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TOPOLOGICAL K-THEORY OF THE INTEGERS AT THE PRIME 2

LUKE HODGKIN

(*communicated by James Stasheff*)

Abstract

Recent results of Voevodsky and others have effectively led to the proof of the Lichtenbaum-Quillen conjectures at the prime 2, and consequently made it possible to determine the 2-homotopy type of the K-theory spectra for various number rings. The basic case is that of $BGL(\mathbb{Z})$; in this note we use these results to determine the 2-local (topological) K-theory of the space $BGL(\mathbb{Z})$, which can be described as a completed tensor product of two quite simple components; one corresponds to a real 'image of J' space, the other to $BBSO$.

1. Introduction

As a result of Voevodsky's solution of the Milnor conjecture [V] and related work by Bloch, Lichtenbaum, Voevodsky and Suslin [B-L], [S-V], Weibel in [W] calculated the algebraic K-theory of the integers $\pi_i(BGL(\mathbb{Z})^+)$ at the prime 2 in terms which essentially confirmed the appropriate version of the Lichtenbaum-Quillen conjectures [L,D-F]. (Much stronger and more general versions of the prime-2 conjectures have since been proved, see in particular [RWK], [R-W].) This result, since it expresses the space $BGL(\mathbb{Z})^+$ in terms of rather well-known spaces, makes it relatively easy to deduce other invariants. Arlettaz et al. in [A-M-N-Y] have done this for the mod 2 cohomology; in this paper, I shall do the same for the (topological) 2-local K-theory; the result, which is, perhaps predictably, quite different from that for cohomology is stated in theorem 4.1 and corollary 5.1 below. The use of 2-local, rather than the more usual 2-complete theory requires a little more work, but perhaps can be considered as giving a more interesting result.

While Weibel's results are more general in character, and could lead to similar calculations for various other rings e.g. $\mathbb{Z}[\sqrt{-1}]$, I shall here confine my attention to the integers, partly because of their 'historical' interest, and partly because of the link with the stable mapping class group $B\Gamma = \varinjlim B\Gamma_n$ via the composite $B\Gamma \to BSp(\mathbb{Z}) \to BGL(\mathbb{Z})$, which arises from the action of surface homeomorphisms on H_1.

Without attempting a complete survey of recent related work, I should draw attention to the most important:

(i) The corresponding decomposition of spectra for the p-adics $BGL(\mathbb{Z}_p\hat{}\ )$ has been known for some time — for arbitrary p — through work of Bökstedt, Madsen and Rognes [B-M], [R2]. (Here the Milnor conjecture is not needed.)

(ii) Dwyer and Mitchell, in a sequence of papers, [D-M], [Mi1], [Mi2], have attacked precisely the problem of finding the K-theory of the spectrum associated with $BGL(R)_p\hat{}\ )$ when R is a ring of algebraic integers, which they have (essentially) solved in terms of the 'Iwasawa module' M_∞ of R. The remaining dificulties are those associated with the structure of M_∞, and are not trivial.

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(iii) In an important special case, Østvær [Ø] has found the homotopy type of $BGL(R)\hat{_2}$ when R is '2-regular and non-exceptional'. Examples include cyclotomic rings $R = \mathbb{Z}[\zeta_{2^r}]$, $r \geqslant 2$, (in particular the Gaussian integers); but no rings of integers in totally real fields. The homotopy type is very simple, a product of a $K(\mathbb{F}_p)$ (finite field) and copies of U, SU.

Having said this, the result presented here is perhaps principally of interest in the way that it displays the intertwining of the factors which go to make up $BGL(\mathbb{Z})$. The justification which Dwyer and Mitchell give for studying the topological K-theory of algebraic spectra (that it sheds light on the K-localization of the spectra, and hence can provide evidence about the Lichtenbaum-Quillen conjectures) is not really at issue at the prime 2 where the conjectures are solved; however, the structure has independent interest.

