Homology, Homotopy and Applications, vol.2, No.10, 2000, pp.127–145 ISSN 1512–0139

KK-THEORY AS THE K-THEORY OF C^*-CATEGORIES

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(*communicated by Jonathan Rosenberg*)

Abstract

Let complex C^* algebras be endowed with a norm-continuous action of a fixed compact second countable group. From a separable C^*-algebra A and a σ-unital C^*-algebra B, we construct a C^*-category $\mathrm{Rep}(A,B)$ and an isomorphism

$$\kappa : K^{i+1}(\mathrm{Rep}(A,B)) \to KK^i(A,B), \quad i \in \mathbb{Z}_2,$$

where on the left-hand side are Karoubi's topological K-groups, and on the right-hand side are Kasparov's equivariant bivariant K-groups.

1. Introduction

The purpose of this article is to study the possibility of calculation of Kasparov KK-theory by K-theory. Some partial results are known in this direction, for example: Paschke's result on K-homology of nuclear C^*-algebras [**13**], the generalization of Paschke's theorem for Kasparov KK-groups when the first argument is nuclear [**7**], [**17**], Higson's modification of Paschke's result for K-homology of separable C^*-algebras [**6**], and Künneth type theorem results for KK-theory [**15**]. In all these situations, the algebras are trivially graded.

Let us present briefly the idea of this paper. The main objects of our study are additive C^*-categories $\mathit{Rep}(A,B)$ and $\mathrm{Rep}(A,B)$, where A and B are trivially graded C^*-algebras with fixed compact group actions. In the first category, objects are equivariant A,B-bimodules and morphisms are invariant B-homomorphisms which commute, up to the ideal of compact homomorphisms, with the action of A. After definition of the first category, we define the category $\mathrm{Rep}(A,B)$ as the universal pseudoabelian C^*-category of $\mathit{Rep}(A,B)$. (The notation 'universal pseudoabelian' is slightly different from Karoubi's analogous definition [**8**], [**9**]). After small modification of Karoubi's K-theory of a Banach category for C^*-categories, we study properties of the K-groups of $\mathrm{Rep}(A,B)$. Then we apply this to prove our main result, that the K-groups of $\mathrm{Rep}(A,B)$ are essentially isomorphic to Kasparov's equivariant KK-groups, up to a dimension shift, when A is a separable C^*-algebra and B is a σ-unital C^*-algebra with fixed compact group actions.

This article is organized as follows. In Section 1 we review the basic definitions and properties of C^*-categories [**4**]. We give a construction of the universal pseudoabelian C^*-category of an additive C^*-category, and a characterization of a cofinal subcategory of $\mathcal{H}(B \otimes C^{(1,0)})$, that are used in the next sections. In Section 2 we review Karoubi's results ([**8**], [**9**]) on K-theory of Banach categories, adapted specially for C^*-categories. In Section 3 we give some remarks on the definition of the KK-groups in the form that is used in the sequel, and especially a characterization of the KK-groups in the case when the algebras are trivially graded [**2**], [**10**], [**16**]. In Section 4 we prove our main theorem.

Note that our main result shows that the category $\mathrm{Rep}(A,B)$ is an interesting object to be studied from various points of view (particularly, for the study of algebraic K-theory, cyclic

The work was partially supported by Grant No GM1-115 of the U.S. CRDF, INTAS Georgia Grant No 213.
Received 19 February 1999, revised 7 November 2000; published on 14 November 2000.
2000 Mathematics Subject Classification: 19K35, 19K99, 46L89, 46M99.
Key words and phrases: K-theory, KK-theory, C^*-category.

homology and connections with K-theory, i.e., connections with KK-theory).

I would like to express thanks to Prof. H. Inassaridze for his care during the working on this article, Prof. J. Rosenberg for his interest and attention. Especially I am thankful to the referee for his valuable comments and his hard work on the improving my language and which made the article readable.

2. Remarks on C^*-categories

In this section we recall the definition of a C^*-category, and the main properties and examples, which are used in the next sections.

2.1. Definition and some properties of a C^*-category

By $\mathbb{C}$ is denoted the field of complex numbers.

Definition 1. A $\mathbb{C}$-linear category A is called a C^*-category if :

a) $\hom(a,b)$ is a complex Banach space, the composition of morphisms is bilinear and $\|fg\| \leqslant \|f\| \cdot \|g\|$ for every pair of composable morphisms f and g;

b) There is an involutive antilinear contravariant endofunctor $* : A \to A$, which preserves objects.

c) $\|f\|^2 = \|f^* f\|$ for each morphism f, where $f^* = *(f)$;

d) The morphism $f^* f$ is a positive element of the C^*-algebra $\hom(a,a)$ for each $f \in \hom(a,b)$.

Example 2. 1) The category with Hilbert spaces as objects and all bounded linear maps as morphisms is a C^*-category, which will be denoted by $\mathcal{H}$.

2) Let B be a C^*-algebra. The category with right Hilbert B-modules as objects and all bounded B-homomorphisms, which have an adjoint, as morphisms is a C^*-category. We denote it by $\mathcal{H}(B)$. If E and F are modules from $\mathcal{H}(B)$ then $\mathcal{L}(E,F)$ or $\hom(E,F)$ denotes the space of morphisms from E to F, and $\mathcal{K}(E,F)$ denotes the ideal of compact B-homomorphisms from E to F.

3) A unital C^*-algebra is a C^*-category with one object and the elements of the algebra themselves as morphisms.

Definition 3. Let A and B be C^*-categories. A functor $\mathcal{F} : A \to B$ is said to be a $*$-functor if

a) $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$;
b) $\mathcal{F}(\lambda f) = \lambda \mathcal{F}(f)$;
c) $\mathcal{F}(f^*) = \mathcal{F}(f)^*$.

$*$-functors between C^*-categories, like $*$-homomorphisms of C^*-algebras, are norm-decreasing. For the following theorem we refer to [**4**].

Theorem 4. *Every C^*-category A may be realized as a concrete C^*-category, i.e., there is a faithful $*$-embedding $\mathcal{F} : A \to \mathcal{H}$*

Let A be a C^*-category and $I \subset \operatorname{Hom} A$ a subset. Put $\hom(a,b)_I = \hom(a,b) \cap I$. Then I is called a *left ideal* if $\hom(a,b)_I$ is linear subspace of $\hom(a,b)$ and $f \in \hom(a,b)_I$, $\quad g \in \hom(b,c)$ imply $gf \in \hom(a,c)_I$. A right ideal is defined similarly. I is two-sided ideal if it is both left and right ideal. An ideal I is closed if $\hom(a,b)_I$ is closed in $\hom(a,b)$ for each pair of objects.

I determines an equivalence relation on the morphisms of A: $f \sim g$ if $f - g \in I$. If $I = I^*$ is an ideal of A, the set of equivalence classes A/I can be made into a C^*-category in a unique way by requiring that the canonical map $f \mapsto \hat{f}$ give rise to a $*$-functor $A \to A/I$. A/I can be made into a normed $*$-category, by defining $\|\hat{f}\| = \sup_{g \in \hat{f}} \|g\|$. Arguing as for C^*-algebras, one can show

Proposition 5. *Let A be a C^*-category and I a be closed, two-sided ideal of A. Then $I = I^*$ and A/I is a C^*-category.*

Example 6. ¿From Example 2(2) it follows that there exists a C^*-category $\mathrm{Cal}(B) = \mathcal{H}(B)/\mathcal{K}(B)$ which sometimes will be called the Calkin C^*-category over B.

A word about $\mathbb{Z}_2$-graded C^*-categories: Let A be a C^*-category. A $\mathbb{Z}_2$-grading on A is a direct sum decomposition, for any pair of objects $a, b \in A$, $\hom(a,b) = \hom^{(0)}(a,b) \oplus \hom^{(1)}(a,b)$, with $\hom^{(0)}(a,b)$ and $\hom^{(1)}(a,b)$ two closed linear subspaces of $\hom(a,b)$, such that

a) if $f \in \hom^{(i)}(a,b)$ and $g \in \hom^{(j)}(b,c)$, then $gf \in \hom^{(i+j)}(a,c)$;

b) if $f \in \hom^{(i)}(a,b)$, then $f^* \in \hom^{(i)}(b,a)$.

A *morphism* from $\hom^{(i)}(a,b)$ is called *homogeneous of degree* i. The degree of a homogeneous element f is denoted ∂f.

A $*$-functor $\mathcal{F} : A \to B$ of graded C^*-categories A and B is *graded* if $\mathcal{F}(\hom^{(i)}(a,b)) \subset \hom^{(i)}(\mathcal{F}(a), \mathcal{F}(b))$ for any pair of objects a, b from A.

Let $\Gamma : A \to A$ be a $*$-functor, which is the identity map on objects and such that $\Gamma^2 = id_A$. Then $\hom^{(0)}(a,b) = \{f : \mathcal{F}(f) = f\}$ and $\hom^{(1)}(a,b) = \{f : \mathcal{F}(f) = -f\}$ gives a $\mathbb{Z}_2$-grading on A. Conversely, given a grading, one can define a corresponding $*$-functor Γ by the identity $\Gamma(f^{(0)} + f^{(1)}) = f^{(0)} - f^{(1)}$.

