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HOMOLOGICAL PERTURBATION THEORY AND ASSOCIATIVITY

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(*communicated by Larry Lambe*)

Abstract

In this paper, we prove various results concerning DGA-algebras in the context of the Homological Perturbation Theory. We distinguish two class of contractions for algebras: full algebra contractions and semi-full algebra contractions. A full algebra contraction is, in particular, a semi-full algebra contraction. Taking a full algebra contraction and an "algebra perturbation" as data of the Basic Perturbation Lemma, the Algebra Perturbation Lemma (or simply, F-APL) of [**20**] and [**29**] appears in a natural way. We establish here a perturbation machinery, the Semi-Full Algebra Perturbation Lemma (or, simply, SF-APL) that is a generalization of the previous one in the sense that the application range of SF-APL is wider than that of F-APL. We show four important applications in which this result is essential for the construction of algebra or coalgebra structures in various chain complexes.

1. Introduction

Homological Perturbation Theory [**52, 14, 8, 9, 18, 36, 19, 20, 29**] is a set of techniques for the transference of structures from one object to another up to homotopy. An essential notion in this framework is that of *contraction*. A contraction $r : \{N, M, f, g, \phi\}$ [**14, 15**] is a special homotopy equivalence determined by three morphisms $f : N_* \to M_*$ (*projection*), $g : M_* \to N_*$ (*inclusion*) and $\phi : N_* \to N_{*+1}$ (*homotopy*) between two DG-modules N (the "big" one) and M (the "small" one). The most important element of HPT is the Basic Perturbation Lemma (BPL) which can be considered as an actual algorithm: its input is a contraction c and a "perturbation" δ of the differential of N and its output is a new contraction r_δ in which the graded modules N and M remain unchanged. Contractions are also known in the literature as SDR-data [**36, 19, 20**] or Eilenberg-Zilber data [**22**].

Perturbation results regarding preservation of additional structures (DG-algebras, DG-coalgebras, Lie algebras, ...) have been largely considered. The technique for obtaining these "homological perturbation machines" is the introduction of hypotheses such that BPL allows to establish preservation results for the additional data structures. Let us now consider the case of DG-algebras. Let A and A' be two DG-algebras with respective products $\mu_A : A \otimes A \to A$ and $\mu_{A'} : A' \otimes A' \to A'$, and let $r : \{A, A', f, g, \phi\}$ be a contraction. In [**20**] and [**29**], an Algebra Perturbation Lemma is described in which the algebra **laws** μ_A and $\mu_{A'}$ are preserved. In other words, taking as input a special class of algebra contraction (that we will name here a *full algebra contraction*, in the sense that its component morphisms f, g and ϕ have the

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"maximum" degree of compatibility with regard to the algebra structures on A and A') and a derivation as perturbation datum, it is proved that the perturbed contraction is also a full algebra contraction.

In this paper, assuming weaker hypotheses for the data of the Basic Perturbation Lemma, we prove analogous theorems, where the resulting contraction, of course, has weaker properties too. More precisely, we analyze conditions under which BPL preserves strict associativity. This analysis consists in studying the A_∞-algebra structure for the small complex of a perturbed contraction and leads us to define a special class of contractions: *semi-full algebra contractions*. In spite of the fact that the component morphisms of a contraction r of this class have a "lower" degree of compatibility with regard to the algebra structures than that of full algebra contractions, the semi-fullness property is sufficient to guarantee that the A_∞-algebra structure on M given by the contraction r (see [**23**]) reduces to the original associative product μ_M. And, what is more important, semi-fullness is a hereditary property by composition, tensor product and perturbation of contractions. Moreover, it is not difficult to find semi-full (non-full!!) algebra contractions in Algebraic Topology and Homological Algebra (all the examples of semi-fullness that we show here are contractions already defined by the firm Eilenberg-Mac Lane in [**14, 15**]):

- The explicit contraction given in [**15**] from the reduced bar construction $\bar{B}(A\otimes A')$ to the tensor product $\bar{B}(A) \otimes \bar{B}(A')$, where A and A' are augmented commutative differential graded algebras.
- The explicit contractions established in [**15**] for the normalized reduced bar constructions of the algebras $\mathbb{Z}[\mathbb{Z}]$ and $\mathbb{Z}[\mathbb{Z}_p]$.
- An Eilenberg-Zilber contraction EZ (see the Apendix of the present paper for a explicit combinatorial description) from $C_*^N(X \times Y)$ to $C_*^N(X) \otimes C_*^N(Y)$, where X and Y are simplicial groups and $C_*^N(K)$ means the normalized chain complex canonically associated to the simplicial set K . The fact that the projection and the inclusion of this contraction are morphisms of DGA-algebras was proved by Eilenberg and Mac Lane in [**15**]. In [**3**], it is proved that this contraction is a semi-full (non-full) algebra contraction.
- In [**14**], Eilenberg and Mac Lane obtained a reduction g (a morphism inducing an isomorphim in homology) from the normalized reduced bar construction $\bar{B}(C_*^N(G))$ to the normalized classifying construction $\bar{W}_N(C_*(G))$, where G is a simplicial group. They conjectured that this relationship could be extended to a contraction R_{WB}. Having in hand the homological perturbation machinery, it is an elementary work to obtain a homotopy equivalence R_{WB} (see [**57, 44**]). A more complex task is to determining if a contraction of this type is a full algebra contraction or not. An answer is given in [**3**], where it is proved that a contraction R_{WB} is a semi-full (non-full) algebra contraction.
- Combining the previous result with those of Section 5 of this paper, it is easy to conclude that the algebra homology of Eilenberg-Mac Lane spaces can be seen from the viewpoint of *semi-full algebra contractions*. There are explicit semi-full algebra contractions from the normalized chain complex of a $K(\pi, n)$ (π being a finitely generated abelian group) to a (non-twisted) tensor product of Cartan's elementary complexes. This treatment of the homology of these prime spaces is extremely important in the design of a general algorithm computing the homology of twisted cartesian products of two Eilenberg-Mac Lane spaces via perturbation (see [**2**]).

Apart from the intrinsic interest of the transference problem dealing with DG-algebras or DG-coalgebras, in several important situations (see Section 5) our results allow us to determine and construct essential algebra or coalgebra structures in various chain complexes which have been constructed via perturbation. For example, the homology theory of commutative DGA-algebras is entirely examined in terms of semi-full algebra contractions in [**1**], and "small" p-local homological models of reduced bar constructions of twisted tensor product of Cartan's elementary complexes are obtained in [**6**], due to the fact that all the contractions appearing there are semi-full.

Here is a summary of the present paper. Notation and terminology are introduced in Section 2. In Section 3, the basic ingredients in Homological Perturbation Theory are reviewed. Our contribution starts in Section 4 which is devoted to introducing the notions of semi-full, almost-full and full algebra contractions and to giving general perturbation results of preservation of the (co)algebra category. Four important applications of this technique are established in Section 5, in which the perturbation machinery of algebras given in [**20**] and in [**29**] cannot be applied. The first one deals with the transference problem of the Hopf algebra structure in the contraction $\bar{B}(r)$, constructed using an initial contraction r from a commutative DGA-algebra A to a simple DGA-module M. The contraction $\bar{B}(r)$ connects the reduced bar construction $\bar{B}(A)$ to the bar tilde construction of the DGA-module M [**23**]. An interesting consequence concerning A_∞-structures is Corollary 5.7 in which Kadeishvili's definition [**31**] of commutative A_∞-algebra appears. The second application is a multiplicative analysis of the contraction from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A')$, where A and A' are both commutative DGA-algebras, described in [**15**]. The third one is about the computation, via perturbation, of small p-local homological models of Cartan's elementary complexes (see [**10**]). The elementary complexes are small commutative DGA-algebras appearing in the Cartan's method for computing the homology algebra of Eilenberg-Mac Lane spaces. The fourth application, is dedicated to analyze the p-local n-homology algebra of a concrete class of commutative DGA-algebras. Finally, an appendix, written by Frederic Morace, is added. It is devoted to prove that the homotopy operator of the Eilenberg-Zilber contraction EZ, recursively described in [**15**], is defined by an explicit formula discovered experimentally by Julio Rubio and Francis Sergeraert. I wish to express my sincere thanks to Frederic for having written down this result.

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2. Notations and definitions

The purpose of this section is to give a reasonably complete account of the notions of homological algebra necessary for this paper -for more complete details the reader is referred to Mac Lane [**38**] and Weibel [**58**].

Let Λ be a commutative ring with $1 \neq 0$, taken henceforth as ground ring and fixed throughout. Graded modules are graded by the non-negative integers. The *degree* of an (homogeneous) element x of a graded module will be denoted by $|x|$. We denote $\sum_{n>0} M_n$ by M^+. A morphism $f : M \to N$ of graded modules has degree i, and it is denoted by $|f| = i$ if it satisfies $f(M_n) \subset N_{n+i}$ for all n.

All tensor products are over Λ. Given a graded module M, we will denote $M^{\otimes 0} = \Lambda$ and $M^{\otimes n} = M \otimes \overset{n \text{ times}}{\cdots} \otimes M$. The identity map of a graded module M will be denoted by 1_M. Throughout this paper, we adopt *Koszul's convention*, which defines the tensor product of two (homogeneous) graded morphisms as:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y).$$

If $f : M \to N$ is a morphism of graded modules and n is a positive integer, the morphism $f \otimes \overset{n \text{ times}}{\cdots} \otimes f; M^{\otimes n} \to N^{\otimes n}$ will be denoted by $f^{\otimes n}$.

Let M and N be two graded modules. The morphism of graded modules $T : M \otimes N \to N \otimes M$ is defined by $T(x \otimes y) = (-1)^{|x| \cdot |y|} y \otimes x$.

If $f : M^{\otimes i} \to M$ is a morphism of graded modules and n is a positive integer, it is possible to define the following morphism

$$f^{[n]} = \sum_{j=0}^{n-i} 1_M^{\otimes j} \otimes f \otimes 1_M^{\otimes n-i-j}; \tag{1}$$

in the summands for $j = 0$ and $j = n - i$, the morphism $1_M^{\otimes 0}$ is understood to be 1_Λ. We define the morphism $f^{[\,]} : \oplus_{j \geqslant i} M^{\otimes j} \to \oplus_{k \geqslant 1} M^{\otimes k}$ as $f^{[\,]}|_{M^{\otimes n}} = f^{[n]}$.

Let (M, d_M) be a DG-module. The suspension of M, denoted by $S(M)$, is defined by $S(M)_{n+1} = M_n$ and its differential is $-d_M$. Let $f : N \to M$ be a morphism of DG-modules (morphism of graded modules compatible with the differential structures) of degree i. We define the morphism $S(f) : S(N) \to S(M)$ by $S(f)(a) = (-1)^i f(a)$.

A DGA-module $(M, d_M, \epsilon_M, \eta_M)$ is a DG-module (M, d_M) equipped with two morphisms of DG-modules: an *augmentation* $\epsilon_M : M \to \Lambda$ and a *coaugmentation* $\eta_M : \Lambda \to M$, so that $\epsilon_M \eta_M$ is the identity map of Λ; morphisms are then required to preserve this structure. The DG-module $\overline{M} = \text{Ker } \epsilon_M$ is defined in a natural way. Given a morphism of DGA-modules $f : M \to N$, it is defined $\bar{f} : \overline{M} \to \overline{N}$ by $\bar{f}(x) = f(x)$.

We shall use without further explanation the concepts of *DGA-algebra*, *DGA-coalgebra*, *derivation*, *coderivation*, *DG Hopf algebra*, etc. The structure maps of a DGA-algebra $(A, d_A, \mu_A, \epsilon_A, \eta_A)$ and a DGA-coalgebra $(C, d_C, \Delta_C, \epsilon_C, \eta_C)$ will be

$$\mu_A : A \otimes A \to A \ \ \text{(product)}, \qquad \text{and} \qquad \eta_A : \Lambda \to A \ \ \text{(unit)}$$

and

$$\Delta_C : C \to C \otimes C \ \ \text{(coproduct)}, \qquad \text{and} \qquad \epsilon_C : C \to \Lambda \ \ \text{(counit)},$$

respectively. Moreover, the augmentation ϵ_A and the coaugmentation η_C preserve the respective structures.

We will denote the unit of a DGA-algebra A by $\theta_A = \eta_A(1)$.

Henceforth, if no possibility of confusion exists for the structure maps, DG-modules, DGA-modules, DGA-algebras and DGA-coalgebras will be denoted only by the capital letter which defines the underlying graded module.

If $X = \sum_{n \geqslant 0} X_n$ is a graded set and M is the free graded module with basis X, we denote by ΛM the free commutative graded algebra generated by M; it is the tensor product of the exterior algebra over X^{odd} and the polynomial algebra over X^{even}.

Let A and C be a DGA-algebra and a DGA-coalgebra, respectively. The morphism $t : C \to A$ of degree -1 is called a *twisting cochain* if it satisfies that $dt + td = \mu_A(t \otimes t)\Delta_C$, $\epsilon_A t = 0$. A *twisted tensor product* $A \otimes_t C$ is a DGA-module, such that, as an augmented graded module, it coincides with $A \otimes C$ and its differential is $1_A \otimes d_C + d_A \otimes 1_C + (\mu_A \otimes 1)(1_A \otimes t \otimes 1_C)(1_A \otimes \Delta_C)$.

A *resolution* of Λ over the DGA-algebra A is a DG-A-module X which is projective as an A-module and such that the homology of X is zero except in degree 0 where it is Λ. If X is actually a free DG-A-module, then X is called a *free resolution*.

Given a DGA-algebra A, the *reduced bar construction* on A is the DGA-coalgebra

$$(\bar{B}(A),\ d_{\bar{B}},\ \Delta_{\bar{B}},\ \epsilon_{\bar{B}},\ \eta_{\bar{B}}) \quad = \quad (T(S(\overline{A})),\ d_t + d_s,\ \Delta_T,\ \epsilon_T,\ \eta_T),$$

where $(T(M), d_t, \Delta_T, \epsilon_T,\ \eta_T)$ is the tensor coalgebra of a DGA-module M.

An homogeneous element $S\,\overline{a_1} \otimes \cdots \otimes S\,\overline{a_n} \in \bar{B}(A)$ will be written in the form $[a_1|\cdots|a_n]$. We also adopt the convention $1 = [\,]$. The morphisms d_t and d_s are called *tensor differential* and *simplicial differential*, respectively. The simplicial differential is defined by:

$$d_s[a_1|\cdots|a_n] \;=\; \sum_{i=1}^{n-1}(-1)^{e_i}[a_1|\cdots|\mu_A(a_i,a_{i+1})|\cdots|a_n],$$

where $e_i = i+|a_1|+\cdots+|a_i|$. Both differentials are coderivations with respect to the canonical coproduct. Given an element $z=[a_1|\ldots|a_n]$, the *tensor degree* is defined by $|z|_t=\sum_{i=1}^{n}|a_i|$, and the *simplicial degree* is defined by $|z|_s=n$. It is clear that

$$|z| \;=\; |z|_t \;+\; |z|_s.$$

Let $u=[a_1|\ldots|a_m]$ and $v=[b_1|\ldots|b_n]$ be two elements of $\bar{B}(A)$. We define the following (non-commutative) product:

$$u\bullet v=[a_1|\ldots|a_m|b_1|\ldots|b_n] \tag{2}$$

which is extended to $\bar{B}(A)$ by linearity.

If A is a commutative DGA-algebra, it is well known that it is possible to define a commutative product $*$ on $\bar{B}(A)$ (called *shuffle product*).

In this way, $\overline{B}(A)$ enjoys a Hopf DGA-algebra structure and both simplicial and tensor differentials are derivations with respect to the shuffle product. Let us recall that the homology (or more precisely, the 1-homology) of a DGA-algebra A is defined as the homology of $\bar{B}(A)$. Due to the fact that the reduced bar construction can be iterated in the case in which A is commutative, the n-homology of A is defined as the homology of $\bar{B}^n(A)=\bar{B}(\overset{n\ \text{times}}{\cdots}\bar{B}(A))\ldots)$.