We begin by describing the space which we shall study. Following [Bö], but working in the category of 2-local spaces, we define the 'étale K-space' $JK(\mathbb{Z})$ as the (2-local) homotopy fibre of the composite map:

$$c(\psi^3 - 1) : BO \xrightarrow{\psi^3-1} BSpin \xrightarrow{c} BSU \tag{1}$$

(Here ψ^3 is the Adams operation and c denotes complexification.) This space can be realized through a number of other fibrations, of which we shall note particularly (cf [R2], (2.3))

$$JR_2 \to JK(\mathbb{Z}) \to BBSO \tag{2}$$

where JR_2 is the real image of J space at 2, defined as the fibre of $\psi^3 - 1 : BO \to BSpin$, localized at 2. (See e.g. [Ma].) The 2-completion of $JK(\mathbb{Z})$ is equivalent to the space which is named $K^{\acute{e}t}(\mathbb{Z})$ in [D-F] and elsewhere. Bökstedt defined a map on 2-completions from $((BGL(\mathbb{Z})^+)\hat{_2}$ to $JK(\mathbb{Z})\hat{_2}$; it is a consequence of Voevodsky's theorem and subsequent work that this is a homotopy equivalence. However, it is not obvious that this map is an equivalence — even that it exists — in the localized sense. [I am grateful to the referee for pointing this fact out.] I shall therefore, in sections 2-4, find the K-theory of $JK(\mathbb{Z})$ with coefficients in $\mathbb{Z}\hat{_2}$ and $\mathbb{Z}_2$, by a simple application of the Rothenberg-Steenrod spectral sequence. Having done that, in §5 I shall deduce the corresponding results for $BGL(\mathbb{Z})^+$; the completed case is easy, by the above remarks, but the local case requires a special investigation, using Bousfield's K-localization functor L_K [Bou] to identify $JK(\mathbb{Z})$ with $L_K(BGL(\mathbb{Z})^+)$.

Where not otherwise stated, all spaces are supposed localized at 2 in what follows.

2. The 2-complete theory

The natural procedure is in any case to begin with the 2-completed theory, and proceed to integrate it with the rational to obtain a 2-local statement. With this in mind, we begin with the following commutative diagram:

$$\begin{array}{ccccc} O & \xrightarrow{c(\psi^3-1)} & SU & \xrightarrow{\eta} & JK(\mathbb{Z}) \\ \| & & \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle j} \\ O & \xrightarrow[\psi^3-1]{} & Spin & \xrightarrow[\theta]{} & JR_2 \end{array} \tag{3}$$

Both rows in this diagram are fibrations; the top row derived in the obvious way from the fibration (1), the second similarly from the definition of JR_2. The two squares are commutative by construction (compare the diagram on p.8 of [R2]); and the right hand square is fibred. The most 'natural' approach for such K-theory computations is usually via the geometric spectral sequence of Rothenberg-Steenrod (see e.g. [A-H]), which gives the K-theory of a quotient of groups (for example) in terms of those of the group and subgroup; and we can apply this spectral sequence to the fibre square provided that one of the *Spin*-actions is free. It will be convenient to suppose this for the action on SU (of course nearly free, but not quite...), by the usual device of replacing SU by $SU \times ESpin$. We are accordingly using a homotopy equivalence of $JK(\mathbb{Z})$ with $(SU \times ESpin) \times_{Spin} JR_2$; and we need the cohomology version of the Rothenberg-Steenrod sequence, which uses derived functors in the category of comodules over a coalgebra — in our case the coalgebra $K^*(Spin; \hat{\mathbb{Z}_2})$.

Much of the argument can be simplified in this case, as we shall see, since we are dealing with a trivial comodule. To clarify the details of the application we want, i.e. to the 2-complete K-cohomology, one or two technical points should be made. First, there is (as usual in K-cohomology of large spaces) the question of topology on the coalgebra and comodules $K^*(\ \ ; \hat{\mathbb{Z}_2})$. Second, the coefficients are not a field, and the modules may not be free or even projective. We need to deal with these objections together so as to obtain a reasonable cotensor product functor. To begin with, the category $\mathcal{C}_2$ of *profinite* modules over $\hat{\mathbb{Z}_2}$ is abelian, and the appropriate tensor product is the completed one, '$\hat{\otimes}$'. Because $\varprojlim$ is exact in the category, the functor $-\hat{\otimes}A$ is exact if A is an inverse limit of finitely generated free $\hat{\mathbb{Z}_2}$-modules. We shall call such a module 'flat', by analogy with the usual case. In particular, this applies to the Hopf algebras $K^*(Spin; \hat{\mathbb{Z}_2})$ and $K^*(SU; \hat{\mathbb{Z}_2})$. (This is a consequence of [H], but the detail will be given later.) Hence $K^*(\ \ ; \hat{\mathbb{Z}_2})$ translates products of spaces into completed tensor products of $\hat{\mathbb{Z}_2}$-algebras, when one of the spaces is SU or $Spin$.