Let $\mathcal{F} : A \to B$ and $\mathcal{G} : A \to B$ be graded $*$-functors. A set

$$\alpha = \{\alpha_a : \mathcal{F}(a) \to \mathcal{G}(a)\}_{a \in \mathrm{Ob}\, A}$$

of morphisms is called a *natural transformation of degree* i, if $\partial\alpha_a = i$ for all α_a and

$$\mathcal{G}(f)\alpha_a = (-1)^{\partial\alpha\partial f}\alpha_b\mathcal{F}(f)$$

for any homogeneous morphism $f : a \to b$ from A.

A natural transformation $\alpha : \mathcal{G} \to \mathcal{F}$ is called *bounded* if $\sup_a \|\alpha_a\| < \infty$. Hereafter, by 'transformation' we will always mean 'bounded transformation.'

Example 7. Let B be a $\mathbb{Z}_2$-graded σ-unital algebra and let $\mathbb{H}(B)$ be the $\mathbb{Z}_2$-graded C^*-category with countably generated $\mathbb{Z}_2$-graded right Hilbert B-modules as objects, and B-homomorphisms between Hilbert modules of degree $i \in \mathbb{Z}_2$, that have an adjoint, as the morphisms of degree i. Let $E = E^{(0)} \oplus E^{(1)}$ be a module from $\mathbb{H}(B)$. Denote by $\check{E} = \check{E}^{(0)} \oplus \check{E}^{(1)}$ the opposite graded module to E, $\check{E}^{(0)} = E^{(1)}$ and $\check{E}^{(1)} = E^{(0)}$. Next we need the following endofunctor and natural transformation of degree 1.

Let $\mathbb{V} : \mathbb{H}(B) \to \mathbb{H}(B)$ be the covariant functor defined by the formula $\mathbb{V}(E) = \check{E}$ and $\mathbb{V}(f) = (-1)^{\partial f} f$, and consider natural transformation $\tau : id_{\mathbb{H}(B)} \to \mathbb{V}$ of degree 1 given by morphisms $\tau_E : E \to \check{E}$:

$$\tau_E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

in the decomposition $\check{E} = \check{E}^{(0)} \oplus \check{E}^{(1)}$. One checks that $\tau_E^* = -\tau_{\check{E}}$.

2.2. Additive and pseudoabelian C^*-categories

First we recall that a *projection* in a C^*-category is a morphism with the properties $p^* = p$ and $p^2 = 1$, i.e., a projection is a self-adjoint idempotent.

Definition 8. An additive C^*-category is said to be a *pseudoabelian C^*-category* if each projection has a kernel.

Remark. The main difference from the analogous definition of [**8**] and [**9**] is that here idempotents in addition are self-adjoint, i.e., are projections.

The following theorem describes how an additive C^*-category can be embedded in a pseudoabelian C^*-category (cf. [**8**], [**9**]).

Theorem 9. *Let A be an additive C^*-category. There exists a pseudoabelian C^*-category $\tilde{A}$ and an additive $*$-functor $\phi : A \to \tilde{A}$ with the following universal property. For any pseudoabelian C^*-category D and any additive $*$-functor $\psi : A \to D$ there exists a unique additive $*$-functor $\psi' : \tilde{A} \to D$ such that $\psi = \phi \cdot \psi'$. The pair $(\phi, \tilde{A})$ is unique up an additive $*$-equivalence of additive C^*-categories.*

Proof. We only give here the constructions, because the proofs are precisely analogous to those in [**8**], [**9**]. An object of $\tilde{A}$ has the form (E, p), where $E \in \mathrm{Ob}(A)$ and $p \in \hom(E, E)$ is a projection. A morphism from (E, p) to (F, q) is defined as a morphism $f : E \to F$ of A such that $fp = qf = f$. The composition of morphisms is defined as a composition of morphisms in A. The sum of objects is given by formula $(E, p) \oplus (F, q) = (E \oplus F, p \oplus q)$, and the norm of morphisms is inherited from A. □

As suggested in [**8**] and [**9**], the construction of K-theory is based on the notion of pseudoabelianness of an additive category, and is slightly different from the similar definition given here. We carry out the construction using notion of a pseudoabelian C^*-category. Consider A as a Banach category and denote by ξA the pseudoabelian category A in Karoubi's sense. We have following:

Lemma 10. *Let A be an additive C^*-category. Then the category $\tilde{A}$ is additively equivalent to ξA.*

Proof. Let $i : \tilde{A} \to \xi A$ be the faithful functor that is the identity on objects and morphisms. To define $j : \xi A \to \tilde{A}$ firstly note that if $q \in \hom(F, F)$ is an idempotent then

$$\bar{q} = \sqrt{(2q^* - 1)(2q - 1) + 1} \cdot q \cdot \sqrt{((2q^* - 1)(2q - 1) + 1)^{-1}}$$

is a projection [**10**] and the pairs (F, q) and $(F, \bar{q})$ are isomorphic by

$$u_q = \sqrt{(2q^* - 1)(2q - 1) + 1}.$$

Then define j by the formulas $(E, q) \mapsto (E, u_q q u_q^{-1})$ on objects and $j(f) = u_{q'} f u_q$ for a morphisms, where $f : (E, q) \to (E', q')$. Now it is easy to show that the isomorphism $i \cdot j \simeq id_{\xi A}$ is given by the essential isomorphisms $\{u_q\}$ and $j \cdot i = id_{\tilde{A}}$. □

2.3. On the main examples of C^*-categories

Here we assume that all C^*-algebras are trivially graded and they have a norm-continuous action of a fixed compact group.

Having treated pseudoabelian C^*-categories, we now proceed to one of the main examples of this paper.

Example 11. 1) Firstly we define the C^*-category $\mathcal{H}(B)$ over fixed compact second countable group G. The objects of this category are all countable generated right Hilbert B-modules equipped with a B-linear, norm-continuous G-action such that $g(xb) = g(x)g(b)$ and $< g(x), g(y) >= g < x, y >$, for all $g \in G$. A morphism $f : E \to E'$ is B-homomorphism such that there exists $f^* : E' \to E$ satisfying the conditions: $< T(x), y >=< x, T^*(y) >$ where $x \in E$ and $y \in E'$. The norm of a morphism is defined as the norm of linear bounded map. It is easy to check that $\mathcal{H}(B)$ is an additive C^*-category with respect to the sum of the Hilbert modules . Finally, note that compact group acts on the morphisms by the following way: if $f : E \to E'$ then morphism $gf : E \to E'$ is defined by the formula $gf(x) = g(f(g^{-1}(x)))$. (Note that this action generally is not norm-continuous). A morphism is called *invariant* if $gf = f$. In the next, under $\mathcal{H}(B)$ we mean above constructed additive C^*-category with this action of G (cf. [**11**])

2) Now, we define the C^*-category $rep(A, B)$. The objects of this category are pairs of form (E, ϕ), where E is a countably generated right Hilbert B-module and $\phi : A \to \mathcal{L}(E)$ is an equivariant $*$-homomorphism. Objects of this type are said to be A, B-bimodules (cf. [**16**]).

A morphism $f : (E, \phi) \to (E', \phi')$ is an invariant B-homomorphism $f : E \to E'$ in $\mathcal{H}(B)$ such that $f\phi(a) = \phi'(a)f$ for all $a \in A$. The structure of C^*-category is inherited from the C^*-category structure of $\mathcal{H}(B)$ and it is easy to show that $rep(A, B)$ is an additive C^*-category (in fact, a pseudoabelian C^*-category). The following property of $rep(A, B)$ will be used to calculate the K-groups of $rep(A, B)$. Note that there exists a $*$-functor $\infty : \mathcal{H}(B) \to \mathcal{H}(B)$ and a natural isomorphism of functors $id_{\mathcal{H}(B)} \oplus \infty \simeq \infty$, where $E^\infty = E \oplus E \oplus \cdots$. This structure induces a corresponding structure on $rep(A, B)$ via the formulas $(E, \phi)^\infty = (E^\infty, \phi^\infty)$, where $\phi^\infty(a) = (\phi(a))^\infty$ for all $a \in A$. This structure will be called an ∞-structure.

3) Consider the additive C^*-category $\mathrm{Cal}(B)$ which is the quotient C^*-category $\mathcal{H}(B)/\mathcal{K}(B)$. It has an essential compact group action inherited from the action of a compact group on $\mathcal{H}(B)$. Denote by $\pi : \mathcal{H}(B) \to \mathrm{Cal}(B)$ the canonical additive $*$-functor. We need also following C^*-category denoted by $\mathrm{Cal}(A, B)$. By definition objects of this category have the form (E, ψ), where E is a Hilbert B-module and $\psi : A \to \hom_{\mathrm{Cal}B}(E, E)$ is a equivariantly liftable $*$-homomorphism, i.e., there exists an A, B-bimodule (E, ϕ) such that $\pi\phi = \psi$. A morphism $f : (E, \psi) \to (E', \psi')$ is a morphism $f : E \to E'$ of the category $\mathrm{Cal}(B)$ such that $f\psi(a) = \psi'(a)f$ for all $a \in A$, and has a invariant lifting in $\mathcal{H}(B)$. Define the $*$-functor $\Theta : rep(A, B) \to \mathrm{Cal}(A, B)$ by $(E, \phi) \mapsto (E, \pi\phi)$ and $f \mapsto \pi(f)$.

