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Global regularity of the regularized Boussinesq equations with zero diffusion

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Abstract. In this paper, we consider the n-dimensional regularized incompressible Boussinesq equations with a Leray-regularization through a smoothing kernel of order α in the quadratic term and a β-fractional Laplacian in the velocity equation. We prove the global regularity of the solution to the n-dimensional logarithmically supercritical Boussinesq equations with zero diffusion. As a direct corollary, we obtain the global regularity result for the regularized Boussinesq equations with zero diffusion in the critical case $\alpha+\beta = \frac{1}{2}+\frac{n}{4}$. Therefore, our results settle the global regularity case previously mentioned in the literatures.

Contents

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1. Introduction and main results

The classical n-dimensional ($n \geq 3$) incompressible Boussinesq equations with zero diffusion take the following form

$$(1.1) \qquad \begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = \theta e_n, \quad x \in \mathbb{R}^n, \quad t > 0, \\ \partial_t \theta + (u \cdot \nabla)\theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases}$$

where $u(x, t) = (u_1(x, t), u_2(x, t), \cdots, u_n(x, t))$ is a vector field denoting the velocity, $\theta = \theta(x, t)$ is a scalar function denoting the temperature, p is the scalar pressure and e_n is the unit vector $(0, 0, \cdots, 1)$. u_0 and θ_0 are the given initial data, with $\nabla \cdot u_0 = 0$. The Boussinesq equations arise from a zeroth order approximation to the coupling between Navier-Stokes equations and the thermodynamic equations. The Boussinesq equations model large scale atmospheric and oceanic flows responsible for cold fronts and the jet stream (see e.g. [**6, 17, 9, 20**]).

The system (1.1) first arose as a model to study natural convection phenomena in geophysics [**20**], as for example in the very important Rayleigh-Bénard problem [**6**]. From the mathematical point of view, full inviscid case is analogous to the incompressible axi-symmetric swirling three dimensional Euler equations (see e.g. [**17**]). For that reason, the global well-posedness of this model has recently attracted a lot of attention. When the spatial dimensions $n = 2$, almost at the same time Chae [**5**] and Hou-Li [**13**] successfully established the global well-posedness of the problem (1.1) by estimating the vorticity and using the Brezis-Wainger inequality (see for example [**1, 11, 8**] for rough initial data). It is worthwhile to mention that when $-\Delta$ was weakened to $\sqrt{-\Delta}$, Hmidi-Keraani-Rousset [**12**] successfully established the global well-posedness result to the system (1.1) by further studying the structure of the coupled system. However, when the spatial dimensions $n \geq 3$, the global well-posedness to the system (1.1) still remains open, which is a very challenging open problem in the fluid mechanics, due in large part to the supercritical nature of the equation with respect to the energy. In fact, because the system at $\theta \equiv 0$ is reduced to the classical three dimensional Navier-Stokes equations, such an issue seems to be extremely challenging. Available global-in-time results in three dimensions require the initial data to be small [**8**]. Remarkably, some interesting models have been proposed to bypass this supercriticality of the multi-dimensional case. For example, if we impose some more dissipation on the velocity equation of the system (1.1), then one may expect to obtain the global well-posedness result. As a matter of fact, for the case $n = 3$, Xiang-Yan [**22**] strengthened the dissipation by replacing $-\Delta$ with $(-\Delta)^\beta$ for $\beta > \frac{5}{4}$ and proved the global well-posedness for the corresponding system. This result was sharped to the case $\beta = \frac{5}{4}$ by several works [**25, 14, 23**], which is consistent with the result of the fractional Navier-Stokes equations (see [**16, 18, 21, 3**]).

Inspired by Olson and Titi [**19**], another interesting model is to regularize the velocity through a smoothing kernel of order α in the nonlinear term and a β-fractional Laplacian, where the key idea is that a weaker nonlinearity and a strong viscous dissipation could work together to imply the regularity. More precisely, we are interested in considering the following regularized n-dimensional incompressible

Boussinesq equations with zero diffusion

$$(1.2)\qquad \begin{cases} \partial_t v + (u\cdot\nabla)v + \Lambda^{2\beta}v + \nabla p = \theta e_n, \quad x\in\mathbb{R}^n, \quad t>0,\\ \partial_t\theta + (u\cdot\nabla)\theta = 0,\\ v = u + \Lambda^{2\alpha}u,\\ \nabla\cdot u = \nabla\cdot v = 0,\\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases}$$

where the fractional Laplacian operator $\Lambda^\gamma := (-\Delta)^{\frac{\gamma}{2}}$ denotes the Zygmund operator which is defined through the Fourier transform, namely

$$\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \hat{f}(\xi).$$

The physical motivation of this regularization defined in terms of smoothing kernels is very related to a sub-grid length scale in the model and these kernels work as a kind of filter with certain widths (see [**19**] for more details). When $\theta = 0$, the system (1.2) reduces to the Leray-α Navier-Stokes equations with fractional dissipation. Very recently, Barbato, Morandin and Romito [**3**] investigated this Leray-α Navier-Stokes equations and established the global well-posedness result when $\alpha \geq 0$, $\beta \geq 0$ and $\alpha+\beta \geq \frac{1}{2}+\frac{n}{4}$, which is true even some optimal logarithmic corrections ([**3**, Theorem 1.2] for details). Thus, this also solved a conjecture formulated by Tao in [**21**]. For the case $n = 3$, Bessaih and Ferrario [**4**] proved that the above system (1.2) is globally well-posed when $\alpha+\beta \geq \frac{5}{4}$ and $\frac{1}{2} < \beta < \frac{5}{4}$. The main object of this paper is to remove the unnatural restriction $\frac{1}{2} < \beta < \frac{5}{4}$ and to extend the result to arbitrary spatial dimensions. To begin with, for the system (1.2) with $\alpha = 0$, we will show the following result, which settles the global regularity case suggested by the author in [**24**] (see Remark 1.1 (2) for details). More precisely, the first result can be stated as follows.

THEOREM 1.1. *Consider the following n-dimensional ($n \geq 3$) incompressible Boussinesq equations with zero diffusion*

$$(1.3)\qquad \begin{cases} \partial_t u + (u\cdot\nabla)u + \mathcal{L}^2 u + \nabla p = \theta e_n, \quad x\in\mathbb{R}^n, \quad t>0,\\ \partial_t\theta + (u\cdot\nabla)\theta = 0,\\ \nabla\cdot u = 0,\\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases}$$

where the operator $\mathcal{L}$ is defined by

$$\widehat{\mathcal{L}u}(\xi) = \frac{|\xi|^\beta}{g(|\xi|)}\widehat{u}(\xi)$$

for some non-decreasing, radially symmetric function $g(\tau) \geq 1$ defined on $\tau \geq 0$. Let $(u_0,\, \theta_0) \in H^s(\mathbb{R}^n)$ with $s > 1+\frac{n}{2}$. If $\beta \geq \frac{1}{2}+\frac{n}{4}$ and g satisfies

$$(1.4)\qquad \int_e^\infty \frac{d\tau}{\tau g^4(\tau)} = \infty,$$

then the system (1.3) admits a unique global solution $(u,\, \theta)$ such that for any given $T > 0$,

$$(u,\, \theta) \in L^\infty([0,T]; H^s(\mathbb{R}^n)), \quad \mathcal{L}u \in L^2([0,T]; H^s(\mathbb{R}^n)).$$

REMARK 1.1. The typical examples of g are as follows

$$g(\tau) = \big[\ln(e+\tau)\big]^{\frac{1}{4}};$$
$$g(\tau) = \big[\ln(e+\tau)\ln(e+\ln(e+\tau))\big]^{\frac{1}{4}};$$
$$g(\tau) = \big[\ln(e+\tau)\ln(e+\ln(e+\tau))\ln(e+\ln(e+\ln(e+\tau)))\big]^{\frac{1}{4}}.$$

For the case $\beta > 0$, we have the following logarithmical result.