If A is a (profinite) flat cocommutative $\hat{\mathbb{Z}_2}$-coalgebra, and B, C are compact comodules over A, we define the *completed* cotensor product $B \,\hat{\square}_A\, C$ to make the sequence (cf [M-M])

$$0 \longrightarrow B \underset{A}{\hat{\square}} C \longrightarrow B\hat{\otimes}C \overset{\Delta\otimes 1 - 1\otimes\Delta}{\longrightarrow} B\hat{\otimes}A\hat{\otimes}C$$

exact. (Δ denotes the structural morphisms for the comodules.) This bifunctor is left exact on sequences of A-comodules which are split-exact over $\hat{\mathbb{Z}_2}$. Its derived functors will be written $\widehat{Cotor}_p^A(B, C)$. Recall the spectral sequence — stated here in the appropriate form for our purpose.

Proposition 2.1. *Let G be a group, and let X, Y be G-spaces with either X or Y free (all in a suitably small category, e.g. 2-local CW-complexes). If G, X have $K^*(\ \ ; \hat{\mathbb{Z}_2})$ flat in $\mathcal{C}_2$, then there is a strongly convergent spectral sequence with*

$$E_2^p = \widehat{Cotor}^p_{K^*(G;\hat{\mathbb{Z}_2})}(K^*(X; \hat{\mathbb{Z}_2}), K^*(Y; \hat{\mathbb{Z}_2}))$$
$$E_\infty \sim K^*(X \times_G Y; \hat{\mathbb{Z}_2})$$

Its edge homomorphism is the 'standard' map

$$\eta : K^*(X \times_G Y; \hat{\mathbb{Z}_2}) \to K^*(X; \hat{\mathbb{Z}_2}) \underset{K^*(G;\hat{\mathbb{Z}_2})}{\hat{\square}} K^*(Y; \hat{\mathbb{Z}_2})$$

(which follows from the definitions).

The proof is the usual geometric one, using the bar resolution. Again because the inverse limit is exact in $\mathcal{C}_2$ there are no convergence problems.

Note. Since we are interested in the 2-local theory, we shall also need a local version of this. Here arguments using profiniteness naturally break down, and alternative methods must be used. The best option is to use the corresponding sequence for K-homology, which involves the ordinary tensor product and the ordinary Tor groups over $K_*(G; \mathbb{Z}_2)$. Again (since homology theories behave well with respect to direct limits) the sequence is strongly convergent; in this case the proof is clearly simpler. We then need to dualize the results in the appropriate way to derive the K-cohomology.

We are now ready to state the structure theorem for the map $c : Spin \to SU$; for maximum generality we shall need the local version.

Proposition 2.2. *(i) The Hopf algebras $K^*(Spin; \mathbb{Z}_2)$ resp. $K^*(SU; \mathbb{Z}_2)$ are completed exterior algebras on submodules of primitive generators, say P_R, P_C respectively; and the map*

c induces an epimorphism from P_C to P_R, whose kernel Q is a direct summand. Accordingly, writing $\hat{E}(\ )$ for the completed exterior algebra on primitive elements, we have:

$$K^*(Spin;\mathbb{Z}_2) = \hat{E}(P_R)$$
$$K^*(SU;\mathbb{Z}_2) = \hat{E}(P_C) \cong \hat{E}(P_R \oplus Q) = \hat{E}(P_R)\hat{\otimes}E(Q)$$

as a tensor product of Hopf algebras.