4) Now, we want to define the additive C^*-category $Rep(A, B)$. The objects of this category are also A, B-bimodules, i.e., objects are pairs (E, ϕ), where E is a countably generated right Hilbert B-module and $\phi : A \to \mathcal{L}(E)$ is a equivariant $*$-homomorphism. Also, a morphism $f : (E, \phi) \to (E', \phi')$ is a invariant morphism $f : E \to E'$ in $\mathcal{H}(B)$ such that

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, E')$$

for all $a \in A$. The structure of C^*-category is inherited from $\mathcal{H}(B)$. It is easy to show that $Rep(A, B)$ is an additive C^*-category but it isn't a pseudoabelian C^*-category. There is a canonical additive $*$-functor $\Pi_{A,B} : Rep(A, B) \to \mathrm{Cal}(A, B)$ defined by $(E, \phi) \mapsto (E, \pi\phi)$ and $f \mapsto \pi f$. From the definition follows easily that the canonical linear map

$$\hom((E, \phi), (E', \phi')) \mapsto \hom((E, \pi\phi), (E, \pi\phi')$$

is surjective, i.e., Π is a Serre functor (see for the definition [**8**]). $\square$

Now we come to our main C^*-category, that is, $\mathrm{Rep}(A, B)$.

Definition 12. The C^*-category $\mathrm{Rep}(A, B)$ is the universal pseudoabelian C^*-category of $Rep(A, B)$. Using the definition of a pseudoabelian C^*-category, we have the following description of $\mathrm{Rep}(A, B)$. Objects of it are triples (E, ϕ, p), where (E, ϕ) is an object and $p : (E, \phi) \to (E, \phi)$ is a morphism of $Rep(A, B)$ such that $p^* = p$ and $p^2 = p$. A morphism $f : (E, \phi, p) \to (E', \phi', p')$ is a morphism $f : (E, \phi) \to (E', \phi')$ of $Rep(A, B)$ such that $fp = p'f = f$. In detail, f has the properties

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, F), \quad fp = p'f = f. \tag{1}$$

The structure of C^*-category of $\mathrm{Rep}(A, B)$ comes from the corresponding structure of $Rep(A, B)$. In particular, the sum of triples is given by formula $(E, \phi, p) \oplus (F, \psi, q) = (E \oplus F, \phi \oplus \psi, p \oplus q)$.

2.4. On a cofinal subcategory of $\mathbb{H}(B \otimes C^{(1,0)})$

When A and B are trivially graded C^*-algebras, for an interpretation of $KK^1(A, B)$, we need information on the following subcategory of $\mathbb{H}(B \otimes C^{(1,0)})$. We begin this subsection recalling a definition from [**8**], [**9**].

Definition 13. An additive $*$-functor $\mathcal{F} : A \to B$ of additive C^*-categories is said to be *quasi-surjective* if for any object b from B there are objects a and b' from A and B respectively, and a unitary isomorphism $b \oplus b' \simeq \mathcal{F}(a)$. In particular, an additive C^* sub-category A of B is cofinal iff the canonical inclusion is quasi-surjective.

Let $C^{(1,0)}$ be the Clifford algebra with one generator g ($g^* = g$ and $g^2 = 1$), with trivial action of a compact group. Consider the cofinal full subcategory $\mathbb{H}_B(B\otimes C^{(1,0)})$ of $\mathbb{H}(B\otimes C^{(1,0)})$ which contains modules isomorphic to modules of form

$$E_{n+1,n} = E \otimes_{B\otimes C^{(1,0)}} (C^{n,n} \otimes_{\mathbb{C}\otimes C^{(1,0)}} C^{(1,0)})$$

where E is a trivially graded equivariant B-module.

Also,denote by $\mathbb{H}_{C^{(1,0)}}(B)$ the full subcategory of $\mathbb{H}(B)$ with objects isomorphic to B-modules of the form

$$E'_{n+1,n} = E \otimes_{B\otimes\mathbb{C}} (C^{n,n} \otimes C^{(1,0)}).$$

There is a canonical additive $*$-functor

$$S : \mathbb{H}_B(B \otimes C^{(1,0)}) \to \mathbb{H}_{C^{(1,0)}}(B)$$

defined by formulas $S(E_{n+1,n}) = E'_{n+1,n}$ and $S(f) = f$ for every $B \otimes C^{(1,0)}$-homomorphism f, because f may be considered also as a B-homomorphism. This functor is injective, i.e., the linear maps on hom sets are injective. Note that a B-homomorphism $f : E'_{n+1,n} \to F'_{n+1,n}$ defines a $B\otimes C^{(1,0)}$-homomorphism iff the morphism f is invariant under the action of $1\otimes 1\otimes\varepsilon$, i.e.,

$$(1 \otimes 1 \otimes \varepsilon)f(1 \otimes 1 \otimes \varepsilon) = f.$$

But the C^*-category $\mathbb{H}_{C^{(1,0)}}(B)$ coincides with the full subcategory of $\mathbb{H}(B)$ generated by modules isomorphic to $E \oplus \check{E}$, where E is trivially graded equivariant B-module. For each $E \oplus \check{E}$ consider the element

$$\varepsilon_E = \begin{pmatrix} 0 & -\tau_{\check{E}} \\ \tau_E & 0 \end{pmatrix} \tag{2}$$

where $\tau_E : E \to \check{E}$ is the canonical isomorphism of degree 1. (See example 7.) Then $\varepsilon_E^* = \varepsilon_E$ and $\varepsilon_E^2 = 1$. Consider the essential $\mathbb{Z}_2$-action on the $\mathbb{H}_{C^{(1,0)}}(B)$ defined as follows. Let $f : E \oplus \check{E} \to F \oplus \check{F}$ be a B-homomorphism. Then we have

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \mapsto \begin{pmatrix} \check{f}_{22} & \check{f}_{21} \\ \check{f}_{12} & \check{f}_{11} \end{pmatrix} \tag{3}$$

where $\check{f} = \mathbb{V}(f)$. (See example 7.) In particular, if f is a homomorphism of degree 0 invariant under this action, then

$$f = \begin{pmatrix} f_0 & 0 \\ 0 & \check{f}_0 \end{pmatrix} \tag{4}$$

and if f has degree 1 under this action then

$$f = \begin{pmatrix} 0 & \check{f}_1 \\ f_1 & 0 \end{pmatrix}. \tag{5}$$

Denote by $\mathbb{H}_*(B)$ this invariant subcategory of $\mathbb{H}_{C^{(1,0)}}(B)$. Thus we have the following:

Proposition 14. *Let B be a trivially graded C^*-algebra. Then the graded additive C^*-category $\mathbb{H}_B(B \otimes C^{(1,0)})$ is graded additively isomorphic to $\mathbb{H}_*(B)$.*

3. K-groups of an additive category

The purpose of this section is to transform some main results of K-theory of Banach categories, introduced by M. Karoubi in [**8**], [**9**], to a C^*-category. There are some minor changes which will be needed in sequel sections.

3.1. K^0 and K^1 groups for an additive C^*-category

Definition 15. The group $K^0(A)$ of an additive C^*-category is the Grothendieck group of the abelian monoid of unitary isomorphism classes of objects of A.

Note that this definition coincides with usual definition because in a C^*-category, objects are isomorphic if and only if they are unitarily isomorphic. Indeed, if $u : E \to F$ is an isomorphism, then $u_0 = u\sqrt{(u^*u)^{-1}}$ is a unitary isomorphism. So from lemma 10 we get the following:

Proposition 16. *Let A be an additive C^*-category. The canonical functor induces an isomorphism $i_* : K^0(\tilde{A}) \to \mathbb{K}^0(\xi A)$, where the left-hand K-group is as in the definition above, and the right one as in* [**8**], [**9**].

Now we discuss analogous questions for the K^{-1} group (cf. [**8**], [**9**]).

Definition 17. Let A be an additive C^*-category. Consider the set of pairs (E, α), where $E \in \mathrm{Ob}A$ and $\alpha \in \hom(E, E)$ is a unitary automorphism.

a). The pairs (E, α) and (E', α') are said to be *unitarily isomorphic* if there exists a unitary isomorphism $u : E \to E'$ such that diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow^{\alpha} & & \downarrow^{\alpha'} \\ E & \xrightarrow{u} & E' \end{array}$$

is commutative.

b). The pairs (E, α) and (E, α') are said to be *homotopic* if α and α' are homotopic in $\mathrm{Aut}\, E$.

c). A pair (E, α) is said to be *elementary* if it is homotopic to $(E, 1_E)$.

d). The sum is defined by the formula

$$(E, \alpha) \oplus (E', \alpha') = (E \oplus E', \alpha \oplus \alpha').$$

e). The pairs (E, α) and (E, α') are said to be *stably isomorphic* if there exist elementary pairs $(\bar{E}, \bar{e})$ and $(\hat{E}, \hat{e})$, and a unitary isomorphism

$$(E, \alpha) \oplus (\bar{E}, \bar{e}) \simeq (E', \alpha') \oplus (\hat{E}, \hat{e}).$$

f). The abelian monoid $K^{-1}(A)$ is defined as the monoid of classes of stably isomorphic pairs. Denote by $d(E, \alpha)$ the class of (E, α) in $K^{-1}(A)$.

