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A metric Ricci flow for surfaces and its applications

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Motivated largely by Perleman's work, the Ricci flow has become lately an object of interest and study in Graphics and Imaging. Various approaches have been suggested previously, ranging from classical approximation methods of smooth differential operators to discrete, combinatorial methods.

In this paper we introduce a metric Ricci flow for surfaces and we investigate its properties: existence, uniqueness and singularities formation. We show that the positive results that exist for the smooth Ricci flow also hold for the metric one and that, moreover, the same results hold for a more general, metric notion of curvature. Furthermore, using the metric curvature approach, we show the existence of the Ricci flow for polyhedral 2-manifolds of piecewise constant curvature. We also study the problem of the realizability of the said flow in $\mathbb{R}^3$.

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1. Introduction and background

Diffusion-type geometric flows have become important building blocks of modern Image Processing, Vision, Graphics and related fields. Accordingly, such methods as the Beltrami and mean curvature flows belong by now to the basic repertoire of methods available to the Imaging, Vision and Graphics communities. Recently, largely motivated by Perleman's work [52], [53], the Ricci flow has become an object of interest and study in Graphics and Imaging.

Various approaches have been suggested previously, ranging from classical approximation methods of smooth differential operators to discrete,

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combinatorial methods. Amongst these, the most successful by far is the one based on the discrete Ricci flow of Chow and Luo [20], due to Gu (see, e.g. [43] and, for more details, [35]). In fact, the present paper was motivated largely by our desire to better understand the discrete, circle-packing based Ricci flow of Chow and Luo, and its relation with the Ricci flow for smooth surfaces introduced by Hamilton [36] and Chow [19].

The metric approach we propose here is based on the notion of the so called *Wald-Berestovskii embedding curvature* that, in conjunction with *smoothings*, allows us to study the metric Ricci flow not directly, but rather via the "classical" Ricci flow (i.e. the Ricci flow for smooth surfaces). While this approach is certainly less elegant than the combinatorial approach of Chow and Luo (and its various implementations due to Gu et al.), it allows us to obtain, by making appeal to the power of the classical theory (see [36], [84], [85]), a number of results that appear not to have been proved previously for the combinatorial flow (and not even considered in the context of Imaging, Vision and Graphics): existence of the reverse flow, uniqueness, singularities formation and the issue of embeddability (realizability) in $\mathbb{R}^3$. (Such results are not less important from the viewpoint of the applications of the combinatorial flow – see, e.g. [35], [43].)

Remark 1.1. We have introduced elsewhere [73], [74] a combinatorial Ricci curvature, based upon the theoretical work of Forman [26]; and we have also developed a fitting Ricci flow for images [3], [4]. We do not discuss here in detail this very different approach, where quite general weighted CW complexes are considered. However, although much more general, the combinatorial approach, in its implementation for images, essentially reduces – due to the necessity of choosing natural, expressive weights, namely to gray-scale level, resolution, etc. – to a very special case of PL surfaces embedded in $\mathbb{R}^3$. In view of this fact we shall comment, throught the paper, on the relevance of our results herein to the combinatorial flow for images.

We consider first compact surfaces without boundary and concentrate mainly on polyhedral objects, since these arise more naturally and, moreover, are of most interest in applications. (Some typical examples are given in Figures 1 and 2.) We shall also show that by considering a more general notion of *metric curvature*, we easily obtain an extension of these results to a larger class of geometric objects and, in particular, to CW complexes, not necessarily regular. (The class of CW complexes for which this generalization holds is larger than the one considered in [20], since we do not have to restrict ourselves to complexes such that each two cell has 3 vertices and each vertex has degree > 2.)

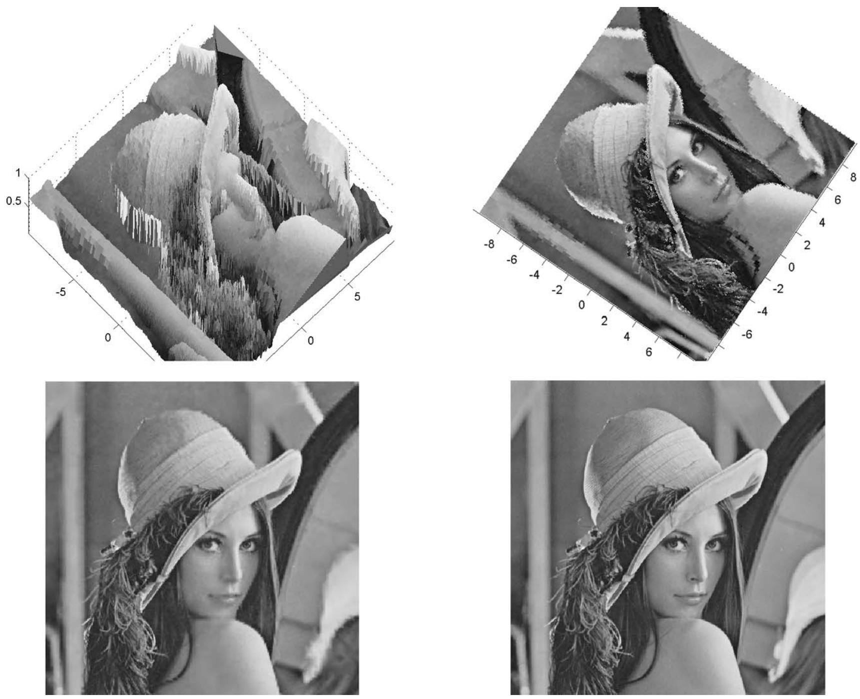

Figure 1: A piecewise flat surface (top, left), and the reconstruction it allows (top, right and bottom, left) of a classical test image (bottom, right).

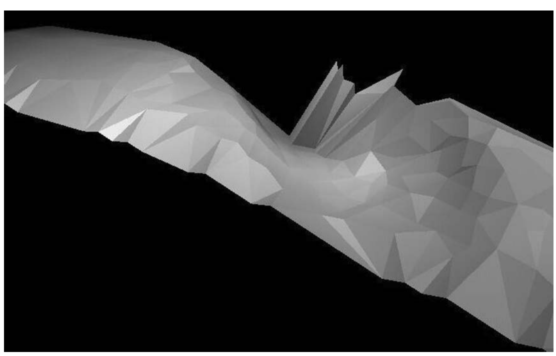

Figure 2: Detail of the piecewise flat surface obtained from the CT scan image obtained from 7 slices of human colon scan. (Courtesy of Dr. Doron Fisher from Rambam Madical Center in Haifa.)

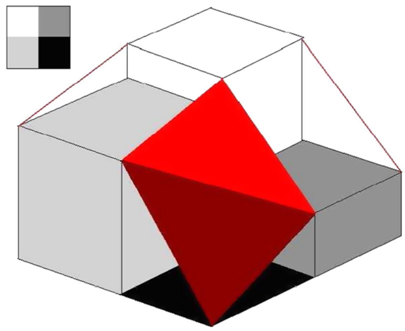

Figure 3: The simplest (but obviously not the most precise) way of passing from the "sticks" model of an image, to a PL surface. Better numerical results were obtained by replacing each quadruple of adjacent pixels with 8 triangles having a common vertex.

Remark 1.2. As we have noted above, in the special case of images, the combinatorial Ricci curvature (and flow) reduces to the study of an analogous type of curvature for PL 2-manifolds. Such manifolds arise due to the need to pass from the piecewise flat, but discontinuous, model of Digital Image Processing, namely of prisms over a square grid with heights equal to the gray-scale level of the pixels (see Figure 3), on which Forman's methods are easily implemented, to a continuous model. The simplest – and most natural from both the geometric and imaging viewpoints – approach in this context is to pass to the dual complex (which is also just a square grid) and to divide the resulting cells into triangles – see Figure 3. (For further details and discussion, see the relevant bibliographical entries.) The naturalness of this passage to PL manifolds enables us to infer, from results regarding the metric flow, a number of corresponding consequences for the combinatorial one, mainly regarding the existence of "good" approximations, convergence rate of the flow and embeddability in $\mathbb{R}^3$. We shall signal out such facts as we encounter them in the sequel.

In straightforward analogy with the classical flow

$$\frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t)\,, \tag{1.1}$$

we define the *metric* Ricci flow by

$$\frac{dl_{ij}}{dt} = -2K_i l_{ij}\,, \tag{1.2}$$

where $l_{ij} = l_{ij}(t)$ denote the edges (1-simplices) of the triangulation (PL or piecewise flat surface) incident to the vertex $v_i = v_i(t)$, and $K_i = K_i(t)$ denotes the curvature at the same vertex. We shall discuss below in detail what proper notion of curvature should be chosen to render this type of metric flow meaningful.

We also consider the close relative of (1.1), the normalized flow

$$\frac{dg_{ij}(t)}{dt} = (K - K(t))g_{ij}(t)\,, \tag{1.3}$$

and its metric counterpart

$$\frac{dl_{ij}}{dt} = (\bar{K} - K_i)l_{ij}\,, \tag{1.4}$$

where $K, \bar{K}$ denote the average classical, respectively metric, sectional (Gauss) curvature of the initial surface S_0: $K = \int_{S_0} K(t)dA \big/ \int_{S_0} dA$, and $\bar{K} = \frac{1}{|V|}\sum_{i=1}^{|V|} K_i$, respectively. (Here $|V|$ denotes, as usually, the cardinality of the vertex set of S_{Pol}.)

Before continuing further on, it is important to remark the asymmetry in Equation 1.2, which is caused by the fact that the curvature on two different vertices acts, so to say, on the same edge. We shall see that in the approach considered in the present paper, this asymmetry is automatically dealt with. Furthermore, in our final remarks, we shall indicate another method of dealing with this issue.

Note also that, if starting with a polyhedral surface S_{Pol}, i.e. the underlying topological space of a locally finite 2-dimensional simplicial complex, in order to ensure that (1.1), (1.3) will hold (and, indeed, make sense as a flow), it is important to approximate S_{Pol} both in the metric sense and also so that curvature is well approximated, hence the interest in – and the need for – such an approximation result. First, let us recall a few definitions regarding *polyhedral metrics*, following [30] (and, to a lesser degree, [31]):

Definition 1.3. Given a simplex σ, a metric on σ is *linear* iff it coincides with the metric induced by some linear embedding of σ in some $\mathbb{R}^N$ (equipped with the usual Euclidean metric).

