = \frac{1}{2}[\mathrm{D}_M, \mathrm{D}_M] = 0$. □

For the Lie module $\bigwedge(s\mathfrak{g}) \otimes \mathbf{B}$, the differential $\mathrm{D}_{\mathbf{B}} = \widehat{m_2}$.

Homotopy Chevalley-Eilenberg cohomology

Definition 2.18. *A* homotopy Chevalley-Eilenberg pair (M, L) *consists of an shLie algebra L and an L-shLie module M.*

A multi-linear graded alternating function $F_n : L^{\times n} \to M$ can be seen as a linear function $F_n : (sL)^{\wedge n} \to M$. So the homotopy Chevalley-Eilenberg algebra $Alt_k(L, M)$ is isomorphic to $\mathrm{Hom}_k(\bigwedge(sL), \bigwedge(sL) \otimes M)$ with the differential $\delta_{hCE} = \langle \mathrm{D}_M, \ \rangle$

PROPOSITION 2.19. $\delta_{hCE} \circ \delta_{hCE} = 0$.

Proof of Proposition 2.19. $\delta_{hCE} \circ \delta_{hCE} = \langle \mathrm{D}_M, \langle \mathrm{D}_M, \ \rangle\rangle = \left\langle \frac{1}{2}[\mathrm{D}_M, \mathrm{D}_M], \ \right\rangle = 0$. □

The cohomology with respect to δ_{hCE} is the *homotopy Chevalley-Eilenberg cohomology* of sL with coefficients in M. A quick check shows that the Chevalley-Eilenberg cohomology complex for the pair $(\mathbf{B}, s\mathfrak{g})$ is simply a degenerate form of the homotopy version where $\delta_{CE} = \delta_{hCE} = \langle \widehat{m_2}, \;\; \rangle$ because $\widehat{m_i}$ and $\widehat{l_i}$ are zero for all $i \neq 2$.

When M is a differential graded commutative algebra, the homotopy Chevalley-Eilenberg complex $\mathrm{Hom}_k(\bigwedge(sL), \bigwedge(sL) \otimes M)$ is a differential graded commutative algebra. The differential δ_{hCE} acts as a derivation with respect to this multiplication.

Homotopy Lie-Rinehart pairs and homotopy Rinehart cohomology

The *homotopy Rinehart complex* is a straightforward generalization of the Rinehart complex in the ungraded setting. Since the Rinehart complex is defined only for Lie-Rinehart pairs $(\mathbf{B}, s\mathfrak{g})$, we must define what constitutes a homotopy Lie-Rinehart pair (M, sL). The subset $Alt_M(sL, M)$ of $Alt_k(sL, M)$ consisting of all M-linear graded alternating functions is isomorphic to $\mathrm{Hom}_M(\bigwedge(sL), \bigwedge(sL) \otimes M)$.

Definition 2.20. *A* homotopy Lie-Rinehart pair (M, sL) *consists of a differential graded commutative algebra* (M, m_i) *which is an shLie module over the shLie algebra* (sL, l_i)*, which in turn is an* M*-module. Moreover, the following two homotopy* Lie-Rinehart *relations must be satisfied for all* $i \geqslant 1$*:*

$(hLRa_i)$: The shLie module structure map m_i is M-linear.

$(hLRb_i)$: The shLie structure map l_i rests on m_i (definition 2.5).

An $\mathcal{L}(p)$-Lie-Rinehart pair $((M, m_i), (sL, l_i))$ *has maps* m_i *and* l_i *which satisfy* $(hLRa_i)$ *and* $(hLRb_i)$ *for* $1 \leqslant i \leqslant p$.

The proposition below follows from the fact that, for every i, the image of F under the map $\langle \widehat{m_i}, \;\; \rangle$ is again a map in $\mathrm{Hom}_M(\bigwedge(sL), \bigwedge(sL) \otimes M)$ for all i [**Kje01**].

Proposition 2.21. *If* F *is* M*-linear, then so is* $\delta_{hCE}F$*.*

We conclude that the subset of all M-linear functions in $\mathrm{Hom}_k(\bigwedge(sL), \bigwedge(sL) \otimes M)$ forms a subcomplex $\mathcal{R} = \mathrm{Hom}_M(\bigwedge(sL), \bigwedge(sL) \otimes M)$ with differential $\delta_{\mathcal{R}} = \langle \mathrm{D}_M, \;\; \rangle$. The differential, together with the cup product, provides the homotopy Rinehart complex $\mathcal{R}$ with the structure of a differential graded commutative algebra. The cohomology of $\mathcal{R}$ with respect to $\delta_{\mathcal{R}}$ is the *homotopy Rinehart* cohomology of sL with coefficients in M.

The following two propositions for general homotopy Lie-Rinehart pairs (M, sL) are observations based on the fact that both $\mathrm{Coder}_M^M(\bigwedge(sL) \otimes M)$ and $\mathrm{Coder}_M^M(\bigwedge(sL))$ are graded Lie algebras.

Proposition 2.22. *The maps* $\mathcal{ACT}_{\mathrm{ID}n}$ *and* $\mathcal{ACT}_n$ *are are* $\mathcal{J}_{\mathrm{ID}n}$ *and* $\mathcal{J}_n$*-subordinate, respectively; both are* M*-linear and* M*-derivation sources.*

The significance of this proposition lies in recognizing that, as a result, each of the maps $\mathcal{ACT}_{\mathrm{ID}n}$ and $\mathcal{ACT}_n$ are completely determined by their image on a generating set for $\bigwedge(sL)$ as an M-module and a generating set for M, so long as the maps $\mathcal{J}_{\mathrm{ID}n}$ and $\mathcal{J}_n$ are available.

PROPOSITION 2.23. *The maps $\mathcal{J}_{\mathrm{ID}_n}$ and $\mathcal{J}_n$ rest on $\mathcal{A}CT_{\mathrm{ID}_n}$ and $\mathcal{A}CT_n$, respectively.*

Here again, the implication is that the maps $\mathcal{J}_{\mathrm{ID}_n}$ and $\mathcal{J}_n$ are completely determined by their image on a generating set of $\bigwedge(sL)$ as an M-module.

Homotopy Lie-Rinehart resolutions of Lie-Rinehart pairs

Let $(\mathbf{B}, s\mathfrak{g})$ be a Lie-Rinehart pair over a k-algebra A with module structure maps $\widehat{\mu} : \mathbf{B} \otimes s\mathfrak{g} \to s\mathfrak{g}$ and $\widehat{m_2} : s\mathfrak{g} \otimes \mathbf{B} \to \mathbf{B}$. Let $\widehat{l_2}$ denote the bracket on $s\mathfrak{g}$ and $\pi_{\mathbf{B}}$ denote the multiplication on $\mathbf{B}$. The basic ingredients of a homotopy Lie-Rinehart resolution for a Lie-Rinehart pair $(\mathbf{B}, s\mathfrak{g})$ are (sL, l_i) and (M, m_i, π_M) where $\mathrm{H}_{l_1}(sL) = s\mathfrak{g}$ and $\mathrm{H}_{m_1}(M) = \mathbf{B}$.

Definition 2.24. *A* homotopy Lie-Rinehart resolution *of a Lie-Rinehart pair* $(\mathbf{B}, s\mathfrak{g})$ *over an algebra A is a homotopy Lie-Rinehart pair (M, sL) over A, such that*

(a) the shLie algebra (sL, l_i), seen as the coalgebra $\bigwedge(sL)$, resolves $\bigwedge(s\mathfrak{g})$, i.e.,

$$\mathrm{H}_{l_1}(\bigwedge(sL)) = \bigwedge(s\mathfrak{g}), \tag{6}$$

where (following the physicists' notation) H_{l_1} denotes the homology with respect to the differential l_1,

(b) the differential graded commutative algebra (M, m_i, π_M) satisfies

$$\mathrm{H}_{m_1}(\bigwedge(sL) \otimes M) = \bigwedge(s\mathfrak{g}) \otimes \mathbf{B}, \tag{7}$$

(c) the dgca (M, m_i, π_M) also satisfies

$$\mathrm{H}_{m_1}(M \otimes M) = \mathbf{B} \otimes \mathbf{B}. \tag{8}$$

Furthermore, the following conditions hold:

i. $\mathrm{H}_{l_1}(l_2) = \widehat{l_2}$ *on* $\bigwedge(s\mathfrak{g})$.

ii. $\mathrm{H}_{m_1}(m_2) = \widehat{m_2}$ *on* $\bigwedge(s\mathfrak{g}) \otimes \mathbf{B}$.

iii. $\mathrm{H}_{m_1}(\pi_M) = \pi_{\mathbf{B}}$ *on* $\mathbf{B} \otimes \mathbf{B}$.

In the proposition below, we state conditions on the homotopy Lie-Rinehart pair that guarantee it satisfies definition 2.24. The proof is found in [**Kje01**].

PROPOSITION 2.25. *Let (M, sL) be a homotopy Lie-Rinehart pair. Suppose the differential graded algebra (M, m_i, π_M) is a projective resolution of $\mathbf{B}$ over A which respects the algebra structure on $\mathbf{B}$, i.e., condition iii of definition 2.24 is satisfied. Suppose (sL, l_i) is a projective resolution $s\mathfrak{g}$. Furthermore,*

(a) $\mathrm{H}_{l_1}(l_2) = \widehat{l_2}$ *on* $s\mathfrak{g} \wedge s\mathfrak{g}$ *and*

(b) $\mathrm{H}_{m_1}(m_2) = \widehat{m_2}$ *on* $s\mathfrak{g} \otimes \mathbf{B}$.

Then (M, sL) is a homotopy Lie-Rinehart resolution of $(\mathbf{B}, s\mathfrak{g})$.

We used a spectral sequence argument in [**Kje01**] to prove the following theorem.

THEOREM 2.26. *If the pair (M, sL) is a projective homotopy Lie-Rinehart resolution of the Lie-Rinehart pair $(\mathbf{B}, s\mathfrak{g})$, then $(\mathcal{R}, \langle \mathrm{D}_M, \ \rangle)$ is a cohomological model for $(R, \langle \widehat{m_2}, \ \rangle)$.*

3. Constructing the homotopy Lie-Rinehart pair $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ and classical BRST cohomology

Constructing homotopy Lie-Rinehart resolutions for a Lie-Rinehart pair is not always easy, as we will see as we construct a homotopy Lie-Rinehart resolution $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ for the Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ from classical BRST cohomology. Details which are not instructive (especially sign arguments) are ommitted, but can be found in [**Kje96**].

The Lie algebra $\mathcal{I}/\mathcal{I}^2$ has already been suspended, i.e., every element of $\mathcal{I}/\mathcal{I}^2$ has degree +1 and the bracket $\widehat{l_2}$ is a coderivation on $\bigwedge \mathcal{I}/\mathcal{I}^2$. However, we will suppress the suspension indicator "s" both here and in the shLie algebra $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$.

