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MORE ABOUT HOMOLOGICAL PROPERTIES OF PRECROSSED MODULES

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Abstract

Homology groups modulo q of a precrossed P-module in any dimensions are defined in terms of nonabelian derived functors, where q is a nonnegative integer. The Hopf formula is proved for the second homology group modulo q of a precrossed P-module which shows that for $q = 0$ our definition is a natural extension of Conduché and Ellis' definition [CE]. Some other properties of homologies of precrossed P-modules are investigated.

Introduction

The homology of precrossed modules was introduced by Conduché and Ellis in [CE]. The aim of this paper is to pursue their line of investigation homological properties of precrossed modules.

Let P be a group. A precrossed P-module (M, μ) is a group homomorphism $\mu : M \to P$ together with an action of P on M denoted by ${}^p m$ for $p \in P$ and $m \in M$, which satisfies the following condition:

$$\mu({}^p m) = p\mu(m)p^{-1}.$$

If in addition the following Peiffer identity holds

$${}^{\mu(m)}m' = mm'm^{-1},$$

(M, μ) is a crossed P-module (see e.x. [BH]).

A morphism $\varphi : (M, \mu) \to (N, \nu)$ of (pre)crossed P-modules is a commutative triangle

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\mu}\searrow & & \swarrow{\scriptstyle\nu} \\ & P & \end{array},$$

where φ is a P-equivariant group homomorphism i.e. $\varphi({}^p m) = {}^p\varphi(m)$ for all $m \in M$, $p \in P$. Let us denote the category of precrossed (crossed) P-modules by $\mathcal{PCM}(P)$ ($\mathcal{CM}(P)$).

Further we shall occasionally supress explicit mention of the homomorphism μ in a precrossed P-module (M, μ) and write simply M.

Precrossed modules form a model of homotopy type in dimensions 1 and 2 for connected CW-complexes. Precisly Kan's $\mathbb{G}$ functor establishes an equivalence relation between the category of connected CW-complexes and the category of free simplicial groups [K] and the first two terms of the Moore chain complex associated to the simplicial group gives a precrossed module.

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Let (M, μ) be a precrossed P-module. The following type elements in M

$$\langle m, m' \rangle = mm'm^{-1\mu(m)}m'^{-1}, \quad m, m' \in M$$

are called Peiffer commutators, and now we give some identities for them from [BC]

$$\langle m, m'm'' \rangle = \langle m, m' \rangle^{\mu(m)}m' \langle m, m'' \rangle^{\mu(m)}m'^{-1}, \tag{1}$$

$$\langle mm', m'' \rangle = m\langle m', m'' \rangle m^{-1} \langle m, {}^{\mu(m')}m'' \rangle, \tag{2}$$

$${}^{p}\langle m, m' \rangle = \langle {}^{p}m, {}^{p}m' \rangle, \tag{3}$$

$$\langle k, m \rangle = kmk^{-1}m^{-1}, \tag{4}$$

$$\langle k, m \rangle \langle m, k \rangle = k^{\mu(m)}k^{-1} \tag{5}$$

for all $m, m', m'' \in M$, $p \in P$ and $k \in Ker\mu$.

The Peiffer commutator subgroup $\langle M, M \rangle$, which is a subgroup of the group M generated by the Peiffer commutators, plays the same role for precrossed modules as the commutator subgroup plays for groups. Analogously as a lower central series in a group, a lower Peiffer central series in a precrossed P-module is defined by Baues and Conduché [BC]

$$M^{(1)} = M \supset M^{(2)} \supset \dots \ .$$

This series has properties like classical central series giving a hope to generalize some methods of Curtis [CU1, CU2] and Quillen [Q] for nonsimply connected spaces.

The crossed P-module $\mu' : M/\langle M, M \rangle \to P$ associated to the precrossed P-module $\mu : M \to P$, where $M/\langle M, M \rangle$ is a factor group of M by the Peiffer commutator subgroup, the homomorphism μ' and the action of P on $M/\langle M, M \rangle$ are induced by μ and the action of P on M respectively, plays the role of abelianization of a group which we further call Peiffer abelianization. As an analog of the classical first group homology, Conduché and Ellis [CE] defined the first homology of a precrossed P-module (M, μ) by Peiffer abelianization i.e.

$$H_1(M)_P = M/\langle M, M \rangle \ .$$

We point out that despite its name, the Peiffer abelianization can be nonabelian.