Applying this procedure on every simplex of a given simplicial polyhedron K, one obtains a (singular Riemannian) a metric on K, metric that is uniquely determined by the vector $(l_1^2, \ldots, l_{|E|}^2)$, where l_j denotes the length of the edge $e_j \in E$ – the set of vertices of K (and $|E|$ the number of the edges, as it is customary). We can be even more precise in our definition, in the following manner:

Definition 1.4. Given a polyhedron K, a *presentation of a Riemannian metric* on K is given by:

1) The polyhedron K;
2) A triangulation $\mathcal{T}$ of K;
3) A collection $\{d_\sigma\}$ of linear metrics on the set of simplices $\{\sigma\}$ of the given triangulation of K, satisfying the following consistency requirement:

$$\sigma < \tau \Longrightarrow d_\sigma = d_\tau|_\sigma .$$

(Here $\sigma < \tau$ is the standard notation for the fact that σ is a face of τ.)

One can extend in a straightforward manner the definition above to any subdivision $\mathcal{T}'$ of a given triangulation $\mathcal{T}$. Moreover, if two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ have a common subdivision $\mathcal{T}'$, then, if the presentations of $\mathcal{T}_1$, $\mathcal{T}_2$, with respect to $\mathcal{T}'$ are identical, they are called *equivalent* presentations. (It is easy to check that the equivalence is, indeed, independent of the particular subdivision $\mathcal{T}'$.) We are thus conducted to the following definition:

Definition 1.5. Given a polyhedron K, a *Riemannian metric* K (or a *polyhedral metric*) is the equivalence class of presentations of Riemannian metrics. A *Riemannian polyhedron* consist of a polyhedron K together with a choice for a Riemmanian metric on K.

As expected, an approximation theorem of the type required indeed exists, assuring that, given a polyhedral surface S^2_{Pol}, we can find a smooth surface S^2 arbitrarily close to it, both in the Hausdorff metric and as far as Gaussian curvature is concerned. More precisely, we have the following result due to Brehm and Kühnel [12]:

Proposition 1.6 ([12])**.** *Let S^2_{Pol} be a compact polyhedral surface without boundary embedded in $\mathbb{R}^3$. Then there exists a sequence $\{S^2_m\}_{m\in\mathbb{N}}$ of smooth surfaces in $\mathbb{R}^3$, (homeomorphic to S^2_{Pol}), such that it assures good approximation of S^2_{Pol} in the*

1) metric sense*, that is*
   a) *$S^2_m = S^2_{Pol}$ outside the $\frac{1}{m}$-neighbourhood of the 1-skeleton of S^2_{Pol},*
   b) *The sequence $\{S^2_m\}_{m\in\mathbb{N}}$ converges to S^2_{Pol} in the Hausdorff metric;*
2) curvature sense*, more precisely the (combinatorial) curvatures of S^2_m weakly converge to S^2_{Pol} in the sense of measures. Here the curvature measure in the smooth case is the area measure weighted by Gauss curvature (considered as a function) and in the polyhedral case it is the Dirac measure at the vertices weighted by the combinatorial curvature.*

Remark 1.7. If only convergence in the Gromov-Hausdorff metric is required, then a similar result, holding for all compact *inner metric spaces* has been proven by Cassorla [17]. Moreover, the basic construction he employs is essentially the same as one of [12], so similar curvature estimates can also be obtained. (In fact, the author also states – a seemingly still unpublished result – that one can approximate the given spaces with a series of smooth surfaces having Gaussian curvature bounded from above by -1. Unfortunately, this cames at the price of loosing the embeddability in $\mathbb{R}^3$ of the approximating surfaces.) However, the generality of the spaces that admit smooth approximations comes at a cost, so to say: One cannot ensure that the genus of the approximating surfaces will remain bounded.

Recall that combinatorial Gauss curvature for polyhedral surfaces is defined by the *angular defect*, that is:

$$K(p) = 2\pi - \sum_{i=1}^{m_p} \alpha_i(p) \tag{1.5}$$

where $\alpha_1, \ldots, \alpha_{m_p}$ are the (interior) face angles adjacent to the vertex v_i.

We do not bring here the original demonstration of Brehm and Kühnel, but rather provide a different proof, which also applies to a larger class of surfaces. Moreover, it captures better the meaning of the notion of Hausdorff convergence and its interplay with curvature. Let us note, however, that the original proof also holds for surfaces embedded in $\mathbb{R}^N$, for some $N > 3$. This is important if an abstract surface is given and one must start by considering first an isometric embedding in $\mathbb{R}^N$. (See [67], [66] for a discussion on the feasibility and practicability of such embeddings for PL, as well as more general types of surfaces, both in the applied and purely mathematical contexts.) Moreover, as noted by Brehm and Kühnel, their proof extends to surfaces that are only *locally embedded* in $\mathbb{R}^3$.

Our approach is less elementary than the one of Brehm and Kühnel, but in a sense more natural for geometers and topologists, as well as for image processing applications, and, moreover, as opposed to the original proof, it uses only standard analytic tools: Instead of building the smooth surfaces from a set of "standard elements" (cylinders, etc.), as in [12], we consider instead *smoothings* S^2_m (see, e.g. [49]). (A similar approach of approximating discrete structures by smooth ones is adopted also in theoretical physics [27], [28], the paradigm therein being that the structure of space-time at the smallest scales is, in fact, discrete and that classical models are smooth approximations of these structures.) Since, by [49], Theorem 4.8, such smoothings are δ-*approximations*, and therefore, for δ small enough, also α-approximations of the given piecewise-linear surface S^2_{Pol}, they approximate arbitrarily well both distances and angles on S^2_{Pol}. (Not to encumber the presentation with too many details regarding tools of differential topology, we have concentrated the relevant definitions, and results in an appendix.) Therefore, angles, hence defects, are arbitrarily well approximated as well. While Munkres' results concern PL manifolds, they can be applied to polyhedral ones as well, because, by definition, polyhedral manifolds have simplicial subdivisions (and furthermore, such that all vertex links are combinatorial manifolds) – see, e.g. [13], p. 346. Of course, for different subdivisions, one may obtain different polyhedral metrics. However, by the *Hauptvermutung* Theorem in dimension 2 (and, indeed, for smooth triangulations of diffeomorphic manifolds in any dimension) (see e.g. [48] and [49] and the references therein), any two subdivisions of the same space will be combinatorially equivalent, hence they will give rise to the same polyhedral metric. The fact that 2-dimensional manifolds with a CW complex structure are also smoothable follows from the fact that any manifold of dimension ≤ 3 admits a PL structure (see, e.g. [83]) and that, furthermore, this structure admits a unique smoothing (see, e.g. [49]). In consequence, Gauss curvature of the smooth surface approximates arbitrarily well metric curvature, at the essential common points (i.e. the vertices of the given polyhedral surface). We note that since by [36], Corollary 5.2 (see also [21], Proposition 5.4) the Ricci flow is conformal, it follows that metric curvature approximates arbitrarily well the curvature of the evolved surfaces, at any time t (see Section 2 below for details).

Note that our proof renders in fact a somewhat stronger result than that of [12], since no embedding in $\mathbb{R}^3$ is apriorily assumed, just in some $\mathbb{R}^N$; however, as we have already seen, this represents only a slight improvement. More importantly, no change in the geometry of the 1-skeleton is made, not even in the neighbourhoods of the vertices.

Moreover, it follows that *metric quadruples* (see definition below) on S_{Pol} are also arbitrarily well approximated (including their angles) by the corresponding metric quadruples) on S_m. But, by [88] (see also [10], Theorems 11.2 and 11.3), the *Wald metric curvature* (see below) of S_m, at a point p, $K_W(p)$ equals the classical (Gauss) curvature $K(p)$. Hence the Gauss curvature of the smooth surfaces S_m approximates arbitrarily well the metric one of S_{PL} (and, as in [12], the smooth surfaces differ from polyhedral one only on (say) the $\frac{1}{m}$-neighbourhood of the 1-skeleton of S_{Pol}). This statement can be made even more precise, by assuring that the convergence is in the Hausdorff metric. This follows from results of Gromov [32] – see [70]

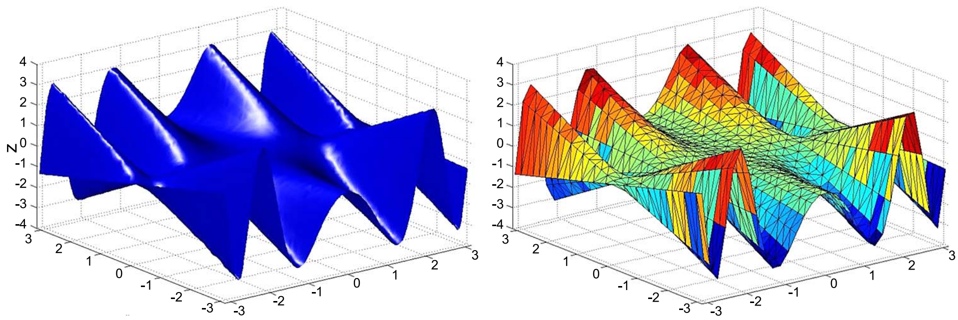

Figure 4: A smooth analytic surface (left) as the smoothing of one of its PL approximations (right).

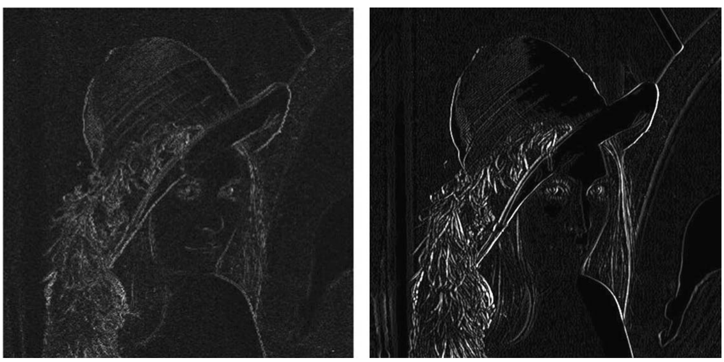

Figure 5: Gaussian curvature of a standard test image, computed using standard discretizations of the differential operators involved (left), and its combinatorial Ricci curvature (right).

for details. That such curvatures converge not only punctually, but as measures as well, i.e. that the so called CCP(K) property of [12], Proposition 1, (that corresponds, essentially, to Condition (2) in Theorem 1.6 above) also holds, follows, as a particular case, from [18], Theorem 5.1, using the fact that polyhedral manifolds represent secant approximations of their own smoothings (see Figure 4).

Remark 1.8. In view of our previous remarks it follows that, given the specific, natural choice of weights for the combinatorial Ricci flow of images, this type of curvature also converges, as resolution increases, to the Gaussian curvature of the smoothed image, viewed as a (smooth) surface (see Figure 5 for a canonical example.)

Another beneficial consequence of the passing to the smooth case, is that the asymmetry in the metric flow that we observed above disappears automatically via the smoothing process.

Here *metric quadruples* are defined as follows:

Definition 1.9. Let (M,d) be a metric space, and let $Q = \{p_1, ..., p_4\} \subset M$, together with the mutual distances: $d_{ij} = d_{ji} = d(p_i, p_j)$; $1 \leq i, j \leq 4$. The set Q together with the set of distances $\{d_{ij}\}_{1\leq i,j\leq 4}$ is called a *metric quadruple*.