THEOREM 1.2. *Consider the following n-dimensional ($n \geq 3$) incompressible Boussinesq equations with zero diffusion*

$$\begin{cases} \partial_t v + (u\cdot\nabla)v + \Lambda^{2\beta} v + \nabla p = \theta e_n, \quad x\in\mathbb{R}^n, \quad t>0, \\ \partial_t \theta + (u\cdot\nabla)\theta = 0, \\ v = u + \mathcal{L}^2 u, \\ \nabla\cdot u = \nabla\cdot v = 0, \\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.5}$$

where the operator $\mathcal{L}$ is defined by

$$\widehat{\mathcal{L}u}(\xi) = \frac{|\xi|^\alpha}{g(|\xi|)}\widehat{u}(\xi)$$

for some non-decreasing, radially symmetric function $g(\tau) \geq 1$ defined on $\tau \geq 0$. Let $(v_0,\,\theta_0) \in H^s(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$. If $\alpha + \beta \geq \frac{1}{2} + \frac{n}{4}$ with $\beta > 0$ and g satisfies

$$\int_e^\infty \frac{d\tau}{\tau g^4(\tau)} = \infty, \tag{1.6}$$

then the system (1.5) admits a unique global solution $(v,\,\theta)$ such that for any given $T > 0$,

$$(v,\,\theta) \in L^\infty([0,T]; H^s(\mathbb{R}^n)), \quad v \in L^2([0,T]; H^{s+\beta}(\mathbb{R}^n)).$$

For the endpoint case $\beta = 0$, we have the following result.

THEOREM 1.3. *Consider the following n-dimensional ($n \geq 3$) incompressible Boussinesq equations with zero diffusion*

$$\begin{cases} \partial_t v + (u\cdot\nabla)v + \nabla p = \theta e_n, \quad x\in\mathbb{R}^n, \quad t>0, \\ \partial_t \theta + (u\cdot\nabla)\theta = 0, \\ v = u + \mathcal{L}^2 u, \\ \nabla\cdot u = \nabla\cdot v = 0, \\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.7}$$

where the operator $\mathcal{L}$ is defined by

$$\widehat{\mathcal{L}u}(\xi) = \frac{|\xi|^\alpha}{g(|\xi|)}\widehat{u}(\xi)$$

for some non-decreasing, radially symmetric function $g(\tau) \geq 1$ defined on $\tau \geq 0$. Let $(v_0,\,\theta_0) \in H^s(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$. If $\alpha \geq \frac{1}{2} + \frac{n}{4}$ and g satisfies

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}\, g^2(\tau)} = \infty, \tag{1.8}$$

then the system (1.7) admits a unique global solution $(v,\, \theta)$ *such that for any given* $T > 0$,

$$(v,\, \theta) \in L^\infty([0,T]; H^s(\mathbb{R}^n)).$$

As a direct corollary of Theorems 1.1, Theorem 1.2 and Theorem 1.3, we have the following global regularity result.

COROLLARY 1.4. *Let* $n \geq 3$ *and* $(v_0,\, \theta_0) \in H^s(\mathbb{R}^n)$ *with* $s > 1 + \frac{n}{2}$. *If* $\alpha \geq 0$ *and* $\beta \geq 0$ *satisfy*

$$\alpha + \beta \geq \frac{1}{2} + \frac{n}{4},$$

then the system (1.2) admits a unique global solution $(v,\, \theta)$ *such that for any given* $T > 0$,

$$(v,\, \theta) \in L^\infty([0,T]; H^s(\mathbb{R}^n)), \quad v \in L^2([0,T]; H^{s+\beta}(\mathbb{R}^n)).$$

Motivated by the proof of Theorem 1.1 and Theorem 1.2, one may show that if $\beta > 1$, then Corollary 1.4 can be improved as follows. For the sake of convenience, we sketch the proof in Appendix B.

THEOREM 1.5. *Consider the following* n*-dimensional* $(n \geq 3)$ *incompressible Boussinesq equations with zero diffusion*

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \mathcal{L}^2 v + \nabla p = \theta e_n, \quad x \in \mathbb{R}^n, \quad t > 0, \\ \partial_t \theta + (u \cdot \nabla)\theta = 0, \\ v = u + \Lambda^{2\alpha} u, \\ \nabla \cdot u = \nabla \cdot v = 0, \\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.9}$$

where the operator $\mathcal{L}$ *is defined by*

$$\widehat{\mathcal{L}u}(\xi) = \frac{|\xi|^\beta}{g(|\xi|)} \widehat{u}(\xi)$$

for some non-decreasing, radially symmetric function $g(\tau) \geq 1$ *defined on* $\tau \geq 0$. *Let* $(v_0,\, \theta_0) \in H^s(\mathbb{R}^n)$ *with* $s > 1 + \frac{n}{2}$. *If* $\alpha + \beta \geq \frac{1}{2} + \frac{n}{4}$ *with* $\beta > 1$ *and* g *satisfies*

$$\int_e^\infty \frac{d\tau}{\tau g^4(\tau)} = \infty, \tag{1.10}$$

then the system (1.9) admits a unique global solution $(v,\, \theta)$ *such that for any given* $T > 0$,

$$(v,\, \theta) \in L^\infty([0,T]; H^s(\mathbb{R}^n)), \quad \mathcal{L}v \in L^2([0,T]; H^s(\mathbb{R}^n)).$$

The rest of the paper is organized as follows: Section 2 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 is presented in Section 3. The simple proof of Theorem 1.3 is carried out in Section 4. Finally, in Appendix A, we present some preparatory results on Besov spaces, and give the proof of some facts used in our proof. Moreover, the proof of Theorem 1.5 is added in Appendix B.

2. The proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. It is worthwhile pointing out that the existence and uniqueness of local smooth solutions in the functional spaces H^s with $s > 1 + \frac{n}{2}$ can be established via a standard approach as in the case of the Euler and Navier-Stokes equations (see [**7, 17**]). Thus, in order to complete the proof of Theorem 1.4, it is sufficient to establish *a priori* estimates that hold for any fixed $T > 0$. In this paper, all constants will be denoted by C that is a generic constant depending only on the quantities specified in the context. We shall write $C(\gamma_1, \gamma_2, \cdots, \gamma_k)$ as the constant C depends on the quantities $\gamma_1, \gamma_2, \cdots, \gamma_k$. We denote $A \approx B$ if there exist two constants $C_1 \leq C_2$ such that $C_1 B \leq A \leq C_2 B$.