(ii) The same statements hold for K-theory with $\mathbb{Z}_{2}\hat{}$ coefficients, and $K^*(Spin;\mathbb{Z}_{2}\hat{}\ )$ resp. $K^*(SU;\mathbb{Z}_{2}\hat{}\ )$ is isomorphic to $K^*(Spin;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\mathbb{Z}_{2}\hat{}$ resp. $K^*(SU;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\mathbb{Z}_{2}\hat{}$

From this will follow:

Proposition 2.3. *For any space X with an action of Spin, the edge homomorphism of the spectral sequence defines a natural isomorphism*

$$\eta : K^*((SU \times ESpin) \underset{Spin}{\times} X;\mathbb{Z}_{2}\hat{}\ ) \to \hat{E}(Q)\hat{\otimes}K^*(X;\mathbb{Z}_{2}\hat{}\ )$$

We postpone the proof of proposition 2.2 to the next section, and show that it implies proposition 2.3.

For this, it is sufficient to identify $K^*(SU;\mathbb{Z}_{2}\hat{}\ )\hat{\Box}_{K^*(Spin;\mathbb{Z}_{2}\hat{}\ )}K^*(X;\mathbb{Z}_{2}\hat{}\ )$. From the splitting of proposition 2.2, we can deduce that

$$\mu^* : K^*(SU;\mathbb{Z}_{2}\hat{}\ ) \to K^*(Spin;\mathbb{Z}_{2}\hat{}\ )\hat{\otimes}K^*(SU;\mathbb{Z}_{2}\hat{}\ )$$

is identified with

$$\hat{E}(P_R)\hat{\otimes}E(Q) \xrightarrow{\Delta\otimes 1} \hat{E}(P_R)\hat{\otimes}\hat{E}(P_R)\hat{\otimes}\hat{E}(Q)$$

Now we know that $(\hat{E}(P_R)\hat{\otimes}\hat{E}(Q))\,\hat{\Box}_{\hat{E}(P_R)}K^*(X;\mathbb{Z}_{2}\hat{}\ ) \cong \hat{E}(Q)\hat{\otimes}K^*(X;\mathbb{Z}_{2}\hat{}\ )$; in fact, this is the dual of the well-known analogous formula for the tensor product, and the isomorphism is natural. However, $\hat{E}(Q)\hat{\otimes}K^*(X;\mathbb{Z}_{2}\hat{}\ )$ is exact as a functor of the comodule $K^*(X;\mathbb{Z}_{2}\hat{}\ )$, since $\hat{E}(Q)$ is flat, and so its derived functors are trivial:

$$\widehat{Cotor}^0_{K^*(Spin;\mathbb{Z}_{2}\hat{}\ )}(K^*(X;\mathbb{Z}_{2}\hat{}\ ), K^*(SU;\mathbb{Z}_{2}\hat{}\ )) = K^*(X;\mathbb{Z}_{2}\hat{}\ )\hat{\otimes}\hat{E}(Q)$$
$$\widehat{Cotor}^p_{K^*(Spin;\mathbb{Z}_{2}\hat{}\ )}(K^*(X;\mathbb{Z}_{2}\hat{}\ ), K^*(SU;\mathbb{Z}_{2}\hat{}\ )) = 0 \qquad (p>0)$$

Using the edge homomorphism of the spectral sequence, proposition 2.3 follows.

3. Structure of $K^*(Spin), K^*(SU)$

We now proceed to the proof of proposition 2.2. Let λ^i_r resp. λ^i_c be the ith 'stabilized' exterior power of the standard representation θ from $Spin(2n+1)$ resp. $SU(2n+1)$ to U, considered as an element of the representation ring. That is, λ^i_r is the result of applying the operation λ^i to $\theta - (2n+1)$. Then it is obvious that under inclusion maps of $Spin(2n+1)$'s and $SU(2n+1)$'s the λ^i's are preserved; and that $c^*(\lambda^i_c) = c^*(\bar{\lambda}^i_c) = \lambda^i_r$.