Remark 1.10. Metric quadruples can be defined in a slightly more abstract manner, without the aid of the ambient space: a metric quadruple being defined, in this approach, as a 4 point metric space; i.e. $Q = \big(\{p_1, ..., p_4\}, \{d_{ij}\}\big)$, where the distances d_{ij} verify the axioms for a metric.

Before we can define the notion of embedding curvature, we first have to introduce some notation: Let S_κ denote the complete, simply connected surface of constant Gauss curvature κ, i.e. $S_\kappa \equiv \mathbb{R}^2$, if $\kappa = 0$; $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$, if $\kappa > 0$; and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$, if $\kappa < 0$. Here $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$ denotes the sphere of radius $R = 1/\sqrt{\kappa}$, and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$ stands for the hyperbolic plane of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the plane disk of radius $R = 1/\sqrt{-\kappa}$.

Definition 1.11. The *embedding curvature* $\kappa(Q)$ of the metric quadruple Q is defined to be the curvature κ of the gauge surface S_κ into which Q can be isometrically embedded. (See Figure 6.)

We are now able to bring the definition of *Wald curvature* [88] (or rather of its modification due to Berestovskii [6]):

Definition 1.12. Let (X,d) be a metric space. An open set $U \subset X$ is called a *region of curvature* $\geq \kappa$ iff any metric quadruple can be isometrically embedded in S_m, for some $m \geq k$. A metric space (X,d) is said to have *Wald-Berestovskii curvature* $\geq \kappa$ iff any $x \in X$ is contained in a region U of curvature $\geq \kappa$.

Remark 1.13. While the second part of the definition above is not needed in the remainder of the paper, we bring it for completeness (and for its importance elsewere – see, e.g. [67], [66]).

The classical (by now) *Wald curvature* at an accumulation point of a metric space (hence on a smooth surface) is defined by considering limits of the Wald-Berestovskii curvatures of nondegenerate regions of diameter converging to 0, more precisely we have

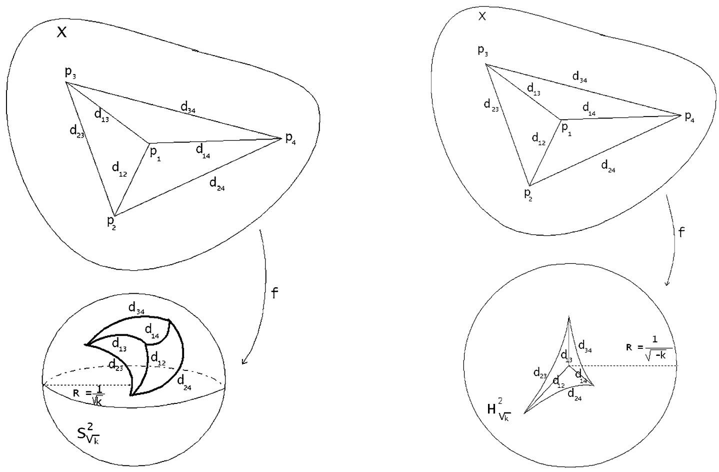

Figure 6: Isometric embedding of a metric quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$ (left) and $\mathbb{H}^2_{\sqrt{\kappa}}$ (right).

Definition 1.14. Let (M,d) be a metric space, and let $p \in M$ be an accumulation point. Then M has (*embedding*) *Wald curvature* $\kappa_W(p)$ at the point p iff

1) Every neighbourhood of p is not contained in a geodesic;
2) For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $Q = \{p_1, ..., p_4\} \subset M$ and if $d(p,p_i) < \delta\,, i = 1, ..., 4$; then $|\kappa(Q) - \kappa_W(p)| < \varepsilon$.

Remark 1.15. Being a generalization of the classical, point-wise Gauss curvature, Wald's definition is less flexible and, in consequence, less powerful than the Wald-Berestovskii comparison curvature.

Remark 1.16. The Wald and Wald-Besetkovskii curvatures can actually be computed, using the following formula for the embedding curvature of a metric quadruple

(1.6)
$$\kappa(Q) = \begin{cases} \quad 0 & \text{if } D(Q) = 0\,; \\ \kappa,\ \kappa < 0 & \text{if } det(\cosh\sqrt{-\kappa}\cdot d_{ij}) = 0\,; \\ \kappa,\ \kappa > 0 & \text{if } det(\cos\sqrt{\kappa}\cdot d_{ij}) \text{ and } \sqrt{\kappa}\cdot d_{ij} \leq \pi \\ & \quad \text{and all the principal minors of order 3 are} \geq 0; \end{cases}$$

where $d_{ij} = d(x_i,x_j), 1 \leq i,j \leq 4$, and $D(Q)$ denotes the so called *Cayley-*

Menger determinant:

$$D(x_1,x_2,x_3,x_4)=\begin{vmatrix} 0 & 1 & 1 & 1 & 1\\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2\\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2\\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2\\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix}. \tag{1.7}$$

However, it should be noted that, as far as the actual computation of $\kappa(Q)$ using Formula (1.6) is concerned, the equations involved are – apart from the Euclidean case – transcendental, therefore not solvable, in general, using elementary methods. Moreover, when solved with computer assisted methods, they display certain numerical instability. For a more detailed discussion and some first numerical results, see [62], [70].

In addition, it is important to notice that Formula (1.6) implies that, in practice, a renormalization might be necessary for some of the vertices of positive Wald-Besetkovskii curvature.

Also, we should point out that while Formulas (1.6) and (1.7) may appear somewhat mysterious, the first line in (1.6) represents nothing else but the condition that the simplex of sides $d_{ij}, 1 \leq i < j \leq 4$ has zero (Euclidean) volume (as given by Formula (1.7)), hence it is degenerate, i.e. planar; while the following two lines of (1.6) are the corresponding conditions for Hyperbolic and Spherical simplices, respectively.

Remark 1.17. The transcendental nature of the Formulas (1.6) in the Spherical and Hyperbolic case represent, in fact, a lighter impediment that it might appear at first view. This is due to an approximation result due to Robinson [60]. To present his method and estimates, we need first to introduce yet another definition:

Definition 1.18. A metric quadruple $Q = Q(p_1,p_2,p_3,p_4)$, of distances $d_{ij} = dist(p_i,p_j),\ i = 1,...,4$, is called *semi-dependent* (or a *sd-quad*, for brevity), iff three of its points are on a common geodesic, i.e. there exist three indices, e.g. 1,2,3, such that: $d_{12} + d_{23} = d_{13}$.

We can now state Robinson's result:

Theorem 1.19 ([60])**.** Given the metric semi-dependent quadruple $Q = Q(p_1,$
$p_2,p_3,p_4)$, of distances $d_{ij} = d(p_i,p_j),\ i,j = 1,...,4$; the embedding curvature $\kappa(Q)$ admits a rational approximation given by:

$$K(Q)=\frac{6(\cos\measuredangle_0 2+\cos\measuredangle_0 2')}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2)+d_{23}\sin^2(\measuredangle_0 2')\big)} \tag{1.8}$$

where: $\measuredangle_0 2 = \measuredangle(p_1p_2p_4)$, $\measuredangle_0 2' = \measuredangle(p_3p_2p_4)$ represent the angles of the Euclidian triangles of sides d_{12}, d_{14}, d_{24} and d_{23}, d_{24}, d_{34}, respectively.

Moreover the *absolute error* R satisfies the following inequality:

$$(1.9) \qquad |R| = |R(Q)| = |\kappa(Q) - K(Q)| < 4\kappa^2(Q)\mathrm{diam}^2(Q)/\lambda(Q)\,,$$

where $\lambda(Q) = d_{24}(d_{12}\sin\measuredangle_0 2 + d_{23}\sin\measuredangle_0 2')/S^2$, and where $S = Max\{p, p'\}$; $2p = d_{12} + d_{14} + d_{24}$, $2p' = d_{32} + d_{34} + d_{24}$.

For the proof of the theorem above we refer the reader to the original [60] or, since this source is less accessible nowadays, to [62] or [70]. The last two bibliographical entries also provide some geometric intuition behind Formula (1.8). Also, we should note that, in special cases (e.g. when $d_{12} = d_{32}$, etc.) simpler formulas are obtained en lieu of (1.8) – again, see [60], or [62], [70].

As expected from the result above, the restriction to sd-quads does not affect the capability of embedding curvature to approximate Gaussian curvature of smooth surfaces, and we have the following theorem (whose full proof can be found in [60]):

Theorem 1.20. *Let S be a smooth (differentiable) surface. Then, for any point $p \in S$:*

$$K_G(p) = \lim_{n\to 0} K(Q_n)\,;$$

for any sequence $\{Q_n\}_{n\geq 1}$ of sd-quads that satisfy the following conditions:

$$Q_n \to Q = \Box p_1pp_3p_4\,;\ \mathrm{diam}(Q_n) \to 0,$$

and

$$(1.10) \qquad \lim \frac{D^2(Q_n)}{\lambda(Q_n)} = 0\,.$$

It is important to underline that the convergence assured by Theorem 1.20 is not just in the sense of measures and, moreover, errors of different signs do not simply cancel each other out. In fact, a stronger approximation holds and $\mathrm{sign}(\kappa(Q)) = \mathrm{sign}(K(Q))$, for any metric quadruple Q.

Even more important is to notice that Formula (1.8) is linear in the distances d_{ij}, a fact of crucial important in regarding the two following related issues: (1) It facilitates a better understanding than the partially transcendental Formula (1.6) of the relation between the Wald-Berestovskii embedding curvature (or, at least, of a good approximation of $\kappa_W(p)$) and the edge lengths; and (2) It suggests that the flow can be implemented in practice in a low complexity, therefore feasible, manner.[1]

Another important issue that should be emphasized is that of the role of $\lambda(Q)$ in the error estimate (1.9) and in the convergence condition (1.10). While its expression might appear somewhat complicated and mystifying, its significance is that of a measure of approach to linearity: Indeed, it is a continuous, nonnegative and homogeneous of order 0 function in the distances in Q. In addition, the right-hand term in inequality (1.9) is finite and bounded away from zero if $\lambda(Q)$ also is. Also, for fixed $\kappa(Q)$ and given shape of Q, we have that $\kappa^2(Q)\mathrm{diam}^2(Q)/\lambda(Q) \sim \mathrm{diam}^2(Q)$ (i.e. in this case $\lambda(Q) \sim \kappa(Q)$). Moreover, $\lambda(Q) = 0$ iff Q is actually linear. Therefore, condition (1.9) guarantees that the sequence $\{Q_n\}_{n\geq 1}$ of sd-quads does not converge to a linear configuration.