A resolution of Λ over a DGA-algebra A is the *bar resolution* $B(A)$. This complex is the twisted tensor product of the DGA-algebra A and the DGA-coalgebra $\bar{B}(A)$, where the twisting cochain t is given by this formula:

$$t([a_1|\cdots|a_n])=\begin{cases} a_1 & n=1\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

We now introduce the notion of twisted tensor product (briefly, TTP) of DGA-algebras, that we will distinguish from that of twisted tensor product of algebra and coalgebra (in Brown's sense) and that of TTP of chain algebras of [**12**].

Let $\{A_i\}_{i\in I}$ be a set of commutative DGA-algebras. A *twisted tensor product* $\tilde{\otimes}^{\rho}_{i\in I}A_i$ is a commutative DGA-algebra satisfying the following conditions:

- **i)** as a graded algebra, $\tilde{\otimes}^{\rho}_{i\in I}A_i$ coincides with the tensor product $\otimes_{i\in I}A_i$,
- **ii)** and its differential is the sum of the differential of the banal tensor product and a derivation ρ.

As an example of TTP of algebras, we can consider the bar resolution $B(A)$ of a commutative DGA-algebra A. In this case, $B(A)$ is the commutative DGA-algebra $A\tilde{\otimes}^{\rho}\bar{B}(A)$, where

$$\rho(a\otimes[a_1|a_2|\cdots|a_n])=\mu_A(a,a_1)\,[a_2|\cdots|a_n]. \tag{4}$$

Hence, in the commutative case, we can regard $B(A)$ as a TTP of algebra and coalgebra as well as a TTP of DGA-algebras.

3. A review of HPT

In this paper, we deal with a special type of homotopy equivalence: a contraction.

A *contraction* is a data set $r : \{N, M, f, g, \phi\}$ where $f : N \to M$ and $g : M \to N$ are morphisms of DGA-modules (called *the projection* and *the inclusion* of the contraction r, respectively) and $\phi : N \to N$ is a morphism of graded modules of degree $+1$ (called *homotopy operator*), and these data are required to satisfy the rules,

$$\begin{array}{ll} \textbf{(c1)} & fg = 1_M; \\ \textbf{(c2)} & \phi d_N + d_N\phi + gf = 1_N; \\ \textbf{(c3)} & \phi g = 0; \\ \textbf{(c4)} & f\phi = 0; \\ \textbf{(c5)} & \phi\phi = 0. \end{array} \tag{5}$$

Given a contraction $r : \{N, M, f, g, \phi\}$, we have $\mathrm{Ker}\ \phi \ = \ \mathrm{Im}\ g \oplus \mathrm{Im}\ \phi$. In fact, the DGA-module N is a direct sum of M and an acyclic DGA-module. N is called the *big DGA-module* of r , and M is called the *small DGA-module* of r.

In this definition we follow Eilenberg-MacLane ([**14**]) terminology; we also find in literature "strong deformation retraction" or SDR (Lambe-Stasheff [**36**], Gugenheim-Lambe [**19**] and Gugenheim-Stasheff [**23**]), "Eilenberg-Zilber data" (Gugenheim-Munkholm [**22**]), or "trivial extension" (Munkholm [**42**]).

The bar resolution $B(A)$ of a DGA-algebra A supports the following contraction:

$$R_{B(A)} : \{B(A), \Lambda, \epsilon_{B(A)}, \eta_{B(A)}, s\}, \tag{6}$$

where the homotopy operator $s : B(A) \to B(A)$ is given by

$$s(a \otimes [a_1|\cdots|a_n]) \ = \ \theta_A \otimes [a|a_1|\cdots|a_n]. \tag{7}$$

An *Eilenberg-Zilber contraction* is a contraction from $(L \times L')_N$ to $L_N \otimes L'_N$, where L and L' are augmented simplicial modules [**37**]. The subscript N means normalization in the simplicial structure. At least, one Eilenberg-Zilber contraction exists:

Theorem 3.1. *[13] Let L and L' be two augmented simplicial Λ-modules. The Alexander-Whitney operator $AW : (L \times L')_N \to L_N \otimes L'_N$, the Eilenberg-MacLane operator $EML : L_N \otimes L'_N \to (L \times L')_N$ and the Shih operator (of degree $+1$) $SHI : (L \times L')_N \to (L \times L')_N$ of L and L' are defined by the following formulas:*

$$AW(x_n, y_n) = \sum_{i=0}^{n} \partial_{i+1} \ldots \partial_n x_n \otimes \partial_0 \ldots \partial_{i-1} y_n, \tag{8}$$

$$EML(x_p \otimes y_q) = \sum_{(\alpha,\beta)\in\{(p,q)-\mathit{shuffles}\}} (-1)^{\sigma(\alpha,\beta)}\ (s_{\beta_q} \ldots s_{\beta_1} x_p, s_{\alpha_p} \ldots s_{\alpha_1} y_q) \tag{9}$$

$$\begin{aligned} SHI(x_n, y_n) &= \\ &= \sum (-1)^{n-p-q+\sigma(\alpha,\beta)} (s_{\beta_q+n-p-q} \ldots s_{\beta_1+n-p-q} s_{n-p-q-1} \partial_{n-q+1} \ldots \partial_n x_n, \\ &\qquad s_{\alpha_{p+1}+n-p-q} \ldots s_{\alpha_1+n-p-q} \partial_{n-p-q} \ldots \partial_{n-q-1} y_n), \end{aligned}$$

where the last sum is taken over all the indices $0 \leqslant q \leqslant n-1$, $0 \leqslant p \leqslant n-q-1$, $(\alpha, \beta) \in \{(p+1, q) - \mathit{shuffles}\}$ and $\sigma(\alpha, \beta) = \sum [\alpha_i - (i-1)]$.

Then, the data

$$EZ_{L,L'} : \{(L \times L')_N,\ L_N \otimes L'_N,\ AW_{L,L'},\ EML_{L,L'},\ SHI_{L,L'}\}$$

defines a contraction.

The first definition of the SHI operator was given in an inductive way in [**15**]. The explicit formula above for the Shih operator is given by J. Rubio in [**48**]. Rubio discovered this formula experimentally by means of a lisp program constructed together with F. Sergeraert. An appendix in this paper, written by F. Morace, is devoted to proving that this explicit formula satisfies the inductive definition given in [**15**]. Having a combinatorial formulation for this operator has been essential for obtaining in [**17**] a simplicial description for cup-i products, which open a door to an extremely general computational treatment of Steenrod cohomology operations.

Let X and Y be two simplicial sets. Using the last contraction, we construct

$$EZ_{X,Y} : \{C_N(X \times Y),\ C_N(X) \otimes C_N(Y),\ AW_{X,Y},\ EML_{X,Y},\ SHI_{X,Y}\},$$

where $C_N(K)$ denotes the normalized chain complex of a simplicial set K.

Several basic and known results ([**36**], [**19**]) about constructions of contractions will be used in this paper. We record that if $r : \{N, M, f, g, \phi\}$ is a contraction, the suspension version $S(r) : \{S(N),\ S(M),\ S(f),\ S(g),\ S(\phi)\}$ and the contraction $\bar{r} : \{\overline{N},\ \overline{M},\ \bar{f},\ \bar{g},\ \bar{\phi}\}$ can be formed. If we have two contractions

$$r_i : \{N_i,\ M_i,\ f_i,\ g_i,\ \phi_i\} \quad i = 1, 2$$

1. and $N_2 = M_1$, the following *composition contraction* can be constructed:

$$r_2 r_1 : \{N_1, M_2,\ f_2 f_1,\ g_1 g_2,\ \phi_1 + g_1 \phi_2 f_1\}$$

2. the *tensor product contraction* can be constructed:

$$r_1 \otimes r_2 : \{N_1 \otimes N_2,\ M_1 \otimes M_2, f_1 \otimes f_2,\ g_1 \otimes g_2,\ \phi_1 \otimes g_2 f_2 + 1_{M_1} \otimes \phi_2\}. \tag{10}$$

Notation 3.2. Of course, if we have a contraction $r : \{N, M, f, g, \phi\}$, it is possible to form:

$$r^{\otimes n} = r \overset{n \text{ times}}{\otimes \ldots \otimes} r : \{N^{\otimes n},\ M^{\otimes n}\ , f^{\otimes n}, g^{\otimes n}, \phi^{[r,n]}\}$$

where its homotopy operator $\phi^{[r,n]} : N^{\otimes n} \to N^{\otimes n}$ is described by

$$\phi^{[r,n]} = \sum_{i=0}^{n-1} \phi^{[r,n,i]} \tag{11}$$

and $\phi^{[r,n,i]} = 1_N^{\otimes i} \otimes \phi \otimes (gf)^{\otimes(n-i-1)}$; in the case $i = 0$, the morphism $1_N^{\otimes i}$ is understood to be the identity 1_Λ, while, for $i = n-1$, $(gf)^{\otimes n-i-1}$ similarly designates the identity 1_Λ.

Let us observe the similarity between this notation $\phi^{[r,n]}$ and $f^{[n]}$ defined in (1). The difference lies in the fact that the dependency of the morphism with regard to the contraction r is indicated in the notation $\phi^{[r,n]}$, whereas the morphism $f^{[n]}$ exclusively depend on f.

Then, using this notation, the *tensor module contraction* of r will be $T(r) : \{T(N), T(M), T(f), T(g), T(\phi)\}$, where $T(f)$, $T(g)$ and $T(\phi)$ are defined in each degree by:

$$T(f)_n = f^{\otimes n}; \quad T(g)_n = g^{\otimes n} \quad \text{and} \quad T(\phi)_n = \phi^{[r,n]}.$$

The Basic Perturbation Lemma (for its genesis, see [**14, 8, 52, 9, 18**]; with regard to later developments see [**27, 36, 28, 19, 29, 20, 34, 35, 32, 21, 55, 26**]) is a systematic and efficient technique for transferring structures from one object to another up to homotopy. This "algebraic fixed point theorem" [**7**] is a powerful device for obtaining chain complexes that represent a given homotopy type.

*Remark 3.3. Historical introductions to HPT can be found in the papers [**29**] and [**20**]. We consider that the germ of this idea of differential perturbation is in [**14**]. In that paper, Eilenberg and Mac Lane defined the notion of contraction and a preliminary and special version of BPL is in Th. 12.1, pages 82-83.*

First we recall the concept of perturbation datum. Let M be a graded module N and let $f : N \to N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all x, being a non-null element of N, a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DGA-module* N is a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\epsilon_N \delta = 0$. For instance, if A is a DGA-algebra, the simplicial differential d_s is a perturbation of $T(S(\overline{A}))$. A *perturbation datum of the contraction* $r : \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N, which satisfies that the composition $\phi\delta$ is pointwise nilpotent.

Theorem 3.4. *(***Basic Perturbation Lemma***) [**52**] Let $r : \{N, M, f, g, \phi\}$ be a contraction and $\delta : N \to N$ a perturbation datum of this contraction. Then a new contraction*

$$r_\delta : \{(N, d_N + \delta, \epsilon_N, \eta_N),\ (M, d_M + d_\delta, \epsilon_M, \eta_M),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

is defined by the following formulas:

$$d_\delta = f\delta\Sigma_r^\delta g; \tag{12}$$

$$f_\delta = f(1 - \delta\Sigma_r^\delta\phi); \tag{13}$$

$$g_\delta = \Sigma_r^\delta g; \tag{14}$$

$$\phi_\delta = \Sigma_r^\delta \phi; \tag{15}$$

where

$$\Sigma_r^\delta \ = \ \sum_{i\geqslant 0}(-1)^i\ (\phi\delta)^i \ = \ 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots$$

Let us note that $\Sigma_r^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DGA-module $(M, d_M, \epsilon_M, \eta_M)$. From now on, the Basic Perturbation Lemma will be called BPL.

For later references, we shall state two special cases.

Proposition 3.5. *[**19**] Under the conditions of Theorem 3.4:*

1. *If the projection f verifies $f\delta\phi = 0$, then $f_\delta = f$ and $d_\delta = f\delta g$.*
2. *If the inclusion g verifies $\phi\delta g = 0$, then $g_\delta = g$ and $d_\delta = f\delta g$.*

It is well known that BPL has a good behaviour when we use classical algebraic constructions.

Proposition 3.6. *([**20**], [**29**]) Let $r : \{N, M, f, g, \phi\}$ be a contraction and let δ, δ' and $\delta + \delta'$ be perturbation data of r. Then, the following commutativity properties are satisfied:*

$$\overline{r_\delta} = \bar{r}_{\bar{\delta}}; \tag{16}$$

$$S(r_\delta) = [S(r)]_{S(\delta)}; \tag{17}$$

$$(r \otimes r)_{(\delta\otimes 1 + 1\otimes\delta')} = r_\delta \otimes r_{\delta'}; \tag{18}$$

$$T(r_\delta) \;=\; [T(r)]_{\delta[\,]}; \tag{19}$$

$$(r_\delta)_{\delta'} \;=\; (r_{\delta'})_\delta \;=\; r_{(\delta+\delta')}. \tag{20}$$

4. Perturbation machinery for algebra and coalgebra categories

In [**20**] and [**29**] it is shown that BPL accepts algebra or coalgebra data without a change in the formulas, provided that the initial contraction satisfies a strong compatibility condition with respect to the underlying structures. In this section, we assume weaker conditions on the morphisms of the initial contraction such that BPL produces a new contraction between two DGA-algebras. The main motivation for doing this is the fact that the (Co)Algebra Perturbation Lemma of [**20**] and [**29**] cannot be applied to several important cases in Homological Algebra and Algebraic Topology (see Section 5).

In this section, we will set up perturbation theorems for DGA-algebras and DGA-coalgebras and give the details for DGA-algebras.

Definition 4.1. Let A and A' be two DGA-algebras and let $r : \{A, A', f, g, \phi\}$ be a contraction. We say that r is an *algebra contraction* if f or g are morphisms of DGA-algebras.

We hereby define the notion of algebra homotopy.

Definition 4.2. *([**19**])* Let A and A' be two DGA-algebras and let $r : \{A, A', f, g, \phi\}$ be a contraction. The homotopy ϕ is *an algebra homotopy* if

$$\mu_A \phi^{[r,2]} = \phi\mu_A.$$

An easy way for determining that a homotopy operator of one contraction is not a (co)algebra homotopy is provided by the following lemma. Its proof follows straight from condition **(c5)** in (5).

Lemma 4.3. *Let $r : \{A, A', f, g, \phi\}$ be an algebra contraction. If the composite*

$$\phi\mu_A\phi^{[r,2]}$$

is not identical to zero, then ϕ is not an algebra homotopy.

Now, we define two notions that will allow us to distinguish algebra contractions.

Definition 4.4. Let $r : \{A, A', f, g, \phi\}$ be an algebra contraction. We say that ϕ is *a quasi algebra homotopy* if the following conditions hold:

$$\begin{aligned} \phi\mu_A(\phi\otimes\phi) &= 0,\\ \phi\mu_A(g\otimes\phi) &= 0,\\ \phi\mu_A(\phi\otimes g) &= 0. \end{aligned} \tag{21}$$

We say that f is a *quasi algebra projection* if the following conditions hold:

$$\begin{aligned} f\mu_A(\phi\otimes\phi) &= 0,\\ f\mu_A(g\otimes\phi) &= 0,\\ f\mu_A(\phi\otimes g) &= 0. \end{aligned} \tag{22}$$

Let us note that the product of the DGA-algebra A' is not involved in the previous definitions.

We shall now characterize various classes of algebra contractions.

Definition 4.5. Let $r : \{A, A', f, g, \phi\}$ be an algebra contraction. We say that r is

- *a full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is an algebra homotopy;
- *an almost-full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is a quasi algebra homotopy;
- *a semi-full algebra contraction* if f is a quasi algebra projection, g is a morphism of DGA-algebras and ϕ is a quasi algebra homotopy.

One can see immediately that in the context of algebra contractions, the designation of full implies almost-full and that almost-full implies semi-full. Let us observe that each of these classes is closed by composition and tensor product of contractions.