Lemma 18. (Cf. [**9**].) *There are the following relations in $K^{-1}(A)$:*

a). $d(E, \alpha) + d(E, \alpha^*) = 0$;

b). If α and α' are homotopic unitary isomorphisms, then $d(E, \alpha) = d(E, \alpha')$.

c). $d(E, \alpha) + d(E, \beta) = d(E, \beta\alpha)$;

In particular, $K^{-1}(A)$ is an abelian group.

Proof. a). The unitary automorphism $\alpha \oplus \alpha^*$ can be written in the form

$$\begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

but each matrix is homotopic to $1_{E\oplus E}$. Thus a) holds.

b). Apply a). We get

$$d(E, \alpha') - d(E, \alpha) = d(E \oplus E, \alpha' \oplus \alpha^*)$$

But $\alpha' \oplus \alpha^*$ is homotopic to $\alpha \oplus \alpha^*$. Thus $d(E \oplus E, \alpha' \oplus \alpha^*) = 0$.

c). Note that $d(E, \alpha) + d(E, \beta) = d(E \oplus E, \alpha \oplus \beta)$ and

$$d(E, \alpha\beta) = d(E \oplus E, \alpha\beta \oplus 1_E).$$

But $(\alpha \oplus \beta)^*(\alpha\beta \oplus 1_E) = \beta \oplus \beta^*$, which is homotopic to $1_{E\oplus E}$. Thus $\alpha \oplus \beta$ is homotopic to $\alpha\beta$. Thus we can apply b). □

The next proposition is analogous to the corresponding property of the group $K^0(A)$.

Proposition 19. *Let A be an additive C^*-category. The canonical homomorphism*

$$i_* : K^{-1}(A) \to \mathbb{K}^{-1}(A),$$

defined by $d(E, \alpha) \mapsto d(E, \alpha)$ is an isomorphism. Here $\mathbb{K}^{-1}(A)$ is Karoubi's group.

Proof. i_* is an epimorphism: Let (E, α) be a pair with α an isomorphism. Consider the unitary isomorphism $\bar{\alpha} = \alpha\sqrt{\alpha^*\alpha}$. It is homotopic to α, because $\alpha^*\alpha$ is homotopic to 1_E. Apply the lemma. We get that $d(E, \alpha) = d(E, \bar{\alpha})$. i is a monomorphism: If $i(d(E, \alpha)) = 0$, then there exists elementary (E', e') such that $(E \oplus E', \alpha \oplus e')$ is elementary. Then $(E \oplus E', \overline{\alpha \oplus e'})$ is also elementary, that is $(E \oplus E', \alpha \oplus \bar{e'})$ elementary. This means $d(E, \alpha) = 0$. □

Thus the properties of $K^{-1}(A)$ are inherited from the corresponding properties of $\mathbb{K}^{-1}(A)$. In particular, we get the following:

Lemma 20. $d(E, \alpha) = 0$ *if there exists an object G such that $\alpha \oplus 1_G$ is homotopic to $1_{E\oplus G}$.*

Theorem 21. *Let A be an additive C^*-category, $\tilde{A}$ be the associated pseudoabelian C^*-category and $i : A \to \tilde{A}$ the canonical additive $*$-functor. Then the induced homomorphism*

$$i_* : K^{-1}(A) \to K^{-1}(\tilde{A}) \tag{6}$$

is an isomorphism.

3.2. The K-group of a $*$-functor

Definition 22. Let A and B be additive C^*-categories and $\mathcal{F} : A \to B$ be an additive $*$-functor. Denote by $\Gamma(\mathcal{F})$ the set of triples (E, F, α), where E and F are objects of A, and $\alpha : \mathcal{F}(E) \to \mathcal{F}(F)$ is a unitary isomorphism from B.

a) Two triples (E, F, α) and (E', F', α') are *unitarily isomorphic* if there exist unitary isomorphisms $f : E \to E'$ and $g : F \to F'$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \stackrel{\alpha}{\to} & \mathcal{F}(F) \\ \downarrow^{\mathcal{F}(f)} & & \downarrow^{\mathcal{F}(g)} \\ \mathcal{F}(E') & \stackrel{\alpha'}{\to} & \mathcal{F}(F') \end{array}$$

is commutative.

b). Two triples (E, F, α) and (E, F, α') are called *homotopic* if α and α' are homotopic in the subspace of unitary isomorphisms of $\hom(E, F)$.

c). The triple $(E, E, 1_E)$ is called trivial. A triple (E, F, α) is said to be *elementary* if this triple is homotopic to the trivial triple.

e). The sum of triples is defined by the formula $(E, F, \alpha) \oplus (E', F', \alpha') = (E \oplus E', F \oplus F', \alpha \oplus \alpha')$.

f). Two triples $\sigma = (E, F, \alpha)$ and $\sigma' = (E', F', \alpha')$ are *stably unitarily isomorphic* if there exist elementary pairs $\tau = (\bar{E}, \bar{E}, \bar{\alpha})$ and $\tau' = (\bar{E}', \bar{E}', \bar{\alpha}')$ such that $\sigma \oplus \tau$ and $\sigma' \oplus \tau'$ are unitarily isomorphic.

The set $K(\mathcal{F})$ of stably isomorphic triples is an abelian monoid with respect to to the sum of triples. Denote by $d(E, F, \alpha)$ the class of (E, F, α) in $K(\mathcal{F})$.

Lemma 23. *The monoid $K(\mathcal{F})$ is an abelian group. Moreover*

$$d(E, F, \alpha) + d(F, E, \alpha^*) = 0.$$

Proof. Note that

$$d(E, F, \alpha) + d(F, E, \alpha^*) = d(E \oplus F, F \oplus E, \alpha \oplus \alpha^*)$$

The last triple is isomorphic to $(E \oplus F, \beta)$, where

$$\beta = \begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}$$

which is homotopic to $1_{\mathcal{F}(E)\oplus\mathcal{F}(F)}$ by $u(t) = \sigma(t)\sqrt{\sigma^*(t)\sigma(t)}$, where

$$\sigma(t) = \begin{pmatrix} 1 & -t\alpha^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^* \\ 0 & 1 \end{pmatrix}$$

□

The following theorem compares our definition of $K(\mathcal{F})$ with the corresponding one of Karoubi.

Theorem 24. *The canonical homomorphism $i : K(\mathcal{F}) \to \mathbb{K}(\mathcal{F})$ defined by*

$$d(E, F, \alpha) \mapsto d(E, F, \alpha)$$

is an isomorphism.

Proof. Let (E, F, α) be a triple which defines an element in $\mathbb{K}(\mathcal{F})$, where α is an isomorphism (but not unitary isomorphism). Let $\bar{\alpha} = \alpha\sqrt{\alpha^*\alpha}$. $\bar{\alpha}$ is unitary and is homotopic to α because $\alpha^*\alpha$ is homotopic to $1_{\mathcal{F}(E)}$. This proves that i is an epimorphism. Now, let $d(E, F, \alpha) \in K(\mathcal{F})$ defines 0 in $\mathbb{K}(\mathcal{F})$. This means by [**9**] that there exist objects G and H and isomorphisms (but after polar decomposition we may suppose they are unitary isomorphisms) $u : E \oplus G \to H$ and $v : F \oplus G \to H$ that $\mathcal{F}(v)(\alpha \oplus 1_{\mathcal{F}}(G))\mathcal{F}(u^*)$ is homotopic to $1_{\mathcal{F}(H)}$ (see [**9**]) by a homotopy $h(t)$. Then $\bar{h}(t) = h(t)\sqrt{h^*(t)h(t)}$ gives homotopy between $(E, F, \alpha) \oplus (G, G, 1_G)$ and $(H, H, 1_H)$. This means $d(E, F, \alpha) = 0$ in $K(\mathcal{F})$. □

This theorem means that all properties of $K(\mathcal{F})$ inherited from the corresponding properties of $\mathbb{K}(\mathcal{F})$. In particular, we get the following results. (Cf. [**8**], [**9**].)

Lemma 25. *There are the following relations in $K(\mathcal{F})$:*
a). If α and α' are homotopic, then $d(E, F, \alpha) = d(E, F, \alpha')$;
b). $d(E, F, \alpha) + d(F, G, \beta) = d(E, G, \beta\alpha)$.

Proposition 26. *Let $\mathcal{F} : A \to B$ be a Serre quasisurjective additive $*$-functor.*
a). If in the definition of $K(\mathcal{F})$ we replace elementary triples by trivial triples then we get the same group.
b). $d(E, F, \alpha) = 0$ iff there exist an object G from A and unitary isomorphism $\beta : E \oplus G \to F \oplus G$ such that $\mathcal{F}(\beta) = \alpha \oplus 1_{\mathcal{F}(G)}$.