Let the set $\mathbf{y}_\alpha = \{y_1, ..., y_s\}$ generate the coisotropic ideal $\mathcal{I}$ as an finitely presented A-module. If $\mathbf{y}_\alpha$ forms a basis of $\mathcal{I}$ over the field k, there is a unique set of structure constants $C^\delta_{\alpha\beta} \in k$, each of which is antisymmetric in its lower indices, such that $\{y_\alpha, y_\beta\} = C^\delta_{\alpha\beta} y_\delta$. (Note: We will use Einstein's summation convention throughout §3 and §4.) With structure constants, the Jacobi identity is a statement about coefficients, namely $C^\delta_{\alpha\beta}C^\epsilon_{\delta\gamma} + C^\delta_{\beta\gamma}C^\epsilon_{\delta\alpha} + C^\delta_{\gamma\alpha}C^\epsilon_{\delta\beta} = 0$ for every ϵ. When the generators do not form a basis for $\mathcal{I}$ over k, we can still find (not necessarily unique) *structure functions* $C^\delta_{\alpha\beta}$ in A such that $\{y_\alpha, y_\beta\} = C^\delta_{\alpha\beta} y_\delta$. (In the BRST context, the elements of the Poisson algebra A are functions on a symplectic manifold, hence the term structure *functions*.) The Jacobi identity becomes $(C^\delta_{\alpha\beta}C^\epsilon_{\delta\gamma} + \{C^\epsilon_{\alpha\beta}, y_\gamma\} + \text{c.p.})y_\epsilon = 0$, where c.p. stands for cyclic permutations of α, β and γ. For convenience we will set $\mathcal{J}^\epsilon$ equal to $(C^\delta_{\alpha\beta}C^\epsilon_{\delta\gamma} + \{C^\epsilon_{\alpha\beta}, y_\gamma\} + \text{c.p.})$, so that the Jacobi identity can be written as $\mathcal{J}^\epsilon y_\epsilon = 0$. The Jacobi identity is now a relation among the generators of $\mathcal{I}$.

Definition 3.1. [**Kim93**], [**FHST89**] *Let $\mathcal{I}$ be finitely generated by the set $\mathbf{y}_\alpha = \{y_1, ..., y_s\}$. The set is* irreducible *if $f^\alpha y_\alpha = 0$ implies $f^\alpha = g^{\alpha\beta} y_\beta$, where each coefficient $g^{\alpha\beta}$ is antisymmetric with respect to its indices. An ideal which has a set of irreducible generators is called* irreducible*.*

Definition 3.2. [**Kim93**], [**FHST89**] *An ideal $\mathcal{I}$ which is finitely generated by the set $\mathbf{y}_{\alpha_1} = \{y_1, y_2, \ldots, y_{s_1}\}$ admits a* complete set of reducibility functions *if there exists $\mathcal{Z}^{\alpha_{n-1}}_{\alpha_n}$ in the complement of $\mathcal{I}$, such that for $n = 2$, $\mathcal{Z}^{\alpha_1}_{\alpha_2} y_{\alpha_1} = 0$ and $f^{\alpha_1} y_{\alpha_1} = 0$ implies that $f^{\alpha_1} = g^{\alpha_2}\mathcal{Z}^{\alpha_1}_{\alpha_2} + \Upsilon^{\alpha_1\beta_1} y_{\beta_1}$, where $\Upsilon^{\alpha_1\beta_1} = -\Upsilon^{\beta_1\alpha_1}$. For $n > 2$, $\mathcal{Z}^{\alpha_{n-1}}_{\alpha_n}\mathcal{Z}^{\alpha_{n-2}}_{\alpha_{n-1}} \equiv 0 \mod \mathcal{I}$ and $f^{\alpha_{n-1}}\mathcal{Z}^{\alpha_{n-2}}_{\alpha_{n-1}} \equiv 0 \mod \mathcal{I}$ implies $f^{\alpha_{n-1}} \equiv g^{\alpha_n}\mathcal{Z}^{\alpha_{n-1}}_{\alpha_n} \mod \mathcal{I}$. An ideal which admits a complete set of reducibility functions is* reducible*.*

We consider only ideals which are either irreducible or reducible. Since the Jacobi identity is a relation among the generators of $\mathcal{I}$, when the ideal is irreducible, we have $\mathcal{J}^\epsilon y_\epsilon = 0$ implies $\mathcal{J}^\epsilon = g^{\epsilon\delta} y_\delta$. When the ideal is completely reducible, we have $\mathcal{J}^\epsilon y_\epsilon = 0$ implies $\mathcal{J}^\epsilon = g^{\alpha_2}\mathcal{Z}^\epsilon_{\alpha_2} + \Upsilon^{\epsilon\beta_1} y_{\beta_1}$.

The quotient $\mathcal{I}/\mathcal{I}^2$ inherits a Lie structure from $\mathcal{I}$ and is both an A and an $A/\mathcal{I}$-module. There is a Lie action of $\mathcal{I}$ on A given by $z \cdot f = \{z, f\}$, which passes to the quotient so that $A/\mathcal{I}$ is a Lie-module over both $\mathcal{I}$ and $\mathcal{I}/\mathcal{I}^2$. Although $(A/\mathcal{I}, \mathcal{I})$ is not a Lie-Rinehart pair, the pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ is. The actions μ and ω are given by

$$\begin{array}{rcl} \mu : A/\mathcal{I} \otimes \mathcal{I}/\mathcal{I}^2 & \to & \underline{\mathcal{I}/\mathcal{I}^2} \\ \underline{f} \otimes \bar{z} & \mapsto & \overline{fz} \end{array} \quad \text{and} \quad \begin{array}{rcl} \omega : \mathcal{I}/\mathcal{I}^2 \otimes A/\mathcal{I} & \to & A/\mathcal{I} \\ \bar{z} \otimes \underline{f} & \mapsto & \underline{\{z, f\}}, \end{array}$$

where $\_$ and $^-$ denote equivalence classes. It is a straightforward exercise to show that the Lie-Rinehart relations are satisfied.

The construction of $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$

Before we describe how we construct the homotopy Lie-Rinehart resolution $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ for the pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$, we skip ahead to STEP 1, where we review the features of the Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$ of $A/\mathcal{I}$ (see [**Tat57**] and [**Joz72**]), a free differential graded commutative algebra over A which is a projective resolution of $A/\mathcal{I}$.

STEP 1: The Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$

The Koszul complex $\mathbf{K}_{(0)}$ is a dgca over A and is isomorphic to $A \otimes \bigwedge\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_s\}$, where the $\mathcal{P}_i$'s are assigned degree 1 and are in one-to-one correspondence with the generators y_i. (Following the physics literature, we call the $\mathcal{P}_i$'s *antighosts* [**HT92**].) The differential m_1 maps each $\mathcal{P}_i$ to y_i and m_1 is extended as an A-linear graded derivation. The zeroth homology of the Koszul complex is $A/\mathcal{I}$ even if higher homologies do not vanish. Using the inductive method Tate introduced in [**Tat57**], we kill an unwanted nontrivial homology class $[z_q]$ of degree q by adjoining a formal variable $\mathcal{P}_{z_q}$ of degree $q+1$ to the existing Koszul complex. The variable $\mathcal{P}_{z_q}$ maps to z_q under the differential, killing the unwanted homology. But these new variables may introduce new nontrivial homology on higher levels, which must in turn be killed. More formally, if $\mathcal{I}$ is reducible and generated by the set $\mathbf{y}_{\alpha_1} = \{y_1, ..., y_{s_1}\}$, the zeroth homology of the Koszul complex $\mathbf{K}_{(0)} = A \otimes \bigwedge\{\mathcal{P}_{\alpha_1}\}$ is $A/\mathcal{I}$, but $\mathrm{H}_1(\mathbf{K}_{(0)})$ may not be zero. The unwanted homology classes form a module generated by the cycles $z_{\alpha_2} = \mathcal{Z}^{\alpha_1}_{\alpha_2}\mathcal{P}_{\alpha_1}$, so we adjoin degree 2 antighosts $\mathcal{P}_{\alpha_2}$ and setting $m_1(\mathcal{P}_{\alpha_2}) = z_{\alpha_2}$. The resulting differential graded commutative algebra $\mathbf{K}_{(1)} = A \otimes \bigwedge\{\mathcal{P}_{\alpha_1}, \mathcal{P}_{\alpha_2}\}$ has homology $\mathrm{H}_0(\mathbf{K}_{(1)}) = A/\mathcal{I}$ and $\mathrm{H}_1(\mathbf{K}_{(1)}) = 0$. Again, $\mathrm{H}_2(\mathbf{K}_{(1)})$ may contain nontrivial classes, either as a result of introducing the level two antighosts or because they were already present in $\mathbf{K}_{(0)}$. Regardless of origin, we choose a generating set z_{α_3} for the nontrivial 2-cycles and adjoin degree 3 antighosts $\mathcal{P}_{\alpha_3}$ to kill them. The third homology of $\mathbf{K}_{(2)}$ may be nonzero, so we continue the process. The complete reducibility of $\mathcal{I}$ guarantees that a representative for each nontrivial n-class can be chosen so that each $z_{\alpha_{n+1}} = f^{\alpha_n}_{\alpha_{n+1}}\mathcal{P}_{\alpha_n} +$ "more". The limit $\mathbf{K}_{(\infty)}$ is the Koszul-Tate projective resolution $\mathbf{K}_{A/\mathcal{I}}$ of $A/\mathcal{I}$ and is isomorphic to $A \otimes \bigwedge P$, where P is the graded vector space with basis $\{\mathcal{P}_{\alpha_1}, \mathcal{P}_{\alpha_2}, \mathcal{P}_{\alpha_3}, \dots\}$. The product $\pi_{\mathbf{K}}$ on $\mathbf{K}_{A/\mathcal{I}}$ is a chain map. It is straightforward to check that $\mathrm{H}_{m_1}(\pi_{\mathbf{K}}) = \pi_{A/\mathcal{I}}$, that is to say, condition (iii) in definition 2.24 is satisfied.

A chain $f^I\mathbf{P}_I$ in $\mathbf{K}_{A/\mathcal{I}}$, where $\mathbf{P}_I = \mathcal{P}_{\alpha_{n_1}} \cdots \mathcal{P}_{\alpha_{n_s}}$, has homogeneous degree, which we again denote by I. In context, we will not confuse the index I with the degree I. The boundary of an antighost $\mathcal{P}_{\alpha_n}$ will be denoted as $\mathcal{Z}^{\alpha_{n-1}}_{\alpha_n}\mathcal{P}_{\alpha_{n-1}} + \mathrm{Z}^{\hat{I}}_{\alpha_n}\mathbf{P}_{\hat{I}}$ whenever the "linear" term plays an important role. Otherwise, we will set

$m_1(\mathcal{P}_{\alpha_n}) = \mathrm{Z}^I_{\alpha_n}\mathbf{P}_I$. Whenever possible, an element of $\mathbf{K}_{A/\mathcal{I}}$ will be denoted simply by $\mathcal{X}$.