Examples of full algebra contractions are given in [**20**], using the "tensor trick". On the other hand, it is very easy to find almost-full (non-full) algebra contractions in Algebraic Topology. For instance, Eilenberg and MacLane determined in [**15**] explicit algebra contractions of this type from $\bar{B}(\mathbb{Z}[\mathbb{Z}])$ to the exterior algebra $E(u, 1)$ on one generator u of degree 1, and from $\bar{B}(\mathbb{Z}[\mathbb{Z}_h])$ to the twisted tensor product $E(u, 1)\tilde{\otimes}^{\delta_h}\Gamma(v, 2)$, where $\Gamma(v, 2)$ is the polynomial power algebra on one "generator" v of degree 2 and $\delta_h(v) = h \cdot u$ is a derivation. An explicit almost-full (non-full) algebra contraction $R_{\bar{B}P}$ can also be established from $\bar{B}(P(u, 2n))$ to $E(v, 2n+1)$, where $P(u, 2n)$ is the polynomial algebra on one generator u of degree $2n$. More precisely, the component morphisms of $R_{\bar{B}P}$ are given in positive degree by:

$$f_{\bar{B}P}[r_1|\cdots|r_m] \;=\; \begin{cases} v & \text{if } m = 1 \text{ and } r_1 = 1 \\ 0 & \text{otherwise;} \end{cases}$$

$$g_{\bar{B}P}(v) \;=\; [1]$$

and

$$\phi_{\bar{B}P}[r_1|r_2|\cdots|r_m] \;=\; [1|r_1 - 1|r_2|\cdots|r_m];$$

where $[r_1|r_2|\cdots|r_m]$ denotes the element $[u^{r_1}|u^{r_2}|\cdots|u^{r_m}]$ of $\bar{B}(P(u, 2n))$.

If the DGA-algebra A is commutative, the contraction $R_{B(A)}$ (see (6)) is an almost-full algebra contraction too.

Finally, it is not difficult either to establish semi-full (non almost-full) algebra contractions. An example is the explicit algebra contraction that can be constructed from $Q_{(2)}(u, 4)\tilde{\otimes}^{\delta}E(v, 5)$ to $E(w, 9)$, where $Q_{(2)}(u, 4) = P(u, 4)/(u^2)$ is the truncated polynomial algebra on one generator u of degree 4 and $\delta(v) = u$. In the following section, we will also deal with several important semi-full algebra contractions that will appear in a natural way in the context of the homology theory of commutative DGA-algebras.

The following proposition gives us a more concise definition of full-algebra contraction.

Proposition 4.6. *An algebra contraction $r : \{A, A', f, g, \phi\}$ in which ϕ is an algebra homotopy, is a full algebra contraction.*

Proof.

Suppose, for instance, that the projection f is a morphism of DGA-algebras. We will prove that g is also a morphism of DGA-algebras.

Making use of the fact that ϕ is an algebra homotopy:

$$\phi\mu_A = \mu_A\phi^{[r,2]}, \tag{23}$$

and composing (23) respectively with d_A from the left and with $d_A^{[2]}$ from the right, we obtain the following relations

$$\begin{aligned} d_A\phi\mu_A &= \mu_A d_A^{[2]}\phi^{[r,2]}, \\ \phi d_A\mu_A &= \mu_A\phi^{[r,2]}d_A^{[2]}. \end{aligned}$$

If we sum these last two equalities, we have:

$$(d_A\phi + \phi d_A)\mu_A = \mu_A(d_A^{[2]}\phi^{[r,2]} + \phi^{[r,2]}d_A^{[2]}) \tag{24}$$

Plugging the rules **(c2)** of (5) for the contractions r and $r \otimes r$ into (24), we get

$$(1 - gf)\mu_A = \mu_A(1_{A^{\otimes 2}} - g^{\otimes 2}f^{\otimes 2})$$

or, simplifying

$$gf\mu_A = \mu_A g^{\otimes 2}f^{\otimes 2}. \tag{25}$$

Taking into account that f is a morphism of DGA-algebras, (25) can be rewritten as

$$g\mu_B f^{\otimes 2} = \mu_A g^{\otimes 2}f^{\otimes 2}.$$

Thus, since the morphism f is onto, the preceeding equality is equivalent to the following one:

$$g\mu_B = \mu_A g^{\otimes 2}.$$

Hence, we have deduced that r is a full algebra contraction.

In an analogous way, we can prove that if g is a morphism of DGA-algebras, then f must also be a morphism of DGA-algebras.

□

The next result describes the failure of compatibility of the component morphisms of a contraction from a DGA-algebra to a simple DGA-module with respect to the "products".

Proposition 4.7. *Let A be a DGA-algebra, M a DGA-module and $r : \{A, M, f, g, \phi\}$ a contraction. We have the following equalities:*

$$f\mu_A - (f\mu_A g^{\otimes 2})f^{\otimes 2}) \;=\; d_M(f\mu_A\phi^{[r,2]}) + (f\mu_A\phi^{[r,2]})d_A^{[2]}, \tag{26}$$

$$\mu_A g^{\otimes 2} - g(f\mu_A g^{\otimes 2}) \;=\; d_A(\phi\mu_A g^{\otimes 2}) + (\phi\mu_A g^{\otimes 2})d_A, \tag{27}$$

$$\phi\mu_A - \mu_A\phi^{[r,2]} \;=\; \phi\mu_A\phi^{[r,2]}d_A^{[2]} - d_A\phi\mu_A\phi^{[r,2]} - gf\mu_A\phi^{[r,2]}. \tag{28}$$

Proof.

We will only verify here the equality (28), since (26) and (27) can be obtained in an analogous way.

First, we use the property **(c2)** of the contraction $r^{\otimes 2}$:

$$1_A^{\otimes 2} - g^{\otimes 2}f^{\otimes 2} \;=\; d_A^{[2]}\phi^{[r,2]} + \phi^{[r,2]}d_A^{[2]}. \tag{29}$$

Composing (29) with μ_A from the left and using the facts that g is multiplicative and μ_A commutes with the differential d_A, we obtain:

$$\mu_A - g\mu_{A'} f^{\otimes 2} \;=\; d_A\mu_A\phi^{[r,2]} + \mu_A\phi^{[r,2]} d_A^{[2]}. \tag{30}$$

Composing again (30) with ϕ from the left and using the annihilation property **(c3)** of r, we have

$$\phi\mu_A \;=\; \phi d_A\mu_A\phi^{[r,2]} + \phi\mu_A\phi^{[r,2]} d_A^{[2]}. \tag{31}$$

Now we apply the property **(c2)** of r to the second term of (31)

$$\phi\mu_A \;=\; (1_A - gf - d_A\phi)\mu_A\phi^{[r,2]} + \phi\mu_A\phi^{[r,2]} d_A^{[2]}. \tag{32}$$

Finally, (32) can be rewritten in the following form

$$\phi\mu_A - \mu_A\phi^{[r,2]} \;=\; \phi\mu_A\phi^{[r,2]} d_A^{[2]} - d_A\phi\mu_A\phi^{[r,2]} - gf\mu_A\phi^{[r,2]}. \tag{33}$$

□

To distinguish almost-full algebra contractions from full algebra contractions, we can use the following result, whose proof is left to the reader.

Proposition 4.8. *Let $r : \{A, A', f, g, \phi\}$ be an almost-full algebra contraction. It is a full algebra contraction if and only if $\phi\mu_A(1\otimes\phi) \;=\; 0$.*

We also leave the proof of the following result to the reader.

Proposition 4.9. *Let $r : \{A, A', f, g, \phi\}$ be a semi-full algebra contraction. It is an almost-full algebra contraction if and only if $f\mu_A(1\otimes\phi) = 0$.*

The following proposition states that a semi-full algebra contraction is an algebra contraction in which Ker ϕ is a sub-algebra of A.

Proposition 4.10. *Let $r : \{A, A', f, g, \phi\}$ be a semi-full algebra contraction. Let u and v be two elements of* Ker ϕ*. Then*

$$f\mu_A(u\otimes v) \;=\; \mu_{A'}(f\otimes f)(u\otimes v) \tag{34}$$

and

$$\phi\mu_A(u\otimes v) \;=\; \mu_A\phi^{[r,2]}(u\otimes v). \tag{35}$$

Proof.
We will only prove here the equality (35), since (34) can be demonstrated in an analogous way.

We shall consider here the equality (28). Since ϕ is a quasi algebra homotopy and f is a quasi algebra projection, the second term of (28) turns out to be:

$$\varphi \;=\; \phi\mu_A(1_A\otimes\phi)d_A^{[2]} - d_A\phi\mu_A(1_A\otimes\phi) - gf\mu_A(1_A\otimes\phi).$$

If u and v are elements of Ker $\phi = \mathrm{Im}\; g \oplus \mathrm{Im}\; \phi$, the relation $\varphi(u\otimes v) = 0$ is immediately proved.

Hence, for these particular arguments, we get (35).

□

In order to determine the pairs of elements for which the projection and the homotopy operator of a semi-full algebra contraction become multiplicative, we may refine the last result.

Proposition 4.11. *Let $r : \{A, A', f, g, \phi\}$ be a semi-full algebra contraction. Let u and v be two elements of A. If u or v are in* $\operatorname{Im} g$, *then*

$$f\mu_A(u \otimes v) \;=\; \mu_{A'}(f \otimes f)(u \otimes v) \tag{36}$$

and

$$\phi\mu_A(u \otimes v) \;=\; \mu_A\phi^{[r,2]}(u \otimes v). \tag{37}$$

We also leave the proof of the following result to the reader. In spite of its simplicity, this proposition will be essential in the proof of our main theorems.

Proposition 4.12. *Let A be a DGA-algebra, M a DGA-module and $r : \{A, M, f, g, \phi\}$ a contraction. Let us suppose that*

$$\begin{array}{c}\phi\mu_A g^{\otimes 2} \;=\; 0,\\ (\textit{resp. } f\mu_A\phi^{[r,2]} \;=\; 0).\end{array}$$

Then the morphism $\upsilon_M = f\mu_A g^{\otimes 2}$ defines a product on M (η_M being its unit) and r is an algebra contraction from $(A, d_A, \mu_A, \epsilon_A, \eta_A)$ to $(M, d_M, \upsilon_M, \epsilon_M, \eta_M)$ with multiplicative inclusion (resp. multiplicative projection).

It is time to introduce the notion of algebra perturbation datum.

Definition 4.13. *[20]* Let A and A' be two DGA-algebras and let $r : \{A, A', f, g, \phi\}$ be a contraction. An *algebra perturbation datum* δ of this contraction is a perturbation datum of r which is also a derivation.

At this point, it is natural to pose this question: if we have an algebra contraction $r : \{A, A', f, g, \phi\}$ and an algebra perturbation datum δ of r, under which conditions can we state that the composition

$$\upsilon_{A'} = f_\delta\mu_A(g_\delta)^{\otimes 2} \tag{38}$$

on A', that is defined by transferring the product on A to A' by means of f_δ and g_δ, is an actual product on $(A')_\delta$?.

In [**20**] and [**29**], the perturbation problem for algebras is solved assuming that the homotopy of the contraction is an algebra homotopy. In this way, they establish a preservation result of (co)algebra **laws**. The following theorem for DGA-algebras (that we will call Full Algebra Perturbation Lemma or, more briefly, F-APL) is a translation, in our language, of this result.

Theorem 4.14 (F-APL). *[20] Let $r : \{A, A', f, g, \phi\}$ be a full algebra contraction and δ an algebra perturbation datum of r.*

Then the contraction r_δ obtained by applying BPL (Th. 3.4) is a full algebra contraction from $(A, d_A + \delta, \mu_A, \epsilon_A, \eta_A)$ to $(A', d_{A'} + d_\delta, \mu_{A'}, \epsilon_{A'}, \eta_{A'})$. Moreover, the perturbation d_δ for the DGA-module A' is a derivation.

The preceding theorem tells us that the class of full algebra contractions is closed by perturbation. The goal of this section and the main aim of this paper is *to establish perturbation machineries for other classes of algebra contractions*. The idea is to *generalize* the following proposition (that we will call Special Algebra Perturbation Lemma or S-APL) which is an enriched version of Proposition 3.5:

Proposition 4.15 (S-APL). *[**19**] Let A and A' be two DGA-algebras, $r : \{A, A', f, g, \phi\}$ a contraction and δ an algebra perturbation datum of r. If f (resp. g) is a morphism of DGA-algebras and*

$$\phi\delta g = 0 \quad (\textit{resp. } f\delta\phi = 0) \tag{39}$$

then the morphism $g_\delta = g$ (resp. $f_\delta = f$) of the contraction r_δ, obtained by applying BPL, is, consequentely, a morphism of DGA-algebras and the perturbation $d_\delta = f\delta g$ obtained in A' is a derivation.

In order to obtain perturbation results *preserving the DGA-algebra category*, we use Proposition 4.12, taking as datum the perturbed contraction r_δ. In this way, we state the following result (that we will call General(inclusion) Algebra Perturbation Lemma or, more briefly *GI*-APL) that guarantees the preservation of the strict associativity in a perturbed contraction.

Theorem 4.16 (*GI*-APL). *Let A and A' be two DGA-algebras, $r : \{A, A', f, g, \phi\}$ a contraction and δ an algebra perturbation datum of r.*

Let us assume that g is a morphism of DGA-algebras and

$$\phi\mu_A(\phi^{[r,2]}\delta^{[2]})^n g^{\otimes 2} \;=\; 0, \qquad \forall n \geqslant 1. \tag{40}$$

Then the inclusion g_δ of r_δ, obtained by applying BPL, is a morphism of DGA-algebras from $(A', d_{A'} + d_\delta, \upsilon_{A'}, \epsilon_{A'}, \eta_{A'})$ to $(A, d_A + \delta, \mu_A, \epsilon_A, \eta_A)$, where the product $\upsilon_{A'}$ is given by:

$$\begin{aligned} \upsilon_{A'} &= f\mu_A(g_\delta)^{\otimes 2} \\ &= f\mu_A[\textstyle\sum_{m\geqslant 0}(-1)^m(\phi^{[r,2]}\delta^{[2]})^m g^{\otimes 2}]. \end{aligned} \tag{41}$$

Moreover, if we assume that f is also a morphism of DGA-algebras, the product $\upsilon_{A'}$ coincides with the initial product $\mu_{A'}$ of A'.

Proof.

Considering Prop. 3.6 (18), we have

$$(g_\delta)^{\otimes 2} \;=\; (g^{\otimes 2})_{\delta^{[2]}} \;=\; \sum_{n\geqslant 0}(-1)^n\ (\phi^{[r,2]}\delta^{[2]})^n g^{\otimes 2}.$$

Hence, condition (40) and the fact that g is a morphism of DGA-algebras imply that $\phi\mu_A(g_\delta)^{\otimes 2} = 0$, and consequently, $\phi_\delta\mu_A(g_\delta)^{\otimes 2} = 0$. If we now apply Prop. 4.12 to the contraction r_δ, we endow A' with an algebra structure

$$\begin{aligned} \upsilon_{A'} &= f_\delta\mu_A(g_\delta)^{\otimes 2} \\ &= (f - f\delta\Sigma_r^\delta\phi)\mu_A(g^{\otimes 2})_{\delta^{[2]}} \\ &= f\mu_A[\textstyle\sum_{m\geqslant 0}(-1)^m(\phi^{[r,2]}\delta^{[2]})^m]g^{\otimes 2}. \end{aligned}$$

Furthermore, the contraction r_δ is an algebra contraction from $(A, d_A + \delta, \mu_A, \epsilon_A, \eta_A)$ to $(A', d_{A'} + d_\delta, \upsilon_{A'}, \epsilon_{A'}, \eta_{A'})$ with multiplicative inclusion.

If f is also a morphism of DGA-algebras, and making use of the properties **(c1)** and **(c4)** of the contraction r, we get

$$\begin{array}{rcl} \upsilon_{A'} & = & f\mu_A[\sum_{m\geqslant 0}(-1)^m(\phi^{[r,2]}\delta^{[2]})^m]g^{\otimes 2} \\ & = & \mu_A f^{\otimes 2}[\sum_{m\geqslant 0}(-1)^m(\phi^{[r,2]}\delta^{[2]})^m]g^{\otimes 2} \\ & = & \mu_{A'} f^{\otimes 2} g^{\otimes 2} \\ & = & \mu_{A'}. \end{array}$$

The previous result was already enunciated in [**45**].

We now give a similar result to the preceding one without proof.