Let (L, λ), (M, μ) and (N, ν) be precrossed P-modules. A short exact sequence of groups

$$1 \longrightarrow L \xrightarrow{\ \varphi\ } M \xrightarrow{\ \psi\ } N \longrightarrow 1$$

is called short exact sequence of precrossed P-modules if φ and ψ are morphisms of the category $\mathcal{PCM}(P)$.

Let X be a set and $\delta : X \to P$ a map to the group P. Then the free precrossed P-module $\partial : F \to P$ with base (X, δ) is defined as follows: F is the free group generated by the set $X \times P$, ∂ is defined on generators by $\partial(x, p) = p\delta(x)p^{-1}$ and the action of P on F is given by ${}^{p}(x, p') = (x, pp')$.

Conduché and Ellis in [CE] also defined the second homology group of a precrossed P-module (M, μ) by the Hopf formula

$$H_2(M)_P = R \cap \langle F, F \rangle / \langle\langle F, R \rangle\rangle \ ,$$

where $1 \to R \to F \to M \to 1$ is a short exact sequence of precrossed P-modules, (F, ∂) is a free precrossed P-module with some base (X, δ) which is called free presentation of the precrossed P-module (M, μ). They studied some properties so defined low dimensional homology groups of precrossed P-modules and hoped that higher homology could be defined analogously using Hopf formulas for higher homology groups (see [BE]). Using this way to define all homology groups of a precrossed P-module (M, μ), $H_n(M)_P$, one should have some difficulties, for $n \geqslant 3$, to prove that the definition does not depend on the free presentation of the precrossed P-module (M, μ).

In the present paper we have another conception to define all homology groups of a precrossed P-module, particulary the use of nonabelian derived functors.

All treatments with homology of precrossed P-modules we consider in the q modular aspect, where q is a nonnegative integer, and for $q = 0$ it gives homology groups of precrossed modules introduced in [CE]. Thus, for nonnegative integer q, we define homology groups modulo q of precrossed P-module (M, μ) in any dimension $n \geqslant 1$, denoted by $H_n(M, q)_P$, and study their properties generalizing the classical homology of groups with coefficients in $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

1. Construction

Let us denote by $\mathrm{Set}(P)$ the category of sets over the group P, whose objects are all sets with a map to P and morphisms are all maps of sets such that the corresponding triangles are commutative.

Consider the functor $\mathcal{F} : \mathrm{Set}(P) \to \mathcal{PCM}(P)$ defined as follows: for an object $X \xrightarrow{\alpha} P$ of the category $\mathrm{Set}(P)$, let $\mathcal{F}(X \xrightarrow{\alpha} P)$ be a free precrossed P-module with base (X, α); for a morphism $X \xrightarrow{\kappa} X'$, let $\mathcal{F}(\kappa)$ be the canonical homomorphism induced by κ.

It is known that the forgetful functor from the category $\mathcal{PCM}(P)$ to the category $\mathrm{Set}(P)$ is a right adjoint of the functor $\mathcal{F}$. This adjunction induces the cotriple $(\mathcal{F}, \tau, \delta)$ in the category $\mathcal{PCM}(P)$. Let $\mathcal{P}$ be the projective class in the category $\mathcal{PCM}(P)$ induced by the cotriple $(\mathcal{F}, \tau, \delta)$ (see [TV], [IH]).

First we describe the projective class $\mathcal{P}$ and the corresponding $\mathcal{P}$-epimorphisms.

Proposition 1.1. *A morphism $M \xrightarrow{\varphi} N$ of the category $\mathcal{PCM}(P)$ is a $\mathcal{P}$-epimorphism if and only if φ is surjective (as map of sets).*

Proposition 1.2. *In the category $\mathcal{PCM}(P)$ the following conditions are equivalent:*

(i) A precrossed P-module (Q, ν) belongs to the projective class $\mathcal{P}$;

(ii) (Q, ν) is a free precrossed P-module with base (X, α) for some object $X \xrightarrow{\alpha} P$ of the category $\mathrm{Set}(P)$

The proof of these propositions is left to the reader.

A precrossed P-module (N, ν) is a precrossed P-submodule of a precrossed P-module (M, μ) if N is a subgroup of M, the action of P on N is induced by the action of P on M and ν is the restriction of μ on N. If, in addition, N is a normal subgroup of the group M then we write $N <_P M$.