When A is a Poisson algebra and the ideal $\mathcal{I}$ is a multiplicative ideal, the Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$ admits a graded Poisson bracket given by $\{f^I\mathbf{P}_I, g^J\mathbf{P}_J\} = \{f^I, g^J\}\mathbf{P}_I\mathbf{P}_J$.

The Procedure

The construction of $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ relies on four features: the Poisson structure on the Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$, the relations among the generators of $\mathcal{I}$, the fact that both $\mathbf{K}_{A/\mathcal{I}}$ and $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ are projective resolutions of $A/\mathcal{I}$ and $\mathcal{I}/\mathcal{I}^2$, respectively, and an A-module coderivation Ψ from $\mathbf{K}_{A/\mathcal{I}}$ to $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$. A general outline of the process follows:

STEP 1: Choose a Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$ for $A/\mathcal{I}$. The differential on $\mathbf{K}_{A/\mathcal{I}}$ is m_1, which must satisfy condition (*iii*) in definition 2.24. (Already done!)

STEP 2: Construct a differential graded $\mathbf{K}_{A/\mathcal{I}}$-module $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ which is a projective resolution of $\mathcal{I}/\mathcal{I}^2$ (See propositions 3.3 and 3.4). The action $\mu : \mathbf{K}_{A/\mathcal{I}} \otimes \mathbf{K}_{\mathcal{I}/\mathcal{I}^2} \to \mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ is free, i.e., $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ is isomorphic to $\mathbf{K}_{A/\mathcal{I}} \otimes \Phi$, where Φ is a graded vector space over k which is isomorphic to the graded vector space P. (We have already suspended the graded vector space Φ. Therefore, $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ is already suspended.) We construct the differential on $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ using the coderivation Ψ. The differential l_1 rests on m_1, so the pair $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ has an $\mathcal{L}(1)$-Lie-Rinehart structure.

STEP 3: The loop—defining the homotopy Lie-Rinehart structure maps m_n and l_n. For $n \geqslant 2$, the loop extends the $\mathcal{L}(n-1)$-Lie-Rinehart structure to an $\mathcal{L}(n)$-Lie-Rinehart structure.

(a_n). Constructing m_n. Since the map m_n must satisfy the homotopy Lie-Rinehart relation $(hLRa_n)$ and be a $\mathbf{K}_{A/\mathcal{I}}$-derivation source, the map m_n is completely determined once we define it on $\Phi^{\wedge n-1} \otimes A$ and the basis elements $\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n$ of $\Phi^{\wedge n-1} \otimes P$, such that $\varphi_1 \leqslant \cdots \leqslant \varphi_n$ with respect to a degree-preserving total ordering of the preghosts:

i. Define m_n on $\Phi^{\wedge n-1} \otimes A$ by setting $m_n(\varphi_{[1 \text{ to } n-1]} \otimes f) =$

$$(-1)^n\{m_{n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}), f\}.$$

Here, the inductive assumption (step a_{n-1}iii "above") guarantees that m_n is well-defined on $\Phi^{\wedge n-1} \otimes A$. The extended Poisson bracket on $\mathbf{K}_{A/\mathcal{I}}$ in the definition guarantees that m_n will be an A-derivation source. (Does not require proof.)

ii. Verify that $\mathcal{ACT}_{\mathrm{ID}n} = 0$ on $\Phi^{\wedge n-1} \otimes A$ by showing that $m_1 m_n = \mathcal{ACT}_{\mathrm{OBST}n}$ on that subspace. (Requires proof. See proposition 3.5.)

iii. Ensure that m_n is well-defined and that $\mathcal{ACT}_{\mathrm{ID}n} = 0$ on $\Phi^{\wedge n-1} \otimes P$ by exploiting the acyclicity of $\mathbf{K}_{A/\mathcal{I}}$ to define m_n on the basis $\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n$ of $\Phi^{\wedge n-1} \otimes P$ with $\varphi_1 \leqslant \cdots \leqslant \varphi_n$. (Requires proof. See proposition 3.6.)

iv. Since $\mathcal{ACT}_{\text{ID}n}$ is completely determined by its image on the generating set $\Phi^{\wedge n-1} \otimes A$ and $\Phi^{\wedge n-1} \otimes P$ (recall Proposition 2.22), it follows that $\mathcal{ACT}_{\text{ID}n} = 0$ on all of $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}^{\wedge n-1} \otimes \mathbf{K}_{A/\mathcal{I}}$ and therefore $\mathcal{ACT}_{\text{ID}n} = 0$ on $\bigwedge \mathbf{K}_{\mathcal{I}/\mathcal{I}^2} \otimes \mathbf{K}_{A/\mathcal{I}}$. (Does not need proof.)

(b_n). Constructing l_n. The map l_n must rest on m_n and is therefore completely determined once we define it on $\Phi^{\wedge n}$.

i. Set $l_n(\varphi_{[1 \text{ to } n]}) = \Psi m_n(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)$. Since m_n is well-defined on $\Phi^{\wedge n-1} \otimes P$, it follows that the map l_n is well-defined on $\Phi^{\wedge n}$. (Does not need proof.)

ii. Verify that $\mathcal{J}_{\text{ID}n} = 0$ on $\Phi^{\wedge n}$. We do so by showing that on the space $\Phi^{\wedge n-1} \otimes P$, the sequence of equalities

$$l_1 l_n(1^{\wedge n-1} \otimes \Psi) = \Psi m_1 m_n = \Psi \mathcal{ACT}_{\text{OBST}\,n} = \mathcal{J}_{\text{OBST}\,n}(1^{\wedge n-1} \otimes \Psi)$$

holds. (Requires proof. See proposition 3.7.)

iii. Since $\mathcal{J}_{\text{ID}n}$ rests on $\mathcal{ACT}_{\text{ID}n}$ (recall Proposition 2.23) and $\mathcal{J}_{\text{ID}n} = 0$ on $\Phi^{\wedge n}$, it follows that $\mathcal{J}_{\text{ID}n} = 0$ on all of $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}^{\wedge n}$ and therefore $\mathcal{J}_{\text{ID}n} = 0$ on $\bigwedge \mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$. (Does not need proof.)

STEP 4: Verify that the conditions (i) and (ii) in definition 2.24 are met.

STEP 2: Constructing $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$

The shLie algebra $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ is the free $\mathbf{K}_{A/\mathcal{I}}$-module $\mathbf{K}_{A/\mathcal{I}} \otimes \Phi$. The graded vector space Φ is spanned by the graded basis $\{\varphi_{\alpha_1}, \varphi_{\alpha_2}, \varphi_{\alpha_3}, \ldots\}$, where there is a one-to-one correspondence between the *preghosts* φ_{α_i}, and the antighosts $\mathcal{P}_{\alpha_i}$. Each φ_{α_n} is assigned degree n. (The vector space Φ is the suspension of the vector space Φ found in [**Kje96**].) A typical element has the form $X^I \mathbf{P}_I^{\alpha_n} \varphi_{\alpha_n}$, where $X^I \mathbf{P}_I^{\alpha_n}$ is an element of $\mathbf{K}_{A/\mathcal{I}}$. As before, the degree of $X^I \mathbf{P}_I^{\alpha_n}$ in $\mathbf{K}_{A/\mathcal{I}}$ will be denoted by I. The degree of the element $X^I \mathbf{P}_I^{\alpha_n} \varphi_{\alpha_n}$, then, is the sum $(I + n)$. Whenever possible, an abbreviated form $\mathcal{X}^{\alpha_n} \varphi_{\alpha_n}$ will be used for a typical element, in which case the degree of $\mathcal{X}^{\alpha_n}$ will be denoted by α_n, again, without confusion.

When $\mathcal{I}$ is irreducible, the map $l_1 = m_1 \otimes 1$ is a differential on $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2} \approx \mathbf{K}_{A/\mathcal{I}} \otimes \Phi$, whose homology is $A/\mathcal{I} \otimes \Phi$, which is isomorphic to $\mathcal{I}/\mathcal{I}^2$ as $A/\mathcal{I}$-modules. The isomorphism is given by sending $\underline{f^{\alpha_1}} \varphi_{\alpha_1} \mapsto \overline{f^{\alpha_1} y_{\alpha_1}}$. The details are left to the reader. It follows that $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ is a resolution of $\mathcal{I}/\mathcal{I}^2$.

We define a degree 0 A-linear map $\Psi : \mathbf{K}_{A/\mathcal{I}} \longrightarrow \mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ by setting $\Psi(\mathcal{P}_{\alpha_n}) = \varphi_{\alpha_n}$ and extending Ψ as a graded derivation, i.e., $\Psi(\mathcal{P}_{\alpha_i}\mathcal{P}_{\alpha_j}) = (-1)^{ij}\mathcal{P}_{\alpha_j}\varphi_{\alpha_i} + \mathcal{P}_{\alpha_i}\varphi_{\alpha_j}$. In the irreducible case, the map Ψ is a chain map.

In the reducible case, the graded module $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ does not come equipped with a differential; we must build one. The tool we need is the map Ψ. We create l_1 on $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ so that Ψ is a chain map. We set $l_1(\varphi_{\alpha_n}) = \Psi m_1(\mathcal{P}_{\alpha_n})$ and extend l_1 to all of $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ so that it rests on m_1. The map Ψ is a chain map provided l_1 is a differential on $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$.

PROPOSITION 3.3. *$l_1 \circ l_1 = 0$ on $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$.*

Proof of Proposition 3.3. We begin by showing that $l_1(l_1(\varphi_{\alpha_n})) = 0$. Recall that Ψ is a chain map. It follows that $l_1(l_1(\varphi_{\alpha_n})) = l_1(\Psi(m_1(\mathcal{P}_{\alpha_n}))) = \Psi(m_1(m_1(\mathcal{P}_{\alpha_n}))) = 0$. Since l_1 rests on m_1, it follows that

$$\begin{aligned} l_1(l_1(\mathcal{X}^{\alpha_n}\varphi_{\alpha_n})) &= l_1(m_1(\mathcal{X}^{\alpha_n})\varphi_{\alpha_n} + (-1)^{\alpha_n}\mathcal{X}^{\alpha_n} l_1(\varphi_{\alpha_n})) \\ &= m_1(m_1(\mathcal{X}^{\alpha_n}))\varphi_{\alpha_n} + (-1)^{\alpha_n - 1} m_1(\mathcal{X}^{\alpha_n}) l_1(\varphi_{\alpha_n}) \\ &\qquad + (-1)^{\alpha_n} m_1(\mathcal{X}^{\alpha_n}) l_1(\varphi_{\alpha_n}) + (-1)^{\alpha_n}\mathcal{X}^{\alpha_n} l_1(l_1(\varphi_{\alpha_n})) \\ &= (-1)^{\alpha_n}\mathcal{X}^{\alpha_n} l_1(l_1(\varphi_{\alpha_n})) \\ &= 0. \quad \square \end{aligned}$$

The spectral sequence argument below proves that $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ resolves $\mathcal{I}/\mathcal{I}^2$ as modules.