Theorem 4.17 (*GP*-APL). *Let A and A' be two DGA-algebras, $r : \{A, A', f, g, \phi\}$ an algebra contraction and δ an algebra perturbation datum of r.*

Let us suppose that f is a morphism of DGA-algebras and

$$f(\delta\phi)^n \;\; \mu_A \;\; \phi^{[r,2]} \; = \; 0, \qquad \forall\, n \;\geqslant\; 1. \tag{42}$$

Then the projection f_δ of the contraction r_δ, obtained by applying BPL, is a morphism of DGA-algebras from $(A,\; d_A + \delta,\; \mu_A,\; \epsilon_A,\; \eta_A)$ to $(A',\; d_{A'} + d_\delta,\; \upsilon_{A'},\; \epsilon_{A'},\; \eta_{A'})$, where the product $\upsilon_{A'}$ is given by the following formula:

$$\upsilon_{A'} \; = \; f_\delta \;\; \mu_A \;\; g^{\otimes 2}. \tag{43}$$

Moreover, if we assume that g is also a morphism of DGA-algebras, the product $\upsilon_{A'}$ coincides with the initial product $\mu_{A'}$ of A'.

The last theorems can be considered as respective generalizations of S-APL Theorem. On the other hand, when we work with a full algebra contraction as initial input, GI-APL and GP-APL theorems only tell us that the perturbed morphims f_δ and g_δ are morphisms of DGA-algebras, giving us less information than F-APL Theorem.

Let us note that, in general, **the perturbation datum δ is involved in an essential way in the last perturbation results**. However, if we take contractions in which its homotopy operator is a quasi-algebra homotopy, we can prove the following perturbation result:

Theorem 4.18 (*SF*-APL). *Let $r : \{A, A', f, g, \phi\}$ be a semi-full algebra contraction and δ an algebra perturbation datum of r. Then the perturbed contraction r_δ is a semi-full algebra contraction from $(A, d_A + \delta, \mu_A, \epsilon_A, \eta_A)$ to $(A', d_{A'} + d_\delta, \mu_{A'}, \epsilon_{A'}, \eta_{A'})$.*

Proof.

Taking into account the formulas of the morphisms of the perturbed contraction r_δ, we see directly that f_δ and ϕ_δ are, respectively, a quasi algebra projection and a quasi algebra homotopy.

With regard to g_δ, taking into account that ϕ is a quasi algebra homotopy and g is a morphism of DGA-algebras, we obtain

$$\phi\mu_A(g_\delta \otimes g_\delta) \; = \; 0. \tag{44}$$

Now, by GI-APL (Th. 4.16), we determine that g_δ is a morphism of DGA-algebras from $(A', d_{A'} + d_\delta, \upsilon_{A'}, \epsilon_{A'}, \eta_{A'})$ to $(A, d_A + \delta, \mu_A, \epsilon_A, \eta_A)$, where the product $\upsilon_{A'}$ is given by (41):

$$\begin{array}{rcl} \upsilon_{A'} & = & f\mu_A(g_\delta \otimes g_\delta) \\ & = & f\mu_A[(\sum_{i\geqslant 0}\,(-1)^i(\phi\delta)^i g) \otimes (\sum_{i\geqslant 0}\,(-1)^j(\phi\delta)^j g)]. \end{array} \tag{45}$$

Taking into account that f is a quasi algebra projection, g is a morphism of DGA-algebras and the property **(c1)** of the contraction r, the formula (45) turns out to be:

$$\upsilon_{A'} \;=\; f\mu_A(g \otimes g) \;=\; fg\mu_{A'} \;=\; \mu_{A'}.$$

□

Hence, semi-fullness in algebra contractions is a hereditary property under homological perturbation. It is important to note that the perturbed differential d_δ is completely determined by its effect on the generators of the algebra A', due to the fact that this morphism is a derivation. This represents a substantial improvement in a later homology computation.

The development of the ideas above for DGA-coalgebras is analogous. Note, however, that some care must be taken in establishing the hypotheses of the coalgebra perturbation results. In this way, GI-CPL, GP-CPL and SF-CPL Theorems can be enunciated.

5. Applications

In this section, we will put forward several examples in which F-A(C)PL cannot be applied, but the hypotheses of SF-A(C)PL or GI-A(C)PL Theorems are verified.

In order to understand the importance of these applications, it is convenient to define our notion of homological model for a commutative DGA-algebra.

Definition 5.1. A *n-homological model* ($n \geqslant 0$) for a commutative DGA-algebra A is a pair (H, r), such that H is a commutative DGA-algebra that is free and of finite type as graded module and r is a semi-full algebra contraction from $\bar{B}^n(A)$ to H.

A more restrictive definition of homological model will be given later, working over $\mathbb{Z}$ localized at a prime.

Let us note that, if the ground ring is $\mathbb{Z}$ or $\mathbb{Z}$ localized at a prime p, the condition of *formality* for a commutative augmented differential graded algebra [**4, 16, 25, 21**] is stronger than that of the mere existence of a small computable homological model.

In this section, we give four applications of the perturbation result SF-APL for semi-full algebra contractions. The first one deals with the transference of the Hopf algebra structure in the contraction $\bar{B}(r)$, which is constructed, via perturbation, using a contraction r from a commutative DGA-algebra A to a simple DGA-module M. The contraction $\bar{B}(r)$ is established from the reduced bar construction of A to the bar tilde construction of M [**53, 23**]. The second one concludes that the contraction given by Eilenberg and Mac Lane in [**15**] from the reduced bar construction of a tensor product of two commutative DGA-algebras to the tensor product of reduced bar constructions of both DGA-algebras is an almost-full algebra contraction. The third one describes p-local homological models of Cartan's elementary complexes. The paper [**6**] makes use of these last results in order to progress in the computation of p-local homological models of TTPs of Cartan's elementary complexes. Taking into account the results of this section, the p-local n-homology theory of commutative DGA-algebras is appropriately developed in terms of semi-full algebra contractions in [**1**]. This application is also an essential step for considering the p-local homology algebra of Eilenberg-Mac Lane spaces from the point of view of Homological Perturbation Theory (see [**3**]). This last study assists us in the search of homological models for some fibre bundles in Topology [**2**].

To extend these positive results to a broader class of spaces is an interesting question. This could be available, for example, by suitably modifying our perturbation technique for DG Hopf algebras. Using a closely related perturbation machinery, Saneblidze constructs a model for a given DG Hopf algebra A [**50**] (with not necessarily commutative product) by the homology $H(A)$. He takes any multiplicative bigraded resolution of $H(A)$, he fix on it an induced coproduct (not necessarily coassociative) from $H(A)$ using the standard Adams-Hilton argument (see [**29**, Th. 3.4]), and then he perturbs simultaneously the resolution differential (compare [**25**]) and the coproduct. This, at least, gives a homotopy Hopf algebra (in the sense of Anick [**5**]) model for A.

Finally, we here analyze the p-local n-homology algebra of an extremely concrete class of commutative DGA-algebras.

Definition 5.2. A *connected minimal Koszul-Sullivan (K-S) algebra of finite type* (or, simply, *minimal K-S algebra*) is a free commutative DG-algebra $(\Lambda M, d)$ where M is the free graded module generated by a strictly positive graded set of finite type $X = \sum_{n>0} X_n$ and

(KS.1) The set X is indexed by a well ordered set I, such that $|x_i| \leqslant |x_j|$ if $i < j$.

(KS.2) $d(X) \subset (\Lambda^+ M)(\Lambda^+ M)$.

In [**24**], the notion of K-S complex is defined. In fact, the differential d of the previous definition satisfies the "nilpotence" and minimality conditions required for K-S complexes of [**24**] and the unique difference is limited to the degree of d that is here -1 (see also [**56**]). We establish in the last subsection that the p-local n-homology algebra of a minimal K-S algebra $(\Lambda M, d)$ can be represented by a homological model that is a (non-twisted) tensor product of Cartan's elementary complexes. More precisely, in the case $n = 1$, we derive that the homological model H is a banal tensor product of exterior and divided power algebras. That is to say, H is the 1-homology algebra of ΛM and, therefore, the DGA-algebra $\bar{B}(\Lambda M)$ is $\mathbb{Z}$-formal.

5.1. Application 1: Hopf algebra structures on the bar construction contraction

Let $r : \{A, A', f, g, \phi\}$ be a contraction where A and A' are two commutative DGA-algebras. We shall here study the multiplicative behavior of the contraction $\bar{B}(r)$ with respect to the shuffle product $*$. Let us recall that the bar construction of a commutative DGA-algebra A is a Hopf DGA-algebra denoted by $(\bar{B}(A),\ d_{\bar{B}},\ *,\ \Delta_{\bar{B}},\ \epsilon_{\bar{B}},\ \eta_{\bar{B}})$.

We begin with the following result:

Theorem 5.3. *[**20**] Let A and M be a DGA-algebra and a DGA-module respectively, and $r : \{A, M, f, g, \phi\}$ be a contraction. Hence, the following full coalgebra contraction can be established:*

$$\bar{B}(r) = [T(S(\bar{r})]_{d_s^A} : \{\bar{B}(A), \tilde{B}(M), \bar{B}(f), \bar{B}(g), \bar{B}(\phi)\} \tag{46}$$

*where d_s^A is the simplicial differential of $\bar{B}(A)$ and the DG-module $\tilde{B}(M)$ is the bar tilde construction of Stasheff [**53**].*

From the contraction $\bar{B}(r)$, it is possible to determine the operations $m_i : M^{\otimes i} \to M$ $(i \geqslant 1)$ of the A_∞-algebra on M (see [**23**]).

In the next result, we state the compatibility of the bar construction contraction and the perturbation machinery.

Theorem 5.4. *Let $r : \{A,\ M,\ f,\ g,\ \phi\}$ be a contraction where A and M are a DGA-algebra and a DGA-module, respectively. If δ is an algebra perturbation datum of r, then the following commutativity property is satisfied:*

$$\bar{B}(r_\delta) \ = \ [\bar{B}(r)]_{S(\bar{\delta})^{[\,]}}$$

Proof.

First, by applying Prop. 5.3 to r_δ, we get the perturbed contraction $\bar{B}(r_\delta) = [T(S(\overline{r_\delta})]_{d_s^A}$.

On the other hand, using the product of A, we define the following morphism of DG-modules of degree -1:

$$\tilde{S}(\overline{\mu_A}) : S(\bar{A}) \otimes S(\bar{A}) \longrightarrow S(\bar{A})$$

defined by $\tilde{S}(\overline{\mu_A})(Sa' \otimes Sa'') = (-1)^{|Sa'|}\mu_A(Sa' \otimes Sa'')$.

It is easy to verify that the simplicial differential d_s^A of $\bar{B}(A)$ is the morphism $\tilde{S}(\overline{\mu_A})^{[\,]}$. Since δ is a derivation, then

$$S(\bar{\delta})^{[\,]}\tilde{S}(\overline{\mu_A})^{[\,]} + \tilde{S}(\overline{\mu_A})^{[\,]}S(\bar{\delta})^{[\,]} = 0.$$

Finally, taking into account the good behaviour of BPL in (16), (17), (19) and (20), we have:

$$\begin{aligned}\bar{B}(r_\delta) &= [T(S(\overline{r_\delta}))]_{d_s}\\ &= [T(S(\bar{r}_{\bar{\delta}}))]_{d_s}\\ &= [T(S(\bar{r})_{S(\bar{\delta})})]_{d_s}\\ &= [T(S(\bar{r}))_{S(\bar{\delta})^{[\,]}}]_{d_s}\\ &= [T(S(\bar{r}))_{d_s}]_{S(\bar{\delta})^{[\,]}}\\ &= \bar{B}(r)_{S(\bar{\delta})^{[\,]}}. \qquad \square\end{aligned}$$

□

If $M = A'$ is a DGA-algebra and r is an algebra contraction, then the contraction $\bar{B}(r)$ "connects" two bar constructions (see [**20**]). For example, if g is a morphism of DGA-algebras, then $\bar{B}(r)$ is a contraction from $\bar{B}(A)$ to $\bar{B}(A')$ and its inclusion $\bar{B}(g)$ coincides with $T(S(\bar{g}))$. Since this morphism preserves shuffle products, $\bar{B}(r)$ is an algebra contraction with multiplicative inclusion. Without making use of any assumption, we demonstrate here that $\bar{B}(g)$ always preserves shuffle products.

Theorem 5.5. *Let $r : \{A, M, f, g, \phi\}$ be a contraction where A is a commutative DGA-algebra and M is a DGA-module. Then*

$$\bar{B}(r) : \{\bar{B}(A), \tilde{B}(M), \bar{B}(f), \bar{B}(g), \bar{B}(\phi)\}$$

is a semi-full algebra contraction.

Proof.

It is known (see [**20**]) that $\bar{B}(r)$ is a full coalgebra contraction. It suffices to prove that the inclusion of $\bar{B}(r)$ is multiplicative with respect to the shuffle product.

First, both differentials d_t^A and d_s^A are $*$-derivations. Secondly, $T(S(\bar{r}))$ is an almost-full algebra contraction. Indeed, it is obvious that both $T(S(\bar{f}))$ and $T(S(\bar{g}))$ are compatible with the shuffle products. It is also a simple exercise to prove that $T(S(\bar{\phi}))$ is a quasi algebra homotopy. Moreover,

$$T(S(\bar{\phi})) * T(S(\bar{\phi}))^{[T(S(\bar{r})),2]}([Sa] \otimes [Sb]) \neq 0$$

for $a, b \in \bar{A}$ with $\phi(a) \neq 0 \neq \phi(b)$. Hence, it follows from Proposition 4.8 that $T(S(\bar{\phi}))$ is not an algebra homotopy.

With these ingredients, the result directly follows from SF-APL.

□

Remark 5.6. *Of course, if r is a semi-full algebra contraction then $\bar{B}(r)$ is a semi-full algebra contraction from $\bar{B}(A)$ to $\bar{B}(M)$.*

From the theorem above, we have in particular that the differential of $\tilde{B}(M)$ is a derivation. This fact means that $\tilde{B}(M)$ is a DGA Hopf algebra and leads us to the notion of commutative A_∞-algebra introduced by Kadeishvili in [**31**].

Corollary 5.7. *Let $r : \{A, M, f, g, \phi\}$ be a contraction where A is a commutative DGA-algebra and M is a DGA-module. Let $(m_1, m_2, m_3, \dots)$ be the A_∞-algebra structure on M derived from the algebra structure on A via the contraction $\bar{B}(r)$ (see [**23**]). If $i \geqslant 2$ and a and b are two elements of $\tilde{B}(M)$, with $|a|_s + |b|_s = i$, being $|a|, |b| \geqslant 1$ then*

$$m_i(a * b) = 0, \tag{47}$$

where the shuffle product $a * b$ is interpreted having its image on M^i.

5.2. Application 2: A particular Eilenberg-Zilber contraction

A contraction $R_{\bar{B}\otimes}$, described by Eilenberg-MacLane in [**15**], from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A')$, where A and A' are two commutative DGA-algebras, is recorded here. Let us note that the reduced bar construction $\bar{B}(A)$ of a commutative DGA-algebra A has a DGA-algebra structure with regard to the shuffle product $*$. In this section, we show that the contraction $R_{\bar{B}\otimes}$ is an almost-full algebra contraction. Moreover, taking this contraction as initial datum, if we perturb the differential of $A \otimes A'$ or we modify the product of $A \otimes A'$ in such a way that we still have a new commutative DGA-algebra, we determine the (co)multiplicative behaviour of the perturbed contraction by making use of SF-APL and GI-CPL Theorems.

The study done in this section can be generalized in a natural way to the contraction from $\bar{B}(\otimes_{i=1}^n A_i)$ to $\otimes_{i=1}^n \bar{B}(A_i)$, for $n = 3, 4, \dots$.

Theorem 5.8. *[**15**, pp. 59-60] If A and A' are two commutative DGA-algebras, then there is an algebra contraction $R_{\bar{B}\otimes}$ with multiplicative projection and inclusion from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A')$.*

$R_{\bar{B}\otimes}$ is also a coalgebra contraction with comultiplicative inclusion.