Let now β be the operation (see [H]) which to any representation ρ of G assigns its class $\beta(\rho)$ in $K^1(G) = [G, U]$ considered as a map from G to U. The basic theorem of [H] gives us that $K^*(SU(2n+1);\mathbb{Z}_2)$ is the exterior algebra

$$E_{\mathbb{Z}_2}(\beta(\lambda^1_c), \ldots, \beta(\lambda^n_c), \beta(\bar{\lambda}^1_c), \ldots, \beta(\bar{\lambda}^n_c))$$

since these can be seen to be equivalent to the basic representations modulo a little manipulation. (The generators are also, as usual, the primitives for the Hopf algebra structure.) The similar result is not quite true for $Spin(2n+1)$, as is well known, the picture being complicated by the Spin representation Δ_n, of dimension 2^n. We have:

$$K^*(Spin(2n+1);\mathbb{Z}_2) = E_{\mathbb{Z}_2}(\beta(\lambda^1_r), \ldots, \beta(\lambda^{n-1}_r), \beta(\Delta_n))$$

However, there is a relation between λ_r^n and Δ_n, since $(\Delta_n)^2 = \lambda_r^n +$ a sum of terms in $\lambda_r^1, \ldots, \lambda_r^{n-1}$. Writing $\Delta_n = 2^n + \tilde{\Delta}_n$, and applying the usual relations for β, we have that $\beta(\Delta_n)^2 = 2^{n+1}\beta(\Delta_n)$. Hence, $\beta(\lambda_r^n) = 2^{n+1}\beta(\Delta_n)$ (mod $\beta(\lambda_r^1), \ldots, \beta(\lambda_r^{n-1})$).

Write M_n for the $\mathbb{Z}_2$-module which generates the exterior algebra $K^*(Spin(2n+1); \mathbb{Z}_2)$ and N_n for the submodule generated by the $\beta(\lambda_r^i)$'s. We can deduce a short exact sequence

$$(E_n) \qquad 0 \to N_n \to M_n \to \mathbb{Z}/2^{n+1} \cdot \beta(\Delta_n) \to 0$$

The restrictions from E_{n+1} to E_n are straightforward if we take into account that Δ_{n+1} restricts to $2.\Delta_n$. Hence the map from $\mathbb{Z}/2^{n+2}$ to $\mathbb{Z}/2^{n+1}$ in the above sequence multiplies the generator by 2. It is easy to deduce that the inverse limit of the $\mathbb{Z}/2^{n+1}$'s is zero; and so (since they are finite) is the $\varprojlim^1$. Hence the map from $\varprojlim\{N_n\}$ to $\varprojlim\{M_n\}$ — the primitives of $K^*(Spin)$ — is an isomorphism, and we have:

Proposition 3.1. *The K-cohomology rings of Spin, SU are as follows:*

$$K^*(Spin; \mathbb{Z}_2) = \hat{E}_{\mathbb{Z}_2}(\beta(\lambda_r^1), \beta(\lambda_r^2), \ldots)$$
$$K^*(SU; \mathbb{Z}_2) = \hat{E}_{\mathbb{Z}_2}(\beta(\lambda_c^1), \beta(\lambda_c^2), \ldots; \beta(\bar{\lambda}_c^1), \beta(\bar{\lambda}_c^2) \ldots)$$

and the restriction c^* from SU to Spin maps $\beta(\lambda_c^i), \beta(\bar{\lambda}_c^i)$ to $\beta(\lambda_r^i)$ $(i = 1, 2, \ldots)$

From this, proposition 2.2 clearly follows.

We next deduce:

Proposition 3.2. *The local K-theory of the quotient is given by*

$$K^*((SU \times ESpin)/Spin; \mathbb{Z}_2) \cong \hat{E}_{\mathbb{Z}_2}(\beta(\lambda_c^1) - \beta(\bar{\lambda}_c^1), \ldots) \cong \hat{E}_{\mathbb{Z}_2}(Q)$$

in the terminology of proposition 2.2.

Proof. As stated above, the best way to prove this is as follows. First, dualize proposition 3.1 to give a result on the local K-homology (the map c now induces a split monomorphism). Next, apply the Rothenberg-Steenrod sequence in local K-homology; this is well-behaved, and strong convergence is easily established, as well as flatness (in the usual sense) for the K-algebras involved. We find a natural isomorphism in K-homology in a form dual to that of proposition 2.3. In the special case where X is a point, this can now simply be dualized back to give the required result.

This procedure is of course roundabout, but seems preferable to developing a theory if topological modules which will deal properly with very large algebras over $\mathbb{Z}_2$ of the kind we are considering here in K-cohomology.