Let (M, μ) be a precrossed P-module, N, N' be two subgroups of M and q be a nonnegative integer. We denote by $\langle N, N' \rangle_{(q)}$ the subgroup of M generated by the elements $\langle n, n' \rangle$ and k^q for all $n \in N$, $n' \in N'$, $k \in N \cap N' \cap Ker\mu$. Let $\langle\langle N, N' \rangle\rangle_{(q)} = \langle N, N' \rangle_{(q)} \langle N', N \rangle_{(q)}$. One has the following

Lemma 1.3. *(i) If N and N' are precrossed P-submodules of M then $\langle N, N' \rangle_{(q)}$ and $\langle\langle N, N' \rangle\rangle_{(q)}$ are precrossed P-submodules of M.*

(ii) If $N <_P M$ then $\langle M, N \rangle_{(q)} <_P M$, $\langle N, M \rangle_{(q)} <_P M$, $\langle\langle M, N \rangle\rangle_{(q)} <_P M$.

Proof. (i) Follows from the relation (3) and the equality ${}^p(k^q) = ({}^pk)^q$, $p \in P$, $k \in N \cap N' \cap Ker\mu$.

(ii) Follows from the relations (1), (2) and the equality $mk^qm^{-1} = (mkm^{-1})^q$, $m \in M$, $k \in N \cap Ker\mu$. □

Using Lemma 1.3 one can define a covariant functor $T_{(q)}$ from the category $\mathcal{PCM}(P)$ to the category $\mathfrak{Gr}$ of groups by the following way: for any precrossed P-module (M, μ), let $T_{(q)}(M) = M/\langle\langle M, M \rangle\rangle_{(q)} = M/\langle M, M \rangle_{(q)}$; for a morphism $(M, \mu) \xrightarrow{\varphi} (M', \mu')$, let $T_{(q)}(\varphi)$ be a group homomorphism induced by φ. Note that for $q = 0$ the functor $T_{(q)}$ is the Peiffer abelianization functor.

In the category $\mathcal{PCM}(P)$ there exist finite limits (easy to show). Let us consider the non-abelian left derived functors $\mathcal{L}_n^{\mathcal{P}}T_{(q)}$, $n \geqslant 0$, of the functor $T_{(q)} : \mathcal{PCM}(P) \to \mathfrak{Gr}$ relative to the projective class $\mathcal{P}$ induced by the cotriple $(\mathcal{F}, \tau, \delta)$ in the category $\mathcal{PCM}(P)$ [IH].

Definition 1.4. *Let P be a group, (M, μ) be a precrossed P-module and q be a nonnegative integer. Define the n-th homology group modulo q of the precrossed P-module (M, μ) by*

$$H_n(M, q)_P = \mathcal{L}_{n-1}^{\mathcal{P}}T_{(q)}(M), \quad n \geqslant 1 .$$

Proposition 1.5. *Let $\mu : M \to P$ be a precrossed P-module such that $\mu(m) = 1$ for all $m \in M$. Then one has*

$$H_n(M, q)_P = H_n(M, \mathbb{Z}_q), \quad n \geqslant 1.$$

Proof. Consider a $\mathcal{P}$-projective pseudosimplicial resolution (see [IH]) of (M, μ) in the category $\mathcal{PCM}(\mathcal{P})$

$$\cdots \;\begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix}\; F_2 \xrightarrow{\kappa_2} Y_1 \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}\; F_1 \xrightarrow{\kappa_1} Y_0 \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix}\; F_0 \to M \quad , \tag{6}$$

where $F_n \in \mathcal{P}$ and Y_n is a simplicial kernel in the category $\mathcal{PCM}(P)$. By Propositions 1 and 2 all F_n are free groups and all κ_n are surjective group homomorphisms, implying that (6) is a projective resolution of the group M in the category $\mathfrak{Gr}$. Since μ is a trivial group homomorphism, $T_{(q)}(F_n) = F_n^{ab}/qF_n^{ab}$. Using [BB] one gets the assertion. □

2. Main Results

In this section we give our main results. We investigate the functor $T_{(q)}$ and prove a Hopf type formula for the second homology modulo q of precrossed P-modules, generalizing the classical one (see [BR], [E]).

Let $\mathcal{C}$ be a category with finite limits, $\mathcal{Q}$ be a projective class in the category $\mathcal{C}$ and T be a covariant functor from the category $\mathcal{C}$ to the category $\mathfrak{Gr}$ of groups.