It is certainly worthwhile to observe that $\lambda(Q)$, as a measure of non-degeneracy, represents nothing but a specific, "augmented" version of *thickness* or *fatness* of triangles (also know in the Graphics milieux as *aspect ration*. The augmentation in question is meant to deal with triangles with a (generalized) median, i.e. with sd-quads. We discussed in detail the connection between metric curvatures and thickness, including the case of some quite general spaces in [69], Section 4, and we have previously explored in depth, in a series of papers – e.g. [63], [65], [69], [75], – the role of curvature in producing thick triangulations (important, amongst other tasks, to the construction of quasi-regular mappings [65], [71] with applications in Medical Imaging [72]). The discussion above highlights yet another connection between metric curvature (in this case, Robinson's version for sd-quads of Wald's embedding curvature) and thickness. In consequence, it underlines again, from another viewpoint (a purely metric one), the crucial importance in practice (to ensure good convergence properties and, indeed, numerical stability) of assuring that the mesh has has good aspect ration (i.e. that the triangulation is thick[2]).

However, the significance of Robinson's work goes beyond these practical issues and has theoretical importance. Indeed, the "augmented" triangles above are in fact identical to one version of the comparison triangles in the definition of Alexandrov curvature (see, e.g. [16]). Thus not only do we have at hand an alternative definition for thickness of comparison triangles, but we also benefit for a concrete, direct method of viewing our results herein in the far larger context of Alexander spaces. (We shall return later on to

[1] For the relative straightforward case of 0 embedding curvature, a study of the optimisation of the algorithm was done by Sippl and Scheraga [79], [80].

[2] For a definition of thickness for general polyhedral meshes, see [34]

the connection between Wald's curvature and the more modern theory of Alexandrov spaces.)

Of course, one would like to estimate the potential of obtaining, via Theorems 1.8 and 1.20, good estimates in concrete, practical applications. One is encouraged in this direction by the following example, due to Robinson himself, that shows that, at least in some cases, the actual computed error is far smaller then the one given by Formula (1.9).

Example 1.21 ([60]). *Let Q_0 be the quadruple of distances $d_{12} = d_{23} = d_{24} = 0.15, d_{14} = d_{34}$ and of embedding curvature $\kappa = \kappa(Q_0) = 1$. Then $\kappa S^2 < 1/16$ and $K(Q_0) \approx 1.0030280$, whereas the error computed using Formula (1.9) is $|R| < 0.45$.*

Regarding the experimental side, one can examine the results on both meshes and images (both natural as well as medical) in [70], pp. 350-352. While the outcomes of the tests therein are only incipient and, perhaps, somewhat naive (since the meshes were simplistic and did not allow good approximation of all directions, and the results on images did not undergo any histogram equalization), they do still comply to the basic expectations regarding the efficiency of the theorems under scrutiny.

Evidently, in the context of polyhedral surfaces, the natural choice for the open set U required in Definition 1.12 is the *open star* of a given vertex v, that is, the set $\{e_{vj}\}_j$ of edges incident to v. Therefore, for such surfaces, the set of metric quadruples containing the vertex v is finite.

We should also emphasize the fact that the two approximations of Gaussian curvature considered therein, namely the combinatorial (defect) and metric (Wald-Berestovskii) notions of curvature, are more closely related than just by having as limit, when the mesh of the polyhedral approximation of a smooth surface tends to zero, the Gauss curvature of the said (smooth) surface. Indeed, Wald-Berestovskii curvature can be characterized in terms of angles' sum at a vertex (hence defect). Before formally stating this result, we first need to introduce some further notation:

Given three points x_i, x_j, x_l in a metric space (X, d), we denote by $\alpha_\kappa(x_i, x_j, x_l) \in [0, \pi]$, the angle $\measuredangle(x_j x_i x_l)$ (that is of apex x_i) of the model triangle in S^2_κ. Let $Q = \{x_1, x_2, x_3, x_4\}$ be a *metric quadruple*. We introduce the following quantity associated with Q:

$$V_\kappa(x_i) = \alpha_\kappa(x_i; x_j, x_l) + \alpha_\kappa(x_i; x_j, x_m) + \alpha_\kappa(x_i; x_l, x_m) \tag{1.11}$$

where $x_i, x_j, x_l, x_m \in Q$ are distinct, and κ is any number (see Figure 8).

We can now bring the promised characterization of Wald-Berestovskii in terms of angle sum:

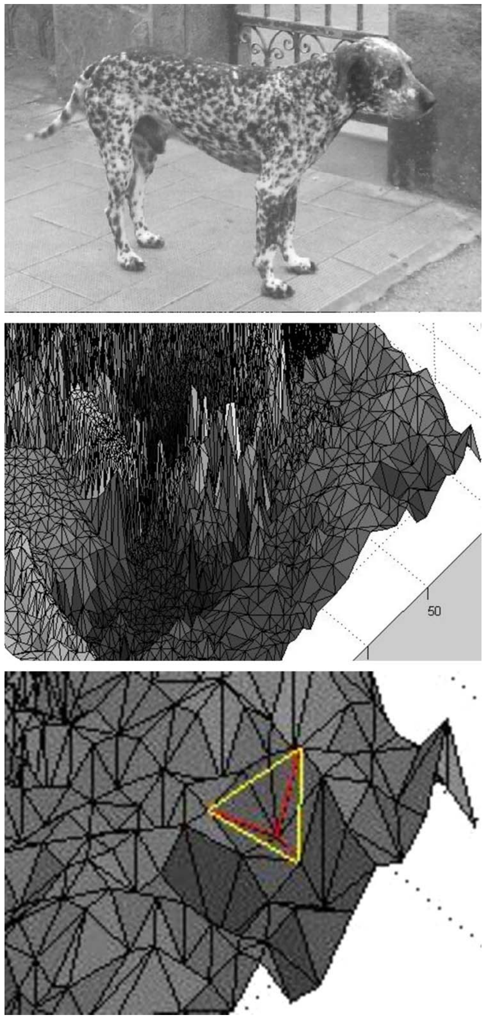

Figure 7: Detail of the triangulation (middle, bottom) corresponding to a natural image (top); and the computation of the Wald curvature at a vertex (bottom). Notice that only the quadruples generated by edges incident to the vertex (red edges) are considered. Note that distances between adjacent vertices (yellow edges) should be considered, even though they are not among the edges of the triangulation.

Proposition 1.22 ([55], Theorem 23)**.** *Let (X, d) be a metric space and let $U \in X$ be an open set. U is a region of curvature $\geq \kappa$ iff $V_\kappa(x) \leq 2\pi$, for any metric quadruple $\{x, y, z, t\} \subset U$.*

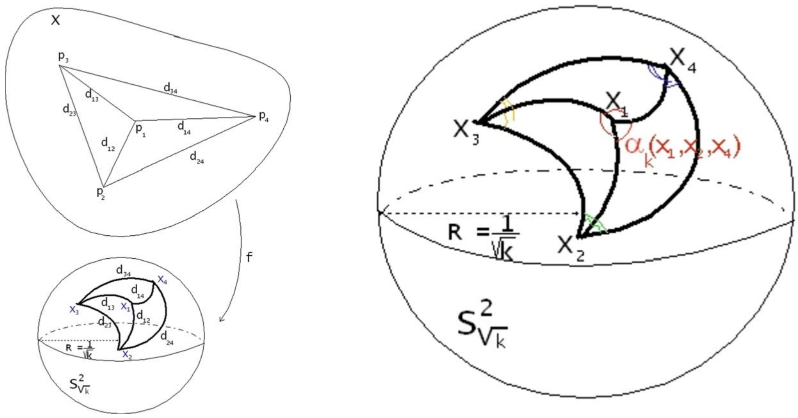

Figure 8: The angles $\alpha_\kappa(x_i, x_j, x_l)$ (right), induced by the isometric embedding of a metric quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$ (left).

The result above shows that, in fact, the metric approach to curvature is essentially equivalent to the combinatorial (angle-based) one, as far as polyhedral surfaces (in $\mathbb{R}^3$) are concerned.[3] In particular, as far as approximations of smooth surfaces in $\mathbb{R}^3$ are concerned, both approaches render, in the limit, the classical Gauss curvature. We should also stress that, in fact, the metric approach is more general, since it can be applied to a very large class of metric spaces (see discussion at the end of this section).

Note that we just gave a positive answer to the question – unposed so far, to the best of our knowledge – whether the metric curvature version of Brehm and Kühnel's basic result also holds, namely we have proved:

Proposition 1.23. *Let S^2_{Pol} be a compact polyhedral surface without boundary. Then there exists a sequence $\{S^2_m\}_{m\in\mathbb{N}}$ of smooth surfaces, (homeomorphic to S^2_{Pol}), such that*

1) a) $S^2_m = S^2_{Pol}$ *outside the* $\frac{1}{m}$*-neighbourhood of the 1-skeleton of* S^2_{Pol},

   b) *The sequence* $\{S^2_m\}_{m\in\mathbb{N}}$ *converges to* S^2_{Pol} *in the Hausdorff metric;*

2) $K(S^2_m) \to K_W(S^2_{Pol})$, *where the convergence is in the weak sense.*

Remark 1.24. The converse implication – namely that Gaussian curvature $K(\Sigma)$ of a smooth surface Σ may be approximated arbitrarily well by the Wald curvatures $K_W(\Sigma_{Pol,m})$ of a sequence of approximating polyhedral

[3]This holds, of course, up to the specific type of convergence for the metric and combinatorial curvature, namely pointwise and in measure, respectively.

surfaces $\Sigma_{Pol,m}$ – is, as we have already mentioned above, quite classical. (For other approaches to curvatures convergence, see, amongst the extensive literature dedicated to the subject, [18] and [11], [22], for the theoretical and applicative viewpoints, respectively.)

Remark 1.25. Since Wald curvature approximates the classical (Gauss) curvature arbitrarily well, the convergence rate of curvature *function* K_W of a series of smooth surfaces approximating a (smooth) given one, is identical to that of the classical Gaussian curvatures as prescribed by Brehm and Kühnels theorem.