We remark that our main efforts are devoted to the proof of the critical case $\beta = \frac{1}{2} + \frac{n}{4}$ as the subcritical case $\beta > \frac{1}{2} + \frac{n}{4}$ can be handled in the same manner with only some suitable modifications. We begin with the basic energy estimates.

Lemma 2.1. *Assume $u_0 \in L^2(\mathbb{R}^n)$ and $\theta_0 \in L^p(\mathbb{R}^n)$ for some $p \in [2, \infty]$. Then the corresponding solution (u, θ) of the system (1.3) admits the following bounds for any $t > 0$*

$$\|u(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{\gamma_1} u(\tau)\|_{L^2}^2 + \|\mathcal{L}u(\tau)\|_{L^2}^2)\, d\tau \leq C(t, u_0, \theta_0), \tag{2.1}$$

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p} \tag{2.2}$$

for any $\gamma_1 \in (0, \frac{n+2}{4})$.

Proof of Lemma 2.1. Let $p \in [2, \infty]$, then we have by multiplying the equation $(1.3)_2$ by $|\theta|^{p-2}\theta$ and integrating it over $\mathbb{R}^n$

$$\frac{d}{dt}\|\theta(t)\|_{L^p} = 0,$$

where we have used

$$\int_{\mathbb{R}^n} (u \cdot \nabla\theta)|\theta|^{p-2}\theta\, dx = 0.$$

We thus obtain

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

Multiplying equation $(1.3)_1$ by u and integrating the resultant over $\mathbb{R}^n$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^n} \mathcal{L}^2 u \cdot u\, dx &= \int_{\mathbb{R}^n} \theta e_n \cdot u\, dx \\ &\leq \|u\|_{L^2}\|\theta\|_{L^2}.\end{aligned}$$

Thanks to Plancherel's theorem, it implies

$$\int_{\mathbb{R}^n} \mathcal{L}^2 u \cdot u\, dx \quad = \quad \int_{\mathbb{R}^n} \frac{|\xi|^{1+\frac{n}{2}}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2\, d\xi = \|\mathcal{L}u\|_{L^2}^2.$$

According to the assumptions on g (more precisely, g grows logarithmically), one may conclude that for any fixed $\delta > 0$, there exists $N = N(\delta)$ satisfying

$$g(r) \leq \widetilde{C} r^\delta, \quad \forall\, r \geq N$$

with the constant $\widetilde{C} = \widetilde{C}(\delta)$. Therefore, we have for any $0 < \sigma < \frac{2+n}{4}$

$$\|\mathcal{L}u\|_{L^2}^2 = \int_{|\xi|<N(\sigma)} \frac{|\xi|^{1+\frac{n}{2}}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2\, d\xi + \int_{|\xi|\geq N(\sigma)} \frac{|\xi|^{1+\frac{n}{2}}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2\, d\xi$$

$$
\begin{aligned}
&\geq \int_{|\xi|\geq N(\sigma)} \frac{|\xi|^{1+\frac{n}{2}}}{\big[\widetilde{C}|\xi|^{\sigma}\big]^2}|\widehat{u}(\xi)|^2\,d\xi \\
&= \int_{\mathbb{R}^n} \frac{|\xi|^{1+\frac{n}{2}}}{\big[\widetilde{C}|\xi|^{\sigma}\big]^2}|\widehat{u}(\xi)|^2\,d\xi - \int_{|\xi|<N(\sigma)} \frac{|\xi|^{1+\frac{n}{2}}}{\big[\widetilde{C}|\xi|^{\sigma}\big]^2}|\widehat{u}(\xi)|^2\,d\xi \\
&\geq C_0\|\Lambda^{\frac{2+n-4\sigma}{4}}u\|_{L^2}^2 - \widetilde{C_0}\|u\|_{L^2}^2, \qquad (2.3)
\end{aligned}
$$

where C_0 and $\widetilde{C_0}$ depend only on σ. This implies for any $\gamma_1 \in (0,\ \frac{n+2}{4})$

$$\|\mathcal{L}u\|_{L^2}^2 \geq C_1\|\Lambda^{\gamma_1}u\|_{L^2}^2 - C_2\|u\|_{L^2}^2. \tag{2.4}$$

Now we obtain that

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\Lambda^{\gamma_1}u\|_{L^2}^2 + \|\mathcal{L}u\|_{L^2}^2 \leq \|u\|_{L^2}\|\theta\|_{L^2} + C\|u\|_{L^2}^2.$$

Using the Gronwall inequality and (2.2), the desired (2.1) follows directly. □

The following estimate plays a crucial role in proving our main result.

LEMMA 2.2. *Assume u_0, θ_0, $\mathcal{L}u_0 \in L^2(\mathbb{R}^n)$. Then the corresponding solution (u,θ) of the system (1.3) admits the following bounds for any $t>0$*

$$\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t \|\mathcal{L}^2u(\tau)\|_{L^2}^2\,d\tau \leq C(t,u_0,\theta_0). \tag{2.5}$$

PROOF OF LEMMA 2.2. Multiplying equation $(1.3)_1$ by $\mathcal{L}^2u$ and integrating the resulting equation over $\mathbb{R}^n$, it follows from integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2 + \|\mathcal{L}^2u\|_{L^2}^2 = -\int_{\mathbb{R}^n}(u\cdot\nabla u)\cdot\mathcal{L}^2u\,dx + \int_{\mathbb{R}^n}\theta e_n\cdot\mathcal{L}^2u\,dx.$$

Applying the same argument used in establishing (2.3), we deduce for any $\gamma_2 \in (0,\ \frac{n+2}{2})$ that

$$\|\mathcal{L}^2u\|_{L^2}^2 \geq C_3\|\Lambda^{\gamma_2}u\|_{L^2}^2 - C_4\|u\|_{L^2}^2. \tag{2.6}$$

By using the Young inequality, we have

$$\int_{\mathbb{R}^n}\theta e_n\cdot\mathcal{L}^2u\,dx \leq \|\theta\|_{L^2}\|\mathcal{L}^2u\|_{L^2} \leq \frac{1}{4}\|\mathcal{L}^2u\|_{L^2}^2 + C\|\theta\|_{L^2}^2.$$

With the aid of the Gagliardo-Nirenberg inequality, one gets

$$
\begin{aligned}
-\int_{\mathbb{R}^n}(u\cdot\nabla u)\cdot\mathcal{L}^2u\,dx &\leq \|u\cdot\nabla u\|_{L^2}\|\mathcal{L}^2u\|_{L^2} \\
&\leq C\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\mathcal{L}^2u\|_{L^2} \\
&\leq C\|\nabla u\|_{L^{\frac{4n}{n+2}}}^2\|\mathcal{L}^2u\|_{L^2}. \qquad (2.7)
\end{aligned}
$$