PROPOSITION 3.4. $\mathrm{H}_{l_1}(\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}) = \mathcal{I}/\mathcal{I}^2$.

Proof of Proposition 3.4. The module $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ is naturally bigraded. The bidegree of the element $\mathcal{X}^{\alpha_n}\varphi_{\alpha_n}$ is (α_n, n). The differential decomposes as $l_1 = (m_1 \otimes 1) + d^{(1)} + d^{(2)} + \cdots$ with respect to the filtration of $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ by preghost degree, where $m_1 \otimes 1$ has bidegree $(-1, 0)$ and $d^{(i)}$ has bidegree $(i-1, i)$. The differential on the E_0 term is $m_1 \otimes 1$. The k^{th} row of the E_0 term is $\mathbf{K}_{A/\mathcal{I}} \otimes \Phi^{(k)}$ where $\Phi^{(k)}$ is the vector space spanned by the φ_{α_k}'s. Each row is exact except in the first slot. Therefore the E_1 term of the spectral sequence is concentrated in the first column:

$$E_1: \qquad \begin{array}{c} \vdots \\ \big\downarrow \widetilde{d^{(1)}} \\ A/\mathcal{I} \otimes \Phi^{(3)} \\ \big\downarrow \widetilde{d^{(1)}} \\ A/\mathcal{I} \otimes \Phi^{(2)} \\ \big\downarrow \widetilde{d^{(1)}} \\ A/\mathcal{I} \otimes \Phi^{(1)} \\ \big\downarrow \\ 0\ . \end{array}$$

Since the E_1 term collapses to one column, the homology of $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ with respect to l_1 will be the homology of the E_1 term.

Let K be the kernel of the map $\pi : A/\mathcal{I} \otimes \Phi^{(1)} \longrightarrow \mathcal{I}/\mathcal{I}^2$ which sends $\underline{f^{\alpha_1}} \otimes \varphi_{\alpha_1}$ to $\overline{f^{\alpha_1} y_{\alpha_1}}$. We need to show that $\widetilde{d^{(1)}}(A/\mathcal{I} \otimes \Phi^{(1)}) = K$. ($\subseteq$): Since $l_1\varphi_{\alpha_n} = \mathcal{Z}^{\alpha_{n-1}}_{\alpha_n}\varphi_{\alpha_{n-1}} +$ "more", it follows that $\widetilde{d^{(1)}}\,\underline{f^{\alpha_n}}\varphi_{\alpha_n} = \underline{f^{\alpha_n}\mathcal{Z}^{\alpha_{n-1}}_{\alpha_n}}\varphi_{\alpha_{n-1}}$. Consider any element $\underline{f^{\alpha_2}}\varphi_{\alpha_2}$ of $A/\mathcal{I} \otimes \Phi^{(2)}$. The image under $\pi\widetilde{d^{(1)}}$ is zero because for all α_2 the sum $\mathcal{Z}^{\alpha_1}_{\alpha_2} y_{\alpha_1} = 0$. ($\supseteq$): Suppose $\pi(\underline{f^{\alpha_1}}\varphi_{\alpha_1}) = \overline{f^{\alpha_1} y_{\alpha_1}} = 0$. Recalling that the ideal

$\mathcal{I}$ admits a complete set of reducibility functions, we see that

$$f^{\alpha_1} y_{\alpha_1} = h^{\alpha_1\beta_1} y_{\beta_1} y_{\alpha_1} \Rightarrow (f^{\alpha_1} - h^{\alpha_1\beta_1} y_{\beta_1}) y_{\alpha_1} = 0$$
$$\Rightarrow f^{\alpha_1} \equiv g^{\alpha_2} \mathcal{Z}^{\alpha_1}_{\alpha_2} \mod \mathcal{I}.$$

It then follows that $\widetilde{d^{(1)}}(\underline{g^{\alpha_2}}\varphi_{\alpha_2}) = \underline{g^{\alpha_2}\mathcal{Z}^{\alpha_1}_{\alpha_2}}\varphi_{\alpha_1} = \underline{f^{\alpha_1}}\varphi_{\alpha_1}$. We conclude that the zeroth cohomology of $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ with respect to l_1 is $(A/\mathcal{I} \otimes \Phi^{(1)})/K$, which is isomorphic to $\mathcal{I}/\mathcal{I}^2$ as modules. For higher cohomologies, suppose $\underline{f^{\alpha_n}}\varphi_{\alpha_n}$ is a cocycle. Then

$$\underline{f^{\alpha_n} \mathcal{Z}^{\alpha_{n-1}}_{\alpha_n}}\varphi_{\alpha_{n-1}} = 0 \Rightarrow f^{\alpha_n} \mathcal{Z}^{\alpha_{n-1}}_{\alpha_n} \in \mathcal{I}$$
$$\Rightarrow f^{\alpha_n} \equiv g^{\alpha_{n+1}} \mathcal{Z}^{\alpha_n}_{\alpha_{n+1}} \mod \mathcal{I}$$
$$\Rightarrow \underline{f^{\alpha_n}}\varphi_{\alpha_n} = \underline{g^{\alpha_{n+1}} \mathcal{Z}^{\alpha_n}_{\alpha_{n+1}}}\varphi_{\alpha_n} = \widetilde{d^{(1)}}\underline{g^{\alpha_{n+1}}}\varphi_{\alpha_{n+1}}.$$

Hence, the higher cohomologies are zero. □

The map Ψ is quite useful. We constructed l_1 so that Ψ was a chain map. In the notation of shLie algebras and shLie modules, we were able to show that $\mathcal{J}_{\mathrm{ID}1} = 0$ on Φ because $\Psi \mathcal{A}CT_{\mathrm{ID}1} = \mathcal{J}_{\mathrm{ID}1}\Psi$ and $\mathcal{A}CT_{\mathrm{ID}1} = 0$ on $\Psi^{-1}\Phi$. We shall see this pattern of proof again in proposition 3.7.

Let $\mathbf{P}_I = \mathcal{P}_{I_1} \cdots \mathcal{P}_{I_s} \in \mathbf{K}_{A/\mathcal{I}}$. Then $\Psi(\mathbf{P}_I) = \kappa(I_i)\mathbf{P}^{I_i}_I \varphi_{I_i}$, where $\mathbf{P}^{I_i}_I = \mathcal{P}_{I_1} \cdots \widehat{\mathcal{P}_{I_i}} \cdots \mathcal{P}_{I_s}$ and $\widehat{\mathcal{P}_{I_i}}$ indicates that this factor is omitted. The sign $\kappa(I_i)$ is the sign produced by moving $\mathcal{P}_{I_i}$ past $\mathcal{P}_{I_1} \cdots \mathcal{P}_{I_{i-1}}$. Recalling that $m_1 \mathcal{P}_{\alpha_n} = \mathrm{Z}^I_{\alpha_n}\mathbf{P}_I$, we can write $l_1(\varphi_{\alpha_n}) = \kappa(I_i)\mathrm{Z}^I_{\alpha_n}\mathbf{P}^{I_i}_I \varphi_{I_i}$.

STEP 3: The loop

We prove the three propositions needed to complete the inductive step of the loop. Then we will return to $n = 2$ and define m_2 at the one location where both $\mathbf{K}_{A/\mathcal{I}}$ and $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ are not exact.

PROPOSITION 3.5. (3$a_n$ii) *If we set* $m_n(\varphi_{[1 \text{ to } n-1]} \otimes f) = (-1)^n \{m_{n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}), f\}$, *for* $\varphi_1 \leqslant \cdots \leqslant \varphi_{n-1}$, *then*

$$m_1(m_n(\varphi_{[1 \text{ to } n-1]} \otimes f)) = \mathcal{A}CT_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n-1]} \otimes f).$$

Proof of Proposition 3.5. We omit sign arguments and specific signs, opting instead for $\pm$. Using the definition, $m_1 m_n(\varphi_{[1 \text{ to } n-1]} \otimes f)$

$$= \pm m_1\{m_{n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}), f\}$$
$$= \pm\{m_1(m_{n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1})), f\} \pm \mathrm{Z}^A_{[1 \text{ to } n-1]}\{m_1(\mathbf{P}_A), f\},$$

where $m_{n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}) = \mathrm{Z}^A_{[1 \text{ to } n-1]}\mathbf{P}_A$. Now, $\mathcal{A}CT_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n-1]} \otimes f)$ contains one term $\pm m_2(l_{n-1}(\varphi_{[1 \text{ to } n-1]}) \otimes f)$ which uses the $(n-1, 0)$-unshuffle. But since

$$l_{n-1}(\varphi_{[1 \text{ to } n-1]_{)}}) = \Psi m_{n-1}(\varphi_{[1 \text{ to } n-1]_{\otimes}}\mathcal{P}_{n-1}) = \mathrm{Z}^A_{[1 \text{ to } n-1]} P^\beta_A \varphi_\beta$$

and m_2 is $\mathbf{K}_{A/\mathcal{I}}$-linear,

$$m_2(\mathrm{Z}^A_{[1\text{ to }n-1]}P^{\beta}_{A}\varphi_\beta \otimes f) = \pm \mathrm{Z}^A_{[1\text{ to }n-1]}P^{\beta}_{A}\{m_1\mathcal{P}_\beta, f\} = \pm \mathrm{Z}^A_{[1\text{ to }n-1]}\{m_1(\mathbf{P}_A), f\}.$$

Since $m_1 m_{n-1}$ is equal to $\mathcal{ACT}_{\mathrm{OBST}\,n-1}$, we may rewrite $\pm\{m_1 m_{n-1}(\varphi_{[1\text{ to }n-2]}\otimes \mathcal{P}_{n-1}), f\}$ as $\pm\{\mathcal{ACT}_{\mathrm{OBST}\,n-1}(\varphi_{[1\text{ to }n-2]}\otimes\mathcal{P}_{n-1}), f\}$, which must equal the remaining terms in the map $\mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1\text{ to }n-1]}\otimes f)$.