However, in practice one deals with the metric curvatures of polyhedral surfaces, rather than with smooth ones. One possible approach to estimating the convergence rate in this discrete setting (as opposed to the rather theoretical one of ideal smoothings), would be to make appeal once again to Robinson's results. Indeed, since by Formula (1.9), $R < 4\kappa^2(Q)\mathrm{diam}^2(Q)/\lambda(Q)$ and since, as we have already noted, for given $\kappa(Q)$ and fixed thickness (of Q), $\kappa^2(Q)\mathrm{diam}^2(Q)/\lambda(Q) \sim \mathrm{diam}^2(Q)$, and, moreover, for given thickness $\varphi_0 > 0$, $\mathrm{diam}^2(Q) = C \cdot \mathrm{Area}(Q), C = C(\varphi_0)$, it follows that $R < C_1 \cdot \mathrm{Area}(Q)$. Therefore, for finite triangular meshes, where both thickness and curvature, as well as diameter of the triangles, are uniformly bounded, the error in the estimation of Wald curvature, hence, for sufficiently fine approximations (triangulations) of smooth surfaces, of Gaussian curvature, curvature approximation converges at the same rate as the area of the triangles or, equivalently, as η^2, where η denotes the *mesh* of the triangulation, i.e. maximum of the diameters of the simplices of the triangulation. (Of course, in dealing with applications such as those encountered in Graphics and Imaging, the non-degeneracy of the shape of the triangles, i.e. boundedness away from zero of thickness, might have to be elevated (by mash improvement) since, in numerical computations, a practical bound usually has to be far greater than the theoretical one.)

Using the facts above one can presumably determine the convergence rate of the curvature *measure* associated to the Wald curvature function, namely $\mu_W(v) = K_W(v) \cdot \mathrm{Area}(St(v))$, where $St(v)$ denotes the star of the vertex v. We postpone the full analysis of this problem for future study.

We should also stress again the properties of the Gromov-Hausdorff convergence of finite ε-*nets* in any sequence of approximating surfaces S^2_m (polyhedral or smooth) of a given surface S^2 (again, smooth or not). (Just for the record, recall that, given a metric space (X, d), a $A \subset X$ is called an ε-*net* iff $d(x, A) \leq \varepsilon$, for any $x \in X$.) In particular, by considering ε-nets on surfaces, one automatically ensures (see [32], [16]) any *intrinsic* geometric property of an approximation to the respective geometric property of the limiting geometric object. Most important for us, this holds for the intrinsic metric, hence for the metric curvatures – see [62], [70] for a more detailed discussion and some numerical experiments.

Also, one can consider simultaneously the metric Ricci flow on a polyhedral surface S^2_{Pol}, as well its classical counterpart on its smoothing S^2. Since, as we have already noted above, the metric and the classical curvatures, $K_W(S^2_{Pol})$ and $K(S^2_m)$, respectively, are arbitrarily close to each other, and since the equations of the two respective flows (metric and classical/smooth) contain the same curvature term, the ensuing metrics at each time during the flow will coincide on the common set, i.e the 1-skeleton of the polyhedral manifold and in the exterior of an arbitrarily small neighbourhood of it. Therefore, the limit surfaces for both flows – $S^2_{0,Pol}$ and S^2_0, respectively – will be isometric on the said set, but perhaps only arbitrarily close to being isometric in the considered neighbourhood. (One can ensure actual isometry by imposing a certain additional constraint on the so called "*volume density*" of the surface – for details, see [82].) Moreover, by considering dense enough ε-nets (of arbitrarily small mesh), the intrinsic metric of a polyhedral approximation g_{Pol} of a smooth manifold, and the (smooth) metric g of the later will be arbitrarily close to each other. If one does not consider a limiting process, then the distortion of g by g_{Pol}, both of the metric per se, as well as that of curvature, can be computed, at each time "t" during the flow, using such Formulae as (2.2) and (2.6) in the proof of Theorm 2.6 below, in conjunction with the computations in, say, [51], Lemma 3.19. For manifolds with boundary, similar distortion estimates follow from the results in [63]. Of course, in this case, one still obtains isometry when restricting to the 1-skeleton. (This answers to a question posed to us by D. X. Gu.)

Remark 1.26. Since for PL surfaces (hence for their smoothings) $K_W(p) = 0$ for all points apart from vertices, the ensuing Ricci flow is stationary, except at vertices where the change rate is quite drastic. In this aspect, the metric Ricci flow introduced here resembles the combinatorial, rather than the smooth (classical) one. This should not be too surprising, given that, as already stated, the initial motivation of considering the metric flow (and, a fortiori, of smoothings) was to gain a better understanding of some of the properties of the combinatorial flow.

Before concluding this section, it is important that we set the Wald curvature in its proper, more general (and modern) context:

While largely forgotten (except by a number of researchers in the field) Wald's curvature is not an esoteric notion. In fact, it essentially equivalent with the much more modern – and widely employed in a vast array of mathematical fields – notion of *Alexandrov curvature*, at least for spaces in which there exists "sufficiently many" minimal geodesics (see, for instance, [55], Corollary 40), a condition that certainly is fulfilled in PL surfaces. We do not bring here the full technical definition of Alexandrov curvature, since this would take us to far afield, and we refer the reader to, e.g. [16]. However, it is important to recall that, in defining Alexandrov curvature, one makes appeal to *comparison triangles* in the model space (i.e. gauge surface S_κ), rather than quadrangles, as in the definition of Wald curvature. We do not elaborate further on the extensive, and by now classical, subject of Alexandrov curvature, and refer the reader again to [16]. Also, for a discussion between the the comparison quadrangles, respective triangles, in the definition of Wald and Alexandrov curvatures, as well as for the practical consequences of the similarities and differences between the two approaches, see [69].

The reasons we prefer working with the Wald curvature, are that it is computable and, moreover, that it has even simpler, more practical approximations – see [70]). For further theoretical relative advantages of the types of curvature discussed above, see [54]. (In fact, we have first considered Wald's curvature – and the metric approach to curvature in general – as means of of computing, in a direct and applicable manner, Alexandrov's curvature.) For computational advantages of this approach, see Remarks 1.17 and 1.25.

However, a few further comments regarding the Wald vs. Alexandrov curvature choice are quite necessary at this point. The most important of these would be that one has take into account the "discrete" nature of the types of spaces considered, hence to compute solely the Wald curvature of the 1-star neighbourhood of a vertex, as already stressed above, and not to consider (ever) smaller neighbourhoods, as perhaps natural in other contexts. This, however, agrees with the method of computing discrete curvature as angular defect, as employed in the Brehm-Kühnel theorem above and in the Chow-Luo discrete Ricci flow (as well as in many other instances – see the bibliography for some of them). A positive consequence of this fact is that any such neighbourhood becomes a region having the same Alexandrov curvature bounded from below as the computed Wald one. Moreover, by the Alexandrov-Topogonov Theorem (see, e.g. [55], Theorem 43 and its proof, pp. 837–840), the whole surface becomes a space of curvature (Wald or Alexandrov) bounded from below.

In fact, due to the finiteness of its set of vertices, it also becomes a space of Alexandrov curvature bounded from above, thus satisfying the double curvature bounds, therefore enjoying a whole range of important properties (see [55] and the bibliography therein, as well as [69] for some of their applications). This property follows from a theorem proved independently by Kirk [40] and Berestovskii [5], asserting that a space in which the existence of geodesics is assured and, moreover, each point has a neighbourhood such that any four points of it can be mapped into some S^3_κ, $\kappa_1 \leq \kappa \leq \kappa_2$, is a space of curvature bounded from below by κ_1 and from above by κ_2. Indeed, the first condition is trivial, while the second one follows from our definition of neighbourhoods of vertices, the finiteness of the set of vertices and from the (obvious) fact that a metric quadruple emebeddable in S^2_κ is, a fortiori, embeddable in S^3_κ (where S^3_κ denotes the 3-dimensional analogue of S^2_κ).

On the other hand, considering only these "discrete" neighbourhoods is very important when equating the Wald and Alexandrov curvature, since it allows to avoid the blow-up of Alexandrov curvature at the vertices during smoothing. However, if one still wishes to consider smaller-and-smaller neighbourhoods of the vertices (motivated, perhaps, by other applications then Imaging and Graphics, such as those in Regge calculus [18]), one can resort to the basic approach of Brehm and Kühnel, that is "rounding" the edges by cylinders of radius ε (without any change in curvature) and replacing the polyhedral cones at the vertices by smooth "caps", up to a predetermined admissible error of, say, ε_1. Moreover, such a "filtration" of K_W by Gaussian curvature (of the approximating smooth surfaces) is in concordance with common practices in Imaging, Vision and, indeed, in many applicative fields.

Remark 1.27. Before proceeding to the main results, we should mention that the anonymous reviewer of an earlier, much more restricted version of the paper, brought to our attention the fact that Simon used [77], [78] a somewhat similar approach to define Ricci flow on spaces with a non-smooth metric tensor. Also, quite some time after this version of the paper was essentially finished, we noted that there are other works regarding the Ricci flow on surfaces with conical singularities [58], [89], as well as on (compact) Alexandrov surfaces [59]. However, we wish to emphasize that the approach herein is totaly independent of the works above, including Simon's (and, indeed, it was developed largely for other, more applications oriented ends).

2. Main results

From the "good", i.e. metric and curvature, approximations results above, it follows that one can study the properties of the metric Ricci flow via those of its smooth counterpart, by passing to a smoothing of the polyhedral surface. The heavier machinery of metric curvature considered above pays off, in the sense that, by using it, the "duality" between the combinatorics of the packings (and angles) and the metric disappears: The flow is purely metric and, moreover, the curvature at each stage (that is, for every "t") is given – as in the classical context – in an intrinsic manner, i.e. solely in terms of the metric.

A number of important properties now follow immediately.

2.1. Existence and uniqueness

In particular, the (local) existence and uniqueness of the *forward* (i.e. given by $dg_{ij}(t)/dt = 2K(t)g_{ij}(t)$) Ricci flows hold, on some *maximal* time interval $[0, T]; 0 < T \leq \infty$ (see, e.g. [21], as well as [36], for the original, different proof[4]). Moreover, the backward uniqueness of a solution (if existing) has been proven by Kotschwar [45] (see also [84] for a sketch of the proof). Beyond the theoretical importance, the existence and uniqueness of the backward flow would allow us to find surfaces in the conformal class of a given circle packing (Euclidean or Hyperbolic). More importantly, the use of the purely metric approach (based on the Wald curvature or any of other equivalent metric curvatures), rather than the combinatorial (and metric) approach of [20], allows us to give a first, tentative, purely theoretical at this point, answer to Question 2, p. 123, of [20], namely whether there exists a Ricci flow defined on the space of all piecewise constant curvature metrics (obtained via the assignment of lengths to a given triangulation of 2-manifold). Since, by Hamilton's results [36] (and those of Chow [19], for the case of the sphere), the Ricci flow exists for all compact surfaces, it follows from our arguments above that the fitting metric flow exits for surfaces of piecewise constant curvature. In consequence, given a surface of piecewise constant curvature (e.g. a mesh with edge lengths satisfying the triangle inequality for each triangle), one can evolve it by the Ricci flow, either forward, as in the works discussed above, to obtain, after the suitable area normalization, the polyhedral surface of constant curvature conformally equivalent to it; or backwards – if possible – to find the "primitive" family of surfaces (including the "original" surface) conformally equivalent to the given one. (Here, by "original", we mean the surface obtained via the backwards Ricci flow, at time T.) We shall dwell again upon the backward existence issue in the concluding remarks. At this point, we do, however, emphasize the fact

[4]See also [84], Theorems 5.2.1 and 5.2.2 and the discussion following them for short exposition of the main steps of the proof.

that it is not necessarily true, however, that all the surfaces obtained via the backwards flow are embedded (or, indeed, embeddable) in $\mathbb{R}^3$ – for details see Section 3 below. We can summarize the discussion above as

Proposition 2.1. *Let (S^2_{Pol}, g_{Pol}) be a compact polyhedral 2-manifold without boundary, having bounded metric curvature. Then there exists $T > 0$ and a smooth family of polyhedral metrics $g(t), t \in [0, T]$, such that*

$$\left\{ \begin{array}{ll} \frac{\partial g}{\partial t} = -2K_W(t)g(t) & t \in [0,T]\,; \\ g(0) = g_{Pol}\,. \end{array} \right. \tag{2.1}$$

(Here $K_W(t)$ denotes the Wald curvature induced by the metric $g(t)$.)