From the high-low frequency technique, it leads to

$$\|\nabla u\|_{L^{\frac{4n}{n+2}}} \leq \|S_N\nabla u\|_{L^{\frac{4n}{n+2}}} + \sum_{j\geq N}\|\Delta_j\nabla u\|_{L^{\frac{4n}{n+2}}},$$

where the operators S_j and Δ_j are defined in the Appendix and N will be specialized later. By Plancherel's theorem and the Sobolev embedding $\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n) \hookrightarrow \dot{W}^{1,\frac{4n}{n+2}}(\mathbb{R}^n)$, we obtain

$$\|S_N\nabla u\|_{L^{\frac{4n}{n+2}}} \leq C\|S_N\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}$$

$$
\begin{aligned}
=&C\|\chi(2^{-N}\xi)|\xi|^{\frac{1}{2}+\frac{n}{4}}\widehat{u}(\xi)\|_{L^2}\\
=&C\Big\|\chi(2^{-N}\xi)g(|\xi|)\frac{|\xi|^{\frac{1}{2}+\frac{n}{4}}}{g(|\xi|)}\widehat{u}(\xi)\Big\|_{L^2}\\
\leq&Cg(2^N)\|\mathcal{L}u\|_{L^2},
\end{aligned}
$$

where χ and φ are associated with the definition of Besov spaces (see Appendix for details). Here and in what follows, the following embedding facts will be used frequently: For $s_1>s_2$, $p_1,p_2\in(1,\infty)$ satisfying $s_1-\frac{n}{p_1}=s_2-\frac{n}{p_2}$, it holds

$$
\dot{W}^{s_1,p_1}(\mathbb{R}^n)\hookrightarrow \dot{W}^{s_2,p_2}(\mathbb{R}^n).
$$

We also use the fact

$$
W^{s,p}(\mathbb{R}^n)\hookrightarrow \dot{W}^{s,p}(\mathbb{R}^n),\quad s>0.
$$

By Lemma A.1 and (A.4), it yields

$$
\begin{aligned}
\sum_{j\geq N}\|\Delta_j\nabla u\|_{L^{\frac{4n}{n+2}}}\leq&C\sum_{j\geq N}2^{-\frac{1}{4}j}2^{\frac{(n+3)j}{4}}\|\Delta_j u\|_{L^2}\\
\leq&C\sum_{j\geq N}2^{-\frac{1}{4}j}\|\Delta_j\Lambda^{\frac{n+3}{4}}u\|_{L^2}\\
\leq&C\sum_{j\geq N}2^{-\frac{1}{4}j}\|\Lambda^{\frac{n+3}{4}}u\|_{L^2}\\
\leq&C2^{-\frac{N}{4}}\|\Lambda^{\frac{n+3}{4}}u\|_{L^2}\\
\leq&C2^{-\frac{N}{4}}\|\Lambda^{\gamma_1}u\|_{L^2}^{\frac{4\gamma_2-n-3}{4\gamma_2-4\gamma_1}}\|\Lambda^{\gamma_2}u\|_{L^2}^{\frac{n+3-4\gamma_1}{4\gamma_2-4\gamma_1}},
\end{aligned}
$$

where

$$
0<\gamma_1<\frac{n+2}{4},\quad \frac{n+3}{4}<\gamma_2<\frac{n+2}{2}. \tag{2.8}
$$

For the sake of simplicity, we denote

$$
\sigma:=\frac{n+3-4\gamma_1}{4\gamma_2-4\gamma_1}.
$$

Therefore, we deduce

$$
\begin{aligned}
\|\nabla u\|_{L^{\frac{4n}{n+2}}}\leq&Cg(2^N)\|\mathcal{L}u\|_{L^2}+C2^{-\frac{N}{4}}\|\Lambda^{\gamma_1}u\|_{L^2}^{1-\sigma}\|\Lambda^{\gamma_2}u\|_{L^2}^{\sigma}\\
\leq&Cg(2^N)\|\mathcal{L}u\|_{L^2}+C2^{-\frac{N}{4}}(\|\mathcal{L}u\|_{L^2}+\|u\|_{L^2})^{1-\sigma}(\|\mathcal{L}^2u\|_{L^2}+\|u\|_{L^2})^{\sigma}\\
\leq&Cg(2^N)\|\mathcal{L}u\|_{L^2}+C2^{-\frac{N}{4}}\|\mathcal{L}u\|_{L^2}^{1-\sigma}\|\mathcal{L}^2u\|_{L^2}^{\sigma}\\
&+C2^{-\frac{N}{4}}\|\mathcal{L}u\|_{L^2}^{1-\sigma}\|u\|_{L^2}^{\sigma}+C2^{-\frac{N}{4}}\|u\|_{L^2}^{1-\sigma}\|\mathcal{L}^2u\|_{L^2}^{\sigma}+C2^{-\frac{N}{4}}\|u\|_{L^2},
\end{aligned}
$$

where we have used (2.4) and (2.6). Inserting the above estimate into (2.7) yields

$$
\begin{aligned}
-\int_{\mathbb{R}^n}(u\cdot\nabla u)\cdot\mathcal{L}^2u\,dx\leq&Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2\|\mathcal{L}^2u\|_{L^2}+C2^{-\frac{N}{2}}\|\mathcal{L}u\|_{L^2}^{2(1-\sigma)}\|\mathcal{L}^2u\|_{L^2}^{1+2\sigma}\\
&+C2^{-\frac{N}{2}}\|\mathcal{L}u\|_{L^2}^{2(1-\sigma)}\|u\|_{L^2}^{2\sigma}\|\mathcal{L}^2u\|_{L^2}\\
&+C2^{-\frac{N}{2}}\|u\|_{L^2}^{2(1-\sigma)}\|\mathcal{L}^2u\|_{L^2}^{1+2\sigma}+C2^{-\frac{N}{2}}\|u\|_{L^2}^2\|\mathcal{L}^2u\|_{L^2}\\
\leq&\frac{1}{4}\|\mathcal{L}^2u\|_{L^2}^2+Cg^4(2^N)\|\mathcal{L}u\|_{L^2}^4+C\|u\|_{L^2}^4\\
&+C2^{-\frac{N}{2}}\|\mathcal{L}u\|_{L^2}^{2(1-\sigma)}\|\mathcal{L}^2u\|_{L^2}^{1+2\sigma}
\end{aligned}
$$

$$+ C2^{-\frac{N}{2}} \|\mathcal{L}u\|_{L^2}^{2(1-\sigma)} \|u\|_{L^2}^{2\sigma} \|\mathcal{L}^2 u\|_{L^2}$$
$$+ C2^{-\frac{N}{2}} \|u\|_{L^2}^{2(1-\sigma)} \|\mathcal{L}^2 u\|_{L^2}^{1+2\sigma}.$$

We thus conclude

$$\begin{aligned}\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2 \leq & Cg^4(2^N)\|\mathcal{L}u\|_{L^2}^4 + C2^{-\frac{N}{2}} \|\mathcal{L}u\|_{L^2}^{2(1-\sigma)} \|\mathcal{L}^2 u\|_{L^2}^{1+2\sigma} \\ & + C2^{-\frac{N}{2}} \|\mathcal{L}u\|_{L^2}^{2(1-\sigma)} \|u\|_{L^2}^{2\sigma} \|\mathcal{L}^2 u\|_{L^2} \\ & + C2^{-\frac{N}{2}} \|u\|_{L^2}^{2(1-\sigma)} \|\mathcal{L}^2 u\|_{L^2}^{1+2\sigma} + C\|u\|_{L^2}^4 + C\|\theta\|_{L^2}^2. \qquad (2.9)\end{aligned}$$