Eilenberg and Mac Lane in [**14**, pp. 75-76] assign to each commutative DGA-algebra A, a construction $\mathcal{B}(A)$ that may be regarded, in particular, as an augmented simplicial Λ-algebra (that we will call $B^{smp}(A)$) and, hence, they establish the normalized construction $B_N^{smp}(A)$. This last construction endowed with the degree, differential, product and coproduct induced by $\mathcal{B}(A)$ coincides with our definition of bar construction $\bar{B}(A)$. Eilenberg and Mac Lane exploit the Eilenberg-Zilber Theorem (Th. 3.1) and they establish a contraction $EZ_{\bar{B}^{smp}(A), \bar{B}^{smp}(A')}$ from $(\bar{B}^{smp}(A) \times \bar{B}^{smp}(A'))_N$ to $\bar{B}_N^{smp}(A) \otimes \bar{B}_N^{smp}(A')$. The desired contraction

$$R_{\bar{B}\otimes} : \{\bar{B}(A \otimes A'),\ \bar{B}(A) \otimes \bar{B}(A'),\ f_{\bar{B}\otimes},\ g_{\bar{B}\otimes},\ \phi_{\bar{B}\otimes}\}.$$

is obtained by composing the preceding contraction with iso-contractions which amount largely to sign changes.

Let us recall the explicit formulas of the projection and inclusion of this contraction given by Eilenberg-Mac Lane in [**15**]:

- $f_{\bar{B}\otimes}[a_1 \otimes a'_1 | \cdots | a_n \otimes a'_n]$
$$= \sum_{i=0}^{n} \xi_A(\mu_A(a_{i+1} \cdots a_n)) \xi_{A'}(\mu_{A'}(a'_1 \cdots a'_i)) [a_1 | \cdots | a_i] \otimes [a'_{i+1} | \cdots | a'_n] \,.$$
- $g_{\bar{B}\otimes}([a_1 | \cdots | a_n] \otimes [a'_1 | \cdots | a'_m])$
$$= [a_1 \otimes \theta_{A'} | \cdots | a_n \otimes \theta_{A'}] * [\theta_A \otimes a'_1 | \cdots | \theta_A \otimes a'_n] \,.$$

The formula of the homotopy operator is:

$$\phi_{\bar{B}\otimes} \;=\; \varsigma^{-1}\ SHI_{\bar{B}^{smp}A, \bar{B}^{smp}A'}\ \varsigma.$$

The morphism $SHI_{\bar{B}^{smp}A,\bar{B}^{smp}A'}$ is the homotopy operator of the contraction $EZ_{\bar{B}^{smp}A,\bar{B}^{smp}A'}$ which is explicitly described in Theorem 3.1 (face and degeneracy operators of $\bar{B}^{smp}A$ are defined in [**14**, p. 75]), and the isomorphism of DG-modules $\varsigma : \bar{B}(A\otimes A') \to (\bar{B}^{smp}(A)\times \bar{B}^{smp}(A'))_N$ is defined in [**15**, p. 60].

Hence, the image $\phi_{\bar{B}\otimes}[a_1\otimes a'_1|\cdots|a_n\otimes a'_n]$ is a sum taken over all $(p+1,q)$-shuffles (with $0\leqslant p\leqslant n-q-1\leqslant n-1$) of elements of this kind (up to sign):

$$\begin{aligned}\epsilon_A(\mu_A(a_{n-q+1}\cdots a_n))[a_1\otimes a'_1|\cdots|a_{n-p-q-1}\otimes a'_{n-p-q-1}|\mu_{A'}(a'_{n-p-q}\cdots a'_{n-q})|\\ |([a_{n-p-q}|\cdots|a_{n-q}]*[a'_{n-q+1}|\cdots|a'_n])].\end{aligned}$$

Now, it is easy to see that $\phi_{\bar{B}\otimes}$ is not an algebra homotopy.

As counterexample, we take $u=[a\otimes a']$ and $u'=[b\otimes b']$, where $a,b\in\overline{A}$, $a',b'\in\overline{A'}$ and $|x|>0,\quad \text{for}\quad x=a,b,a',b'$.

On one hand, we have,

$$\phi_{\bar{B}\otimes}(u*u')=-[a\otimes a'|b'|b]+[\mu_{A'}(a',b')|a|b]+[b\otimes b'|a'|a]-[\mu_{A'}(a',b')|b|a];$$

On the other hand, we get

$$u*\phi_{\bar{B}\otimes}(u')+\phi_{\bar{B}\otimes}(u)*g_{\bar{B}\otimes}f_{\bar{B}\otimes}(u')=[b'|b|a\otimes a']-[b'|a\otimes a'|b]+[a\otimes a'|b'|b].$$

In addition, the morphism $\phi_{\bar{B}\otimes}$ is not a coalgebra homotopy. In fact, the projection $f_{\bar{B}\otimes}$ is not a morphism of DGA-coalgebras. If we take $[a|a']\in\bar{B}(A\otimes A')$, where a and a' are elements of $\overline{A}$ and $\overline{A'}$ respectively, we have

$$\begin{aligned}(f_{\bar{B}\otimes}\otimes f_{\bar{B}\otimes})\Delta_{\bar{B}\otimes}[a|a'] \quad = \quad &([\,]\otimes[\,])\otimes([a]\otimes[a'])+([a]\otimes[\,])\otimes([\,]\otimes[a'])+\\ &+([a]\otimes[a'])\otimes([\,]\otimes[\,])\end{aligned}$$

and on the other hand,

$$\begin{aligned}&(1\otimes T\otimes 1)(\Delta_{\bar{B}}\otimes\Delta_{\bar{B}})\ f_{\bar{B}\otimes}[a|a']=\\ &\qquad([\,]\otimes[a'])\otimes([a]\otimes[\,])+([a]\otimes[\,])\otimes([\,]\otimes[a'])+\\ &\qquad+([\,]\otimes[\,])\otimes([a]\otimes[a'])+([a]\otimes[a'])\otimes([\,]\otimes[\,])\end{aligned}$$

Since $g_{\bar{B}\otimes}$ is a morphism of DGA-coalgebras and taking into account Proposition 4.6, we deduce that $\phi_{\bar{B}\otimes}$ is not a coalgebra homotopy.

The rest of this subsection is devoted to proving that $R_{\bar{B}\otimes}$ is an almost-full algebra contraction.

First, we introduce the following definitions:

Definition 5.9. Let A and A' be two DGA-algebras. An element a of $\bar{A}$ (resp. a' of $\bar{A}'$) can be regarded as an element of the tensor product $A\otimes A'$ by the morphism of DGA-algebras:

$$\begin{gathered}a\mapsto c=a\otimes\theta_{A'},\\ (\text{resp. } a'\mapsto c'=\theta_A\otimes a').\end{gathered}$$

The elements of $c\in A\otimes A'$ (resp. $c'\in A\otimes A'$) which are obtained in this way are called *simple* elements of $A\otimes A'$.

Remark 5.10. *We can consider the DGA-algebras A and A' as sub-DG-algebras of $A\otimes A'$. In the same way, the constructions $\bar{B}(A)$ and $\bar{B}(A')$ can be viewed as sub-DG-algebras of*

$\bar{B}(A \otimes A')$ *and of* $\bar{B}(A) \otimes \bar{B}(A')$*. From now on, we will make use of these identifications when they may be considered appropriate.*

Definition 5.11. Let A and A' be two DGA-algebras. Let $u = [c_1|\cdots|c_m]$ be an element of $\bar{B}(A \otimes A')$. We say that u is *simple* if for all $1 \leqslant i \leqslant m$, c_i is a simple element of $A \otimes A'$. In this case, this element *has s inversions* if there are exactly s elements $c_{r_i} \in \overline{A'}$ with $i = 1, \dots, s$ and $r_1 < r_2 < \dots < r_s$, such that $c_{r_s+1} \in \overline{A}$. The graded module generated by the simple elements of $\bar{B}(A \otimes A')$ will be denoted by $S\bar{B}(A \otimes A')$.

Of course, if u is a simple element of Ker $f_{\bar{B}\otimes}$, this element has at least 1 inversion.

Definition 5.12. Let A and A' be two DGA-algebras. Let $u = [c_1|\cdots|c_m]$ be an element of $\bar{B}(A \otimes A')$. We say that u is 1-*simple* if there is only one subscript $k \in \{1, \dots, m\}$ such that c_i is simple, for $i \neq k$, and $c_k \in \overline{A} \otimes \overline{A'}$. In this case, this element *has $s+1$ inversions* if there are exactly s elements $c_{r_i} \in \overline{A'}$ with $i = 1, \dots, m$ and $r_1 < r_2 < \dots < r_s$, such that $c_{r_s+1} \in \overline{A}$ or $c_{r_s+1} = c_k \in \overline{A} \otimes \overline{A'}$. The graded module generated by the 1-simple elements of $\bar{B}(A \otimes A')$ will be denoted by $S1\bar{B}(A \otimes A')$.

Definition 5.13. Let A and A' be two DGA-algebras. Let $u = [c_1|\cdots|c_m]$ be an element of $\bar{B}(A \otimes A')$. We say that u *finishes with an inversion* if there is a subscript $k \in \{1, \dots, m\}$ such that $[c_k|c_{k+1}|\cdots|c_m]$ is an element of $S\bar{B}(A \otimes A')$ having one inversion. The graded module spanned by these elements will be denoted by $I\bar{B}(A \otimes A')$.

The proof of the following proposition follows from the explicit definition of the morphisms $f_{\bar{B}\otimes}$, $g_{\bar{B}\otimes}$ and $\phi_{\bar{B}\otimes}$ of the contraction $R_{\bar{B}\otimes}$.

Proposition 5.14. *Let A and A' be two commutative DGA-algebras. Then,*

- **a)** $S1\bar{B}(A \otimes A') \subset$ Ker $f_{\bar{B}\otimes}$.
- **b)** Im $g_{\bar{B}\otimes} \subset S\bar{B}(A \otimes A') \subset$ Ker $\phi_{\bar{B}\otimes}$.
- **c)** $S\bar{B}(A \otimes A') * S\bar{B}(A \otimes A') \subset S\bar{B}(A \otimes A')$.
- **d)** $\Delta_{\bar{B}}(S\bar{B}(A \otimes A')) \subset S\bar{B}(A \otimes A') \otimes S\bar{B}(A \otimes A')$.
- **e)** *The homotopy operator $\phi_{\bar{B}\otimes}$ carries 1-simple elements with s inversions to zero or to sums of simple elements of* Ker $f_{\bar{B}\otimes}$ *with at least s inversions.*
- **f)** $I\bar{B}(A \otimes A') =$ Im $\phi_{\bar{B}\otimes} \subset$ Ker $\phi_{\bar{B}\otimes}$.

From the proposition above, it is easy to obtain:

Theorem 5.15.

(1) The contraction $R_{\bar{B}\otimes}$ is an almost-full algebra contraction.

(2) The contraction $R_{\bar{B}\otimes}$ is a coalgebra contraction with comultiplicative inclusion. Neither is $f_{\bar{B}\otimes}$ a quasi coalgebra projection nor is $\phi_{\bar{B}\otimes}$ a quasi coalgebra homotopy.

The following result directly follows from SF-APC and Theorem 5.15 (1).

Corollary 5.16. *Let A and A' be two commutative DGA-algebras and δ an algebra perturbation datum of the contraction $R_{\bar{B}\otimes}$ from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A)$. Then the perturbed contraction $(R_{\bar{B}\otimes})_\delta$ is a semi-full algebra contraction.*

Now, we define two special types of perturbation data of the contraction $R_{\bar{B}\otimes}$.

Definition 5.17. Let A and A' be two commutative DGA-algebras and δ a perturbation datum of the contraction $R_{\bar{B}\otimes}$. We say that δ is a *simple* (resp. *1-simple*) perturbation datum if it is a derivation, a coderivation and carries simple elements into simple or 1-simple (resp. into 1-simple) elements.

The perturbations for $R_{\bar{B}\otimes}$ arising from perturbations of the differential of $A\otimes A'$ (for examples, PTTs of algebras) or from "modifications" of the product of $A\otimes A'$ so that the new modified tensor product is still a commutative DGA-algebra, are simple perturbation data. We will find specific perturbation data of this type in the next section.

The use of the coalgebra perturbation result GI-CPL is essential in the following theorem.

Theorem 5.18. *Let A and A' be two commutative DGA-algebras. Assume that δ is a simple perturbation datum of $R_{\bar{B}\otimes}$.*

Then the inclusion of the perturbed contraction $(R_{\bar{B}\otimes})_\delta$ is a morphism of Hopf DGA-algebras from

$$(\bar{B}(A)\otimes\bar{B}(A'), d_{\bar{B}}^{[2]}+d_\delta, (*_A\otimes *_{A'})(1\otimes T\otimes 1), \tilde{\Delta}_{\otimes\bar{B}}, \epsilon_{\bar{B}}^{\otimes 2}, \eta_{\bar{B}}^{\otimes 2})$$

to

$$(\bar{B}(A\otimes A'), d_{\bar{B}}+\delta, *_{A\otimes A'}, \Delta_{\bar{B}}, \epsilon_{\bar{B}}, \eta_{\bar{B}}),$$

where

$$\tilde{\Delta}_{\otimes\bar{B}} \;=\; (f_{\bar{B}\otimes}\otimes f_{\bar{B}\otimes})\Delta_{\bar{B}}(g_{\bar{B}\otimes})_\delta.$$

Moreover, the perturbation d_δ over $\bar{B}(A)\otimes\bar{B}(A')$ is a derivation and a coderivation.

Proof.

With regard to algebra structures, all have been proved in Theorem 5.16 which is valid for every algebra perturbation datum δ.

To complete the proof, it suffices to verify that the following condition holds:

$$(1\otimes\phi_{\bar{B}\otimes}+\phi_{\bar{B}\otimes}\otimes g_{\bar{B}\otimes}f_{\bar{B}\otimes})\Delta_{\bar{B}}(\phi_{\bar{B}\otimes}\delta)^n g_{\bar{B}\otimes} \;=\; 0, \qquad \forall\, n\geqslant 0. \tag{48}$$

First, by Prop. 5.14 **b)** we have that $\mathrm{Im} g_{\bar{B}\otimes}\subset S\bar{B}(A\otimes A')$. From the fact that δ is a simple perturbation datum and Prop. 5.14 **e)**, the composite $(\phi_{\bar{B}\otimes}\delta)^n$ carries any element of $S\bar{B}(A\otimes A')$ into an element of this same graded module. Finally, (48) follows from Prop. 5.14 **d)** and **b)**.

Thus, we are able to apply *GI*-CPL and this proves the theorem.

□

5.3. Application 3: p-minimal homological models of Cartan's elementary complexes

Let us recall that in 1954, Henri Cartan [**10**] determined *the integer homology algebra of Eilenberg-MacLane spaces*. To do this, he introduced the notion of *construction* and several homological operations (suspension, transpotence and divided powers) constructing, thanks to them, tensor products of a certain number of "elementary complexes", one for each appropriated homological operation (admissible words). The homology of this tensor product reduced by several identifications gave the desired integer homology of an Eilenberg-MacLane space. On the other hand, an extensive study of the p-local (p prime) homology of Cartan's little constructions was given by John C. Moore in [**40**]. A short account of the work of Moore on this subject is given in [**54**].

As a third illustration, taking $\Lambda=\mathbb{Z}_{(p)}$ ($\mathbb{Z}$ localized at a prime p) as the ground ring, we here show a method for representing the homology algebras of Cartan's elementary complexes in terms of semi-full algebra contractions.

We must point out that the relation of the methods of the Séminaire Cartan to Homological Perturbation Theory is studied in [**33**], [**34**] and [**35**], in which it is proved that Cartan's little constructions can be considered as small resolutions which split off of the bar resolution. In these papers, a constructive version of the comparison theorem for resolutions is used for establishing a splitting (contraction) of the bar resolution. Here, we use the techniques of Eilenberg and Mac Lane in [**14**] and [**15**] for obtaining homological information of Cartan's elementary complexes, under the form of semi-full algebra contractions. Thanks to Theorem 3.3 of [**6**], it is possible to translate all this information to resolutions in an appropriate way. In our process, a case appears in which SF-APL will be the unique algebra perturbation theorem that could be applied. In the method developed by Cartan, the generators of the elementary algebras appearing in this iterative process admit an interpretation in terms of homological operations. Here, we maintain this notation in order to provide a comparison with Cartan's method. Finally, let us note that this work is extremely important in the design of algorithms, based on perturbation techniques, for computing the homology of simplicial fibre bundles in Topology [**2**] or the n-homology (with $n \geqslant 2$) of commutative DGA-algebras [**1, 11**].