4. The K-theory of $JK(\mathbb{Z})$

We are now in a position to put the pieces together. The key point is that JR_2 is a 2-adic space, so the local theory and the 2-adic theory coincide for it.

Theorem 4.1. *There is a natural isomorphism:*

$$K^*(JK(\mathbb{Z}); \mathbb{Z}_2) \cong \hat{E}_{\mathbb{Z}_2}(Q)\hat{\otimes}K^*(JR_2; \mathbb{Z}_2)$$
$$\cong K^*(BBSO; \mathbb{Z}_2)\hat{\otimes}K^*(JR_2; \mathbb{Z}_2)$$

with an analogous isomorphism for $\mathbb{Z}_2\hat{}$ coefficients.

Proof. We'd like to use a basepoint in JR_2, but of course can't suppose there is one which is fixed under *Spin*. Consider instead the equivariant embedding of JR_2 in the unreduced cone C^+JR_2. If we can prove the result for C^+JR_2 and for the pair $P = (C^+JR_2, JR_2)$

separately, then it will follow for JR_2 by the 5-lemma. Now for C^+JR_2 it is already proved (by proposition 3.2). For P, we consider the commutative diagram:

$$\begin{array}{ccc} K^*(SU \times_{Spin} P; \mathbb{Z}_2) & \xrightarrow{\eta} & K^*(SU;\mathbb{Z}_2)\,\hat{\square}_{K^*(Spin;\mathbb{Z}_2)}\,K^*(P;\mathbb{Z}_2) \\ \downarrow \alpha & & \beta \downarrow \\ K^*(SU \times_{Spin} (P); \mathbb{Z}_2\hat{}\ ) & \xrightarrow[\eta]{} & K^*(SU;\mathbb{Z}_2\hat{}\ )\,\hat{\square}_{K^*(Spin;\mathbb{Z}_2\hat{}\ )}\,K^*(P;\mathbb{Z}_2\hat{}\ ) \end{array}$$

The arrow η in the lower row is an isomorphism by proposition 2.3. The two vertical arrows are induced by the coefficient homomorphism. Since the reduced homology of JR_2 is finite in every dimension, the same is true for the pair $SU \times_{Spin} P$; so $K^*(SU \times_{Spin} P; \mathbb{Z}_2)$ is a 2-adic group. Hence the arrow marked α is an isomorphism. On the other hand, we can embed $K^*(SU;\mathbb{Z}_2)\,\hat{\square}_{K^*(Spin;\mathbb{Z}_2)}\,K^*(P;\mathbb{Z}_2)$ in $K^*(SU;\mathbb{Z}_2)\hat{\otimes}K^*(P;\mathbb{Z}_2)$ and identify the latter with

$$K^*(SU;\mathbb{Z}_2)\hat{\otimes}(\mathbb{Z}_2\hat{}\ \hat{\otimes}K^*(P;\mathbb{Z}_2))$$

(again because $K^*(P;\mathbb{Z}_2)$ is 2-adic). Using this, and the definition of the cotensor product, we find that the right hand vertical arrow β is also an isomorphism. Hence the upper arrow η is one.

Now by the argument used in proposition 2.3, this implies that $K^*(SU \times_{Spin} P; \mathbb{Z}_2)$ is isomorphic to $\hat{E}_{\mathbb{Z}_2}(Q)\hat{\otimes}K^*(P;\mathbb{Z}_2)$. This proves the first line of the theorem. The second results from

Lemma 4.1. *The fibre $SU/Spin$ of $c : BSpin \to BSU$ can be identified with the Hopf map $\eta : BBSO \to BSpin$.*

Proof. The composite $c \circ \eta$ is trivial and so lifts to a map $\tilde{\eta} : BBSO \to Fib(c)$. A check on the homotopy sequence shows that this is a homotopy equivalence. □

A comparison with the fibre sequence (2) shows that the sequence splits from the viewpoint of 2-local K-theory. Finally, it is worth noting that the K-theory of JR_2 has been known for a long time, see [H-S]; it is essentially the completed representation ring of the infinite symmetric group Σ_∞.