We organize the remaining terms by unshuffle. Let σ be an $(i, j-2)$-unshuffle. Applying σ to $\varphi_{[1\text{ to }n-1]}$ produces two hands $\varphi_{\sigma[1\text{ to }i]}$ and $\varphi_{\sigma[i+1\text{ to }n-1]}$. In $\mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1\text{ to }n-1]}\otimes f)$, there are two terms in which the φ_i's appear in the σ order. They are

$$(\mathbb{M}_j\mathbb{L}_i\sigma) = \pm m_j(l_i(\varphi_{\sigma[1\text{ to }i]})\wedge\varphi_{\sigma[i+1\text{ to }n-1]}\otimes f)$$

and

$$(\mathbb{M}_{i+1}\mathbb{M}_{j-1}\sigma) = \pm m_{i+1}(\varphi_{\sigma[1\text{ to }i]}\otimes m_{j-1}(\varphi_{\sigma[i+1\text{ to }n-1]}\otimes f)).$$

There is a unique $(j-2, i)$-unshuffle ρ which switches the σ hands, i.e., $\varphi_{\rho[j-1\text{ to }n-1]} = \varphi_{\sigma[1\text{ to }i]}$ and $\varphi_{\rho[1\text{ to }j-2]} = \varphi_{\sigma[i+1\text{ to }n-1]}$. The two terms in which the φ_i's appear in the ρ order, written in terms of σ are

$$(\widetilde{\mathbb{M}}_{i+2}\widetilde{\mathbb{L}}_{j-2}\sigma) = \pm m_{i+2}(l_{j-2}(\varphi_{\sigma[i+1\text{ to }n-1]})\wedge\varphi_{\sigma[1\text{ to }i]}\otimes f)$$

and

$$(\widetilde{\mathbb{M}}_{j-1}\widetilde{\mathbb{M}}_{i+1}\sigma) = \pm m_{j-1}(\varphi_{\sigma[i+1\text{ to }n-1]}\otimes m_{i+1}(\varphi_{\sigma[1\text{ to }i]}\otimes f)).$$

Organized in this way, the terms remaining in $\mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1\text{ to }n-1]}\otimes f)$ can be rewritten as

$$\frac{1}{2}\sum_{\substack{i+j=n+1\\ 2<j<n}}\sum_{\substack{\sigma\\ (i,j-2)}} \mathbb{M}_j\mathbb{L}_i\sigma + \mathbb{M}_{i+1}\mathbb{M}_{j-1}\sigma + \widetilde{\mathbb{M}}_{i+2}\widetilde{\mathbb{L}}_{j-2}\sigma + \widetilde{\mathbb{M}}_{j-1}\widetilde{\mathbb{M}}_{i+1}\sigma.$$

Without loss of generality, we now compute the sum

$$\mathbb{M}_j\mathbb{L}_i e + \mathbb{M}_{i+1}\mathbb{M}_{j-1}e + \widetilde{\mathbb{M}}_{i+2}\widetilde{\mathbb{L}}_{j-2}e + \widetilde{\mathbb{M}}_{j-1}\widetilde{\mathbb{M}}_{i+1}e$$

for the identity $(i, j-2)$-unshuffle e and show that the sum equals the two terms in

$$\pm\{\mathcal{ACT}_{\mathrm{OBST}\,n-1}(\varphi_{[1\text{ to }n-2]}\otimes\mathcal{P}_{n-1}), f\}$$

for which $\varphi_{[1\text{ to }n-2]}\otimes\mathcal{P}_{n-1}$ is split into the hands $\varphi_{[1\text{ to }i]}$ and $\varphi_{[i+1\text{ to }n-2]}\otimes\mathcal{P}_{n-1}$. The terms of the sum $\mathbb{M}_j\mathbb{L}_i e + \mathbb{M}_{i+1}\mathbb{M}_{j-1}e + \widetilde{\mathbb{M}}_{i+2}\widetilde{\mathbb{L}}_{j-2}e + \widetilde{\mathbb{M}}_{j-1}\widetilde{\mathbb{M}}_{i+1}e$ expand to become

$$
\begin{aligned}
(\mathbb{M}_j\mathbb{L}_i e) &= \pm\{m_{j-1}(l_i(\varphi_{[1 \text{ to } i]}) \wedge \varphi_{[i+1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}, f\} && (A)\\
&\qquad \pm\{\mathrm{Z}^A_{[1 \text{ to } i]}, f\} m_{j-1}(\varphi_{[i+1 \text{ to } n-1]} \otimes \mathbf{P}_A), && (B)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\mathbb{M}}_{j-1}\widetilde{\mathbb{M}}_{i+1} e) &= \pm\{\mathrm{Z}^B_{[i+1 \text{ to } n-1]}, \{\mathrm{Z}^A_{[1 \text{ to } i]}, f\}\}\mathbf{P}_B\mathbf{P}_A && (C)\\
&\qquad \pm\{\mathrm{Z}^A_{[1 \text{ to } i]}, f\} m_{j-1}(\varphi_{[i+1 \text{ to } n-1]} \otimes \mathbf{P}_A), && (D)
\end{aligned}
$$

$$
\begin{aligned}
(\mathbb{M}_{i+1}\mathbb{M}_{j-1} e) &= \pm\{\mathrm{Z}^A_{[1 \text{ to } i]}, \{\mathrm{Z}^B_{[i+1 \text{ to } n-1]}, f\}\}\mathbf{P}_A\mathbf{P}_B && (E)\\
&\qquad \pm\{\mathrm{Z}^B_{[i+1 \text{ to } n-1]}, f\} m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes \mathbf{P}_B), && (F)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\mathbb{M}}_{i+2}\widetilde{\mathbb{L}}_{j-2} e) &= \pm\{\mathrm{Z}^B_{[i+1 \text{ to } n-1]} m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes \mathbf{P}_B), f\} && (G)\\
&\qquad \pm\{\mathrm{Z}^B_{[i+1 \text{ to } n-1]}, f\} m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes \mathbf{P}_B), && (H)
\end{aligned}
$$

Terms which are identical up to sign cancel, i.e., (B) cancels with (D), (F) cancels with (H). After exchanging $\mathbf{P}_B$ and $\mathbf{P}_A$ in (C), we can combine (C) with (E) and use the Jacobi identity to produce $\pm\{\{\mathrm{Z}^A_{[1 \text{ to } i]}, \mathrm{Z}^B_{[i+1 \text{ to } n-1]}\}\mathbf{P}_A\mathbf{P}_B, f\}$, where we have brought $\mathbf{P}_A\mathbf{P}_B$ inside the bracket. Since $m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes \mathrm{Z}^B_{[i+1 \text{ to } n-1]}) = \pm\{\mathrm{Z}^A_{[1 \text{ to } i]}, \mathrm{Z}^B_{[i+1 \text{ to } n-1]}\}\mathbf{P}_A$, the term above becomes $\pm\{m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes \mathrm{Z}^B_{[i+1 \text{ to } n-1]})\mathbf{P}_B, f\}$. To this we add (G), yielding one of the desired terms: $\pm\{m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes m_{j-1}(\varphi_{[i+1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1})), f\}$. This term and (A) are the two terms in $\pm\{\mathcal{ACT}_{\mathrm{OBST}\,n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}), f\}$ for which $\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}$ is split into the hands $\varphi_{[1 \text{ to } i]}$ and $\varphi_{[i+1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}$. Therefore, for each $(i, j-2)$-unshuffle σ, we produce a term of the form $\pm\{m_{j-1}(l_i(\text{first } \sigma \text{ hand}) \otimes (\text{second } \sigma \text{ hand})), f\}$ and a term of the form $\pm\{m_{i+1}((\text{first } \sigma \text{ hand}) \otimes m_{j-1}(\text{second } \sigma \text{ hand})), f\}$ with the appropriate signs. As j runs from 2 to n, $j-1$ runs from 1 to $n-1$, so as we run through all $(i, j-2)$-unshuffles and divide by two, we produce $\pm\{\mathcal{ACT}_{\mathrm{OBST}\,n-1}(\varphi_{[1 \text{ to } n-2]} \otimes \mathcal{P}_{n-1}), f\}$. □

PROPOSITION 3.6. $(3a_n\text{iii})$ *For $n \geqslant 3$, one can define $m_n(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)$ for each element of the totally ordered basis for $\Phi^{\wedge n-1} \otimes P$ so that m_n is well-defined and so that*

$$m_1(m_n(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)) = \mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n).$$

Proof of Proposition 3.6. We ensure that m_n is well-defined on $\Phi^{\wedge n-1} \otimes P$ by defining m_n on the basis elements $\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n$ with $\varphi_1 \leqslant \cdots \leqslant \varphi_n$ with respect to the total ordering of the preghosts. The image of $\varphi_{[1 \text{ to } n-1]_\otimes} \mathcal{P}_n$ under $\mathcal{ACT}_{\mathrm{OBST}\,n}$ is in $\mathbf{K}_{A/\mathcal{I}}$. The map $\mathcal{ACT}_{\mathrm{OBST}\,n}$ has degree -2, so for $n \geqslant 3$, the degree of $\mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)$ is at least one, at which level $\mathbf{K}_{A/\mathcal{I}}$ is exact. So if we can show that

$$m_1 \mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n) = 0,$$

then a pre-image exists which we can set equal to $m_n(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)$. Since $\mathcal{ACT}_{\mathrm{OBST}\,n} = \mathcal{ACT}_n - m_n m_1$ and that $\mathcal{ACT}_n$ is a symmetric chain map, we find that

$$\begin{aligned} m_1\mathcal{ACT}_{\mathrm{OBST}n} &= m_1(\mathcal{ACT}_n - m_n m_1) \\ &= m_1\mathcal{ACT}_n - m_1 m_n m_1 \\ &= \mathcal{ACT}_n m_1 - m_1 m_n m_1 \\ &= (\mathcal{ACT}_n - m_1 m_n)m_1. \end{aligned}$$

Let us begin by examining m_n on an ordered basis element $\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n$ with $|\varphi_i| = 1$ for all i. For such a basis element we have

$$\begin{aligned} m_1\mathcal{ACT}_{\mathrm{OBST}n}(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n) &= (\mathcal{ACT}_n - m_1 m_n)m_1(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n) \\ &= (\mathcal{ACT}_n - m_1 m_n)(\varphi_{[1 \text{ to } n-1]} \otimes y_n). \end{aligned}$$

Since we already know that $\mathcal{ACT}_{\mathrm{ID}n}(\varphi_{[1 \text{ to } n-1]} \otimes y_n) = 0$, we can replace $\mathcal{ACT}_n - m_1 m_n$ with $(-1)^{n-1}m_n m_1$, yielding $m_n m_1(\varphi_{[1 \text{ to } n-1]} \otimes y_n)$, which equals 0.