Proposition 27. *Let $\mathcal{F} : A \to B$ be a quasi-surjective additive $*$-functor. Then the sequence of abelian groups*

$$K^{-1}(A) \xrightarrow{f_1} K^{-1}(B) \xrightarrow{\partial} K^0(\mathcal{F}) \xrightarrow{i} K^0(A) \xrightarrow{\partial} K^0(B) \tag{7}$$

*is exact, where $i(d(E, F, \alpha)) = d(E) - d(F)$ (for the definition of ∂ see [**9**]). In addition, if there exists a functor $\Psi : B \to A$ such that $\mathcal{F} \cdot \Psi \simeq Id_B$, then there exists a split exact sequence*

$$0 \to K^0(\mathcal{F}) \xrightarrow{i} K^0(A) \xrightarrow{j} K^0(B) \to 0. \tag{8}$$

Example 28. 1)Recall that an object of $rep(A, B)$ has the form (E, ϕ), where E is a right Hilbert B-module and $\phi : A \to \mathcal{L}(E)$ is supposed equivariant. A morphism from (E, ϕ) to (E', ϕ') is by definition an invariant B-homomorphism $f : E \to E'$ such that $f\phi(a) = \phi'(a)f$ (see example 11). Note that $rep(A, B)$ is a pseudoabelian C^*-category. To show that $K^i(rep(A, B)) = 0$ for all $i \in \mathbb{Z}_2$, consider the ∞-structure of $rep(A, B)$ $E^\infty = E \oplus E \oplus \cdots$, $\alpha^\infty = \alpha \oplus \alpha \oplus \cdots$, and $\phi^\infty(a) = (\phi(a))^\infty$. Let

$$\infty : rep(A, B) \to rep(A, B)$$

be the $*$-functor defined by the formula $\infty(E) = E^\infty$, $\infty(\phi) = \phi^\infty$, and if α is a morphism in $rep(A, B)$, then $\infty(\alpha) = \alpha^\infty$. There exists a natural isomorphism $id_{rep(A,B)} \oplus \infty \simeq \infty$. From this it follows that the groups $K^i(rep(A, B))$ (and the cancellation monoid $C(rep(A, B))$ of isomorphism classes of objects of $rep(A, B)$) have an automorphism I with property that

$$id_{K^i(rep(A,B))} + I = I.$$

From this fact it follows that $K^i(rep(A, B)) = 0$ (resp., $C(rep(A, B) = 0)$.

2) Consider the canonical quasi-surjective functor

$$\Theta_{A,B} : rep(A, B) \to \mathrm{Cal}(A, B).$$

Applying the exact sequence (7) of K-groups and result of example 1, one gets that the canonical homomorphism

$$\partial : K^{-1}(\mathrm{Cal}(A, B)) \to K^0(\Theta_{A,B}) \tag{9}$$

is an isomorphism. □

4. Remarks on the definition of KK-groups

In this section we review definitions of Kasparov KK-groups in the form that will be needed for our purposes.

Let A and B be $\mathbb{Z}_2$-graded C^*-algebras, assuming that A is separable and B is σ-unital. Let A^+ be obtained from A by adjoining a unit of degree 0. We also assume that all C^*-algebras have fixed compact group actions as above.

4.1. Operatorial homotopy and KK-theory

Let $\mathcal{E}(A, B)$ be the set of Kasparov A, B-bimodules. Denote by $KK(A, B)$ the Kasparov group obtained by dividing $\mathcal{E}(A, B)$ by the equivalence relation generated by unitary isomorphism and operator homotopy, modulo degenerate bimodules [**10**], [**16**], [**3**].

We need the following elementary properties of $KK(A, B)$:

a) "Cofinality principle". Let $\mathcal{F}$ be a cofinal full additive subcategory of $\mathbb{H}(B)$. If in the definition of $KK(A, B)$ the Kasparov A, B-bimodules are replaced by Kasparov A, B-bimodules defined by $\mathcal{F}$, then we get the same group.

b) "Unitization principle". There exists a split exact sequence

$$0 \to KK(A, B) \xrightarrow{j} KK(A^+, B) \overset{p^*, i^*}{\leftrightarrow} KK(\mathbb{C}, B) \to 0.$$

Remark. Let $s : A \to A^+$ be the canonical inclusion, $p : A^+ \to \mathbb{C}$ the canonical projection, and $i : \mathbb{C} \to A^+$ the canonical inclusion. One has following split exact sequence

$$0 \to KK(A^+, A) \xrightarrow{s^*} KK(A^+, A^+) \overset{i_*, p_*}{\leftrightarrow} KK(A^+, \mathbb{C}) \to 0.$$

Consider the element $1 - ip$ in $KK(A^+, A^+)$. One checks that $p_*(1 - ip) = 0$. From this follows that there exists a unique $j \in KK(A^+, A)$ such that

$$js = 1_A \text{ in } KK(A, A) \text{ and } sj + ip = 1_{A^+} \text{ in } KK(A^+, A^+).$$

c) "Unitality principle". Let A be unital C^*-algebra. Then in the definition of $KK(A, B)$ it is possible to take Kasparov A, B-bimodules of form (E, ϕ, F) where $\phi(1) = 1$.

The following definition is motivated by b) and c).

Definition 29. A Kasparov A, B-bimodule $e = (E, \varphi, F)$ is said to be *almost unital* if e has the following properties:

$$[\varphi(a), F] \in \mathcal{K}(E), \quad F - F^* \in \mathcal{K}(E), \quad F^2 - 1 \in \mathcal{K}(E) \tag{10}$$

for all $a \in A$. An almost unital A, B-bimodule (E, φ, F) is degenerate if

$$[\varphi(a), F] = 0, \quad F = F^*, \quad F^2 = 1 \tag{11}$$

Denote by $\mathcal{E}^+(A, B)$ set of almost unital Kasparov A, B-bimodules.

Define the group $KK^+(A, B)$ by analogy with the definition of $KK(A, B)$, using almost unital A, B-bimodules. By principles b) and c), $KK^+(A, B)$ is essentially isomorphic to $KK(A^+, B)$.

Remark. a) Let $e = (E, \varphi, F)$ be an almost unital A, B-module, and $\pi : \mathcal{L}(E) \to \mathrm{Cal}(E)$ be a canonical $*$-homomorphism. The morphism $\pi(F)$ is a unitary morphism in $\mathrm{Cal}(E)$, and thus $||\pi(F)|| = 1$. Apply the lifting theorem from [**1**], which confirms the existence of norm-preserving lifting of an element, we get an element $G' \in \mathcal{L}(E)$ such that $G' - F \in \mathcal{K}(E)$ and $||G'|| = 1$. Therefore, (E, φ, G') is also an almost unital A, B-module with $||G'|| = 1$. Replacing G' by $G = \frac{G'+G'^*}{2}$, one get the A, B-module $e' = (E, \varphi, G)$ with the following properties:

$$[\varphi(a), G] \in \mathcal{K}(E), \quad G = G^*, \quad G^2 - 1 \in \mathcal{K}(E), \quad ||G|| \leqslant 1 \tag{12}$$

and $G - F \in \mathcal{K}(E)$. This fact implies that e is operatorial homotopy to e', connected by segment. The A, B-modules with properties 12 will be called *fine* A, B-modules.

b) Let (E, φ, G_0) and (E, φ, G_1) be two fine A, B-modules connected by an operatorial homotopy (E, φ, G_t) of almost unital A, B-modules. Using the same technic as in a), one gets an operatorial homotopy (E, φ, G'_t) of fine A, B-modules. When $i = 0, 1$, the A, B-modules (E, φ, G_i) and $(E.\varphi, G'_i)$ are trivially operatorial homotopic. Thus (E, φ, G_0) and (E, φ, G_1) are operatorial homotopic in the set of fine A, B-modules.

The above remark shows that $KK^+(A, B)$ (resp., $E_+(A, B)$, CE^+A, B, $KE_+(A, B)$) does not change if one replaces almost unital A, B-modules by the fine A, B-modules in the constructions.

Now, consider $\mathcal{G}(A, B)$ the set of A, B-bimodules (E, ϕ, G) with the following properties:

$$[\phi(a), G] \in \mathcal{K}(E), \quad G = G^*, \quad G^2 = 1. \tag{13}$$

which will be called *best* A, B-modules. Let $G(A, B)$ be the abelian monoid of equivalence classes of A, B-bimodules from $\mathcal{G}(A, B)$, equivalence being generated by unitary isomorphism and operatorial homotopy. Let $CG(A, B)$ be the cancellation monoid (resp., $KG(A, B)$ be the Grothendieck group) of $G(A, B)$. The essential map $\mathcal{G}(A, B) \to \mathcal{E}(A, B)$ induces homomorphisms

$$\mu : CG(A, B) \to KK^+(A, B)$$

and

$$\alpha : CG(A, B) \to KK^+(A, B)$$

since $KK^+(A, B)$ is abelian group. According on the last remark, one has

Proposition 30. *Let A and B be as above. Then the following sequence of groups is split exact*

$$0 \to CG(A, B) \overset{\mu}{\to} KK^+(A, B) \overset{p^*, i^*}{\leftrightarrow} KK(\mathbb{C}, B) \to 0. \tag{14}$$

Also, the canonical homomorphisms $CG(A, B) \to KG(A, B)$ and

$$\alpha : CG(A, B) \to KK(A, B)$$

are isomorphisms.