5. The results for $BGL(\mathbb{Z})^+$

As was remarked in §1, theorem 4.1 immediately gives us the 2-adic K-theory of $BGL(\mathbb{Z})^+$, since $(BGL(\mathbb{Z})^+)_2\hat{}$ is homotopy equivalent to $JK(\mathbb{Z})_2\hat{}$. To deal with the localizations, we shall prove the following result:

Theorem 5.1. *Let L_K denote Bousfield's K-theory localization functor on spaces* [Bou]*. There is a homotopy equivalence of $JK(\mathbb{Z})$ with $L_K(BGL(\mathbb{Z})^+)$.*

Since $K^*(L_K(X;\mathbb{Z}_2)) \cong K^*(X;\mathbb{Z}_2)$ for any X, this shows:

Corollary 5.1. *The K-theory of $BGL(\mathbb{Z})^+$ with $\mathbb{Z}_2$ or $\mathbb{Z}_2\hat{}$ coefficients is computed by theorem 4.1.*

For the proof of theorem 5.1, we shall simplify notation by writing $K(\mathbb{Z})$ for the ring $\mathbb{Z}\times BGL(\mathbb{Z})^+$. We follow the arguments of [Bö] and (§2 of) [R1]. In [Bö], the rational component is considered as well as the 2-adic, but the discussion is essentially concerned with $\Omega K(\mathbb{Z})$; while in [R1], the argument is at the level of spectra, but is purely 2-adic. Our concern is to use K-localization to circumvent these restrictions.

We first define a map s as the composite $BSO \xrightarrow{i} BSG \xrightarrow{\eta} SG$, where i is the 'forgetful' map from bundles to spherical fibrations, and η is multiplication by the Hopf map. Both maps are 2-locally defined, and are infinite loop maps. If $r : SG \to K(\mathbb{Z})_1$ is the map into the 'one-component' induced by $QS^0 \to K(\mathbb{Z})$, then r is again a local infinite loop map. In [R1] it is shown that the 2-completion of $r \circ s$ is nullhomotopic at the level of spectra. However, since SG has finite homotopy groups, $r \circ s$ is 2-locally trivial if it is trivial after completion; and as in [R1], the nullhomotopy deloops to give a map $g : Bfib(s) \to K(\mathbb{Z})_1$.

Lemma 5.1. *The localizations $L_K(Bfib(s))$ and $L_K(JK(\mathbb{Z}))$ are homotopy equivalent.*

Proof. This follows directly from diagram 2.8 of [R1], since the fibre sequence $C_\otimes \to Bfib(s) \to JK(\mathbb{Z})$ can be constructed 2-locally using ρ^3. Since $L_K(C_\otimes)$ is a point by [H-S], the fibration becomes an equivalence after K-localization. □

Finally, we need:

Lemma 5.2. *The K-theory localization of g, $L_K(g)$, is a homotopy equivalence from $L_K(Bfib(s))$ to $L_K(K(\mathbb{Z})_1)$.*

Proof. Since the homotopy groups in each case are finitely generated, it will be enough to show that g induces isomorphisms on homotopy (a) when 2-completed and (b) when tensored with $\mathbb{Q}$. For the 2-completion, we again use Rognes' diagram 2.8. The preceding lemma implies that the map there called 'h' exists after K-localization; we shall call it '$L_K(h)$', ignoring the question of whether h exists. And $L_K(g)$ is an equivalence if and only if $L_K(h)$ is. But by the subsequent arguments of Rognes, $L_K(h)$ is a right inverse for the more usual map $\Phi : L_K(K(\mathbb{Z})_1)\hat{_2} \to L_K JK(\mathbb{Z})\hat{_2}$. (As pointed out in [R1], this involves an essentially 2-adic argument.) We now know, however, that Φ is a 2-adic equivalence, and so $L_K(h)$ is. For the rational version we use the argument on pp.31-2 of [Bö]; the homotopy groups of $fib(s)\hat{_2}$ and $(\Omega K(\mathbb{Z})_1)\hat{_2}$ are the same after inverting 2, and hence Ωg is an isomorphism on rational homotopy. The same result therefore follows for g and so for $L_K(g)$. □

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Luke Hodgkin `luke.hodgkin@kcl.ac.uk`

King's College
London