We begin with the following definitions:

Definition 5.19. *[10]* Let A be a commutative DGA-algebra. We define the following morphisms of graded modules:

- The suspension,

$$\sigma : A \to \bar{B}(A) \text{ defined by}$$
$$\sigma(a) = [a], \text{ where } a \in A.$$

- The p-transpotence, where p is a prime number,

$$\varphi_p : A \to \bar{B}(A), \text{ defined by}$$
$$\varphi_p(a) = [a|a^{p-1}] \text{ where } a \in \text{A}.$$

Let us notice that Cartan defines the additive functions suspension and transpotence with values in $H_*(\bar{B}(A))$, whereas here we define analogous functions from a DGA-algebra A to its reduced bar construction.

Now we shall define Cartan's elementary complexes.

Definition 5.20. *[10]* Let p be a prime number. An *elementary complex* is a DGA-algebra of the form:

- Type I: an exterior algebra $E(u, 2n-1)$;
- Type II: a divided power algebra $\Gamma(u, 2n)$;
- Type III: a twisted tensor product $E(u, 2n-1)\tilde{\otimes}^{\delta_{\pm p^r}}\Gamma(v, 2n)$;
- Type IV: a twisted tensor product $\Gamma(u, 2n)\tilde{\otimes}^{\delta_{\pm p^r}}E(v, 2n+1)$;

where $n \geqslant 1$ and the derivation $\delta_{\pm p^r}$ is given by the formula

$$\delta_{\pm p^r}(v) \ = \ \pm p^r u, \qquad r \geqslant 1.$$

Definition 5.21. *([29])* Let p be a prime number and let $\Lambda = \mathbb{Z}_{(p)}$ be our ground ring. A DGA-module $(M, d_M, \epsilon_M, \eta_M)$ is called *p-minimal* if it is free, of finite type as a graded module and $d_M(M) \subset p \cdot M$.

Definition 5.22. Let p be a prime number and let $\Lambda = \mathbb{Z}_{(p)}$ be our ground ring. We say that H is a *p-minimal n-homological model* of a commutative DGA-algebra A if there is an algebra contraction from the iterated bar construction $\bar{B}^n(A)$ to the p-minimal DGA-algebra H.

Working over $\mathbb{Z}_{(p)}$, our goal in this subsection is to obtain by perturbation semi-full algebra contractions from bar constructions of Cartan's elementary complexes to p-minimal DGA-algebras.

A first result on the homology of elementary complexes is given by Eilenberg-MacLane:

Theorem 5.23. *[15] Let Λ be a commutative ring with $1 \neq 0$ and let n be a natural number. There is an isomorphism of Hopf DGA-algebras between the bar construction of an exterior algebra $\bar{B}(E(u, 2n-1))$ and the divided power algebra $\Gamma(\sigma(u), 2n)$.*

In an obvious way, the last isomorphism can be considered as a full algebra contraction:

$$R_{\bar{B}I} : \{\overline{B}(E(u, 2n-1)),\ \Gamma(\sigma(u), 2n),\ f_{\bar{B}I},\ g_{\bar{B}I},\ 0\} \tag{49}$$

where

$$f_{\bar{B}I}([u| \overset{k \text{ times}}{\cdots} |u]) = \gamma_k(\sigma(u));$$

and

$$g_{\bar{B}I}(\gamma_k(\sigma(u)) = [u| \overset{k \text{ times}}{\cdots} |u].$$

From now on, in order to compare later our method with the one developed by Cartan, we will denote the generators of the small DGA-algebras of the contractions that may appear, by making use of the suspension, transpotence and divided power morphisms . For example, the generator v of the divided power algebra in the last contraction has been denoted by $\sigma(u)$. The notation $\sigma(u)$ means that $g_{\bar{B}I}(v) = \sigma(u)$.

Now, it is about to find an explicit contraction for the bar construction of a divided power algebra (complex of type II). This case is quite complicated. This was the principal problem Eilenberg and MacLane encountered when studying the homology of $K(\pi, n)$s.

Here, we can solve this question if we work with coefficients in $\mathbb{Z}_{(p)}$ (p prime). We will only deal with odd primes p. For $p = 2$, similar results can be obtained. In the sequel of this subsection, we will take as the ground ring $\mathbb{Z}_{(p)}$.

First, we establish the following result

Proposition 5.24. *Let n and p be a natural number and a prime number, respectively. There is an isomorphism of DGA-algebras between the divided power algebra $\Gamma(u, 2n)$ and*

$$\tilde{\otimes}_{i \geqslant 0} Q_{(p)}(u_i, 2np^i).$$

As $\mathbb{Z}_{(p)}$-module, this last DGA-algebra is equal to the ordinary tensor product $\otimes_{i \geqslant 0} Q_{(p)}(u_i, 2np^i)$. Its multiplicative law is

$$u_i^k\, u_j^h = \begin{cases} u_i^k \otimes u_j^h & \text{if } i \neq j, \\ u_i^{k+h} & \text{if } i = j \text{ and } k+h < p, \\ -pu_i^t u_{i+1} & \text{if } i = j \text{ and } k+h = p+t. \end{cases} \tag{50}$$

Proof.

We can consider the above isomorphism as a full algebra contraction:

$$R_\Gamma : \{\Gamma(u, 2n),\ \tilde{\otimes}_{i \geqslant 0} Q_{(p)}(u_i, 2np^i),\ f_\Gamma,\ g_\Gamma,\ 0\} \tag{51}$$

We take the function $S_p(n) = \frac{p^n - 1}{p-1}$, $\forall n = 0, 1, 2, \ldots$. The explicit morphisms of R_Γ are defined by:

$$f_\Gamma : \Gamma(u, 2n) \to \tilde{\otimes}_{i \geqslant 0} Q_{(p)}(u_i, 2np^i)$$

$$f_\Gamma(\gamma_k(u)) = \frac{(-p)^{\sum_{i=1}^{r} k_i S_p(i)}}{k!}\, u_0^{k_0} \ldots u_r^{k_r},$$

where $k = k_0 + k_1 p + \ldots + k_r p^r$ $(0 \leqslant k_i < p)$ is the p-adic development of k, and

$$g_\Gamma : \tilde{\otimes}_{i \geqslant 0} Q_{(p)}(u_i, 2np^i) \to P(u, 2n)$$

$$g_\Gamma(u_n^k) = \frac{(-1)^{nk} k(kp^n - 1)!}{p^{kS_p(n)-n}} \gamma_{kp^n}(u), \quad 0 \leqslant k < p.$$

Notice that the numbers $\dfrac{(-p)^{\sum_{i=1}^r k_i S_p(i)}}{k!}$ and $\dfrac{(-1)^{nk} k(kp^n - 1)!}{p^{kS_p(n)-n}}$ are invertible elements in $\mathbb{Z}_{(p)}$.

□

Remark 5.25. *From now on, we will identify the generators u_i of the truncated algebras with the elements $\gamma_{p^i}(u)$ of $\Gamma(u, 2n)$; in fact, $g_\Gamma(u_i)$ coincides with $\gamma_{p^i}(u)$, excluding the coefficient.*

This simplification of the DGA-algebra $\Gamma(u, 2n)$ tells us that knowing the homology of the truncated algebras is essential for obtaining by perturbation the homology algebra of a divided power algebra. In order to get an explicit contraction for the bar construction of a truncated algebra, we have slightly modified an argument used by Eilenberg-MacLane in [**15**].

Proposition 5.26. *Let n and p be a natural number and an odd prime number, respectively. There is an almost-full algebra contraction:*

$$R_{BQ} : \{\bar{B}(Q_{(p)}(u, 2n)), \;\; E(\sigma(u), 2n+1) \otimes P(\varphi_p(u), 2np+2), \\ f_{BQ}, g_{BQ}, \phi_{BQ}\} \tag{52}$$

Proof.

We denote an element of $\bar{B}(Q_{(p)}(u, 2n))$ with the form $[u^{r_1}|\ldots|u^{r_m}]$ by $[r_1|\ldots|r_m]$, where $0 \leqslant r_i < p$.

The explicit morphisms of R_{BQ} are the following:

$$f_{BQ}[r_1|t_1|\ldots|r_m|t_m] = \{\textstyle\prod_{k=1}^n \delta_{p, r_k + t_k}\}\gamma_m(\varphi_p(u)),$$

$$f_{BQ}[r_1|t_1|\ldots|r_m|t_m|l] = \delta_{1,l}\{\textstyle\prod_{k=1}^n \delta_{p, r_k + t_k}\}\sigma(u) \cdot \gamma_m(\varphi_p(u)),$$

where the symbols $\delta_{i,j}$ are defined by:

$$\delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

The morphism $g_{BQ} : E(\sigma(u), 2n+1) \otimes \Gamma(\varphi_p(u), 2np+2) \to \bar{B}(Q_{(p)}(u, 2n))$ is defined over the generators as follows:

$$g_{BQ}(\sigma(u)) = [1],$$

$$g_{BQ}(\gamma_k(\varphi_p(u))) = [1|p-1| \overset{\text{k times}}{\cdots} |1|p-1].$$

The homotopy operator ϕ_{BQ} is defined by:

$$\phi_{BQ}1 = 0; \qquad \phi_{BQ}[1] = 0;$$

$$\phi_{BQ}[x] = -[1|x-1] \quad 1 < x < p;$$

$$\phi_{BQ}[x|y] = -[1|x-1|y];$$

$$\phi_{BQ}[x|y|z] = -[1|x-1|y|z] - \delta_{p,x+y}[1|p-1|\phi(z)]$$

where $z \in \bar{B}(Q_{(p)}(u, 2n))$.

Finally, proving the almost-fullness of this algebra contraction is a simple exercise that it is left to reader.

□

Remark 5.27. *For $p = 2$, the truncated algebra $Q_{(2)}(u, 2n)$ coincides with $E(u, 2n)$ as DGA-algebras.*

We are ready to establish:

Theorem 5.28. *Let n and p be a natural number and an odd prime number, respectively. There is a semi-full algebra contraction $R_{\bar{B}II}$ from $\overline{B}(\Gamma(u, 2n))$ to the DGA-algebra*

$$E(\sigma(u), 2n+1) \otimes (\otimes_{i\geqslant 1}[E(\sigma\gamma_{p^i}(u), 2np^i + 1)\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(u), 2np^i + 2)])$$

Proof.

For the sake of clarity, we will write the DGA-algebras without noting the degree of the generators.

Proposition 5.24 tells us that there is an isomorphism of DGA-algebras between $\Gamma(u)$ and a tensor product $\tilde{\otimes}_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u))$ (note that we are using the identifications of (5.25)). This then enables us to state that the respective bar constructions of these two algebras are isomorphics too. This isomorphism of Hopf DGA-algebras is denoted by $\bar{B}(R_\Gamma)$, where the construction $\bar{B}(\,)$ over a contraction is defined in Theorem 5.3 and the contraction R_Γ is determined by (51).

Then, for the construction of $R_{\bar{B}II}$, we need to establish a contraction for the Hopf DGA-algebra $\bar{B}(\tilde{\otimes}_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$. To this end, we first consider the ordinary tensor product, excluding the modification of the product of the truncated algebras $\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u))$ (50), and we take:

- the almost-full algebra contraction:

$$R_{\bar{B}\otimes Q} : \{\bar{B}(\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u))), \otimes_{i\geqslant 0}\bar{B}(Q_{(p)}(\gamma_{p^i}(u))), f_{\bar{B}\otimes Q},\ g_{\bar{B}\otimes Q},\ \phi_{\bar{B}\otimes Q}\},$$

  which is constructed thanks to Theorem 5.8,

- and the almost-full algebra contraction:

$$R_{\otimes\bar{B}Q} : \{\otimes_{i\geqslant 0}\bar{B}(Q_{(p)}(\gamma_{p^i}(u))),\ \otimes_{i\geqslant 0} E(\sigma\gamma_{p^i}(u)) \otimes \Gamma(\varphi_p\gamma_{p^i}(u)),\ f_{\otimes\bar{B}Q},\ g_{\otimes\bar{B}Q},\ \phi_{\otimes\bar{B}Q}\},$$

  which is constructed thanks to Proposition 5.26 and (10). In fact, $R_{\otimes\bar{B}Q}$ is the contraction $\otimes_{i\geqslant 0}R^i_{BQ}$, where

$$R^i_{BQ} : \{\bar{B}(Q_{(p)}(\gamma_{p^i}(u)),\ E(\sigma\gamma_{p^i}(u)) \otimes \Gamma(\varphi_p\gamma_{p^i}(u)),\ f^i_{BQ},\ g^i_{BQ},\ \phi^i_{BQ}\}$$

  is a contraction of type (52).

It is time to apply the machinery of homological perturbation to the composition contraction $R_{BP} = R_{\otimes\bar{B}Q}R_{\bar{B}\otimes Q}$, where the perturbation datum δ is the difference between the differentials of the bar constructions $\bar{B}(\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$ and $\bar{B}(\tilde{\otimes}_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$. It is clear that δ is a derivation and represents the perturbation induced in the simplicial differential of $\bar{B}(\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$ by the modification produced in the product of the algebra $\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u))$. The projection, inclusion and homotopy operator of the contraction R_{BP} will be denoted by f_{BP}, g_{BP} and ϕ_{BP}, respectively.

From SF-APL, we directly deduce that $(R_{\bar{B}P})_\delta$ is a semi-full algebra contraction. Moreover, S-APL can be applied in this case. Our aim here is to verify the following relation:

$$\phi_{BP}\delta g_{BP} = 0,$$

or, written in a more developed form:

$$(\phi_{\bar{B}\otimes Q} + g_{\bar{B}\otimes Q}\phi_{\otimes\bar{B}Q}f_{\bar{B}\otimes Q})\,\delta\,(g_{\bar{B}\otimes Q}g_{\otimes\bar{B}Q}) = 0. \tag{53}$$

First, the composite $g_{\bar{B}\otimes Q}g_{\otimes\bar{B}Q}$ carries any generator w into an element z of the form $[1]$ or $[1|p-1]$ of $\bar{B}(Q_{(p)}(\gamma_{p^{r_i}}(u)) \subset \bar{B}(\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$. Obviously, z is a simple element of $\bar{B}(\otimes_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$.

We shall now study the image of z under δ. It is not difficult to see that this image is zero if $z = [1]$ and $[\gamma_{p^{i+1}}(u)]$ if $z = [1|p-1]$. Then the image of any generator under $\delta g_{\bar{B}\otimes Q}g_{\otimes\bar{B}Q}$ lies in $\bar{S}(\otimes_{i\geqslant 0}Q^i_{(p)})$ and, hence, in Ker $\phi_{\bar{B}\otimes Q}$ (see Prop. 5.14).

Thus, it is clear that $f_{\bar{B}\otimes Q}\delta g_{BP}(w)$ is zero or an element $[1] \in \text{Ker}\ \phi^{r_i}_{BQ} \subset \text{Ker}\ \phi_{\otimes\bar{B}Q}$. This completes the proof of (53).

Therefore, Theorem S-APL tells us that the perturbed contraction $R_{\bar{B}II} = (R_{BP})_\delta$ from $\bar{B}(\tilde{\otimes}_{i\geqslant 0}Q_{(p)}(\gamma_{p^i}(u)))$ to $E(\sigma(u)) \otimes (\otimes_{i\geqslant 1}[E(\sigma\gamma_{p^i}(u))\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(u))])$ can be constructed, where its inclusion $g_{\bar{B}II}$ is precisely $g_{BP} = g_{\bar{B}\otimes Q}g_{\otimes\bar{B}Q}$. Consequently, $g_{\bar{B}II}$ is a morphism of DGA-algebras.