The proof is completed by strong induction. Let $\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k}$ be the k^{th} basis element in the ordered list and suppose $\mathcal{ACT}_{\mathrm{ID}n}(\varphi_{[1 \text{ to } n-1]_i} \otimes \mathcal{P}_{n_i}) = 0$ for all basis elements before the k^{th} one, where once again,

$$m_1\mathcal{ACT}_{\mathrm{OBST}n}(\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k}) = (\mathcal{ACT}_n - m_1 m_n)m_1(\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k}).$$

The element $m_1(\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k})$ can only have pieces in $\mathbf{K}^{\wedge p}_{\mathcal{I}/\mathcal{I}^2} \otimes \mathbf{K}_{A/\mathcal{I}}$ for $0 \leqslant p \leqslant n-1$ and, moreover, the pieces in $\mathbf{K}^{\wedge n-1}_{\mathcal{I}/\mathcal{I}^2} \otimes \mathbf{K}_{A/\mathcal{I}}$ contain only basis elements of order less than that of $\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k}$. So for $m_1(\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k})$, we can replace $\mathcal{ACT}_n - m_1 m_n$ with $m_n m_1$. It follows that

$$(\mathcal{ACT}_n - m_1 m_n)m_1(\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k}) = m_n(m_1(m_1(\varphi_{[1 \text{ to } n-1]_k} \otimes \mathcal{P}_{n_k}))) = 0. \quad \Box$$

PROPOSITION 3.7. ($3b_n$ii) *If we set* $l_n(\varphi_{[1 \text{ to } n]}) = \Psi m_n(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)$, *then* $\mathcal{J}_{\mathrm{ID}n} = 0$ *on* $\Phi^{\wedge n}$.

Proof of Proposition 3.7. As outlined above, we will prove that $\mathcal{J}_{\mathrm{ID}n} = 0$ on $\Phi^{\wedge n}$ by showing that on $\Phi^{\wedge n-1} \otimes P$, the following equalities hold:

$$l_1 l_n(1^{\wedge n-1} \otimes \Psi) = \Psi m_1 m_n = \Psi\mathcal{ACT}_{\mathrm{OBST}n} = \mathcal{J}_{\mathrm{OBST}n}(1^{\wedge n-1} \otimes \Psi).$$

The first equality holds because the map Ψ is a chain map. Since $\mathcal{ACT}_{\mathrm{ID}n} = 0$, it follows that $m_1 m_n = \mathcal{ACT}_{\mathrm{OBST}n}$. The second equality holds because $\Psi m_1 m_n = \Psi\mathcal{ACT}_{\mathrm{OBST}n}$. Showing that $\Psi\mathcal{ACT}_{\mathrm{OBST}n} = \mathcal{J}_{\mathrm{OBST}n}(1^{\wedge n-1} \otimes \Psi)$ is more difficult because Ψ does not commute with m_i for $i > 1$. We will need to organize $\mathcal{ACT}_{\mathrm{OBST}n}$ and $\mathcal{J}_{\mathrm{OBST}n}$ by unshuffle. Let σ be any $(i, j-2)$-unshuffle. The two terms in the sum $\mathcal{ACT}_{\mathrm{OBST}n}$ for which $\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n$ is split into the hands $\varphi_{\sigma[1 \text{ to } i]}$ and $\varphi_{\sigma[i+1 \text{ to } n-1]} \otimes \mathcal{P}_n$ are

$$\begin{aligned} &\pm m_j\left(l_i(\varphi_{\sigma[1 \text{ to } i]}) \wedge \varphi_{\sigma[i+1 \text{ to } n-1]} \otimes \mathcal{P}_n\right) \\ &\qquad \pm m_{i+1}\left(\varphi_{\sigma[1 \text{ to } i]} \otimes m_{j-1}(\varphi_{\sigma[i+1 \text{ to } n-1]} \otimes \mathcal{P}_n)\right). \end{aligned}$$

We will show that when Ψ is applied to each of the terms above, their sum becomes

$$\begin{aligned} &\pm l_j\left(l_i(\varphi_{\sigma[1 \text{ to } i]}) \wedge \varphi_{\sigma[i+1 \text{ to } n-1]} \wedge \varphi_n\right) \\ &\qquad \pm l_{i+1}\left(\varphi_{\sigma[1 \text{ to } i]} \wedge l_{j-1}(\varphi_{\sigma[i+1 \text{ to } n-1]} \wedge \varphi_n)\right). \end{aligned}$$

This fact completes the proof because we can expand $\mathcal{J}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n]})$ so that φ_n is never moved. For any $(i, j-1)$-unshuffle which moves φ_n, we will use the graded symmetry of the shLie structure maps to move φ_n back to the last position. With this approach,

$$\mathcal{J}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n]}) = \sum_{\substack{i+j=n+1 \\ i>0, j>1}} \sum_{\substack{\sigma \\ (i,j-2)}} \Big(l_j \left(l_i(\varphi_{\sigma[1 \text{ to } i]}) \wedge \varphi_{\sigma[i+1 \text{ to } n]} \right) \pm l_{i+1} \left(\varphi_{\sigma[1 \text{ to } i]} \wedge l_{j-1}(\varphi_{\sigma[i+1 \text{ to } n]}) \right) \Big).$$

Without loss of generality, we shall show that the desired equality holds for the identity $(i, j-2)$-unshuffle e. The e-terms of $\mathcal{ACT}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n-1]} \otimes \mathcal{P}_n)$ are

$$m_j \left(l_i(\varphi_{[1 \text{ to } i]}) \wedge \varphi_{[i+1 \text{ to } n-1]} \otimes \mathcal{P}_n \right) = \pm \mathrm{Z}^A_{[1 \text{ to } i]} \mathbf{P}^{A_a}_A m_j(\varphi_{A_a} \wedge \varphi_{[i+1 \text{ to } n-1]} \otimes \mathcal{P}_n) \qquad (\mathbb{M}_j \mathbb{L}_i)$$

and

$$m_{i+1} \left(\varphi_{[1 \text{ to } i]} \otimes m_{j-1}(\varphi_{[i+1 \text{ to } n-1]} \otimes \mathcal{P}_n) \right) = \pm \{ \mathrm{Z}^A_{[1 \text{ to } i]}, \mathrm{Z}^B_{[i+1 \text{ to } n]} \} \mathbf{P}_A \mathbf{P}_B \qquad (\{\}_{(i,j-2)})$$

$$\pm \mathrm{Z}^B_{[i+1 \text{ to } n]} \mathbf{P}^{B_b}_B m_{i+1}(\varphi_{[1 \text{ to } i]} \otimes \mathcal{P}_{B_b}). \qquad (\mathbb{M}_{i+1} \mathbb{M}_{j-1})$$

Applying Ψ to $\mathbb{M}_j \mathbb{L}_i$ produces

$$\pm \mathrm{Z}^A_{[1 \text{ to } i]} \mathbf{P}^{A_a}_A l_j(\varphi_{A_a} \wedge \varphi_{[i+1 \text{ to } n]}) \pm \mathrm{Z}^A_{[1 \text{ to } i]} m_j(\varphi_{[i+1 \text{ to } n]} \otimes \mathbf{P}^{A_a}_A) \varphi_{A_a}.$$

The map Ψ acts on $\mathbf{P}_A \mathbf{P}_B$ in the $\{\}_{(i,j-2)}$ term to produce two terms $\pm \mathbf{P}_B \mathbf{P}^{A_a}_A \varphi_{A_a} \pm \mathbf{P}_A \mathbf{P}^{B_b}_B \varphi_{B_b}$. Using the derivational property of the bracket and the definition of m_j, the first term of $\{\}_{(i,j-2)}$ equals $\pm m_j(\varphi_{[i+1 \text{ to } n]} \otimes \mathrm{Z}^A_{[1 \text{ to } i]}) \mathbf{P}^{A_a}_A \varphi_{A_a}$. Adding this term to $\Psi(\mathbb{M}_j \mathbb{L}_i)$, we have

$$\pm \mathrm{Z}^A_{[1 \text{ to } i]} \mathbf{P}^{A_a}_A l_j(\varphi_{A_a} \wedge \varphi_{[i+1 \text{ to } n]}) \pm m_j(\varphi_{[i+1 \text{ to } n]} \otimes \mathrm{Z}^A_{[1 \text{ to } i]} \mathbf{P}^{A_a}_A) \varphi_{A_a},$$

which equals $\pm l_j \left(l_i(\varphi_{[1 \text{ to } i]}) \wedge \varphi_{[i+1 \text{ to } n]} \right)$. Similarly, the sum of $\Psi(\mathbb{M}_{i+1} \mathbb{M}_{j-1})$ and the second term of $\Psi(\{\}_{(i,j-2)})$ is the second desired term $\pm l_{i+1} \left(\varphi_{[1 \text{ to } i]} \wedge l_{j-1}(\varphi_{[i+1 \text{ to } n]}) \right)$. We produce all the terms in $\mathcal{J}_{\mathrm{OBST}\,n}(\varphi_{[1 \text{ to } n]})$ as we run through all $(i, j-2)$-unshuffles with $2 < j < n$. □

Return to $n = 2$:
We can define $m_2(\varphi_{\alpha_n} \otimes \mathcal{P}_{\beta_m})$ inductively on $\varphi_{\alpha_n} \otimes \mathcal{P}_{\beta_m}$ with $\varphi_{\alpha_n} \leqslant \varphi_{\beta_m}$, just as in proposition 3.6, except for $\varphi_{\alpha_1} \otimes \mathcal{P}_{\beta_1}$ with $\varphi_{\alpha_1} \leqslant \varphi_{\beta_1}$. Since m_2 must be a chain map, $m_1(m_2(\varphi_{\alpha_1} \otimes \mathcal{P}_{\beta_m})$ should equal $m_2(\varphi_{\alpha_1} \otimes m_1(\mathcal{P}_{\beta_1}))$. Having chosen structure functions $C^{\gamma_1}_{\alpha_1 \beta_1}$ in A, we find that $m_2(\varphi_{\alpha_1} \otimes m_1(\mathcal{P}_{\beta_1})) = \{y_{\alpha_1}, y_{\beta_1}\} = C^{\gamma_1}_{\alpha_1 \beta_1} y_{\gamma_1}$, which is in $\mathcal{I}$. Therefore a pre-image exists under m_1; we set $m_2(\varphi_{\alpha_1} \otimes \mathcal{P}_{\beta_1}) = C^{\gamma_1}_{\alpha_1 \beta_1} \mathcal{P}_{\gamma_1}$.

We have completed the definition of m_2 and in doing so, have finished the definition of l_2 on $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ as well.

STEP 4:
To complete the construction of the homotopy Lie-Rinehart resolution $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$, we must verify that $\mathrm{H}_{m_1}(m_2) = \widehat{m_2}$ on $\mathcal{I}/\mathcal{I}^2 \otimes A/\mathcal{I}$ and that $\mathrm{H}_{l_1}(l_2) = \widehat{l_2}$ on $\mathcal{I}/\mathcal{I}^2 \wedge \mathcal{I}/\mathcal{I}^2$, i.e., conditions (ii) and (i) in definition 2.24 are satisfied.