Definition 2.1 ([P]). *The functor T is called a cosheaf over $(\mathcal{C}, \mathcal{Q})$ if for any $\mathcal{Q}$-epimorphism $X \to A$ the sequence of groups*

$$T(X \times_A X) \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix}\; T(X) \longrightarrow T(A) \longrightarrow 1 \ ,$$

is simplicially exact, where $X \times_A X$ is the pullback of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ X & \longrightarrow & A \end{array} .$$

Lemma 2.2. *Let P be a group and q be a nonnegative integer. Then the functor $T_{(q)} : \mathcal{PCM}(P) \to \mathfrak{Gr}$ is a cosheaf over $(\mathcal{PCM}(P), \mathcal{P})$, where $\mathcal{P}$ is the projective class induced by the cotriple $(\mathcal{F}, \tau, \delta)$ (see above).*

Proof. It is easy to verify that for a short exact sequence of precrossed P-modules

$$1 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 1$$

there is an exact sequence of groups

$$T_{(q)}(L) \longrightarrow T_{(q)}(M) \longrightarrow T_{(q)}(N) \longrightarrow 1 \ . \tag{7}$$

Consider a $\mathcal{P}$-epimorphism $Q \xrightarrow{\alpha} M$ in the category $\mathcal{PCM}(P)$. We have to show that the diagram of groups

$$T_{(q)}(Q \times_M Q) \;\underset{d_1}{\overset{d_0}{\rightrightarrows}}\; T_{(q)}(Q) \xrightarrow{T_{(q)}(\alpha)} T_{(q)}(M) \longrightarrow 1$$

is exact. In effect, there is the following commutative diagram of groups

$$\begin{array}{ccccccc} T_{(q)}(R) & \longrightarrow & T_{(q)}(Q) & \xrightarrow{T_{(q)}(\alpha)} & T_{(q)}(M) & \longrightarrow & 1 \\ \downarrow \lambda & & \| & & \| & & \\ \operatorname{Ker} d_0 & \xrightarrow{d_1} & T_{(q)}(Q) & \xrightarrow{T_{(q)}(\alpha)} & T_{(q)}(M) & \longrightarrow & 1 \end{array},$$

where R is the kernel of $\alpha : Q \to M$, λ is a homomorphism induced by the inclusion $R \hookrightarrow Q \times_M Q$, $r \mapsto (r, 1)$, and the top row is exact by (7). Hence the bottom row of this diagram is also exact. □

Proposition 2.3. *Let P be a group, (M, μ) be a precrossed P-module and q be a nonnegative integer. Then there is a natural isomorphism*

$$H_1(M, q)_P \approx M/\langle M, M\rangle_{(q)}$$

Proof. Follows by Lemma 2.2 and [P or IH, Proposition 2.26].□

Theorem 2.4 (Hopf Formula). *Let P be a group, (M, μ) be a precrossed P-module and q be a nonnegative integer. Then there is an isomorphism*

$$H_2(M, q)_P \approx R \cap \langle F, F\rangle_{(q)}/\langle\langle F, R\rangle\rangle_{(q)} \ ,$$

where $1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 1$ *is any free presentation of the precrossed P-module (M, μ) i.e. using Propositions 1 and 2, F is an object of the projective class $\mathcal{P}$ and φ is a $\mathcal{P}$-epimorphism.*

Proof. Consider the Čech resolution of $(M, \mu) \in \mathcal{PCM}(P)$ for $\varphi : F \to M$ [P or IH, Definition 2.31, Examples]

$$\cdots \begin{array}{c}\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\end{array} F \times_M F \times_M F \ \begin{array}{c}\xrightarrow{d_0}\\\xrightarrow{d_1}\\\xrightarrow{d_2}\end{array} \ F \times_M F \ \begin{array}{c}\xrightarrow{d_0}\\\xrightarrow[d_1]{}\end{array} \ F \xrightarrow{\varphi} M.$$

By Lemma 2.2 $T_{(q)}$ is a cosheaf over $(\mathcal{PCM}(P), \mathcal{P})$ and using [P or IH, Theorem 2.39(ii)] there is an isomorphism

$$\mathcal{L}_1^{\mathcal{P}} T_{(q)}(M) \approx \pi_1 C_*,$$

where C_* is the following simplicial group

$$C_* \equiv \cdots \begin{array}{c}\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\end{array} T_{(q)}(F \times_M F \times_M F) \ \begin{array}{c}\xrightarrow{T_q(d_0)}\\\xrightarrow{T_q(d_1)}\\\xrightarrow{T_q(d_2)}\end{array} \ T_{(q)}(F \times_M F) \ \begin{array}{c}\xrightarrow{T_q(d_0)}\\\xrightarrow[T_q(d_1)]{}\end{array} \ T_{(q)}(F) \ .$$

The Moore complex NC_* of the simplicial group C_* has length 1 i.e. $(NC_*)_n = 0$, $n \geqslant 2$. This follows from the fact that the Moore complex of the Čech resolution has length 1.