We can easily check that the formula given by BPL for the twisted differential of the pairs

$$E(\sigma\gamma_{p^i}(u), 2np^i + 1)\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(u), 2np^i + 2)$$

is the following:

$$\begin{aligned} \delta_p(\gamma_k\varphi_p\gamma_{p^{i-1}}(u)) &= f_{B\Gamma}\delta g_{BP}(\gamma_k\varphi_p\gamma_{p^{i-1}}(u)) \\ &= p\,(\sigma\gamma_{p^i}(u) \otimes \gamma_{k-1}\varphi_p\gamma_{p^{i-1}}(u)), \end{aligned}$$
$$\forall k \geqslant 1 \quad \text{and} \quad \forall i \geqslant 1.$$

This yields the desired result.

□

Therefore, a complex of type II presents as p-minimal 1-homological model a tensor product of complexes of type I and III. We are now concerned with an explicit algebra contraction for the bar construction of a complex of type III.

Theorem 5.29. *Let n and p be a natural number and an odd prime number, respectively. There is a semi-full algebra contraction $R_{\bar{B}III}$ from $\bar{B}(E(u, 2n+1)\tilde{\otimes}^{\delta_{p^r}}\Gamma(v, 2(n+1))$ to the DGA-algebra*

$$\begin{gathered} [\Gamma(\sigma(u), 2(n+1))\tilde{\otimes}^{\delta_{-p^r}}E(\sigma(v), 2n+3)]\otimes \\ (\otimes_{i\geqslant 1}[E(\widetilde{\sigma\gamma_{p^i}}(v), 2(n+1)p^i + 1)\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(v), 2(n+1)p^i + 2)]). \end{gathered} \tag{54}$$

The notation $\widetilde{\sigma\gamma_{p^i}(v)}$ means that a change of basis has taken place in which the generator of the algebra $E(\sigma\gamma_{p^i}(v))$ has been modified.

Proof.

For brevity, we shall write the algebras omitting the degree of the generator.

First, we shall consider:

- the almost-full algebra contraction:

$$R^{E\Gamma}_{\bar{B}\otimes} : \{\bar{B}(E(u)\otimes\Gamma(v)), \bar{B}(E(u))\otimes\bar{B}(\Gamma(v)), f_{\bar{B}\otimes}, g_{\bar{B}\otimes}, \phi_{\bar{B}\otimes}\}$$

  which is constructed using Theorem 5.8,

- and the semi-full algebra contraction $R^{E,\Gamma}_{\otimes\bar{B}} = R_{\bar{B}I}\otimes R_{\bar{B}II}$ from $\bar{B}(E(u))\otimes\bar{B}(\Gamma(v))$ to the DGA-algebra

$$\Gamma(\sigma u)\otimes E(\sigma v)\otimes[\otimes_{i\geqslant 1}E(\sigma\gamma_{p^i}v)\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}v)].$$

  This contraction is obtained from the contraction (49) and that of Th. (5.28). We shall denote the projection, inclusion and homotopy operator of the contraction $R^{E,\Gamma}_{\otimes\bar{B}}$ by $f_{\otimes\bar{B}}$, $g_{\otimes\bar{B}}$ and $\phi_{\otimes\bar{B}}$, respectively.

Hence, we are interested in the semi-full algebra contraction $R_{BE\Gamma} = R^{E,\Gamma}_{\otimes\bar{B}}R^{E\Gamma}_{\bar{B}\otimes}$. The projection, inclusion and homotopy operator of this last contraction will be denoted as $f_{BE\Gamma}$, $g_{BE\Gamma}$ and $\phi_{BE\Gamma}$, respectively.

Here, the perturbation datum ρ for $R_{BE\Gamma}$ will be the modification generated by δ_{p^r} that is induced in the tensor differential of the bar construction $\bar{B}(E(u)\tilde{\otimes}^{\delta_{p^r}}\Gamma(v))$. The morphism ρ is a derivation because it is the difference between the differentials of the constructions $\bar{B}(E(u)\tilde{\otimes}^{\delta_{p^r}}\Gamma(v))$ and $\bar{B}(E(u)\otimes\Gamma(v))$. Additionally, it is not hard to prove the pointwise nilpotency of the composite $\phi_{BE\Gamma}\delta$.

Hence, the SF-APL perturbation machine guarantees that $(R_{BE\Gamma})_\rho = R_{BIII}$ is a semi-full algebra contraction from

$$\bar{B}(E(u)\tilde{\otimes}^{\delta_{p^r}}\Gamma(v))$$

to

$$(\Gamma(\sigma(u))\otimes E(\sigma(v))\otimes\otimes_{i\geqslant 1}[E(\sigma\gamma_{p^i}(v))\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(v))],\ \ d')$$

where d' denotes the differential obtained by perturbation.

We shall now compute the differential d' in the small DGA-algebra of the perturbed contraction R_{BIII}. First at all, we analyze the inclusion of R_{BIII}:

$$(g_{BE\Gamma})_\rho = \sum_{n\geqslant 0}(-1)^n(\phi_{BE\Gamma}\rho)^n g_{BE\Gamma}. \tag{55}$$

The morphism $g_{BE\Gamma}$ is defined by:

$$g_{BE\Gamma}\ =\ g_{\bar{B}\otimes}g_{\otimes\bar{B}}\ =\ g_{\bar{B}\otimes}(g_{\bar{B}I}\otimes g_{\bar{B}II})\ =\ g_{\bar{B}\otimes}(g_{\bar{B}I}\otimes g_{B\Gamma}).$$

The previous morphism carries any generator of the small DGA-algebra (54) to an element of $\bar{B}(E(u))$ or an element w of the form $[1]$ or $[1|p-1]$ of $\bar{B}(Q_{(p)}(\gamma_{p^i}(v)))\subset\bar{B}(\Gamma(v))\subset\bar{B}(E(u)\otimes\Gamma(v))$. Obviously, this image lies in Ker $\phi_{\bar{B}\otimes}$.

We can now concentrate our study on determining the image under ρ of an argument w of the form $[1]$ or $[1|p-k]$ (with $0<k<p$) of $\bar{B}(Q_{(p)}(\gamma_{p^i}(v)))$. This image is, in general, a sum of two 1-simples elements of $\bar{B}(E(u)\otimes\Gamma(v))$. The only term with 1 inversion in this sum is

$$\bar{w} = [u \otimes b] \bullet \bar{z} \tag{56}$$

where $b \in \Gamma(v)$ and $\bar{z}$ is $[\,]$ if $w = [1]$ or $[p-k]$ if $w = [1|p-k] \in \bar{B}(Q_{(p)}(\gamma_{p^i}(v)))$. Let us note that in the case $i = 0$, $\bar{w}$ is the simple element $[u] \bullet \bar{z}$.

Now, the homotopy operator of the contraction $R_{BE\Gamma}$ has the form:

$$\phi_{BE\Gamma} \;=\; \phi_{\bar{B}\otimes} \;+\; g_{\bar{B}\otimes}\phi_{\otimes\bar{B}} f_{\bar{B}\otimes} \;=\; \phi_{\bar{B}\otimes} \;+\; g_{\bar{B}\otimes}(1 \otimes \phi_{\bar{B}II}) f_{\bar{B}\otimes}.$$

By Proposition 5.14 **g)**, the image of a 1-simple element of $\bar{B}(E(u)\otimes\Gamma(v))$ with r inversions ($r \geqslant 2$) under $\phi_{\bar{B}\otimes}$ is zero or a sum of simple elements of Ker $f_{\bar{B}\otimes}$ with at least r inversions. It is easy to check that if we take a simple element of Ker $f_{\bar{B}\otimes}$ with r inversions ($r \geqslant 2$), the image of this element under ρ is a sum of 1-simple elements with at least r inversions or simple elements of Ker $f_{\bar{B}\otimes}$. On the other hand, $f_{\bar{B}\otimes}$ carries a 1-simple element into zero. Furthermore, from Prop. 5.14 **b)**, the morphism $\phi_{B\otimes}$ carries a simple element of Ker $f_{\bar{B}\otimes}$ into zero.

Now we pick $w' = [u \otimes b] \bullet [u| \overset{r-1\text{ times}}{\cdots} |u] \bullet z'$ with $r \geqslant 1$, $b \in \overline{\Gamma(v)}$ and $z' = [\,]$ or $z' = [h] \in \bar{B}(Q_{(p)}(\gamma_{p^i}(v)))$.

In $\phi_{\bar{B}\otimes}(w')$, the unique simple element with 1 inversion (up to sign):

$$[b] \bullet [u| \overset{r\text{ times}}{\cdots} |u] \bullet z'. \tag{57}$$

Now the morphism ρ maps an argument of the form (57) onto an element of $\bar{B}(E(u)\otimes\Gamma(v))$ which only has one term with at most 1 inversion:

$$[u \otimes b'] \bullet [u| \overset{r\text{ times}}{\cdots} |u] \bullet z', \tag{58}$$

where $\delta_{p^r}(b) = p^r\,(u \otimes b')$.

In fact, if $b = \gamma_1(v)$, the element (58) is the simple element $x = [u| \overset{r+1\text{ times}}{\cdots} |u] \bullet z'$. Since $\phi_{BI} = 0$, we determine that $\phi_{BE\Gamma}$ carries the element x into zero if $z' = [\,]$ or $z' = [1]$, and into the shuffle product $y = [u| \overset{r+1\text{ times}}{\cdots} |u] * [1|h-1]$ if $z' = [h]$ with $h > 1$. On the other hand, $f_{BE\Gamma}$ maps x to $\gamma_{r+1}\sigma(u)$ if $z' = [\,]$, to zero if $z' = [h]$, with $h > 1$ and to $\gamma_{r+1}\sigma(u)\ \otimes \sigma\gamma_{p^i}(v)$ if $z' = [1]$.

Taking into account that ρ is a derivation and Proposition 4.11 (37), we have

$$(\phi_{BE\Gamma}\rho)^n(y) \;=\; [u| \overset{r+1\text{ times}}{\cdots} |u] * (\phi_{BE\Gamma}\rho)^n([1|h-1]).$$

Summarizing, we can determine the image (up to invertible coefficient in $\mathbb{Z}_{(p)}$) of different generators of (54) under $d' = f_{\otimes B}\,\rho\,\Sigma^{\rho}\,g_{\otimes B}$:

$$d'(\sigma(v)) \;= -\,p^r\,\sigma(u); \tag{59}$$

$$d'(\sigma\gamma_{p^i}(v)) = -\,p^{r\,p^i}\gamma_{p^i}\sigma(u) \tag{60}$$

$$d'(\varphi_p\gamma_{p^{i-1}}(v)) = -\,p^{rp^{i-1}(p-1)}\gamma_{p^{i-1}(p-1)}\sigma(u)\otimes\sigma\gamma_{p^{i-1}}(v). \tag{61}$$

That is to say, a scheme of the behaviour of the differential d' could be the following,

$$d'(\Gamma(\sigma(u))) = 0;$$

$$d'(E(\sigma(v))) \subset \Gamma(\sigma(u)); \tag{62}$$

$$d'(E(\sigma\gamma_{p^i}(v))) \subset \Gamma(\sigma(u)); \tag{63}$$

$$d'(\Gamma(\varphi_p\gamma_{p^{i-1}}(v))) \subset \Gamma(\sigma(u))\otimes E(\sigma\gamma_{p^{i-1}}(v)). \tag{64}$$

We can see that (62) corresponds to (59), (63) corresponds to (60), and (64) corresponds to (61).

Now (63) and (64) can be eliminated, making a basis change. This basis change can be seen as a full algebra contraction R_{norm} (with homotopy operator zero) from

$$([\Gamma(\sigma(u)) \otimes E(\sigma(v))] \otimes (\otimes_{i \geqslant 1}[E(\sigma\gamma_{p^i}(v))\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^i}(v))]), d'),$$

to

$$(\Gamma(\sigma(u))\tilde{\otimes}^{\delta_{-p^r}} E(\sigma(v))) \otimes (\otimes_{i \geqslant 1}[E(\widetilde{\sigma\gamma_{p^i}(v)})\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^i}(v))]).$$

As we have mention before, the notation $\widetilde{\sigma\gamma_{p^i}(v)}$ means that a change of basis has been carried out in which the generator of the algebra $E(\sigma\gamma_{p^i}(v))$ has been modified.

In fact, this process of "normalization" was determined by Moore [**40**] in his study of the p-adic homology of the Eilenberg-MacLane spaces.

The expression (62) coincides with the derivation δ_{-p^r}. In the same way, the differential δ_p of the couples $E(\sigma\gamma_{p^i}(v))\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(v))$ remains unchanged too.

Therefore, the desired contraction is:

$$R_{BIII} = R_{norm}(R_{BE\Gamma})_\rho.$$

□

Thus, an elementary complex of type III has a tensor product of elementary complexes of type III and IV as p-minimal 1-homological model.

Proceeding as we did in the previous proof, we obtain an explicit semi-full algebra contraction for an elementary complex of type IV. In this case, it is possible to apply S-APL Theorem. Its p-minimal 1-homological model will be a tensor product of elementary complexes of type III. We leave the details to the reader.

Theorem 5.30. *Let n and p be a natural number and an odd prime number, respectively. There is a semi-full algebra contraction $R_{\bar{B}IV}$ from $\bar{B}(\Gamma(u, 2n)\tilde{\otimes}^{\delta_{p^r}} E(v, 2n + 1))$ to*

$$[E(\sigma(u), 2n + 1)\tilde{\otimes}^{\delta_{-p^r}}\Gamma(\sigma(v), 2n + 2)]\otimes$$
$$(\otimes_{i \geqslant 1}[E(\sigma\gamma_{p^i}(u), 2np^i + 1)\tilde{\otimes}^{\delta_p}\Gamma(\varphi_p\gamma_{p^{i-1}}(u), 2np^i + 2)])$$

Remark 5.31. *For $p = 2$, all the previous results are valid, with the sole modification in their statements of changing the symbol φ_p for the composition $\gamma_2\sigma$ and the inclusions of contractions which are, in this case, morphisms of Hopf DGA-algebras.*

The behavior of elementary complexes of any type when we apply the bar construction to them, corresponds to symbol rules for its generators; that is to say, as we have denoted the generators in relation to the operations of suspension σ, transpotence ϕ_p and k-th divided power γ_k, these generators are in one-to-one correspondence with "admissible words" in the alphabet composed by the indicated three letters. These results agree with those described by Cartan (see [**10**]).

We are able to translate the Cartan-Moore approach (for obtaining the homology algebras of Eilenberg-Mac Lane spaces) to the homological perturbation framework. Working over $\mathbb{Z}$ localized at a prime p, we here obtain explicit semi-full algebra contractions from bar constructions of Cartan's elementary complexes to tensor products of these same DGA-algebras. On the other hand, an old conjecture established by Eilenberg-Mac Lane (see [**14**]) can be immediately solved by means of Homological Perturbation Theory [**57**]. An explicit contraction R_{WB} from $C_*^N(\bar{W}(G))$ to $\bar{B}(C_*^N(G))$ is described in that paper, where $C_*^N(\ )$ means the normalized chain complex canonically associated to a simplicial set and $\bar{W}(G)$ is the classifying simplicial set of the simplicial group G. It is proved in [**3**] that R_{WB} is a semi-full algebra contraction. Appropriately combining this result with those showed in this section, it is possible to determine the homology algebra of the $K(\pi, n)$ by semi-full algebra contractions from $C_*^N(K(\pi, n))$ to (non-twisted) tensor product of Cartan's elementary complexes. That is, we have a p-minimal homological model of these simplicial groups. From these data, it is very simple to design algorithms, via perturbation, for computing the p-local homology of some fibre bundles (see [**2**]). That work is influenced by the organization of effective homology versions of various important spectral sequences given in [**51, 48, 49, 45**]. Significant improvements of these algorithms can be done whenever we deal with twisted cartesian products of simplicial groups which are themselves simplicial groups. These improvements rest on the fact that the Eilenberg-Zilber contraction (Th. 3.1), in the case of working with simplicial groups, is an almost-full algebra contraction [**3**].