Proof. Firstly we show that μ is a monomorphism. Let (E, φ, G) and (E', φ', G') be best A, B-modules, and let them define the same element of $KK^+(A, B)$. Then there exist degenerate A, B-modules $(\hat{E}, \hat{\varphi}, \hat{G})$ and $(\hat{E}', \hat{\varphi}', \hat{G}')$ such that

$$(E, \varphi, G) \oplus (\hat{E}, \hat{\varphi}, \hat{G})$$

is operatorial homotopic in the set of fine A, B-modules to

$$(E', \varphi', G') \oplus (\hat{E}', \hat{\varphi}', \hat{G}').$$

Let (R, ϕ, L_t) be this operatorial homotopy. Since $||L_t|| \leqslant 1$ for $t \in [0; 1]$, one has the following operatorial homotopy in the set of best A, B-modules:

$$(R \oplus R, \phi \oplus 0, L'_t)$$

where

$$L'_t = \begin{pmatrix} L_t & (1 - L_t^2)^{1/2} \\ (1 - L_t^2)^{1/2} & -L_t \end{pmatrix} ,$$

which gives operatorial homotopy in the set of best A, B-modules. This means that the elements (E, φ, G) and (E', φ', G') are operatorial homotopic in the set of best A, B-modules up to adding of degenerate A, B-modules. But from the example 28 follows that degenerate A, B-modules define zero element in $CG(A, B)$. Thus (E, φ, G) is equal to (E', φ', G') in $CG(A, B)$. Now we prove that $\ker(KK^+(A, B) \to KK(A, B)) \subset im(\mu)$. Let (E, φ, G) be fine A, B-module such that the induced $\mathbb{C}, B$-module defines zero element in $KK(\mathbb{C}, B)$. Then A, B-module $(E, 0, G)$ defines zero element in $KK^+(A, B)$ too. Thus the best A, B-module $(E \oplus E, \varphi \oplus 0, D)$, where

$$D = \begin{pmatrix} G & (1 - G^2)^{1/2} \\ (1 - G^2)^{1/2} & -G \end{pmatrix} , \tag{15}$$

defines the same element as (E, φ, G) in $KK^+(A, B)$. To prove the other statements of exactness is trivial and left to the reader. As a consequence we get that $CG(A, B)$ is an abelian group, and thus it is isomorphic to $KG(A, B)$. Finally, comparing our split exact sequence with the exact sequence of "unitization principle", one gets that α is an isomorphism. □

4.2. Fredholm picture for the trivially graded case

In this subsection we consider the case when A and B are trivially graded C^*-algebras with compact group actions. Let $E = E^0 \oplus E^1$ be a graded Hilbert B-module. Then E^0 is a trivially graded B-module and $E^1 = \check{M}$, where M is a trivially graded Hilbert B-module. Consider $e = (E, \phi, F)$, an almost unital Kasparov (A, B)-bimodule. Then in the decomposition $E = E^{(0)} \oplus E^{(1)}$, one has

$$\phi = \begin{pmatrix} \phi^{(0)} & 0 \\ 0 & \phi^{(1)} \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & F^{(0)} \\ F^{(1)} & 0 \end{pmatrix}$$

where $(E^{(0)}, \phi^{(0)})$ and $(E^{(1)}, \phi^{(1)})$ are A, B-bimodules and $F^{(i)} : (E^{(i)}, \phi^{(i)}) \to (E^{(i+1)}, \phi^{(i+1)})$ are bimodule morphisms of degree 1. This interpretation motivates the following:

Definition 31. a). Let A and B be C^*-algebras with compact group actions. A Fredholm A, B-bimodule is $\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F)$, where $E^{(i)}$, $i = 0, 1$, are trivially graded Hilbert B-modules, and $(E^{(0)}, \phi^{(0)})$ and $(E^{(1)}, \phi^{(1)})$ are A, B-bimodules, and $F : (E^{(0)}, \phi^{(0)}) \to (E^{(1)}, \phi^{(1)})$ is a morphism of A, B-bimodules of degree 0, i.e.,

$$F\phi^{(0)}(a) - \phi^{(1)}(a)F \in \mathcal{K}(E^{(0)}, E^{(1)})$$

such that $F^*F - 1 \in \mathcal{K}(E^{(0)})$ and $FF^* - 1 \in \mathcal{K}(E^{(1)})$. The set of Fredholm A, B-bimodules is denoted by $\mathcal{F}(A, B)$.

b). Fredholm bimodules

$$\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F)$$

and

$$\bar{\mu} = (\bar{E}^{(0)}, \bar{\phi}^{(0)}, \bar{E}^{(1)}, \bar{\phi}^{(1)}, \bar{F})$$

are called *unitarily isomorphic* if there exist unitary B-isomorphisms $u : E^{(0)} \to \bar{E}^{(0)}$ and $v : E^{(1)} \to \bar{E}^{(1)}$ such that $u\phi^{(0)}u^* = \bar{\phi}(0)$, $v\phi^{(1)}v^* = \bar{\phi}^{(1)}$ and $uFv^* = \bar{F}$.

c). An operator homotopy through Fredholm bimodules is

$$\mu_t = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F_t)$$

which is a Fredholm bimodule for all $t \in [0,1]$, such that $t \mapsto F_t$ is norm continuous.

d). The addition of Fredholm bimodules is defined by the formula

$$\mu \oplus \bar{\mu} = (E^{(0)} \oplus \bar{E}^{(0)}, \phi^{(0)} \oplus \bar{\phi}^{(0)}, E^{(1)} \oplus \bar{E}^{(1)}, \phi^{(1)} \oplus \bar{\phi}^{(1)}, F \oplus \bar{F})$$

e). Let $F(A,B)$ be the monoid of equivalence classes of Fredholm bimodules, where equivalence is generated by unitary isomorphism and operator homotopy, and let $CF(A,B)$ (resp., $KF(A,B)$) be its cancellation semigroup (resp., Grothendieck group).

Now our concern is to compare the notion of Fredholm bimodule and Kasparov bimodule.

Construction A

Let $\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F)$ be a Fredholm A,B-bimodule and $\check{E}^{(1)}$ be the opposite graded Hilbert B-module of $E^{(1)}$. Let $\check{\phi}^{(1)} : A \to \mathcal{L}(\check{E}^{(1)})$ be the opposite to ϕ^1. Define an almost unital Kasparov A,B-bimodule $\bar{\mu}$ as the triple $(\bar{E}, \bar{\phi}, \bar{F})$, where

$$\bar{E} = E^{(0)} \oplus \check{E}^{(1)}, \quad \bar{\phi} = \begin{pmatrix} \phi^{(0)} & 0 \\ 0 & \check{\phi}^{(1)} \end{pmatrix}, \quad \bar{F} = \begin{pmatrix} 0 & \check{F}^* \\ \check{F} & 0 \end{pmatrix}$$

and $\check{F}$ is the composition of F with the canonical B-homomorphism

$$\tau_{E^{(1)}} : E^{(1)} \to \check{E}^{(1)}$$

of degree 1 (see example 7).

Proposition 32. *Let $\chi : \mathcal{F}(A,B) \to \mathcal{E}^+(A,B)$ be defined by $\mu \mapsto \bar{\mu}$. Then the induced homomorphism of semigroups $\chi : F(A,B) \to E^+(A,B)$ is an isomorphism. Therefore the groups $CF(A,B)$, $KF(A,B)$ and $KK^+(A,B)$ are canonically isomorphic.*

Proof. The first part is easy to check and the second part follows using proposition 30. □

Now consider the canonical quasi-surjective additive $*$-functor

$$\Theta : Rep(A,B) \to \mathrm{Cal}(A,B).$$

We can define Karoubi's group $K(\Theta)$. There exists a canonical homomorphism $\eta : CF \to K(\Theta)$ which maps the Fredholm A,B-bimodule

$$\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F) \tag{16}$$

to

$$\eta(\mu) = (E^{(0)}, \pi\phi^{(0)}, E^{(1)}, \pi\phi^{(1)}, \pi(F)).$$

(See example 11 for the definition of π.)

Proposition 33. *The homomorphism $\eta : CF(A,B) \to K^0(\Theta_{A,B})$ is an isomorphism.*

Proof. It is enough to show that η is a monomorphism. Let (16) be a Fredholm bimodule such that $\eta(\mu) = 0$. Then there exist elementary Fredholm bimodules $\varepsilon = (E, \phi, E, \phi, H)$ and $\omega = (\bar{E}, \bar{\phi}, \bar{E}, \bar{\phi}, \bar{H})$ and an operator homotopy $\mu \oplus \varepsilon \simeq \omega$. But the cancellation monoid of the abelian monoid of isomorphic elementary Fredholm bimodules is 0 (cf. example 28). Thus $\mu = 0$ in $CF(A,B)$. □

Combining the above propositions we have the following:

Corollary 34. *Let A and B be as above. Then the canonical homomorphism*

$$\eta\chi^{-1} : KK^{+}(A,B) \to K(\Theta_{A,B})$$

is an isomorphism.

4.3. An interpretation of $KK^1(A,B)$

In this subsection we assume that all C^*-algebras are trivially graded and equipped with a compact group action. We use the property of the subcategory of $\mathbb{H}(B \otimes C^{(1,0)})$, given in subsection 2.4, for characterization of $KK^1(A,B)$.