Hence $\pi_1 C_* = \operatorname{Ker} T_q(d_0) \cap \operatorname{Ker} T_q(d_1)$.

Furthermore, one has the following isomorphism of precrossed P-modules $F \times_M F \xrightarrow{\approx} R \rtimes F$, defined by $(r, f) \mapsto (rf, f)$, where the precrossed P-module structure on the group $R \rtimes F$ is given by the following way: a homomorphism $R \rtimes F \to P$ is defined by $(r, f) \mapsto \mu\varphi(f)$ and an action of P on $R \rtimes F$ by ${}^p(r, f) = ({}^pr, {}^pf)$ for all $p \in P$, $r \in R$, $f \in F$. One gets $R \rtimes F \ \begin{array}{c}\xrightarrow{d_0}\\\xrightarrow[d_1]{}\end{array} \ F$, $d_0(r, f) = f$, $d_1(r, f) = rf$.

It only remains and easy to prove that the homomorphism

$$\alpha : (R/\langle\langle F, R\rangle\rangle_{(q)}) \times F/\langle F, F\rangle_{(q)} \longrightarrow T_{(q)}(R \rtimes F) \ ,$$

defined by $\alpha([r], [f]) = [(r, f)]$, is an isomorphism. □

Remark 2.5. *For $\mu = 0$ Theorem 2.4 generalizes the classical Hopf Formula from [BR] and for $q = 0$ Proposition 2.3 and Theorem 2.4 show that one can get the first and the second homology of precrossed P-modules of Conduché and Ellis [CE] as nonabelian derived functors of the Peiffer abelianization functor.*

Conjecture. *Let P be a group, (M, μ) be a precrossed P-module and q be a nonnegative integer. Choose a precrossed P-module (F, ∂) and $R_1, \ldots, R_n <_P F$ such that: $F/\prod_{1 \leqslant i \leqslant n} R_i \approx M$, $H_2(F, q)_P = 1$ and $H_r(F/\prod_{i \in A} R_i, q)_P = 1$ for every proper subset $A \neq \emptyset$ of $\langle n \rangle = \{1, \ldots, n\}$, $r = |A| + 1$ and $|A| + 2$ (for example, the precrossed P-modules $F/\prod_{i \in A} R_i$ are free for $A \neq \langle n \rangle$). Then there is an isomorphism*

$$H_{n+1}(M, q)_P \approx \{\cap_{i=1}^{n} R_i \cap \langle F, F \rangle_{(q)}\} / \{\prod_{A \subseteq \langle n \rangle} \langle\langle \cap_{i \in A} R_i, \cap_{i \notin A} R_i \rangle\rangle_{(q)}\}$$

3. Some other results

In this section we investigate low dimensional, first and second, homologies modulo q of precrossed P-modules, always have in mind Proposition 2.3 and Theorem 2.4 and give some results generalizing in q modular aspect the results of Conduché and Ellis [CE].

Proposition 3.1. *Let P be a group, q be a nonnegative integer and*

$$1 \to L \to M \to N \to 1$$

a short exact sequence of precrossed P-modules. Then there is an exact sequence of groups

$$\begin{array}{c} H_2(M, q)_P \to H_2(N, q)_P \to L/\langle\langle M, L \rangle\rangle_{(q)} \\ \to H_1(M, q)_P \to H_1(N, q)_P \to 1 \end{array} . \tag{8}$$

Proof. Suppose $1 \to R \to F \to M \to 1$ is a free presentation of the precrossed P-module M, and hence $1 \to R' \to F \to N \to 1$ is a free presentation of the precrossed P-module N. Therefore $R \subset R'$ implying $R \cap \langle F, F \rangle_{(q)} \subset R' \cap \langle F, F \rangle_{(q)}$, $\langle\langle F, R \rangle\rangle_{(q)} \subset \langle\langle F, R' \rangle\rangle_{(q)}$ and there is the canonical group homomorphism $H_2(M, q)_P \to H_2(N, q)_P$.