Denoting

$$A(t) := \|\mathcal{L}u(t)\|_{L^2}^2, \qquad B(t) := \|\mathcal{L}^2 u(t)\|_{L^2}^2, \qquad f(t) := C + C\|u(t)\|_{L^2}^4 + C\|\theta(t)\|_{L^2}^2$$

and taking N as

$$2^N \approx e + A(t),$$

it follows from (2.9) that

$$\begin{aligned}\frac{d}{dt}A(t) + B(t) \leq & C\|\mathcal{L}u\|_{L^2}^2 g^4(2^N)A(t) + C2^{-\frac{N}{2}} + C\big(e + A(t)\big)^{\frac{1}{2}-\sigma} B(t)^{\frac{1}{2}+\sigma} \\ & + C\|u\|_{L^2}^{2\sigma}\big(e + A(t)\big)^{\frac{1}{2}-\sigma} B(t)^{\frac{1}{2}} + C\|u\|_{L^2}^{2(1-\sigma)} B(t)^{\frac{1}{2}+\sigma} + f(t).\end{aligned}$$

We further take σ such that

$$\sigma < \frac{1}{2} \Leftrightarrow \gamma_1 + \gamma_2 > \frac{n+3}{2}.$$

Keeping in mind (2.8), it is easy to show that such γ_1 and γ_2 that satisfies all the above restrictions do exist. Now we may get

$$\frac{d}{dt}A(t) + B(t) \leq \frac{1}{2}B(t) + C\|\mathcal{L}u\|_{L^2}^2 g^4\big(e + A(t)\big)\big(e + A(t)\big) + C\big(e + A(t)\big) + f(t).$$

Therefore, it gives

$$\frac{d}{dt}A(t) + B(t) \leq C(1 + \|\mathcal{L}u\|_{L^2}^2) g^4\big(e + A(t)\big)\big(e + A(t)\big) + f(t).$$

Noticing

$$g^4\big(e + A(t)\big)\big(e + A(t)\big) \geq 1,$$

we have

$$\frac{d}{dt}A(t) + B(t) \leq C(1 + \|\mathcal{L}u\|_{L^2}^2 + f(t)) g^4\big(e + A(t)\big)\big(e + A(t)\big). \qquad (2.10)$$

Consequently, we deduce from (2.10) that

$$\int_{e+A(0)}^{e+A(t)} \frac{d\tau}{\tau g^4(\tau)} \leq C \int_0^t \big(1 + \|\mathcal{L}u(\tau)\|_{L^2}^2 + f(\tau)\big)\, d\tau.$$

Recalling (1.4), namely,

$$\int_e^\infty \frac{d\tau}{\tau g^4(\tau)} = \infty$$

and

$$\int_0^t \big(1 + \|\mathcal{L}u(\tau)\|_{L^2}^2 + f(\tau)\big)\, d\tau \leq C(t, u_0, \theta_0),$$

we deduce that

$$A(t) \leq C(t, u_0, \theta_0).$$

Coming back to (2.13), we also get

$$\int_0^t B(\tau)\,d\tau \le C(t,\,u_0,\,\theta_0).$$

We thus have

$$\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t \|\mathcal{L}^2 u(\tau)\|_{L^2}^2\,d\tau \le C.$$

This completes the proof of the lemma. □

With the above estimate (2.5) at our disposal, we are now ready to deduce the global H^s-estimate.

THE GLOBAL H^s-ESTIMATE. Applying Λ^s to (1.3) and taking the L^2 inner product with $\Lambda^s u$ and $\Lambda^s\theta$ respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\mathcal{L}\Lambda^s u\|_{L^2}^2\\ &= -\int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx - \int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla\theta)\Lambda^s\theta\,dx + \int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s u\,dx.\end{aligned}$$

According to the proof of (2.3), it follows for any $\gamma_3 \in (0,\,\frac{n+2}{4})$ that

$$\|\mathcal{L}\Lambda^s u\|_{L^2}^2 \ge C_5\|\Lambda^{s+\gamma_3}u\|_{L^2}^2 - C_6\|\Lambda^s u\|_{L^2}^2. \tag{2.11}$$

It follows from the commutator (A.2), we obtain

$$\begin{aligned}-\int_{\mathbb{R}^n}\Lambda^s\big(u\cdot\nabla u\big)\Lambda^s u\,dx &= -\int_{\mathbb{R}^n}[\Lambda^s, u\cdot\nabla]u\,\Lambda^s u\,dx\\ &\le C\|[\Lambda^s, u\cdot\nabla]u\|_{L^2}\|\Lambda^s u\|_{L^2}\\ &\le C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2.\end{aligned}$$

By the divergence-free condition and the commutator (A.3), it thus gives

$$\begin{aligned}&-\int_{\mathbb{R}^n}\Lambda^s\big(u\cdot\nabla\theta\big)\Lambda^s\theta\,dx + \int_{\mathbb{R}^n}\big(u\cdot\Lambda^s\nabla\theta\big)\Lambda^s\theta\,dx\\ &= -\int_{\mathbb{R}^n}\Lambda^s\partial_i\big(u_i\theta\big)\Lambda^s\theta\,dx + \int_{\mathbb{R}^n}\big(u_i\Lambda^s\partial_i\theta\big)\Lambda^s\theta\,dx\\ &= -\int_{\mathbb{R}^n}[\Lambda^s\partial_i, u_i]\theta\,\Lambda^s\theta\,dx\\ &\le C\|[\Lambda^s\partial_i, u_i]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}\\ &\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\theta\|_{L^\infty}\|\Lambda^{s+1}u\|_{L^2})\|\Lambda^s\theta\|_{L^2}\\ &\le C\Big(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\theta\|_{L^\infty}(\|\Lambda^s u\|_{L^2} + \|\mathcal{L}\Lambda^s u\|_{L^2})\Big)\|\Lambda^s\theta\|_{L^2}\\ &\le \frac{1}{2}\|\mathcal{L}\Lambda^s u\|_{L^2}^2 + C(1+\|\nabla u\|_{L^\infty})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2),\end{aligned}$$

where we have used the estimate (2.11). The Young inequality ensures

$$\int_{\mathbb{R}^n}\Lambda^s\theta e_n\cdot\Lambda^s v\,dx \le C(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).$$

Combining the above estimates together yields

$$\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\mathcal{L}\Lambda^s u\|_{L^2}^2 + \|\Lambda^{s+\gamma_3}u\|_{L^2}^2$$

$$\leq C(1 + \|\nabla u\|_{L^\infty})(1 + \|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2).$$