Moreover, both the forwards and the backwards (when existing) Ricci flows have the uniqueness of solutions property, that is, if $g_1(t), g_2(t)$ are two Ricci flows on S^2_{Pol}, such that there exists $t_0 \in [0, T]$ such that $g_1(t_0) = g_2(t_0)$, then $g_1(t) = g_2(t)$, for all $t \in [0, T]$.

In fact, the existence and uniqueness of the Ricci flow hold even if we do not restrict to compact surfaces, but we still require that the manifold is complete. Indeed, by applying the ideas employed in the proof above to a result of Shi [76] (see also [41]), we obtain the following

Proposition 2.2. *Let (S^2_{Pol}, g_{Pol}) be a complete polyhedral surface, such that $0 < K_W \le K_0$. Then there exists a (small) T as above, such that there exists a unique solution of (1.1) for any $t \in [0, T]$.*

Remark 2.3. Shi's result (hence the proposition above) does not necessarily hold for noncomplete surfaces – see [86] (See, however, Remark 2.7.) This is most pertinent for the case of images, as emphasized below:

Remark 2.4. By the same arguments as in Remarks 1.2 and 1.4 above, it follows that the forward existence, as well as uniqueness (both forwards and backwards) are also guaranteed for the combinatorial Ricci flow of images.

Before proceeding further, we should stress that, at this point, the metric approach introduced above is purely theoretical and, while it allows for a number of (theoretical) results to be inferred from the classical theory, it lacks (at least for now) the simple algorithmic capability of the combinatorial one of Chow and Luo, and certainly of its subsequent development – see [35]).

2.2. Convergence rate

A further type of result, highly important both from the theoretical viewpoint and for computer-driven applications, is that of the convergence rate.

Definition 2.5. A solution of (1.4) is said to be *convergent* iff

1) $\lim_{t\to\infty} K_i(t) = K_i(\infty)$, for all $1 \leq u \leq |V|$, where $K_i(\infty) \in (0, 2\pi)$;
2) $\lim_{t\to\infty} l_{ij}(t) = l_{ij}(\infty)$, $l_{ij}(\infty) > 0$.

A convergent solution is said to *converge exponentially fast* iff there exists constants c_1, c_2, such that, for any $t \geq 0$, the following inequalities hold:

1) $|K_i(t) - \bar{K}_i| \leq c_1 e^{-c_2 t}$;
2) $|l_{ij}(t) - \bar{l}_{ij}| \leq c_1 e^{-c_2 t}$

(The fitting definition for the flow (1.2) is immediate.)

For the combinatorial flow, it is shown in [20] that, in the case of background Euclidean (Theorem 1.1) or Hyperbolic (Theorem 1.2) metric, the solution – if it exists – converges, without singularities, exponentially fast to a metric of constant curvature. Using the classical results of [36] and [19], we can do slightly better, since we already know that the solution exists and it is unique (see the subsection below for the nonformation of singularities). Moreover, we can control the convergence rate of the curvature:

Theorem 2.6. *Let (S^2_{Pol}, g_{Pol}) be a compact polyhedral 2-manifold without boundary. Then the normalized metric Ricci flow converges to a surface of constant metric curvature. Moreover, the convergence rate is*

1) *exponential, if $\bar{K} = \bar{K}_W < 0$ (i.e. $\chi(S^2_{Pol}) < 0$);*
2) *uniform; if $\bar{K} = 0$ (i.e. $\chi(S^2_{Pol}) = 0$);*
3) *exponential, if $\bar{K} > 0$ (i.e. $\chi(S^2_{Pol}) > 0$).*

Proof. As already noted, a unique solution for the Ricci flow exists for all $0 < t \leq T$ and, again, these solutions are uniformly (conformally) equivalent. Indeed, by [36], Corollary 5.2 and [21], Proposition 5.15, there exists $C = C(\widetilde{g}_{Pol})$, where $\widetilde{g}_{Pol}$ is the smoothing of g_{Pol}, such that

$$\frac{1}{C}\widetilde{g}_{Pol} \leq \widetilde{g}_t \leq C\widetilde{g}_{Pol}\,, \tag{2.2}$$

where $\widetilde{g}_t$ is the smoothing of g_t, and the discussion above shows that the same holds for the polyhedral metrics. (It is in this sense that we say that the convergence is, in the case $\bar{K} = 0$, uniform.)

To estimate the convergence rate for the curvature, we make appeal to the following formulae (see, e.g. [21], Proposition 5.18): There exists $C' = C'(g_{Pol}) > 0$, (in fact, $C' = C'(\widetilde{g}_{Pol})$), such that

1) If $\bar{K} < 0$ then

$$\bar{K} - C' e^{\bar{K}t} \leq K(t) \leq \bar{K} + C' e^{\bar{K}t}\,; \tag{2.3}$$

2) If $\bar{K} = 0$ then

$$-\frac{C'}{1 + C't} \leq K(t) \leq C'\,; \tag{2.4}$$

3) If $\bar{K} > 0$ then

$$-C' e^{\bar{K}t} \leq K(t) \leq \bar{K} + C' e^{\bar{K}t}\,. \tag{2.5}$$

To show that the metric also converges with exponential rate, one has to make appeal to a refinement of (2.2), namely that the constant C therein is, in fact, given by $C = e^{2K_{Max}}$, where $K_{Max} = \max|K(t)|$, $t \in [0,T]$. (This holds, in fact, for the case of the general Ricci flow, with Gaussian curvature K being replaced, of course, by the Ricci curvature Ric – see, e.g. [84], Lemma 5.3.2.) In fact, given that the manifold under investigation is compact, hence of curvature bounded below and above, a stronger form of this improvement of (2.2) can be given, and, moreover, one that is better fitted for the case of polyhedral manifolds (see, e.g. [44] Lemma 27.1, Remark 27.5 and the following material):

$$e^{-K_{Max}t} \leq \frac{\mathrm{dist}_t(x,y)}{\mathrm{dist}_0(x,y)} \leq e^{K_{Max}t}\,, \tag{2.6}$$

where K_{Max} is as above.

By the approximation results above, namely Propositions 1.6 and 1.23, the result follows for the Wald-Berestovskii curvatures, respectively. Alternatively, one can more directly infer the respective convergence rates from (2.6) and (2.2) as far as the metric is concerned, and for the Wald-Berestovskii curvatures from (2.6) and (2.3)–(2.5). □

Remark 2.7. Existence, uniqueness and, furthermore, convergence rate results for polyhedral surfaces can be obtained, using the same techniques as before, for smooth surfaces with (a finite number of) *cusps* and *funnels*, using quite recent results of Isenberg, Mazzeo and Sesum [41] (for finite area surfaces) and Albin, Aldana and Rochon [1] (for surfaces of infinite area). We do not bring here the technical details – however interesting and relevant (see below) they might be – since they would bring us too far afield; for details and further related results, see [42].

Remark 2.8. Again, as in our previous Remarks 1.4 and 2.5, we obtain the corresponding result for the combinatorial Ricci flow of images, in this case for its rate of convergence. However, for the sake of honesty, we should note that a more realistic model for (gray-scale as well as color) images should probably be based on surfaces with boundary. It turns out that similar results can be obtained for this type of surface (see [14], [15]), however we postpone for further study the detailed analysis, in this model, of the metric Ricci flow of images.

2.3. Singularities formation

Another important aspect of any Ricci flow, be it smooth or discrete, is that of singularities formation. By [20], Theorem 5.1, for compact surfaces of genus ≥ 2, the combinatorial Ricci flow evolves without singularities. However, for surfaces of low genus no such result exists. Indeed, in the case of the Euclidean background metric, that is, the one of greatest interest in graphics, singularities do appear [33]. Such singularities are always combinatorial in nature and amount to the fact that, at some t, the edges of at least one triangle do not satisfy the triangle inequality [33]. These singularities are removed in heuristic manner, using the graphics equivalent of ε-moves (see, e.g. [23]). However, by [36], Theorem 1.1, the smooth Ricci flow exists at all times, i.e. no singularities form. By the considerations above, it follows that the metric Ricci flow also exists at all times without the formation of singularities. In fact, by a quite recent result of Topping [85], the same result holds even for unbounded (but complete) Riemannian 2-manifolds (M, g) with bounded curvature and satisfying a certain mild noncollapsing condition, namely that, there exists $r_0 > 0$, such that, for all $x \in M$, the following holds:

$$\mathrm{Vol}_g\left(B_g(x, r_0)\right) \geq \varepsilon > 0. \tag{2.7}$$

(Here, as usual, $B_g(x, r_0)$ denotes the open ball, in the metric g, of center x and radius r_0.)

Again, we can recap the discussion above as

Proposition 2.9. *Let (S^2_{Pol}, g_{Pol}) be a complete polyhedral 2-manifold, with at most a finite number of hyperbolic cusps (punctures), having bounded metric curvature and satisfying the noncollapsing condition (2.7). Then there exists a unique Ricci flow that contracts the cusps. Furthermore, the curvature remains bounded at all times during the flow.*

Remark 2.10. Both the boundedness and the noncollapsing conditions evidently hold for surfaces that appear in graphics, hence the fitting result for the metric flow also holds for this type of application. It follows that we can apply also in this context the cusp contracting result of [85]. One may, however, argue that manifolds with (hyperbolic) cusps do not appear in graphics, only compact manifolds (with or without boundary), but in fact many algorithms are modeled upon surfaces with punctures – see e.g. [35], [43].