The following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R' & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 1 \end{array}$$

induces a homomorphism $H_2(N, q)_P \to L/\langle\langle M, L \rangle\rangle_{(q)}$.

Other homomorphisms are defined naturally and it is easy to check that the sequence (8) is exact. □

Remark 3.2. *One can extend the sequence (8) to any dimensions using the long exact sequence of the nonabelian derived functors and recovering for $\mu = 0$ the eight term exact homology sequence of groups with coefficients in $\mathbb{Z}_q$ [ER].*

The following result generalises the classical group result and uses the standard proof, originally due to [S].

For any precrossed P-module (M, μ) and any nonnegative integer q there is the following family of precrossed P-submodules

$$M_{(q)}^{(1)} = M,\ M_{(q)}^{(2)} = \langle\langle M, M \rangle\rangle_{(q)}, \ldots, M_{(q)}^{(n+1)} = \langle\langle M, M_{(q)}^{(n)} \rangle\rangle_{(q)}.$$

Theorem 3.3. *Let P be a group, q be a nonnegative integer and $\varphi : M \to N$ be a morphism of precrossed P-modules such that the following properties hold:*

(i) the natural homomorphism $H_1(M,q)_P \to H_1(N,q)_P$, induced by φ, is an isomorphism;

(ii) the natural homomorphism $H_2(M,q)_P \to H_2(N,q)_P$, induced by φ, is a surjection.

Then φ induces a natural isomorphism of precrossed P-modules

$$M/M_{(q)}^{(n)} \xrightarrow{\approx} N/N_{(q)}^{(n)} \qquad \text{for } n \geqslant 2\,.$$

Proof. By induction. For $n=2$ the theorem is true. Suppose it is true for n. By Proposition 3.1 and the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{(q)}^{(n)} & \longrightarrow & M & \longrightarrow & M/M_{(q)}^{(n)} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle\varphi} & & \big\downarrow & & \\
1 & \longrightarrow & N_{(q)}^{(n)} & \longrightarrow & N & \longrightarrow & N/N_{(q)}^{(n)} & \longrightarrow & 1
\end{array},$$

one has the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccc}
H_2(M,q)_P & \longrightarrow & H_2(M/M_{(q)}^{(n)},q)_P & \longrightarrow & M_{(q)}^{(n)}/M_{(q)}^{(n+1)} & \longrightarrow \\
\big\downarrow & & \big\downarrow & & \big\downarrow & \\
H_2(N,q)_P & \longrightarrow & H_2(N/N_{(q)}^{(n)},q)_P & \longrightarrow & N_{(q)}^{(n)}/N_{(q)}^{(n+1)} & \longrightarrow
\end{array}$$

$$\begin{array}{cccccc}
\longrightarrow & H_1(M,q)_P & \longrightarrow & H_1(M/M_{(q)}^{(n)},q)_P & \longrightarrow & 1 \\
 & \big\downarrow & & \big\downarrow & & \\
\longrightarrow & H_1(N,q)_P & \longrightarrow & H_1(N/N_{(q)}^{(n)},q)_P & \longrightarrow & 1
\end{array}.$$

Using the "five lemma" $M_{(q)}^{(n)}/M_{(q)}^{(n+1)}$ is isomorphic to $N_{(q)}^{(n)}/N_{(q)}^{(n+1)}$. Then the following commutative diagram of groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{(q)}^{(n)}/M_{(q)}^{(n+1)} & \longrightarrow & M/M_{(q)}^{(n+1)} & \longrightarrow & M/M_{(q)}^{(n)} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & N_{(q)}^{(n)}/N_{(q)}^{(n+1)} & \longrightarrow & N/N_{(q)}^{(n+1)} & \longrightarrow & N/N_{(q)}^{(n)} & \longrightarrow & 1
\end{array}$$

gives the result for $n+1$. □

For any precrossed P-module $\mu : M \to P$, let $M \wedge_P^q M$ be the group generated by the symbols $m \wedge m'$ and $\{k\}$, $m, m' \in M$, $k \in Ker\mu$ subject to the following relations:

$$m \wedge m'm'' = (m \wedge m')(m \wedge m'')(\langle m, m''\rangle^{-1} \wedge {}^{\mu m}m'), \tag{9}$$

$$mm' \wedge m'' = (m \wedge m'm''m'^{-1})({}^{\mu m}m' \wedge {}^{\mu m}m''), \tag{10}$$