Let λ_1 be a splitting from $A/\mathcal{I}$ into $\mathbf{K}_{A/\mathcal{I}}$ and let λ_2 be a splitting of $\mathcal{I}/\mathcal{I}^2$ into $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$. Then for $\overline{z} \otimes \underline{f} \in \mathcal{I}/\mathcal{I}^2 \otimes A/\mathcal{I}$, the map m_2 maps $\lambda_2(\mathcal{I}/\mathcal{I}^2) \otimes \lambda_1(A/\mathcal{I})$ into A. It follows that $\mathrm{H}_{m_1}(\overline{z} \otimes \underline{f}) = \underline{m_2(\lambda_2(\overline{z}) \otimes \lambda_1(\underline{f}))}$. Suppose $\lambda_2(\overline{z}) = g_{\overline{z}}^{\alpha_1}\varphi_{\alpha_1}$, which implies that $\overline{z} = \overline{g_{\overline{z}}^{\alpha_1} y_{\alpha_1}}$. Then

$$\begin{aligned}\underline{m_2(g_{\overline{z}}^{\alpha_1}\varphi_{\alpha_1} \otimes \lambda_1(\underline{f}))} &= \underline{g_{\overline{z}}^{\alpha_1}\{y_{\alpha_1}, \lambda_1(\underline{f})\}} \\ &= \underline{\{g_{\overline{z}}^{\alpha_1} y_{\alpha_1}, \lambda_1(\underline{f})\} - \{g_{\overline{z}}^{\alpha_1}, \lambda_1(\underline{f})\} y_{\alpha_1}} \\ &= \underline{\{g_{\overline{z}}^{\alpha_1} y_{\alpha_1}, \lambda_1(\underline{f})\}} \\ &= \widehat{m_2}(\overline{z} \otimes \underline{f}).\end{aligned}$$

Similarly, for any $\overline{z} \wedge \overline{w}$, there exists an element $g_{\overline{z}}^{\alpha_1}\varphi_{\alpha_1} \wedge h_{\overline{w}}^{\beta_1}\varphi_{\beta_1}$ in $\mathbf{K}_{\mathcal{I}/\mathcal{I}^2} \wedge \mathbf{K}_{\mathcal{I}/\mathcal{I}^2}$ whose class is $\overline{z} \wedge \overline{w}$. Then

$$\begin{aligned}\mathrm{H}_{l_1}(l_2)(\overline{z} \wedge \overline{w}) &= [l_2(g_{\overline{z}}^{\alpha_1}\varphi_{\alpha_1} \wedge h_{\overline{w}}^{\beta_1}\varphi_{\beta_1})] \\ &= [g_{\overline{z}}^{\alpha_1} h_{\overline{w}}^{\beta_1} C_{\alpha_1\beta_1}^{\gamma_1}\varphi_{\gamma_1} + g_{\overline{z}}^{\alpha_1} m_2(\varphi_{\alpha_1} \otimes h_{\overline{w}}^{\beta_1})\varphi_{\beta_1} - h_{\overline{w}}^{\beta_1} m_2(\varphi_{\beta_1} \otimes g_{\overline{z}}^{\alpha_1})\varphi_{\alpha_1}] \\ &= \overline{g_{\overline{z}}^{\alpha_1} h_{\overline{w}}^{\beta_1} C_{\alpha_1\beta_1}^{\gamma_1} y_{\gamma_1} + g_{\overline{z}}^{\alpha_1}\{y_{\alpha_1}, h_{\overline{w}}^{\beta_1}\} y_{\beta_1} - h_{\overline{w}}^{\beta_1}\{y_{\beta_1}, g_{\overline{z}}^{\alpha_1}\} y_{\alpha_1}} \\ &= \overline{\{g_{\overline{z}}^{\alpha_1} y_{\alpha_1}, h_{\overline{w}}^{\beta_1} y_{\beta_1}\}} \\ &= \widehat{l_2}(\overline{z} \wedge \overline{w}).\end{aligned}$$

We conclude that the homotopy Rinehart algebra $\mathcal{R}$ for the homotopy Lie-Rinehart resolution $(\mathbf{K}_{A/\mathcal{I}}, \mathbf{K}_{\mathcal{I}/\mathcal{I}^2})$ is a model for the Rinehart algebra R for the Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$.

4. Comparing the homotopy Rinehart algebra $\mathcal{R}$ with the BRST algebra $\mathcal{A}$

Following [**FV75**], [**BF83**] and [**BV85**], the classical BRST algebra $\mathcal{A}$ for the Poisson reduction of the Poisson algebra A by a finitely presented coisotropic ideal $\mathcal{I}$ is a differential graded Poisson algebra, built from a specific choice of Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$ of $A/\mathcal{I}$. The BRST algebra $\mathcal{A}$ is formed by tensoring the graded commutative algebra $\bigwedge N$ with $\mathbf{K}_{A/\mathcal{I}}$, where N and P are isomorphic as vector spaces over k. The basis elements of N are denoted by $\{\eta^{\alpha_1}, \eta^{\alpha_2}, \eta^{\alpha_3}, ...\}$, and are called *ghosts*. The result is

$$\mathcal{A} \approx \bigwedge N \otimes A \otimes \bigwedge P$$

(see [**FHST89**], [**HT88**], [**Sta92**], [**HT92**], [**Sta96**] etc.). Each η^{α_n} has the same degree as the corresponding $\mathcal{P}_{\alpha_n}$, namely n. If we let N^I denote a string of ghosts $\eta^{\alpha_{n_1}} \wedge \cdots \wedge \eta^{\alpha_{n_i}} = \eta^{\alpha_{n_{[1 \text{ to } i]}}}$, the degree of N^I is the sum $\sum_1^i n_p$, which we shall denote by I without confusion. There are two significant gradings on the BRST algebra $\mathcal{A}$. Let $\mathbf{N}^I f_I^J \mathbf{P}_J$ be an element of $\mathcal{A}$. The first grading is the *internal degree*, which is the difference between the *ghost degree* $gh(\mathbf{N}^I f_I^J \mathbf{P}_J) = I$ and the *antighost* degree $antigh(\mathbf{N}^I f_I^J \mathbf{P}_J) = J$. The second grading is by *ghost number*. The ghost number $gh\#(\mathbf{N}^I f_I^J \mathbf{P}_J) = i$ if $N^I = \eta^{\alpha_{n_{[1 \text{ to } i]}}}$. The multiplication on $\mathcal{A}$ is given by the multiplication in each of its three pieces and obeys the Koszul sign convention, that is to say,

$$(\mathbf{N}^I f_I^J \mathbf{P}_J)(\mathbf{N}^K f_K^L \mathbf{P}_L) = (-1)^{JK} \mathbf{N}^I \mathbf{N}^K f_I^J f_K^L \mathbf{P}_J \mathbf{P}_L.$$

The Poisson bracket on $\mathbf{K}_{A/\mathcal{I}}$ is extended to all of $\mathcal{A}$ by setting $\{N, N\} = \{N, A\} = 0$ and $\{\eta^{\alpha_n}, \mathcal{P}_{\beta_m}\} = \delta^{\alpha_n}_{\beta_m}$ (where $\delta^{\alpha_n}_{\beta_m}$ is the Kronecker delta). If we view N as the dual of P, this bracket formula is the usual symplectic structure on $N \oplus P$ [**Sta92**].

When $\mathcal{I}$ is irreducible, Batalin, Fradkin and Vilkovisky ([**BF83**] and [**BV85**]) were the first to define a differential $\mathcal{D}$ on $\mathcal{A}$ which is a Poisson derivation and whose zeroth cohomology $\mathrm{H}^0(\mathcal{A}, \mathcal{D}) \approx (A/\mathcal{I})^{\mathcal{I}}$ [**BF83**]. Later it was shown that the BRST algebra $(\mathcal{A}, \mathcal{D})$ is a model for the entire Rinehart complex in the irreducible case (see [**HT92**]).

A $\mathbf{K}_{A/\mathcal{I}}$-linear map $F_k : \bigwedge^k \mathbf{K}_{\mathcal{I}/\mathcal{I}^2} \to \mathbf{K}_{A/\mathcal{I}}$ is completely determined by where it sends each element of the ordered basis $\varphi_{\alpha_{n_{[1 \text{ to } k]}}}$. Once we set η^{α_n} equal the k-dual of φ_{α_n}, the map F_k can be represented by

$$\sum_{\alpha_{n_1} \leqslant \cdots \leqslant \alpha_{n_k}} \eta^{\alpha_{n_{[1 \text{ to } k]}}} F_k(\varphi_{\alpha_{n_{[1 \text{ to } k]}}}).$$

It follows that the homotopy Rinehart algebra $\mathcal{R} \approx \mathrm{Hom}_k(\bigwedge(\Phi), \mathbf{K}_{A/\mathcal{I}})$. This fact allows us to compare the homotopy Rinehart algebra with the BRST algebra $\mathcal{A}$.

Theorem 4.1. *Given a Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ and a specific Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$ of $A/\mathcal{I}$, the BRST algebra $\mathcal{A} = \bigwedge N \otimes \mathbf{K}_{A/\mathcal{I}}$ is isomorphic to the homotopy Rinehart algebra $\mathcal{R} = \mathrm{Hom}_{\mathbf{K}_{A/\mathcal{I}}}(\bigwedge(\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}), \bigwedge(\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}) \otimes \mathbf{K}_{A/\mathcal{I}})$ as $\mathbf{K}_{A/\mathcal{I}}$-modules and as algebras (but not necessarily as differential graded algebras).*

Proof of Theorem 4.1. It is straightforward to show that the map sending

$$F_k \mapsto \sum_{\alpha_{n_1} \leqslant \cdots \leqslant \alpha_{n_k}} \eta^{\alpha_{n_{(1 \text{ to } k)}}} F_k(\varphi_{\alpha_{n_{[1 \text{ to } k]}}})$$

is bijective and respects the $\mathbf{K}_{A/\mathcal{I}}$-module structures on both $\mathcal{R}$ and $\mathcal{A}$. We need to show that it is a map of algebras. Let $F_i : \bigwedge^i(\Phi) \to \mathbf{K}_{A/\mathcal{I}}$ and $G_j : \bigwedge^j(\Phi) \to \mathbf{K}_{A/\mathcal{I}}$. Recall that the product $F_i \smile G_j : \bigwedge^{i+j}(\Phi) \to \mathbf{K}_{A/\mathcal{I}}$, evaluated on $\varphi_{\alpha_{p_{[1 \text{ to } i+j]}}}$ is

$$\sum_{\substack{\sigma \\ (i,j)}} \kappa(\sigma)\ \kappa(G_j; \varphi_{\alpha_{p_{\sigma[1 \text{ to } i]}}}) F_i(\varphi_{\alpha_{p_{\sigma[1 \text{ to } i]}}}) G_j(\varphi_{\alpha_{p_{\sigma[i+1 \text{ to } i+j]}}}).$$