3. Embeddability in $\mathbb{R}^3$

In this section we mainly consider a problem regarding smooth surfaces, and we hope that by now, the connection with its version for polyhedral surfaces is clear. It should be noted, in this context that, by [49], Theorem 8.8, any δ-approximation of an embedding is also an embedding, for small enough δ. Since, as we have already mentioned, smoothings represent δ-approximations, the possibility of using results regarding smooth surfaces to infer results regarding polyhedral embeddings is proven. (The other direction – namely from smooth to PL and polyhedral manifolds – follows from the fact that the *secant approximation* (see Appendix) is a δ-approximation if the simplices of the PL approximation satisfy a certain nondegeneracy condition – see [49], Lemma 9.3.) We wish to stress here the importance of the embeddability in graphics and image processing. In the only fully implemented Ricci flow, that is the combinatorial flow [35], [43], the goal is, in fact, to produce, via the circle packing metric, a conformal mapping from the given surface to a surface of constant (Gauss) curvature. Since in the relevant cases (see [20]) the surface in question is a planar region (usually a subset of the unit disk), its embeddability (not necessarily isometric) is trivial. Moreover, in the above mentioned works, there is no interest (and indeed, no need) to consider the (isometric) embeddability of the surfaces S^2_t (see below) for an intermediate time $t \neq 0, T$. However, this aspect is very important if one considers the problem of the Ricci flow for surfaces of piecewise constant curvature; as well as in image processing – see [3], [4]. (In fact, the results below represent an answer to a question, regarding precisely these applicative aspects of the Ricci flow, which was posed to us by Ron Kimmel.)

Let S^2_0 be a smooth surface of positive Gauss curvature, and let S^2_t denote the surface obtained at time t from S^2_0 via the Ricci flow. For all omitted background material (proofs, further results, etc.) we refer to [37].

Proposition 3.1. *Let S_0^2 be the unit sphere $\mathbb{S}^2$, equipped with a smooth metric g, such that $K(g) > 0$. Then the surfaces S_t^2 are (uniquely, up to a congruence) isometrically embeddable in $\mathbb{R}^3$, for any $t \geq 0$.*

Proof. By a now-classical result of Nirenberg [50] and Pogorelov [56] (independently), S_0^2 admits a smooth isometric embedding in $\mathbb{R}^3$.

The metric g_t being conformal to g_0, for any finite t (see [36]), we can write it as $g_t = e^{2\varphi}g_0$, for some smooth function φ. (This represents, in fact, the first, basic step in the proof of the Niremberg-Pogorelov theorem.)

We reparemeterize the flow, by a parameter τ, in such a manner that, if the original parameter t belongs to any specific (given) time interval $[0, t_0]$ it is replaced by $\tau \in [0, 1]$.

Then $K_\tau = e^{-2\varphi}(K_{g_0} - \Delta_g \varphi)$ (see [37], Lemma 2.1.3). From here, by elementary computations (see, e.g. [37], Lemma 9.1.4), we obtain that the Gauss curvature of this metric is

$$K_\tau = K(g_\tau) = e^{-2\varphi}(K_{g_0} - \tau\Delta_{g_0}\varphi) = \tau e^{2(1-t)\varphi}K_g + (1-\tau)e^{-2\tau\varphi} > 0\,.$$

Therefore, again by Nirenberg and Pogorelov's theorem, the surfaces S_t^2 are isometrically embeddable in $\mathbb{R}^3$, for any $t \geq 0$. □

We can, in fact, do somewhat better:

Corollary 3.2. *Let S_0^2 be a compact smooth surface. If $\chi(S_0^2) > 0$, then there exists some $t_0 \geq 0$, such that the surfaces S_t^2 are isometrically embeddable in $\mathbb{R}^3$, for any $t \geq t_0$.*

Proof. By the continuity of the Gauss curvature during the Ricci flow, it follows that, as some time t_0, the $|K_{t_0}| \leq K_0 > 0$. Applying again the arguments in the proof above, the corollary follows. □

We also briefly sketch an alternative proof,[5] of both the proposition and the corollary above, that makes no appeal to facts regarding embeddings, but only on results regarding the Ricci flow.

Alternative Proof In dimension 2, Ricci curvature essentially equates Gaussian curvature, i.e. scalar curvature, and since, for surfaces, $\frac{d\mathrm{scal}}{dt} = \Delta\mathrm{scal} + \mathrm{scal}^2$, it follows from the maximum principle (see e.g. [84], Theorem 3.1.1 and Corollary 3.1.2), that if scal is positive at $t = 0$, it remains positive for ant $t > 0$. From this fact, and from [19], Theorem 1.2, that states that, in the conditions of the corollary, the curvature will eventually become positive, the conclusion of the corollary now follows immediately. □

[5] for which the author is indebted to the anonymous reviewer of an earlier version of the paper

Remark 3.3. In this context it is impossible not to mention Alexandrov's results [2] regarding convex surfaces in $\mathbb{R}^3$: (A) Any convex surface, endowed with its intrinsic metric, is a manifold of nonnegative curvature; and, in essentially the opposite direction, (B) Any complete metric of positive curvature (nonnegative) on the sphere (the plane) represents the metric of a (not necessarily smooth) convex surface.

However, we should underline that requiring that a certain polyhedral sphere actually has (strictly) positive curvature at all its vertices it is quite a strong condition; indeed, it is a well known fact in graphics (see, e.g. [46]) that even the most standard polygonal approximations of the sphere, exhibit, even at high resolution, saddle points at certain vertices.

In contrast with this positive result regarding surfaces uniformized by the sphere, for (complete) surfaces uniformized by the Hyperbolic plane we have the following negative result:

Proposition 3.4. *Let (S^2_0, g_0) be a complete smooth surface, and consider the normalized Ricci flow on it. If $\chi(S^2_0) < 0$, then there exists some $t_0 \geq 0$, such that the surfaces S^2_t are not isometrically embeddable in $\mathbb{R}^3$, for any $t \geq t_0$.*

Proof. Since at time $T = +\infty$, the surface undergoing the flow has constant negative Gauss curvature, it is not smoothly ($\mathcal{C}^4$) embeddable in $\mathbb{R}^3$, by a classical theorem of Hilbert [39]. By the continuity of the Gauss curvature during the (normalized) Ricci flow, it follows that, at some time t_0, $K_{t_0} \leq K_0 < 0$. Therefore, by a result of Efimov [24], it follows that S_{t_0} admits no smooth ($\mathcal{C}^2$) isometric immersion (hence embedding) in $\mathbb{R}^3$. $\square$

Remark 3.5. Efimov [25] also proved that even if only the gradient of the Gaussian curvature is bounded (by some specific constant – see [25], [37]), there exists no smooth ($\mathcal{C}^3$) isometric immersion in $\mathbb{R}^3$, and no $\mathcal{C}^2$ isometric immersion exists if the suprema of K and of its gradient are $< \infty$. On the other hand, Hong [38] showed that smooth isometric immersions in $\mathbb{R}^3$ exist if the decay rate at infinity of K is slower than the inverse square of geodesic distance.

Obviously, the above mentioned results render appropriate versions of Proposition 3.4. However, their applicability is of far lesser interest in the context of the Ricci flow for polyhedral surfaces, so we do not formulate them explicitly.

Remark 3.6. In fact, the the situation is far worse, so to speak, than even Hilbert's and Efimov's theorems might suggest. Indeed, no general existence/nonexistence result is available, even if one restricts oneself to local

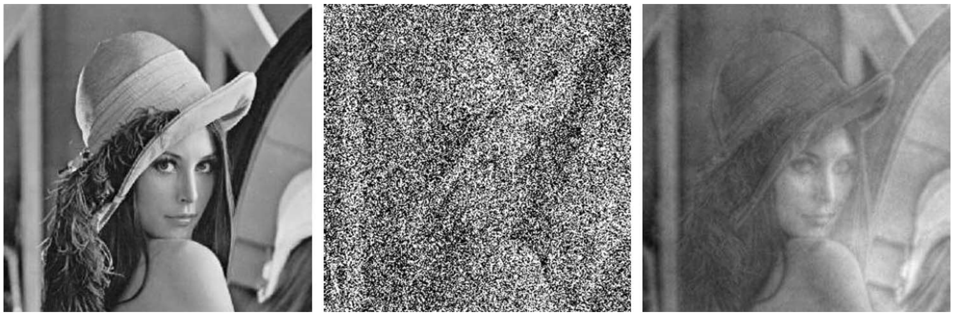

Figure 9: The combinatorial flow applied to a noisy version (middle) of "Lenna" (left), and the denoised image obtained after 3 iterations of the combinatorial Ricci flow. (Here the noise applied is of 3.5 db.) Even though in this case some priors have been used (namely the sign of the image's gradient), results like this one show, experimentally, the existence of the combinatorial Ricci flow and its embeddability in $\mathbb{R}^3$.

isometric embedding. Without going into too many details (since this would bring us out of our scope), it is known that such embeddings are possible if K does not vanish; or when $K(p) = 0$ and $dK(p) \neq 0$ or if $dK(p) \geq 0$ in some neighbourhood of a point p; and again when $K(p) = 0, dK(p) = 0$ and $\mathrm{Hess}K(p) < 0$. (For further details and bibliographical references see e.g. [37].)

However, as in the global case, (for lower differentiability classes), no general results are even possible. This was first shown by Pogorelov [57], who constructed a $\mathcal{C}^{2,1}$ metric on the unit disk $\mathbb{B}^2 = \mathbb{B}^2(0,1) \subset \mathbb{R}^2$, that admits no $\mathcal{C}^2$ isometric imbedding in $\mathbb{R}^3$ of $\mathbb{B}^2(0,r)$, for any $0 < r < 1$.

Remark 3.7. The experimental results obtained with the combinatorial Ricci flow for images show, contrary to the pessimistic outlook conveyed by Hilbert's and Efimov's results and Remark 3.6 above, not only the (short time) existence, but also the embeddability in R^3 of the flow. Indeed, the fact that one can successfully apply the combinatorial flow to an image and still obtain an image, bears testimony both to the existence of the combinatorial Ricci flow and to its embeddability in $\mathbb{R}^3$ (see Figure 9).

Remark 3.8. For other results on the embedding of the Ricci flow on manifolds of revolution, see [61], [81].

We mention in this context that a criterion for the local isometric embedding of polyhedral surfaces in $\mathbb{R}^3$, akin to the classical Gauss fundamental (compatibility) equation in the classical differential geometry of surfaces, was given in [67]. Namely, given a vertex v, with metric curvature $K_W(v)$, the following system of inequalities should hold:

$$\begin{cases} \max A_0(v) \leq 2\pi; \\ \alpha_0(v; v_j, v_l) \leq \alpha_0(v; v_j, v_p) + \alpha_0(v; v_l, v_p), \quad \text{for all } v_j, v_l, v_p \sim v; \\ V_\kappa(v) \leq 2\pi. \end{cases} \tag{3.1}$$

Here

$$A_0 = \max_i V_0\,; \tag{3.2}$$

"$\sim$" denotes incidence, i.e. the existence of a connecting edge $e_i = vv_j$ and, of course, $V_\kappa(v) = \alpha_\kappa(v; v_j, v_l) + \alpha_\kappa(v; v_j, v_p) + \alpha_\kappa(v; v_l, v_p)$, where $v_j, v_l, v_p \sim v$, etc.