$$\langle m, m'\rangle \wedge \langle n, n'\rangle = (m \wedge m')(n \wedge n')(m \wedge m')^{-1}(n \wedge n')^{-1}, \tag{11}$$

$$(\langle m, m'\rangle \wedge m'')(m'' \wedge \langle m, m'\rangle) = (m \wedge m')({}^{\mu m''}m \wedge {}^{\mu m''}m')^{-1}, \tag{12}$$

$$k \wedge k = 1, \tag{13}$$

$$\{k\}(m \wedge m')\{k\}^{-1} = (k^q m \wedge m')(k^q \wedge {}^{\mu m}m')^{-1}, \tag{14}$$

$$\{kk'\} = \{k\} \prod_{i=1}^{q-1} (k^{-1} \wedge k^{1-q+i}(k')^i k^{q-1-i})\{k'\}, \tag{15}$$

$$\{k\}\{k'\}\{k\}^{-1}\{k'\}^{-1} = k^q \wedge k'^q, \tag{16}$$

$$\{\langle m, m'\rangle\} = (m \wedge m')^q \tag{17}$$

for all $m, m', m'', n, n' \in M$ and $k, k' \in Ker\mu$.

Note that (9)-(13) are the defining relations for the group $M \wedge_P M$ defined in [CE]. Furthermore, when $P = 1$ or $\mu = 0$ the group $M \wedge_P^q M$ coincides with the nonabelian exterior product modulo q, $M \wedge^q M$, introduced by Conduché and Rodriguez-Fernandez [CR] (see also [B], [ER], [E], [IN]).

There is an action of the group P on the group $M \wedge_P^q M$ given by ${}^p(m \wedge m') = {}^p m \wedge {}^p m'$ and ${}^p\{k\} = \{{}^p k\}$ for all $m, m' \in M$, $k \in Ker\mu$. Moreover, there exists a P-equivariant group homomorphism $\partial_2^q : M \wedge_P^q M \to M$ defined by $\partial_2^q(m \wedge m') = \langle m, m' \rangle$ and $\partial_2^q(\{k\}) = k^q$. It is clear that $\partial_2^q(M \wedge_M^q M) = M_{(q)}^{(2)}$.

Note that the complex of groups $M \wedge_P^q M \xrightarrow{\partial_2^q} M \xrightarrow{\mu} P$ is a 2-crossed module in the sence of Conduché [C].

Proposition 3.4. *Let (M, μ) be a precrossed P-module, $q > 0$ and*

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 1$$

a short exact sequence of precrossed P-modules, where (F, ν) is a free precrossed P-module. If the homomorphism $\partial_2^q : F \wedge_P^q F \to F$ is injective then the group $M \wedge_P^q M$ is isomorphic to the group $F_{(q)}^{(2)}/\langle\langle F, R\rangle\rangle_{(q)}$.

Proof. Let L_F (resp. L_M) be the free group generated by the set $(F \times F) \cup Ker\nu$ (resp. $(M \times M) \cup Ker\mu$). There is a commutative diagram of groups

$$\begin{array}{ccc} L_F & \longrightarrow & L_M \\ {\scriptstyle \pi_F}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ F \wedge_P^q F & \longrightarrow & M \wedge_P^q M \end{array} \quad ,$$

where the horizontal homomorphisms are surjective and π_F and π_M are canonical homomorphisms defined by $\pi_F(f, f') = f \wedge f'$, $\pi_F(g) = \{g\}$ and $\pi_F(m, m') = m \wedge m'$, $\pi_F(k) = \{k\}$ for all $f, f' \in F$, $g \in Ker\nu$, $m, m' \in M$ and $k \in Ker\mu$. It is easy to get that $Ker(F \wedge_P^q F \to M \wedge_P^q M)$ is the homomorphic image of $Ker(L_F \to L_M)$ by π_F. It is also easy to check that $Ker(L_F \to L_M)$ is the normal subgroup of L_F generated by the elements $(f_1, f_2)(f'_1, f'_2)^{-1}$ and $f_3 {f'}_3^{-1}$ such that $\varphi f_i = \varphi f'_i$, $f_i, f'_i \in F$ $(i = 1, 2)$ and $\varphi f_3 = \varphi f'_3$, $f_3, f'_3 \in Ker\nu$. Thus its image in $F \wedge_P^q F$ is the normal subgroup generated by the elements $(f_1 \wedge f_2)(f'_1 \wedge f'_2)^{-1}$ and $\{f_3\}\{f'_3\}^{-1}$, which by the formulas (9), (10) and (16) coincides with the normal subgroup of $F \wedge_P^q F$ generated by the elements $f \wedge r$, $r \wedge f$ and $\{r\}$, $f \in F$, $r \in R$. Then the image of this subgroup by the isomorphism $F \wedge_P^q F \approx \partial_2^q(F \wedge_P^q F) = F_{(q)}^{(2)}$ is $\langle\langle F, R\rangle\rangle_{(q)}$ and thus $F_{(q)}^{(2)}/\langle\langle F, R\rangle\rangle_{(q)} \approx M \wedge_P^q M$. □