Finally, the results obtained in this subsection tell us that it is possible to "control" the differential structure of the p-minimal 1-homological models of these particular algebras, in the sense that the small complexes are simple tensor products of Cartan's elementary complexes. The problem of generalizing these results to a large set of twisted tensor products (TTP) of exterior and divided power algebras is studied in [**6**]. In that paper, given a twisted tensor product A of n (exterior and divided power) DGA-algebras, it is established that a tensor product of TTPs of i (exterior and divided power) DGA-algebras, with $i \leqslant n$, is a p-minimal 1-homological model of A. In this way, we have a phenomenon of controllability of the differential for this kind of algebras.

Passing from reduced complexes to resolutions can be done using Theorem 8.1.3 of [**34**], but there the multiplicative structures are not studied in the commutative case. Using SF-APL, one can directly prove the following result affecting to Cartan's little constructions.

Theorem 5.32. *Let $\mathbb{Z}_{(p)}$ be the ground ring. Let A be a Cartan's elementary complex. There is a free resolution $A \otimes^{\rho'} H$ that splits off of the bar resolution. More precisely, the splitting is a semi-full algebra contraction.*

The reduced complex H is determined by one of the Theorems 5.23, 5.28, 5.29 or 5.30. A generalization of this construction of small free resolutions is given in [**6**, Th. 3.3].

Finally, the A_∞-coalgebra structure of the 1-homological model $\{H, r\}$ of a Cartan elementary complex can be obtained from the contraction r, via the cobar functor. An interesting question is the determination of the higher homotopy coalgebra structure on H derived from r.

5.4. Application 4: On the p-local n-homology algebra of a minimal K-S algebra

With all the results established in this section at hand, we are able to get n-homological information for minimal K-S algebras (see Def. 5.2). For $n = 1$, we can state the following result:

Theorem 5.33. *Let $(\Lambda M, d)$ be a minimal K-S algebra. There is a semi-full algebra contraction from $\bar{B}(\Lambda M, d)$ to a tensor product of exterior and divided power algebras. That is to say, we obtain a 1-homological model (in fact, its algebra homology) for $(\Lambda M, d)$.*

This result tells us that $\bar{B}(\Lambda M, d)$ is formal over the integers.

We shall not specify full details of the proof, only the ideas involved in it.

Using the almost-full algebra contractions $R_{\bar{B}E}$ (see (49)), $R_{\bar{B}P}$ (see page 4) and $R_{\bar{B}\otimes}$ (see Th. 5.8), we construct an almost-full algebra contraction R from $\bar{B}(\Lambda M)$ (excluding the differential d) to a (eventually infinite) tensor product of exterior and divided power algebras H. Moreover, at graded algebra level, each $E(x_i, 2n-1)$ factor in ΛM contributes a $\Gamma(\sigma(x_i), 2n)$ factor to H and each $P(x_i, 2n)$ factor in ΛM contributes a $E(\sigma(x_i), 2n+1)$ factor to H.

The differential-derivation d produces an algebra perturbation datum δ for R. Now, let us denote by f, g and ϕ the projection, inclusion and homotopy operator of R. We deduce the pointwise nilpotency of the composite $\phi\delta$ from the facts that ϕ increases simplicial degree by 1 and δ does not change simplicial degree (in fact, this morphism decreases tensor degree by 1).

By applying SF-APL, we obtain a semi-full algebra contraction R_δ from $\bar{B}(\Lambda M, d)$ to the connected commutative DG-algebra (H, d_δ). Since the differential d_δ is a derivation we only need to compute this morphism on the generators of this DG-algebra. Due to the minimality condition of the differential d and to the fact that the projection f is the last morphism that is applied when we obtain the image of a generator x_i under $d_\delta = f\delta\Sigma_R^\delta g$, it is easy to deduce that d_δ is zero. In other words and recalling the terminology of Subsection 5.2, δ is a 1-simple perturbation datum (see Def. 5.17) and f is zero over 1-simple elements (see Prop. 5.14 **a)**). All this means that the 1-homological model of the minimal K-S algebra $(\Lambda M, d)$ is a banal tensor product of exterior and divided power algebras.

□

We now proceed to construct a n-homological model for a minimal K-S algebra.

Theorem 5.34. *Let $\mathbb{Z}_{(p)}$ be the ground ring and $(\Lambda M, d)$ be a minimal K-S algebra. There is a semi-full algebra contraction from $\bar{B}^n(\Lambda M)$ ($n \geqslant 2$) to a tensor product of Cartan's elementary complexes. That is, we have a p-minimal n-homological model for $(\Lambda M, d)$.*

Let $X = \sum_{n>0} X_n$ be the graded set generating the free graded module M. Recall that the set X is indexed by a well-ordered set I, such that $d(x_j)$ is a polynomial in those x_i with $i < j$.

The starting point here is the semi-full algebra contraction R_δ obtained in Theorem 5.33 from $\bar{B}(\Lambda M, d)$ to a tensor product $H1$ of exterior and divided power algebras. Working over $\mathbb{Z}_{(p)}$ and applying the contraction construction $\bar{B}()$ (see Subsection 5.1) to R_δ, we determine a semi-full algebra contraction C_1 from $\bar{B}^2(\Lambda M, d)$ to $\bar{B}(H1)$. Since $H1$ can be expressed by a tensor product $\otimes_{i\in I} A_i$, where $A_i = E(\sigma(x_i), |x_i|+1)$ (with $|x_i|$ even) or $A_i = \Gamma(\sigma(x_i), |x_i|+1)$ (with $|x_i|$ odd), using the contractions $R_{\bar{B}\otimes}$, $R_{\bar{B}E}$ and $R_{\bar{B}\Gamma}$ (see Th. 5.28) we construct a semi-full algebra contraction C_2 from $\bar{B}(H1)$ to a tensor product $H2$ of Cartan's elementary complexes. Therefore, the composition C_2C_1 establishes a "small" p-minimal 2-homological model for the K-S algebra. Proceeding in an analogous way and taking into account that the 1-homology of elementary complexes can be represented as a tensor product of elementary complexes (see the previous Subsection), we obtain the result for every homological degree n.

□

6. Appendix: The Shih operator

The complex of simplicial chains (respectively of normalized simplicial chains) of a simplicial set X will be denoted $C(X)$ (respectively $C_N(X)$).

The Eilenberg-Mac Lane operator is the only (see [**43**]) natural transformation $C_N(X) \otimes C_N(Y) \xrightarrow{EML} C_N(X \times Y)$; it has been defined in (9). The Alexander-Whitney natural transformation, $C_N(X \times Y) \xrightarrow{AW} C_N(X) \otimes C_N(Y)$, has been defined in (8).

Recall that for any simplicial operator $D = s_{i_r}...s_{i_1}\partial_{j_s}...\partial_{j_1}$ of degree $r - s$ and initial dimension p, one can associate its derived simplicial operator D' of degree $r - s$ and initial dimension $p + 1$, defined by

$$D' = s_{i_r+1}...s_{i_1+1}\partial_{j_s+1}...\partial_{j_1+1}.$$

We call Shih natural transformation to the natural transformation $C(X \times Y) \xrightarrow{\Phi} C(X \times Y)$

satisfying the inductive formula

$$\begin{cases} \Phi_0 = 0 \\ \Phi_n = -(\Phi_{n-1})' + (EML \circ AW)'\, s_0. \end{cases}$$

It induces a natural transformation on normalized complexes.

Let us recall Rubio's formula for $C(X \times Y) \xrightarrow{SHI} C(X \times Y)$

$$\begin{aligned} SHI(x_n, y_n) = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (65) \\ & \sum (-1)^{n-p-q+\sigma(\alpha,\beta)} (s_{\beta_q+n-p-q} \ldots s_{\beta_1+n-p-q} s_{n-p-q-1} \partial_{n-q+1} \ldots \partial_n x_n, \\ & \qquad s_{\alpha_{p+1}+n-p-q} \ldots s_{\alpha_1+n-p-q} \partial_{n-p-q} \ldots \partial_{n-q-1} y_n). \end{aligned}$$

where the sum is taken over all $0 \leqslant q \leqslant n-1$, $0 \leqslant p \leqslant n-q-1$ and $(\alpha, \beta) \in \{(p+1, q)-\text{shuffles}\}$.

Remark 6.1. *Note that this formula makes sense whenever $n \geqslant 1$. We set $SHI = 0 : C_0^N(X \times Y) \longrightarrow C_1^N(X \times Y)$.*

This explicit formula has already been used in [**48**], [**49**] and [**47**].

To obtain more information about the previous definitions, the reader may consult [**48**], [**37**] and [**39**].

Theorem 6.2. *The Shih natural transformation is characterized by Rubio's formula.*

Proof of the theorem.

Thanks to remark 6.1, in dimension 0 both definitions correspond to $\Phi_0 = 0$. It remains to demonstrate that Rubio's formula satisfies the inductive formula given by Shih :

$$SHI = -SHI' + (EML \circ AW)'\, s_0.$$

We are going to make calculations in $C(X \times Y)$ and only at the end of this proof, we will show that the expected result is true in $C_N(X \times Y)$.

To begin with, let us apply the first term of the right member of this identity to any n-simplex (x_n, y_n)

$$\begin{aligned} -SHI'(x_n, y_n) = & -\sum (-1)^{(n-1)-p-q+\sigma(\alpha,\beta)} \\ & (s_{\beta_q+(n-1)-p-q+1} \ldots s_{\beta_1+(n-1)-p-q+1} s_{(n-1)-p-q-1+1} \partial_{(n-1)-q+1+1} \ldots \partial_{(n-1)+1} x_n, \\ & \qquad s_{\alpha_{p+1}+(n-1)-p-q+1} \ldots s_{\alpha_1+(n-1)-p-q+1} \partial_{(n-1)-p-q+1} \ldots \partial_{(n-1)-q-1+1} y_n) \end{aligned}$$

where the sum is taken over all $0 \leqslant q \leqslant n-2$, $0 \leqslant p \leqslant n-q-2$ and $(\alpha, \beta) \in \{(p+1, q)-\text{shuffles}\}$.

The previous expression may be simplified and can be rewritten as

$$\begin{aligned} -SHI'(x_n, y_n) = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (66) \\ & \sum (-1)^{n-p-q+\sigma(\alpha,\beta)} (s_{\beta_q+n-p-q} \ldots s_{\beta_1+n-p-q} s_{n-p-q-1} q \partial_{n-q+1} \ldots \partial_n x_n, \\ & \qquad s_{\alpha_{p+1}+n-p-q} \ldots s_{\alpha_1+n-p-q} \partial_{n-p-q} \ldots \partial_{n-q-1} y_n) \end{aligned}$$

where the sum is taken over q, p and (α, β) describing the same sets.

Let us now compute $(EML \circ AW)'\, s_0$ on any n-simplex (x_n, y_n). First,

$$\begin{aligned} EML \circ AW(x_n, y_n) = & \\ & \sum_{i=0}^{n} \sum_{(\alpha,\beta)} (-1)^{\sigma(\alpha,\beta)} (s_{\beta_{n-i}} \ldots s_{\beta_1} \partial_{i+1} \ldots \partial_n x_n, s_{\alpha_i} \ldots s_{\alpha_1} \partial_0 \ldots \partial_{i-1} y_n) \end{aligned}$$

where the second sum is taken over all the $(i, n-i)$-shuffles.

We can derive this operator and apply it on the 0-degenerate simplex (s_0x_n, s_0y_n)

$$(EML \circ AW)'(s_0x_n, s_0y_n) = \sum_{i=0}^{n}\sum_{(\alpha,\beta)} (-1)^{\sigma(\alpha,\beta)}(s_{\beta_{n-i}+1}...s_{\beta_1+1}\partial_{i+2}...\partial_{n+1}s_0x_n, s_{\alpha_i+1}...s_{\alpha_1+1}\partial_1...\partial_i s_0y_n)$$

where the second sum is still taken over all the $(i, n-i)$-shuffles.

Using the identities $\partial_i s_j = sj\partial_{i-1}$ whenever $i > j+1$, since $i+2 > 1$, we can write

$$(EML \circ AW)'(s_0x_n, s_0y_n) = \sum_{i=0}^{n}\sum_{(\alpha,\beta)} (-1)^{\sigma(\alpha,\beta)}(s_{\beta_{n-i}+1}...s_{\beta_1+1}s_0\partial_{i+1}...\partial_n x_n, s_{\alpha_i+1}...s_{\alpha_1+1}\partial_1 s_0\partial_1...\partial_{i-1}y_n)$$

Using the identity $\partial_1 s_0 = \mathbf{1}$, we get

$$(EML \circ AW)'(s_0x_n, s_0y_n) = \sum_{i=0}^{n}\sum_{(\alpha,\beta)} (-1)^{\sigma(\alpha,\beta)}(s_{\beta_{n-i}+1}...s_{\beta_1+1}s_0\partial_{i+1}...\partial_n x_n, s_{\alpha_i+1}...s_{\alpha_1+1}\partial_1...\partial_{i-1}y_n). \tag{67}$$

When, in the previous sum, $i = 0$, there is only one $(0, n)$-shuffle corresponding to the identical permutation of n terms ($\beta_k = k-1$, for all $0 \leqslant k \leqslant n-1$). The single corresponding term in the previous sum $(s_n...s_1s_0\partial_1...\partial_n x_n, s_0y_n)$ is clearly 0-degenerate. Indeed, using the identities $s_{j+1}s_0 = s_0s_j$, whenever $j \geqslant 0$, we can write

$$\begin{aligned}(s_n...s_1s_0\partial_1...\partial_n x_n, s_0y_n) &= (s_0s_{n-1}...s_0\partial_1...\partial_n x_n, s_0y_n)\\ &= s_0(s_{n-1}...s_0\partial_1...\partial_n x_n, y_n).\end{aligned}$$

Let us set $z = (s_{n-1}...s_0\partial_1...\partial_n x_n, y_n)$.

Remark 6.3. *The s_0z class is null in the normalized complex.*

The expression (67) may be written

$$(EML \circ AW)'(s_0x_n, s_0y_n) = s_0z + \sum(-1)^{\sigma(\alpha,\beta)}(s_{\beta_j+1}...s_{\beta_1+1}s_0\partial_{n-j+1}...\partial_n x_n, s_{\alpha_i+1}...s_{\alpha_1+1}\partial_1...\partial_{n-j-1}y_n) \tag{68}$$

where the sum is now taken over all $0 \leqslant j \leqslant n-1$, and all the (i, j)-shuffles (α, β) with $i = n-j$.

Note that when $i = 1$, there are no face operators in the second terms of the couples. In fact, we once again find those couples in Rubio's formula (see (65)) when $q = n-1$ and $p = 0$. This suggests a change of indices. Let us now consider the new indices p and q, defined by $p = i-1$ and $q = j$. The relation (68) becomes

$$(EML \circ AW)'(s_0x_n, s_0y_n) = s_0z + \sum(-1)^{\sigma(\alpha,\beta)}(s_{\beta_q+1}...s_{\beta_1+1}s_0\partial_{n-q+1}...\partial_n x_n, s_{\alpha_{p+1}+1}...s_{\alpha_1+1}\partial_1...\partial_{n-q-1}y_n) \tag{69}$$

where the sum is taken over all $0 \leqslant q \leqslant n-1$, and all the $(p+1, q)$-shuffles with $p = n-q-1$.

If we add the right terms of (66) and (69) and subtract s_0z, we get exactly (65). Indeed, as in (69), $p = n - q - 1$ then $n - p - q = 1$ and (7) becomes

$$\begin{aligned}(EML \circ AW)'(s_0 x_n, s_0 y_n) = \\ s_0 z + \sum (-1)^{\sigma(\alpha,\beta)} (s_{\beta_q + n-p-q} \dots s_{\beta_1+n-p-q} s_0 \partial_{n-q+1} \dots \partial_n x_n, \\ s_{\alpha_{p+1}+n-p-q} \dots s_{\alpha_1+n-p-q} \partial_{n-p-q} \dots \partial_{n-q-1} y_n)\end{aligned} \tag{70}$$

where this last sum is still taken over all $0 \leqslant q \leqslant n-1$, and all the $(p+1, q)$-shuffles with $p = n - q - 1$. The expression of the summands in (70) is now the same as in (66) and (65), except the term $s_0 z$ but this is of no consequence since it is degenerate. It then suffices to note that the set of shuffles over which the sum is taken in (65) is the disjoint union of those over which the sums are taken in (66) and (70).

Due to Remark 6.3, we get the expected result in the normalized complex. □

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