Note that the first two inequalities represent the (extrinsic) embedding condition, while the third one represents the intrinsic curvature (of the PL manifold) at the vertex v.

For details and a fitting global embedding criterion see [67].

Remark 3.9. In Formula 3.1 the correct definition of the open neighbourhood of a vertex is essential (see the relevant discussion in Section 1).

4. Final remarks

We conclude with a number of concluding remarks regarding the feasibility of this approach and sketching some further directions of study.

1) As it is clear by now to the reader, unfortunately we have no experimental data with the metric Ricci flow (with the exception of the related combinatorial curvature and flow). Therefore, the main future task is to convert this into an algorithm and experiment with it on triangular (or even more general polygonal) meshes.
2) In practice, especially in Imaging and Vision, one often wants to use larger neighbourhoods (masks). Therefore, it would be useful to better understand how to efficiently compute the Wald curvature for larger neighbourhoods (2-star, 3-star neighbourhoods, etc.) and how to efficiently implement this calculation in a practical algorithm (see the discussion in (1) above).
3) The previous observations conduct us naturally to muse upon the computation complexity of the flow introduced in the present paper. This

question is inherently divided into two parts: The computational complexity of the calculating Wald curvature and, respectively, that of the flow itself.

We have discussed the first aspect to some extent in Remarks 1.17 and 1.25, at least as far as the problem of computing embedding curvature for triangular meshes is concerned. (In particular, we should recall that, precisely like in the implementation of the combinatorial Ricci flow (see also below) a mesh improvement method needs to be applied first, thus complexity increasing accordingly.) If, however, one wishes to reproduce the theoretical process and deal with smooth surfaces, then one has to make appeal to one of the standard smoothing techniques available in the Graphics or Imaging arsenal of tools (such as making appeal to splines, wavelets, etc.). Of course, in this case, the complexity of the smoothing method compounds the basic one of calculating the metric curvature.

As far as the complexity of the flow is concerned, we must emphasize that in our basic approach, smoothing allows us to pass to the classical flow, thus obtaining almost effortlessly its properties detailed in Sections 2 and 3. If remaining in this rather theoretical setting, it appears quite difficult to determine the complexity of the algorithm. However, if we pass to the purely metric flow (see discussion below), then the complexity of our method equates that of the combinatorial flow of Gu et al. (for reasons that we shall expound upon shortly).

4) One would be tempted to extend this method to higher dimensional manifolds, both for theoretical reasons and because of their applications (Medical Imaging, Video, etc.) However, it is not clear how to correctly define a flow in this case: Indeed, 3-dimensional analogues of all the pertinent results on the Ricci flow for smooth surfaces have, by and large yet to be obtained. Moreover, even defining a metric Ricci curvature in dimension 3 and higher is a daunting task. We have proposed one such PL metric curvature for the case $n = 3$ in [68], and experiments with it, in collaboration with D. Gu, are currently underway.

5) The most far-reaching results in this direction would be to develop a purely metric Ricci flow, that is without making appeal to smoothings. Some first, tentative results in this direction are presented in study. One basic observation that has to be made is that the lack of symmetry that we mentioned when we first introduced the metric flow in Section 1 will not not disappear by passing to the limit, and has to be

dealt with in a different and direct manner. From symmetry reasons, a natural way of defining the flow is (using the same notation as before):

$$\frac{dl_{ij}}{dt} = -\frac{K_i + K_j}{2} l_{ij} \,, \tag{4.1}$$

where in this case, K_i, K_j denote, of course, the Wald curvature at the vertices v_i and v_j, respectively. It is also important to notice that, in fact, this expression appears also in the practical method of computing the combinatorial curvature, where it is derived via the use of a conformal factor (see [35]).

An important consequence of such a purely metric flow would be that, precisely as in the case of combinatorial Ricci flow of Chow and Luo [20], Equations (1.1) and 1.2) become – due to the fact that K_i depends only on the lengths of the edges l_{ij}, and not on their derivatives – ODE's, instead of PDE's, thence they are easier to study and enjoy better properties. In particular (and of importance in applications) the metric flow will have the backward existence property.

Perhaps the most natural way of developing an efficient purely metric flow would be to follow the same basic pattern as in the combinatorial flow of Gu et al., namely by first showing that the initial metric prescribes the conformal class of the flow, i.e. the metric at time is given by $g_t = w_t(x)g_0$, g_0 being the initial metric, thus being able to write the flow as heat type equation in w. In fact, this seems to be a quite straightforward task if one is identifying (as we have done at times in this paper) Wald and Alexandrov curvatures and if, furthermore, one is willing to make appeal to Richard's recent results on Ricci flow on Alexandrov surfaces [59].[6] However, we believe that this task is achievable directly, without making appeal to such "ready made" tools.

Since we have not discussed the relevance of the reversibility of the flow when we first introduced it, in Section 2, it is perhaps better to emphasize in this context its importance in a variety of Imaging and Graphics tasks, such as morphing, registration, deformation estimates in medical images and sampling (or rather resampling). A more detailed study

[6] As already stressed before, our approach here is different from Richard's work (being much more direct and, in a sense, more elementary) and we became aware of his work long after the paper was essentially written. We should also underline (yet again) that our method facilitates concrete, computational treatment of the flow.

of some of these endeavors, at least from the more theoretical point of view, represents work currently in progress.

6) We can hardly conclude this article without drawing a comparison, even if only a brief one, between the metric flow proposed herein and the combinatorial one of Gu et al. As already stressed in the introduction, our approach lacks the elegance and esthetic allure of the circle packing approach. Furthermore, as we have noted above, at this point in time we are only beginning to develop an efficient method of actual computation of the flow. However, as we also remarked, once this is achieved, the two methods will probably perform with similar computational effectiveness.

   On the positive side, we should emphasize again that, even in the absence of a concrete and practical method of computation, by passing to the smooth case, we were able to obtain essentially effortlessly a number of very important properties of the flow (some of which it would seem we considered for the first time in the context of Imaging and Graphics), such as uniqueness, reversibility, singularities formation and embeddability in $\mathbb{R}^3$. This is the point to add, before we proceed further, that the combinatorial flow also has the reversibility property, precisely as our metric one.

   On the other hand, the combinatorial flow is not realisable in $\mathbb{R}^3$ (at least no method of doing this is known to us), while the metric flow has this capability. Therefore, the first one is ideally suited for such tasks as, for instance, registration, where the flow evolves till $T = \infty$, i.e. to the (simply connected) gauge surface of constant curvature; whereas the metric flow is best fitted, due to the parallels with the classical theory, for the study of short time evolution processes that appear in such Imaging tasks as smoothing, sharpening and denoising.

   Moreover, while due to its very definition, the combinatorial flow performs at great efficiency in the context of Graphics where smooth surfaces are subliminally presumed (if not the proclaimed norm). On the other hand, the metric flow, given its definition, needs no smoothness assumptions (even implicit one) and it performs as well on any data, as long as a polygonal structure is assumed. Therefore, it appears to be best suited for settings with “rough” data, such as Ultrasound Imaging and even, perhaps, some Manifold Learning tasks.

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Appendix

We include here the modicum of differential topology needed (mainly in the alternative proof of Proposition 1.6). Our source for this material is [49]. We presume that the reader is familiar with the basic concepts (simplicial complexes, triangulations, etc.) however, as a background text, we warmly recommend Munkres' notes [49].

Definition 4.1. 1) Let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}^*_+$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a *δ-approximation to f* iff:
(i) There exists a subdivision K' of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$, for any $x \in |K|$;
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{eucl}(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.

2) Let K' be a subdivision of K, $U = \mathring{U}$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n)$, $g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a *$\delta$-approximation* of f (on U) iff conditions (ii) and (iii) above hold for any $a \in U$.
(Here $St(a, K)$ denotes, as it standardly does, the *star* of a (in K), i.e. $St(a, K) = \bigcup_{a \in \sigma, \sigma \in K} \sigma$.

Recall that in the PL context the differential (of a map) is defined as follows:

Definition 4.2. Let σ be a simplex, and let $f : \sigma \to \mathbb{R}^n,\ f \in \mathcal{C}^r$. If $a \in \sigma$ we define $df_a : \sigma \to \mathbb{R}^n$ as follows: $df_a(x) = Df(a)\cdot(x-a)$, where $Df(a)$ denotes the Jacobian matrix $Df(a) = (\partial f_i/\partial x^j)_{1\leq i,j\leq n}$, computed with respect to some orthogonal coordinate system contained in $\Pi(\sigma)$, where $\Pi(\sigma)$ is the hyperplane determined by σ. The map $df_a : \sigma \to \mathbb{R}^n$ does not depend upon the choice of this coordinate system.

Moreover, $df_a|_{\sigma\cap\tau}$ is well defined, for any $\sigma, \tau \in \overline{St}(a, K)$. Therefore the map $df_a : \overline{St}(a, K) \to \mathbb{R}^n$ is well-defined and continuous, and it is called – analogous to the case of differentiable manifolds, the differential of f.

Definition 4.3. Let K' be a subdivision of K and let $f \in \mathcal{C}^r(K, \mathbb{R}^n),\ g \in \mathcal{C}^r(K', \mathbb{R}^n)$ be non-degenerate mappings (i.e. $rank(f|_\sigma) = rank(g|_\sigma) = dim\,\sigma$, for any $\sigma \in K$) and let $U = \overset{\circ}{U} \subset |K|$. The mapping g is called an α-*approximation* (of f on U) iff:

$$\angle\big(df_a(x), dg_a(x)\big) \leq \alpha\,;\ \text{for any } a \in U, \text{ and any } x \in \overline{St}(a, K'),\ a \neq x. \tag{4.2}$$

As expected, a fine enough δ-approximation is also an α-approximation:

Lemma 4.4 ([49], Lemma 8.7)**.** *Let K be a (finite) simplicial complex and let $f : K \to \mathbb{R}^n$ be a non-degenerate $\mathcal{C}^r, 1 \leq r \leq \infty$ map. Then, for any $\alpha > 0$, there exists $\delta = \delta(\alpha) > 0$ such that any non-degenerate $\mathcal{C}^r$ map $g : K' \to \mathbb{R}^n$, which is a δ-approximation of f on some open set U, is also an α-approximation of f on U. (Here K' denotes, as above, a subdivision of K.)*

We conclude the appendix with the following definition:

Definition 4.5. Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then the linear map: $L_s : s \to \mathbb{R}^n$, defined by $L_s(v) = f(v)$ where v is a vertex of s, is called the *secant map induced by f*.

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