In detail, using the "cofinality principle" of subsection 4.1, and according to the proposition 14, one can use the category $\mathbb{H}_*(B)$ for the definition of $KK^1(A,B)$. Firstly note that if $E \in \mathcal{H}(B)$ then $E \oplus \check{E} \in \mathbb{H}_*(B)$. Denote by $\mathcal{L}_*(E,E')$ the space of morphisms from $E \oplus \check{E}$ to $E' \oplus \check{E}'$ in the category $\mathbb{H}_*(B)$.

Consider $\mathcal{G}_*(A,B)$ the set of triples $(E \oplus \check{E}, \phi, g)$, where $E \in \mathcal{H}(B)$,

$$\phi : A \to \mathcal{L}_*(E \oplus \check{E})$$

is a $*$-homomorphism, and $g \in \mathcal{L}_*(E \oplus \check{E})$, with

$$[g, \phi(a)] \in \mathcal{L}_*(E \oplus \check{E}),\ g^* = g,\ g^2 = 1,\ \partial(g) = 1.$$

Thus using triples from $\mathcal{G}_*(A,B)$ and the analogue of the construction of $KK(A,B)$, one gets $KK^1(A,B)$. (cf. [**2**])

Denote by $G_*(A,B)$ the abelian monoid of equivalence classes of $\mathcal{G}(A,B)$, where equivalence is generated by operator homotopy and unitary isomorphism. From the properties of the category $\mathbb{H}_*(A,B)$ it follows that

$$\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \check{\phi}_0 \end{pmatrix} \tag{17}$$

because all elements of A have degree 0, and

$$g = \begin{pmatrix} 0 & \check{g}_0 \\ g_0 & 0 \end{pmatrix} \tag{18}$$

(as g has degree 1). From $g^* = g$ and $g^2 = 1$ follows that $g_0^* = \check{g}_0$. Thus $\bar{g} = \tau_{\check{E}} g_0$ has degree 0, and $\bar{g}^* = \bar{g}$, $\bar{g}^2 = 1$.(see example 7). From this construction follows that one can consider $\mathcal{G}(A,B)$ as the set of triples of form (E, ϕ, g) where E is trivially graded Hilbert B-module $\phi : A \to \mathcal{L}(B)$ is equivariant $*$-homomorphism and $g \in \mathcal{L}(B)$ such that

$$[\phi(a), g] \in \mathcal{K}(E),\ g^* = g,\ g^2 = 1 \text{ and } \partial g = 0, \text{ where } a \in A.$$

We conclude that canonical homomorphism $\rho : CG(A,B) \to KK^1(A,B)$ defined by

$$(E,\phi,g) \mapsto \left(E \oplus \check{E}, \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix}, \begin{pmatrix} 0 & g\tau_{\check{E}} \\ -\tau_E g & 0 \end{pmatrix} \right)$$

is an isomorphism.

Construction B

Denote by $\mathcal{P}(A,B)$ the set of triples of the form (E, ϕ, p), where E is a trivially graded B-module, $\phi : A \to L(E)$ is a $*$-homomorphism, $\partial(p) = 0$ and

$$[\phi(a), p] \in \mathcal{K}(E),\ a \in A;$$
$$p^* = p \text{ and } p^2 = p$$

Define a map $\vartheta : \mathcal{P}(A,B) \to \mathcal{G}_*(A,B)$ as follows. For each triple (E, ϕ, p) consider the triple (E, ϕ, g_p), where $g_p = (2p - 1)$. □

Let $P(A,B)$ be the abelian monoid of equivalence class of bimodules of the above form, where equivalence is generated by unitary isomorphism and operator homotopy. Let $CP(A,B)$ (resp,. $KP(A,B)$) be the cancellation monoid (resp., Grothedieck group) of $P(A,B)$.

Lemma 35. *The map $\vartheta : \mathcal{P}(A,B) \to \mathcal{G}_*(A,B)$, defined in construction B, is a bijection.*

Proof. The inverse map is defined by the formula:

$$(E,\phi,g) \mapsto (E,\phi,p_g),$$

where $p_g = \frac{1-g}{2}$ $\square$.

Thus we have

Theorem 36. *Let A be separable and B be σ-unital trivially graded C^*-algebras with compact group action. Then the canonical homomorphism*

$$\theta : KP(A,B) \to KK^1(A,B), \tag{19}$$

defined as the composition $\theta = \rho\vartheta$, is an isomorphism.

5. Main Theorem

In this section we prove the following main result.

Theorem 37. *Let A be separable and B be σ-unital trivially graded C^*-algebras with compact group action. There exists an essential isomorphism*

$$\kappa : K^j(\mathrm{Rep}(A,B)) \xrightarrow{\simeq} KK^{j+1}(A,B), \tag{20}$$

where $\mathrm{Rep}(A,B)$ *is the pseudoabelian C^*-category associated with the additive C^*-category $Rep(A,B)$ and $j = -1, 0$.*

The proof of this theorem consists of two parts, considered in subsections 5.1 and 5.2.

5.1. Proof of theorem in dimension zero

Firstly, recall the definition of $K^0(\mathrm{Rep}(A,B))$. By definition 12, objects of $\mathrm{Rep}(A,B)$ have the form (E,ϕ,p), where E is a Hilbert B-module, $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism and p is a projection, (i.e., $p^2 = p$ and $p^* = p$) such that

$$\phi(a)p - p\phi(a) \in \mathcal{K}(E).$$

Two objects (E,ϕ,p) and (F,ψ,q) are unitarily isomorphic in the C^*-category $\mathrm{Rep}(A,B)$ (or C^*-categorically unitarily isomorphic) if there exists a partial isometry $v : E \to F$ such that

$$v\phi(a) - \psi(a)v \in \mathcal{K}(E,F)$$

for all $a \in A$, $v^*v = p$ and $vv^* = q$. Define by $K^0(\mathrm{Rep}(A,B))$ the Grothendieck group of the abelian monoid of unitarily isomorphic objects. This coincides with Karoubi's analogous definition (see section 2). On the other hand we may define w-unitary isomorphism of objects: (E,ϕ,p) and (F,ψ,q) are w-unitary isomorphic if there exists a unitary $u : E \to F$ such that

$$u\phi(a) - \psi(a)u \in \mathcal{K}(E,F)$$

for all $a \in A$, $up = qu$. Denote by $K^0_w(\mathrm{Rep}(A,B))$ the Grothendieck group of the abelian monoid of w-unitary isomorphic objects. Let us compare these groups. If two objects (E,ϕ,p) and (F,ψ,q) are w-unitarily isomorphic by a unitary u, then qup is an isomorphism between these objects. Conversely, if $v : (E,\phi,p) \to (F,\psi,q)$ is a unitary isomorphism of the given objects, then $v : E \to F$ is a partial isometry with $v^*v = p$ and $vv^* = q$. Consider the objects $(E,\phi,1)$ and $(F,\psi,1)$. Then

$$\omega : (E,\phi,p) \oplus (F,\psi,1) \to (F,\psi,q) \oplus (E,\phi,1)$$

is a w-isomorphism, where

$$\omega = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}. \tag{21}$$

¿From this remark it follows that

$$(E, \phi, p) \oplus (F, \psi, 1) \text{ and } (F, \psi, q) \oplus (E, \phi, 1)$$

are equal in $K^0_w(\mathrm{Rep}(A, B))$, Note that $K^0(rep(A, B)) = 0$, by example 28. From this fact it follows that

$$(E, \phi, 1) = (F, \psi, 1) = 0$$

in $K^0_w(\mathrm{Rep}(A, B))$. Thus

$$(E, \phi, p) = (F, \psi, q)$$

in the last group. Therefore, there is a correctly defined essential isomorphism

$$\xi : K^0(\mathrm{Rep}(A, B)) \to K^0_w(\mathrm{Rep}(A, B)). \tag{22}$$

The next step is to define a homomorphism

$$\mu : K^0_w(\mathrm{Rep}(A, B)) \to KP(A, B). \tag{23}$$

To do this one needs the following:

Lemma 38. *Let (E, ϕ, p) and (E, ψ, p) be two objects such that*

$$\phi(a) - \psi(a) \in \mathcal{K}(E)$$

for all $a \in A$. Then $(E, \phi, p) \oplus (E, \psi, 1)$ and $(E, \phi, 1) \oplus (E, \psi, p)$ are operator homotopic.

Proof. Consider

$$p_t = \begin{pmatrix} 1 - \cos^2 t \cdot (1 - p) & \sin t \cdot \cos t \cdot (1 - p) \\ \sin t \cdot \cos t \cdot (p - 1) & 1 + \sin^2 t \cdot (p - 1) \end{pmatrix}.$$

Then $(E \oplus E, \phi \oplus \psi, p_t)$ is desired operator homotopy.

¿From this lemma follows that the map $(E, \phi, p) \mapsto (E, \phi, p)$ correctly defines the epimorphism μ and also the epimorphism

$$\mu \cdot \xi : K^0(\mathrm{Rep}(A, B)) \to KP(A, B).$$

To prove that $\mu \cdot \xi$ is a monomorphism one needs the following:

Lemma 39. *Let (E, ϕ, p_0) and (E, ϕ, p_1) be operator homotopic triples. Then they are unitarily isomorphic as objects in* $\mathrm{Rep}(A, B)$.