Notice the following logarithmic Sobolev inequality (see Appendix)

$$\|\nabla u\|_{L^\infty} \leq C\|u\|_{L^2} + Cg^2(2^N)\sqrt{N}\|\mathcal{L}^2 u\|_{L^2} + C2^{N(1+\frac{n}{2}-\widetilde{s})}\|\Lambda^{\widetilde{s}} u\|_{L^2}, \tag{2.12}$$

where $\widetilde{s} > 1 + \frac{n}{2}$. Denoting

$$X(t) := 1 + \|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2,$$
$$Y(t) := \|\mathcal{L}\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^{s+\gamma_3} u(t)\|_{L^2}^2$$

and taking $\widetilde{s} = s + \gamma_3 > 1 + \frac{n}{2}$, we further have

$$\begin{aligned}\frac{d}{dt}X(t) + Y(t) \leq & C(1 + \|\nabla u\|_{L^\infty})X(t)\\ \leq & C(1 + g^2(2^N)\sqrt{N}\|\mathcal{L}^2 u\|_{L^2})X(t) + C2^{N(1+\frac{n}{2}-s-\gamma_3)}\|\Lambda^{s+\gamma_3} u\|_{L^2}X(t)\\ \leq & C(1 + g^2(2^N)\sqrt{N}\|\mathcal{L}^2 u\|_{L^2})X(t) + C2^{-\frac{N}{2}}Y^{\frac{1}{2}}(t)X(t),\end{aligned}$$

where in the last line we have taken

$$\gamma_3 \geq \frac{3+n}{2} - s.$$

Choosing N such

$$2^N \approx e + X(t),$$

we have

$$\begin{aligned}\frac{d}{dt}X(t) + Y(t) \leq & C\left(1 + \|\mathcal{L}^2 u\|_{L^2}\right) g^2\big(e + X(t)\big)\sqrt{\ln\big(e + X(t)\big)}\big(e + X(t)\big)\\ & + CY^{\frac{1}{2}}(t)\big(e + X(t)\big)^{\frac{1}{2}}\\ \leq & C\left(1 + \|\mathcal{L}^2 u\|_{L^2}\right) g^2\big(e + X(t)\big)\sqrt{\ln\big(e + X(t)\big)}\big(e + X(t)\big)\\ & + \frac{1}{2}Y(t) + C\big(e + X(t)\big),\end{aligned}$$

which implies

$$\frac{d}{dt}X(t) + Y(t) \leq C\left(1 + \|\mathcal{L}^2 u\|_{L^2}\right) g^2\big(e + X(t)\big)\sqrt{\ln\big(e + X(t)\big)}\big(e + X(t)\big). \tag{2.13}$$

Thanks to

$$g^2\big(e + X(t)\big)\sqrt{\ln\big(e + X(t)\big)}\big(e + X(t)\big) \geq 1,$$

we may deduce from (2.13) that

$$\int_{e+X(0)}^{e+X(t)} \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau)} \leq C\int_0^t \left(1 + \|\mathcal{L}^2 u(\tau)\|_{L^2}\right)\, d\tau.$$

We suppose that the following holds true whose proof is postponed in the Appendix

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau)} = \infty. \tag{2.14}$$

Thanks to

$$\int_0^t \left(1 + \|\mathcal{L}^2 u(\tau)\|_{L^2}\right)\, d\tau \leq C(t,\, u_0,\, \theta_0),$$

we deduce from (2.14) that

$$X(t) \leq C(t,\, u_0,\, \theta_0).$$

Coming back to (2.13), we also get

$$\int_0^t Y(\tau)\,d\tau \le C(t,\,u_0,\,\theta_0).$$

We finally obtain

$$\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2 + \int_0^t \|\mathcal{L}\Lambda^s u(\tau)\|_{L^2}^2\,d\tau \le C(t,u_0,\theta_0),$$

which is the desired global bounds in Theorem 1.1. Since the solution pair $(u,\,\theta)$ belongs to the Lipschitz space, the uniqueness is easy to obtain. Thus, this completes the proof of Theorem 1.1. □

3. The proof of Theorem 1.2

We remark that the proof of Theorem 1.2 can be performed as that of Theorem 1.1 with some certain modification. The details are provided below. As above, it suffices to consider the critical case $\alpha+\beta=\frac{1}{2}+\frac{n}{4}$.

First, we have the following estimates.

LEMMA 3.1. *Assume $v_0\in L^2(\mathbb{R}^n)$ and $\theta_0\in L^p(\mathbb{R}^n)$ for some $p\in[2,\infty]$. Then the corresponding solution (v,θ) of (1.5) admits the following bounds for any $t>0$*

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta v(\tau)\|_{L^2}^2\,d\tau \le C(t,v_0,\theta_0), \tag{3.1}$$

$$\|u(t)\|_{L^2}^2 + \|\mathcal{L}^2 u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \mathcal{L}^2 u(\tau)\|_{L^2}^2\,d\tau \le C(t,v_0,\theta_0), \tag{3.2}$$

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}. \tag{3.3}$$

PROOF OF LEMMA 3.1. Let $p\in[2,\infty]$, then multiplying the equation $(1.5)_2$ by $|\theta|^{p-2}\theta$ and integrating by parts lead to

$$\frac{d}{dt}\|\theta(t)\|_{L^p}=0.$$

We can deduce the result (3.3) by integrating in time. Multiplying both sides of the equation $(1.5)_1$ by v and integrating by parts, one has by using (3.3) with $p=2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\Lambda^\beta v\|_{L^2}^2 \le& \int_{\mathbb{R}^n} |v|\,|\theta|\,dx\\ \le&\|v\|_{L^2}\|\theta\|_{L^2}\\ \le&\|v\|_{L^2}\|\theta_0\|_{L^2},\end{aligned} \tag{3.4}$$

where we have used

$$\int_{\mathbb{R}^n} (u\cdot\nabla v)\cdot v\,dx=0.$$

It follows from (3.4) that

$$\frac{d}{dt}\|v(t)\|_{L^2} \le \|\theta_0\|_{L^2}.$$

Integrating in time yields

$$\|v(t)\|_{L^2} \le \|v_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

Recalling (3.4) and integrating in time imply

$$\begin{aligned}\|v(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\beta v(\tau)\|_{L^2}^2\,d\tau \le& \|v_0\|_{L^2}^2 + 2\int_0^t \|v(\tau)\|_{L^2}\|\theta_0\|_{L^2}\,d\tau\\ \le& \|v_0\|_{L^2}^2 + 2\int_0^t (\|v_0\|_{L^2} + \tau\|\theta_0\|_{L^2})\|\theta_0\|_{L^2}\,d\tau\\ =& (\|v_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.\end{aligned}$$

Notice the fact

$$\int_0^t \|v(\tau)\|_{H^\beta}^2\,d\tau \approx \int_0^t (\|v(\tau)\|_{L^2}^2 + \|\Lambda^\beta v(\tau)\|_{L^2}^2)\,d\tau \le C(t, v_0, \theta_0).$$

The desired bound (3.1) follows directly. Notice that the relation $v = u + \mathcal{L}^2 u$, we have

$$\widehat{v}(\xi) = \widehat{u}(\xi) + \frac{|\xi|^{2\alpha}}{g^2(|\xi|)}\widehat{u}(\xi),$$

which leads to

$$\|u\|_{L^2} = \|\widehat{u}(\xi)\|_{L^2} = \Big\|\frac{1}{1+\frac{|\xi|^{2\alpha}}{g^2(|\xi|)}}\widehat{v}(\xi)\Big\|_{L^2} \le \|\widehat{v}(\xi)\|_{L^2} = \|v\|_{L^2},$$

which gives

$$\|u\|_{L^2} \le \|v\|_{L^2}. \tag{3.5}$$

Finally, (3.2) is an easy consequence of (3.1) and (3.5). This completes the proof of Lemma 3.1. □

We are now in the position to show the key estimate.