This product can be represented in $\mathcal{A}$ by

$$\sum_{\alpha_{p_1} \leqslant \cdots \leqslant \alpha_{p_{i+j}}} \eta^{\alpha_{p[1 \text{ to } i+j]}} (F_i \smile G_j)(\varphi_{\alpha_{p_{[1 \text{ to } i+j]}}}).$$

Similarly, $F_i = \sum_{\alpha_{r_1} \leqslant \cdots \leqslant \alpha_{r_i}} \eta^{\alpha_{r(1 \text{ to } i)}} F_i(\varphi_{\alpha_{r_{[1 \text{ to } i]}}})$ and $G_j = \sum_{\alpha_{q_1} \leqslant \cdots \leqslant \alpha_{q_j}} \eta^{\alpha_{q(1 \text{ to } j)}} G_j(\varphi_{\alpha_{q_{[1 \text{ to } j]}}})$. The product of F_i and G_j in $\mathcal{A}$ is

$$\sum_{\substack{\alpha_{r_1} \leqslant \cdots \leqslant \alpha_{r_i} \\ \alpha_{q_1} \leqslant \cdots \leqslant \alpha_{q_j}}} \kappa(G_j; \varphi_{\alpha_{p_{\sigma[1 \text{ to } i]}}}) \eta^{\alpha_{r[1 \text{ to } i]}} \eta^{\alpha_{q[1 \text{ to } j]}} F_i(\varphi_{\alpha_{r_{[1 \text{ to } i]}}}) G_j(\varphi_{\alpha_{q_{[1 \text{ to } j]}}}),$$

but the terms $\eta^{\alpha_{r[1 \text{ to } i]}} \eta^{\alpha_{q[1 \text{ to } j]}}$ are no longer in the proper order. Returning the η's in each term to the proper order produces the sign $\kappa(\sigma)$ corresponding to the (i, j)-unshuffle σ which places $n_{[1 \text{ to } i+j]}$ into the hands $r_{[1 \text{ to } i]}$ and $q_{[1 \text{ to } j]}$. The sum above becomes

$$\sum_{\alpha_{p_1} \leqslant \alpha_{p_{i+j}}} \eta^{\alpha_{p[1 \text{ to } i+j]}} \left(\sum_{\substack{\sigma \\ (i,j)}} \kappa(\sigma)\, \kappa(G_j; \varphi_{\alpha_{p_{\sigma[1 \text{ to } i]}}}) F_i(\varphi_{\alpha_{p_{\sigma[1 \text{ to } i]}}}) G_j(\varphi_{\alpha_{p_{\sigma[i+1 \text{ to } i+j]}}}) \right),$$

which equals $\sum_{\alpha_{p_1} \leqslant \cdots \leqslant \alpha_{p_{i+j}}} \eta^{\alpha_{p[1 \text{ to } i+j]}} (F_i \smile G_j)(\varphi_{\alpha_{p_{[1 \text{ to } i+j]}}})$. □

The bidegree (ghost number, internal degree) on $\mathcal{A}$ agrees with the bidegree (external degree, (suspended internal degree − external degree)) on $\mathcal{R}$.

The differentials $\{Q, \ \}$ and $\langle \mathrm{D}_M, \ \rangle$ when $\mathcal{I}$ is irreducible

The differential $\mathcal{D}$ is an inner-derivation on $\mathcal{A}$, i.e. $\mathcal{D} = \{Q, \ \}$, where the element $Q \in \mathcal{A}$ has total degree $+1$ and $\{Q, Q\} = 0$. The element Q (called the BRST charge) is a sum $\sum_{n=0}^{\infty} Q_n$, where Q_n has ghost number $n + 1$. The Jacobi identity for the Poisson bracket guarantees that D has square zero. We construct the BRST charge Q using methods from homological perturbation theory. The BRST algebra $\mathcal{A}$ is filtered by ghost degree and Q_n is defined by induction on n once we select a set of generators $\{y_\alpha\}$ for $\mathcal{I}$ and structure functions $C^\gamma_{\alpha\beta}$ [**Sta92**]. We set $Q_0 = \eta^\alpha y_\alpha$ and find that Q_1 must equal $-\frac{1}{2}\eta^\alpha \eta^\beta C^\gamma_{\alpha\beta} \mathcal{P}_\gamma$ in order to kill the nonzero piece of $\{Q_0, Q_0,\}$ with ghost degree 2. This process continues: for every $i > 1$, the terms of $\{\sum_{n=0}^{i-1} Q_n, \sum_{n=0}^{i-1} Q_n\}$ have ghost degree at least $i + 1$. An element Q_i with ghost degree $i + 1$ is selected to kill the terms with ghost degree $i + 1$.

With respect to the filtration of $\mathcal{A}$ by ghost number, $\{Q, \ \}$ decomposes as $1 \otimes m_1 + \widehat{\delta_R} + \delta_2 + \ldots$. In terms of the BRST differential, $1 \otimes m_1$ is $\{Q_0, \ \}_{|\bigwedge P}$ and

$$\widehat{\delta_R} = \{Q_1, \ \}_{|\bigwedge N} + \{Q_0, \ \}_{|A} + \{Q_1, \ \}_{|\bigwedge P}$$

is, loosely speaking, a "lifting" of the Rinehart differential δ_R on R. The E_1 term of the associated spectral sequence is isomorphic to $\bigwedge N \otimes A/\mathcal{I}$ and the differential d_1 for the E_1 term which $\widehat{\delta_R}$ induces on $\mathrm{H}_{1 \otimes m_1}(\mathcal{A})$ is the map δ_R.

The filtration of $\mathcal{R}$ by external degree is the same as the filtration of $\mathcal{A}$ by ghost number. The differential $\langle \mathrm{D}_M, \ \rangle$ breaks up into the sum $\sum_{i=1}^{\infty} \langle m_i, \ \rangle$, where $\langle m_i, \ \rangle$ increases external degree by $i-1$. For any map $F_k \in \mathcal{R}$, the map $\langle m_1, F_k\rangle$ is $m_1 F_k$ because $l_1 = 0$ on $\bigwedge(\Phi)$. Suppressing the sum over the ordered basis (it is henceforth understood), we write F_k as $\eta^{\alpha_{n_{[1 \text{ to } k]}}} F_k(\varphi_{\alpha_{n_{[1 \text{ to } k]}}})$ and the map $m_1 F_k$ is represented in $\mathcal{A}$ by $\eta^{\alpha_{n_{[1 \text{ to } k]}}} m_1 F_k(\varphi_{\alpha_{n_{[1 \text{ to } k]}}})$, which equals $(1 \otimes m_1)(\eta^{\alpha_{n_{[1 \text{ to } k]}}} F_k(\varphi_{\alpha_{n_{[1 \text{ to } k]}}}))$. So $\langle m_1, \ \rangle$, when realized on $\mathcal{A}$, is $1 \otimes m_1$. We conclude that in the irreducible case, the E_1 terms of the spectral sequences associated with $\{Q, -\}$ and $\langle \mathrm{D}_M, \ \rangle$ are isomorphic. The differentials for the E_1 terms are induced by $\widehat{\delta_R}$ and $\langle m_2, \ \rangle$, respectively; we claim that $\widehat{\delta_R} = \langle m_2, \ \rangle$. Because both $\widehat{\delta_R}$ and $\langle m_2, \ \rangle$ are derivations, a quick check on the ghosts, antighosts and the elements of A suffices. For η^α, we see that $\langle m_2, \eta^\alpha\rangle (\varphi_\beta \wedge \varphi_\gamma) = \eta^\alpha l_2(\varphi_\beta \wedge \varphi_\gamma) = \eta^\alpha (C^\delta_{\beta\gamma} \varphi_\delta) = C^\alpha_{\beta\gamma}$ and

$$\left\{-\frac{1}{2}\eta^a \eta^b C^\delta_{ab} \mathcal{P}_\delta, \eta^\alpha\right\}(\varphi_\beta \wedge \varphi_\gamma) = \frac{1}{2}\eta^a\eta^b C^\alpha_{ab}(\varphi_\beta \wedge \varphi_\gamma) = \frac{1}{2}C^\alpha_{\beta\gamma} - \frac{1}{2}C^\alpha_{\gamma\beta} = C^\alpha_{\beta\gamma}.$$

For $\mathcal{P}_\alpha$, we find that $\langle m_2, \mathcal{P}_\alpha\rangle (\varphi_\beta) = m_2 \mathcal{P}_\alpha(\varphi_\beta) = -m_2(\varphi_\beta \otimes \mathcal{P}_\alpha) = C^\delta_{\alpha\beta}\mathcal{P}_\delta$ and

$$\left\{-\frac{1}{2}\eta^a \eta^b C^\delta_{ab} \mathcal{P}_\delta, \mathcal{P}_\alpha\right\}(\varphi_\beta) =$$

$$\frac{1}{2}\eta^a C^\delta_{a\alpha}\mathcal{P}_\delta(\varphi_\beta) - \frac{1}{2}\eta^b C^\delta_{\alpha b}\mathcal{P}_\delta(\varphi_\beta) = -\frac{1}{2}C^\delta_{\beta\alpha}\mathcal{P}_\delta + \frac{1}{2}C^\delta_{\alpha\beta}\mathcal{P}_\delta = C^\delta_{\alpha\beta}\mathcal{P}_\delta.$$

And for $f \in A$, it is straightforward to compute that $\langle m_2, f\rangle (\varphi_\beta) = m_2 f(\varphi_\beta) = m_2(\varphi_\beta \otimes f) = \{y_\beta, f\}$ and $\{\eta^\alpha y_\alpha, f\}(\varphi_\beta) = \eta^\alpha\{y_\alpha, f\}(\varphi_\beta) = \{y_\beta, f\}$. We conclude that $(\mathcal{A}, \{Q, \ \})$ and $(\mathcal{R}, \langle \mathrm{D}_M, \ \rangle)$ are equivalent as models for the Rinehart cohomology of the Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ when the ideal $\mathcal{I}$ is irreducible.

When the ideal is reducible, we know that $(\mathcal{R}, \langle \mathrm{D}_M, \ \rangle)$ is also a model for the Rinehart cohomology, but we do not know, except in cases arising from particularly nice symplectic settings (see [**FHST89**]), whether $(\mathcal{A}, \{Q, \ \})$ is a model for the Rinehart cohomology. This problem will be addressed in a future paper.

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