Lemma 3.5. *Let $h : A \to B$ and $g : B \to C$ be group homomorphisms. If h is surjective then the following sequence of groups*

$$1 \longrightarrow \mathrm{Ker}(h) \longrightarrow \mathrm{Ker}(gh) \longrightarrow \mathrm{Ker}(g) \longrightarrow 1.$$

is exact.

Theorem 3.6. *Let $\mu : M \to P$ be a precrossed P-module, $q > 0$ and*

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 1$$

a short exact sequence of precrossed P-modules, where F is a free precrossed P-module. If the homomorphism $\partial_2^q : F \wedge_P^q F \to F_{(q)}^{(2)}$ is an isomorphism then there is an isomorphism of groups

$$H_2(M, q) \approx Ker(M \wedge_P^q M \to M).$$

Proof. By Lemma 3.5 one has the following exact sequence of groups

$$1 \longrightarrow Ker(\varphi \wedge_P^q \varphi) \longrightarrow Ker(\partial_2^q(\varphi \wedge_P^q \varphi)) \longrightarrow Ker\partial_2^q \longrightarrow 1.$$

From the commutative diagram of groups

$$\begin{array}{ccc} F \wedge_P^q F & \xrightarrow{\varphi \wedge_P^q \varphi} & M \wedge_P^q M \\ \approx\downarrow & & \downarrow \partial_2^q \\ F_{(q)}^{(2)} & \xrightarrow{\varphi_{(q)}^{(2)}} & M_{(q)}^{(2)} \end{array}$$

one gets $Ker(\partial_2^q(\varphi \wedge_P^q \varphi)) \approx Ker\varphi_{(q)}^{(2)} = R \cap F_{(q)}^{(2)}$. Then $Ker(\partial_2^q : M \wedge_P^q M \to M) \approx R \cap F_{(q)}^{(2)}/\langle\langle F, R\rangle\rangle_{(q)} = H_2(M, q)_P$. □

Finally we give an example showing that there exists such a group P and a free precrossed P-module F that the homomorphism $\partial_2^q : F \wedge_P^q F \to F$ is injective.

Lemma 3.7. *Let (M, μ) be precrossed P-module and q be a nonnegative integer. Then there is an exact sequence of groups*

$$M \wedge_P M \xrightarrow{\ \varphi\ } M \wedge_P^q M \longrightarrow \ker\mu/\langle M, M\rangle \longrightarrow 1\ .$$

Proof. The homomorphism φ is given by $\varphi(m \wedge m') = m \wedge m'$. The required exacteness is easy to check. □

Proposition-Example 3.8. *Let P be a free group, $\mu : F \to P$ be a free precrossed P-module and $q > 0$. Then the homomorphism $\partial_2^q : F \wedge_P^q F \to F_{(q)}^{(2)}$ is an isomorphism.*

Proof. Using Lemma 3.7 one has the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F \wedge_P F & \xrightarrow{\varphi} & F \wedge_P^q F & \longrightarrow & \ker\mu/F^{(2)} & \longrightarrow & 1 \\ & & \downarrow \partial_2 & & \downarrow \partial_2^q & & \downarrow \alpha & & \\ 1 & \longrightarrow & F^{(2)} & \longrightarrow & F_{(q)}^{(2)} & \longrightarrow & F_{(q)}^{(2)}/F^{(2)} & \longrightarrow & 1 \end{array},$$

where $F^{(2)} = \langle F, F\rangle$, ∂_2 is an isomorphism [BC] proved applying a theorem of J.H.C. Whitehead [W] on 2-dimensional CW-complexes and the theorem of Kan [K] (see above) and hence φ is injective. One can directly check that α is an isomorphism and so is ∂_2^q. □

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