LEMMA 3.2. *Assume $v_0 \in H^\beta(\mathbb{R}^n)$. Let (v, θ) be the corresponding solution of the system (1.5). If $\alpha + \beta = \frac{1}{2} + \frac{n}{4}$ and $\beta > 0$, then the following estimate holds*

$$\|\Lambda^\beta v(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{2\beta} v(\tau)\|_{L^2}^2\,d\tau \le C(t, v_0, \theta_0), \tag{3.6}$$

or

$$\Big\|\frac{\Lambda^{2\alpha+\beta}}{g^2(\Lambda)}u(t)\Big\|_{L^2}^2 + \int_0^t \Big\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u(\tau)\Big\|_{L^2}^2\,d\tau \le C(t, v_0, \theta_0). \tag{3.7}$$

PROOF OF LEMMA 3.2. Applying Λ^β to equation $(1.5)_1$ and taking the inner product with $\Lambda^\beta v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta v(t)\|_{L^2}^2 + \|\Lambda^{2\beta} v\|_{L^2}^2 = -\int_{\mathbb{R}^n} \Lambda^\beta\big(u\cdot\nabla v\big)\Lambda^\beta v\,dx + \int_{\mathbb{R}^n} \Lambda^\beta \theta e_n \Lambda^\beta v\,dx. \tag{3.8}$$

To bound the first term at the right hand side of (3.8), we split it into two cases, namely, the case $\beta < 1$ and the case $\beta \ge 1$. For the case $\beta < 1$, we need a delicate commutator estimate (A.1). More precisely, thanks to the divergence-free condition and (A.1), it yields for the case $\beta < 1$ that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} \Lambda^\beta\big(u\cdot\nabla v\big)\Lambda^\beta v\,dx\Big|\\ &= \Big|\int_{\mathbb{R}^n} [\Lambda^\beta, u\cdot\nabla]v\ \Lambda^\beta v\,dx\Big|\\ &\le C\|[\Lambda^\beta, u\cdot\nabla]v\|_{H^{-\beta}}\|\Lambda^\beta v\|_{H^\beta}\end{aligned}$$

$$
\begin{aligned}
&\leq C(\|\nabla u\|_{L^{\frac{n}{\beta}}}\|v\|_{B^0_{\frac{2n}{n-2\beta},2}}+\|u\|_{L^2}\|v\|_{L^2})\|\Lambda^{\beta}v\|_{H^{\beta}}\\
&\leq C(\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\Lambda^{\beta}v\|_{L^2}+\|u\|_{L^2}\|v\|_{L^2})(\|\Lambda^{\beta}v\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2})\\
&\leq \frac{1}{16}\|\Lambda^{2\beta}v\|_{L^2}^2+C\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\Lambda^{\beta}v\|_{L^2}(\|\Lambda^{\beta}v\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2})\\
&\quad +C\|u\|_{L^2}^2\|v\|_{L^2}^2.
\end{aligned}
$$

For the case $\beta\geq 1$, the commutator estimate (A.2) would suffice our purpose. Actually, it is not hard to check that

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^n}\Lambda^{\beta}\big(u\cdot\nabla v\big)\Lambda^{\beta}v\,dx\Big| &= \Big|\int_{\mathbb{R}^n}[\Lambda^{\beta},u\cdot\nabla]v\ \Lambda^{\beta}v\,dx\Big|\\
&\leq C\|[\Lambda^{\beta},u\cdot\nabla]v\|_{L^2}\|\Lambda^{\beta}v\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\Lambda^{\beta}v\|_{L^{\frac{2n}{n-2\beta}}}+\|\nabla v\|_{L^{\frac{2n}{n-4\beta+2}}}\|\Lambda^{\beta}u\|_{L^{\frac{n}{2\beta-1}}})\\
&\quad\times\|\Lambda^{\beta}v\|_{L^2}\\
&\leq C\|\Lambda^{\beta}u\|_{L^{\frac{n}{2\beta-1}}}\|\Lambda^{2\beta}v\|_{L^2}\|\Lambda^{\beta}v\|_{L^2},
\end{aligned}
$$

where we used the embedding

$$
\dot{W}^{\beta,\frac{n}{2\beta-1}}(\mathbb{R}^n)\hookrightarrow\dot{W}^{1,\frac{n}{\beta}}(\mathbb{R}^n),\quad \dot{H}^{2\beta}(\mathbb{R}^n)\hookrightarrow\dot{W}^{1,\frac{2n}{n-4\beta+2}}(\mathbb{R}^n),\quad \beta\geq 1.
$$

The Young inequality gives

$$
\int_{\mathbb{R}^n}\Lambda^{\beta}\theta e_n\Lambda^{\beta}v\,dx\leq\|\theta\|_{L^2}\|\Lambda^{2\beta}v\|_{L^2}\leq\frac{1}{16}\|\Lambda^{2\beta}v\|_{L^2}^2+C\|\theta\|_{L^2}^2.
$$

Plugging the above estimates in (3.8), it implies

$$
\begin{aligned}
\frac{d}{dt}\|\Lambda^{\beta}v(t)\|_{L^2}^2+\|\Lambda^{2\beta}v\|_{L^2}^2 \leq& C\|\nabla u\|_{L^{\frac{n}{\beta}}}\chi_{\{\beta<1\}}\|\Lambda^{\beta}v\|_{L^2}(\|\Lambda^{\beta}v\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2})\\
&+C\|\Lambda^{\beta}u\|_{L^{\frac{n}{2\beta-1}}}\chi_{\{\beta\geq 1\}}\|\Lambda^{2\beta}v\|_{L^2}\|\Lambda^{\beta}v\|_{L^2}\\
&+C(\|u\|_{L^2}^2\|v\|_{L^2}^2+\|\theta\|_{L^2}^2)\\
:=& D_1+D_2+D_3. \qquad (3.9)
\end{aligned}
$$

Now making use of the same argument used in proving (2.3), we have for any $\lambda_1\in(0,\,2\alpha+\beta)$ and any $\lambda_2\in(0,\,2\alpha+2\beta)$ that

$$
\|\Lambda^{\beta}v\|_{L^2}^2\geq C_1\|\Lambda^{\lambda_1}u\|_{L^2}^2-C_2\|u\|_{L^2}^2, \qquad (3.10)
$$

$$
\|\Lambda^{2\beta}v\|_{L^2}^2\geq C_3\|\Lambda^{\lambda_2}u\|_{L^2}^2-C_4\|u\|_{L^2}^2. \qquad (3.11)
$$

It follows from the high-low frequency technique that

$$
\|\nabla u\|_{L^{\frac{n}{\beta}}}\leq\|S_N\nabla u\|_{L^{\frac{n}{\beta}}}+\sum_{j\geq N}\|\Delta_j\nabla u\|_{L^{\frac{n}{\beta}}}.
$$

According to Plancherel's theorem and $\dot{H}^{1+\frac{n}{2}-\beta}(\mathbb{R}^n)\hookrightarrow\dot{W}^{1,\frac{n}{\beta}}(\mathbb{R}^n)$, it implies

$$
\begin{aligned}
\|S_N\nabla u\|_{L^{\frac{n}{\beta}}} \leq& C\|S_N\Lambda^{1+\frac{n}{2}-\beta}u\|_{L^2}\\
=& C\|\chi(2^{-N}\xi)|\xi|^{1+\frac{n}{2}-\beta}\widehat{u}(\xi)\|_{L^2}\\
=& C\Big\|\chi(2^{-N}\xi)g^2(|\xi|)\frac{|\xi|^{2\alpha+\beta}}{g^2(|\xi|)}\widehat{u}(\xi)\Big\|_{L^2}\\
\leq& Cg^2(2^N)\|\Lambda^{\beta}v\|_{L^2}.
\end{aligned}
$$