Proof. Let (E, ϕ, p_t) be an operator homotopy. As $[0, 1]$ is a compact space, one can choose finite set of points $t_0, t_1, \ldots, t_n \in [0, 1]$ such that $\|p_{t_{i+1}} - p_{t_i}\| < 1$ for all $i = 0, \ldots, n - 1$. Using lemma 6.4 of [**10**], one gets that $p_{t_{i+1}}$ and p_{t_i} are unitarily isomorphic in the C^*-algebra of morphisms from (E, ϕ) to (E, ϕ) (in the category $Rep(A, B)$). From this it follows that the triples from the lemma are isomorphic objects in $\mathrm{Rep}(A, B)$. □

Now one can define a canonical isomorphism

$$\kappa : K^0(\mathrm{Rep}(A, B)) \to KK^1(A, B) \tag{24}$$

as the composition of isomorphisms ξ, μ and isomorphism θ of 19.

5.2. Proof of theorem in dimension one

Consider the quasi-surjective Serre additive $*$-functor

$$\Pi_{A,B} : Rep(A, B) \to \mathrm{Cal}(A, B).$$

Applying the exact sequence (7), one gets the following exact sequence:

$$K^{-1}(Rep(A, B)) \stackrel{\Pi_*}{\to} K^{-1}(\mathrm{Cal}(A, B)) \stackrel{\partial_\Pi}{\to} K^0(\Pi) \stackrel{i}{\to} K^0(Rep(A, B)).$$

The properties of this sequence are given below.

Lemma 40. *There exists a canonical isomorphism*

$$\tau : K^0(\Pi_{A,B}) \to K^0(\Theta_{O,B}), \tag{25}$$

where O is the zero C^*-algebra.

Proof. Consider $e = (E, \phi, E', \psi, \alpha)$ such that $\alpha\overline{\phi(a)} = \overline{\psi(a)}\alpha$ for all $a \in A$, where $\overline{\phi(a)} = \pi\phi(a)$, $\overline{\psi(a)} = \pi\psi(a)$. Then e defines an element in $K^0(\Pi_{A,B})$. Let $0_\phi : O \to \mathcal{L}(E)$ and $0_\psi : O \to \mathcal{L}(E')$ be the zero homomorphisms. One has $e_0 = (E, 0_\phi, E', 0_\psi, \alpha)$ that gives an element in $K^0(\Theta_{O,B})$. Conversely, let $e_0 = (E, 0_\phi, E', 0_\psi, \alpha)$ define an element from $K^0(\Theta_{O,B})$. Then the corresponding element in $K^0(\Pi_{A,B})$ is defined by the same e_0, with 0_ϕ and 0_ψ the trivial homomorphisms on A. Thus we have two homomorphisms

$$\tau : K^0(\Pi_{A,B}) \to K^0(\Theta_{0,B})$$

and

$$\eta : K^0(\Theta_{0,B}) \to K^0(\Pi_{A,B})$$

such that $\tau\eta = 1$. To prove that τ is an isomorphism, it is enough to show that $\{e\} - \{e_0\} = 0$. In order to show this, consider

$$(E \oplus E', \phi \oplus 0_\psi, E' \oplus E, \psi \oplus 0_\phi, \alpha \oplus \alpha^*).$$

It is isomorphic to

$$(E \oplus E', \phi \oplus 0_\psi, E \oplus E', 0_\phi \oplus \psi, t(\alpha \oplus \alpha^*),$$

where

$$t(\alpha \oplus \alpha^*) = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \alpha^* \\ -\alpha & 0 \end{pmatrix}. \tag{26}$$

Note that $\begin{pmatrix} 0 & \alpha^* \\ -\alpha & 0 \end{pmatrix}$ is homotopic to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $\alpha \oplus \alpha^*$ lifts to an isomorphism from $\phi \oplus 0_\psi$ to $\psi \oplus 0_\phi$. That concludes the proof of the lemma (see subsection 3.2). □

Lemma 41. *Let $e = (E, \phi, \alpha)$ and $e' = (E, \phi, \beta)$ both define elements in $K^{-1}(Rep(A, B))$. If $\alpha - \beta \in \mathcal{K}(E)$, then $\{e\} = \{e'\}$.*

Proof. Consider the triples $(\bar{E}, \bar{\phi}, \bar{\alpha})$ and $(\bar{E}, \bar{\phi}, \bar{\beta})$ where

$$\bar{E} = E \oplus E, \quad \bar{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \bar{\beta} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \tag{27}$$

These triples are operator homotopic by

$$h(t) = \begin{pmatrix} \alpha + \sin^2 t \cdot (\beta - \alpha) & \sin t \cdot \cos t \cdot (\alpha - \beta) \\ \sin t \cdot \cos t \cdot (\beta - \alpha) & \beta + \sin^2 t \cdot (\alpha - \beta) \end{pmatrix}. \qquad \square$$

Lemma 42. *The canonical homomorphism*

$$\Pi_* : K^{-1}(Rep(A, B)) \to K^{-1}(\mathrm{Cal}(A, B))$$

is a monomorphism.

Proof. Let $e = (E, \phi, \alpha)$ represent an element in $K^{-1}(Rep(A, B))$ such that $\Pi_*(e) = 0$. Then by lemma 20 there exists $(E', \psi, 1_\psi)$ such that $\Pi(\alpha) \oplus 1_{\Pi(\psi)}$ is homotopic to $1_{\Pi(\phi)\oplus\Pi(\psi)}$. Let $h(t)$ be this homotopy. Consider its lifting in the group $U(E \oplus E', \phi \oplus \psi)$ of unitary automophisms, such that $h(0) = 1$. Put $\beta = h(1)$, then $\beta - \alpha \oplus 1_\psi \in \mathcal{K}(E)$. Using lemma 18 and lemma 41, we get

$$\{(E, \phi, \alpha) \oplus (F, \psi, 1)\} = \{E \oplus F, \phi \oplus \psi, \beta\} = 0. \qquad \square$$

Let $e = \{(E, \phi, E', \psi, \alpha)\} \in K^0(\Pi_{A,B})$. Consider the unique element $e' = \{(0, \beta)\} \in K^{-1}(\mathrm{Cal}(A, B))$ such that $\partial_\Pi(e') = e$. Then the homomorphism

$$s : K^0(\Pi) \to K^{-1}(\mathrm{Cal}(A, B))$$

defined by $e \mapsto e'$defines a right inverse for ∂_Π.

Corollary 43. *Let A and B be trivial graded C^*-algebras with compact group actions. Then there is a split exact sequence of groups*

$$0 \to K^{-1}(Rep(A, B)) \xrightarrow{\Pi} K^{-1}(\mathrm{Cal}(A, B)) \xrightarrow{\tau \cdot \partial_\Pi} K^0(\Theta_{O,B}) \to 0. \quad (28)$$

Now we are ready to prove the main theorem. Firstly note that one can replace $K^{-1}(\mathrm{Rep}(A, B))$ by $K^{-1}(Rep(A, B))$ (see theorem 21).

We need some homomorphisms:

a) Define the homomorphism

$$\kappa : K^{-1}(Rep(A, B)) \to KK^0(A, B)$$

as follows. Let $e = (E, \phi, \alpha)$ define an element of $K^{-1}(Rep(A, B))$. Consider the triple $\bar{e} = (\bar{E}, \bar{\phi}, \bar{\alpha})$, defined in construction A of section 3, by

$$\bar{E} = E \oplus \check{E},\ \bar{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix},\ \bar{\alpha} = \begin{pmatrix} 0 & \check{\alpha} \\ \alpha & 0 \end{pmatrix}.$$

Then by definition $\kappa(e) = \bar{e}$.

b) Define homomorphism

$$\Delta : K^{-1}(\mathrm{Cal}(A, B)) \to KK^+(A, B)$$

in the following way. Let $e = (E, \pi\phi, \alpha)$ be a triple that defines element of $K^{-1}(\mathrm{Cal}(A, B))$, where (E, ϕ) is object of $Rep(A, B)$. Choose F such that $\pi(F) = \alpha$. Of course, the triple $e' = (E, \phi, F)$ is an almost unital Kasparov A, B-bimodule. By definition $\Delta(e) = \bar{e}'$.

c) Let $\omega : K(\Theta_{O,B}) \to KK^0(\mathbb{C}, B)$ be defined by the equality $\omega = \chi\eta^{-1}$.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K^{-1}(\mathrm{Rep}(A, B)) & \xrightarrow{\Pi} & K^{-1}(\mathrm{Cal}(A, B)) & \xrightarrow{\tau \cdot \partial} & K^0(\Theta_{O,B}) & \to & 0 \\ & & \downarrow^{\kappa} & & \downarrow^{\Delta} & & \downarrow^{\omega} & & \\ 0 & \to & KK^0(A, B) & \xrightarrow{j} & KK^+(A, B) & \xrightarrow{i^*} & KK^0(\mathbb{C}, B) & \to & 0 \end{array}$$

Then Δ and ω are isomorphisms, because $\Delta = \chi\eta^{-1}\partial_\Theta$ is the composition of the isomorphism from corollary 34 and the isomorphism (9) of example 28. Also, $\omega = \chi\eta^{-1}$ is an isomorphism because of lemma 40. Thus κ is an isomorphism.

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