Using Lemma A.1 and (A.4) yields

$$\begin{aligned}\sum_{j\geq N}\|\Delta_j\nabla u\|_{L^{\frac{n}{\beta}}} \leq& C\sum_{j\geq N}2^{-\sigma j}2^{\frac{(n+2-2\beta+2\sigma)j}{2}}\|\Delta_j u\|_{L^2}\\ \leq& C\sum_{j\geq N}2^{-\sigma j}\|\Delta_j\Lambda^{\frac{n+2-2\beta+2\sigma}{2}}u\|_{L^2}\\ \leq& C\sum_{j\geq N}2^{-\sigma j}\|\Lambda^{\frac{n+2-2\beta+2\sigma}{2}}u\|_{L^2}\\ \leq& C2^{-\sigma N}\|\Lambda^{\frac{n+2-2\beta+2\sigma}{2}}u\|_{L^2}\\ \leq& C2^{-\sigma N}\|\Lambda^{\lambda_1}u\|_{L^2}^{1-\varrho}\|\Lambda^{\lambda_2}u\|_{L^2}^{\varrho},\end{aligned}$$

where the positive parameters σ, ϱ, λ_1 and λ_2 should satisfy

$$\varrho=\frac{n+2-2\beta+2\sigma-2\lambda_1}{2\lambda_2-2\lambda_1},$$

$$0<\lambda_1<2\alpha+\beta,\qquad \frac{n+2-2\beta+2\sigma}{2}<\lambda_2<2\alpha+2\beta. \tag{3.12}$$

We remark that in order to guarantee the existence of ϱ or λ_2, it follows from the last restriction of (3.12) and $\alpha+\beta=\frac{1}{2}+\frac{n}{4}$ that

$$\sigma<\beta.$$

Therefore, it gives

$$\|\nabla u\|_{L^{\frac{n}{\beta}}}\leq Cg^2(2^N)\|\Lambda^\beta v\|_{L^2}+C2^{-\sigma N}\|\Lambda^{\lambda_1}u\|_{L^2}^{1-\varrho}\|\Lambda^{\lambda_2}u\|_{L^2}^{\varrho}.$$

Thanks to the following embedding $\dot{H}^{1+\frac{n}{2}-\beta}(\mathbb{R}^n)\hookrightarrow\dot{W}^{\beta,\frac{n}{2\beta-1}}(\mathbb{R}^n)$, it is easy to show that

$$\|\Lambda^\beta u\|_{L^{\frac{n}{2\beta-1}}}\leq Cg^2(2^N)\|\Lambda^\beta v\|_{L^2}+C2^{-\sigma N}\|\Lambda^{\lambda_1}u\|_{L^2}^{1-\varrho}\|\Lambda^{\lambda_2}u\|_{L^2}^{\varrho}.$$

Consequently, one obtains by using (3.10) and (3.11)

$$\begin{aligned}D_1+D_2\leq& C\Big(g^2(2^N)\|\Lambda^\beta v\|_{L^2}+2^{-\sigma N}\|\Lambda^{\lambda_1}u\|_{L^2}^{1-\varrho}\|\Lambda^{\lambda_2}u\|_{L^2}^{\varrho}\Big)\\ &\times\|\Lambda^\beta v\|_{L^2}(\|\Lambda^\beta v\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2})\\ \leq& C\Big(g^2(2^N)\|\Lambda^\beta v\|_{L^2}+2^{-\sigma N}(\|u\|_{L^2}+\|\Lambda^\beta v\|_{L^2})^{1-\varrho}(\|u\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2})^{\varrho}\Big)\\ &\times\|\Lambda^\beta v\|_{L^2}(\|\Lambda^\beta v\|_{L^2}+\|\Lambda^{2\beta}v\|_{L^2})\\ =& Cg^2(2^N)\|\Lambda^\beta v\|_{L^2}^2\|\Lambda^{2\beta}v\|_{L^2}+Cg^2(2^N)\|\Lambda^\beta v\|_{L^2}^3\\ &+C2^{-\sigma N}\|\Lambda^\beta v\|_{L^2}^{2-\varrho}\|\Lambda^{2\beta}v\|_{L^2}^{1+\varrho}+\text{easy terms}\\ \leq&\frac{1}{16}\|\Lambda^{2\beta}v\|_{L^2}^2+Cg^4(2^N)\|\Lambda^\beta v\|_{L^2}^4+C\|\Lambda^\beta v\|_{L^2}^2\\ &+C2^{-\sigma N}\|\Lambda^\beta v\|_{L^2}^{2-\varrho}\|\Lambda^{2\beta}v\|_{L^2}^{1+\varrho}+\text{easy terms},\end{aligned}$$

where "easy terms" means that they can be handled easily compared to the term $2^{-\sigma N}\|\Lambda^\beta v\|_{L^2}^{2-\varrho}\|\Lambda^{2\beta}v\|_{L^2}^{1+\varrho}$. For the sake of short presentation, we may ignore their concrete forms. This along with (3.9) yields

$$\begin{aligned}\frac{d}{dt}\|\Lambda^\beta v(t)\|_{L^2}^2+\|\Lambda^{2\beta}v\|_{L^2}^2\leq& Cg^4(2^N)\|\Lambda^\beta v\|_{L^2}^4+C2^{-\sigma N}\|\Lambda^\beta v\|_{L^2}^{2-\varrho}\|\Lambda^{2\beta}v\|_{L^2}^{1+\varrho}\\ &+\text{easy terms}+C(\|u\|_{L^2}^2\|v\|_{L^2}^2+\|\theta\|_{L^2}^2).\end{aligned}\tag{3.13}$$

Now we denote

$$X(t)=\|\Lambda^{\beta}v(t)\|_{L^2}^2,\qquad Y(t):=\|\Lambda^{2\beta}v(t)\|_{L^2}^2$$

and take N as

$$2^{\sigma N}\approx\sqrt{e+X(t)}.$$

By tedious computations, we thus deduce from (3.13) that

$$\frac{d}{dt}X(t)+Y(t)\le C(1+\|\Lambda^{\beta}v\|_{L^2}^2)(e+X(t))g^4\left[\big(e+X(t)\big)^{\frac{1}{2\sigma}}\right]+f(t), \tag{3.14}$$

where $f(t)$ satisfies

$$\int_0^t f(\tau)\,d\tau\le C(t,\,v_0,\,\theta_0).$$

Thanks to

$$g^4\left[\big(e+X(t)\big)^{\frac{1}{2\sigma}}\right]\big(e+X(t)\big)\ge 1,$$

we deduce from (3.14) that

$$\int_{e+X(0)}^{e+X(t)}\frac{d\tau}{\tau g^4(\tau^{\frac{1}{2\sigma}})}\le C\int_0^t\left(1+f(\tau)+\|\Lambda^{\beta}v(\tau)\|_{L^2}^2\right)\,d\tau.$$

Let us keep in mind the following fact due to variable substitution and (1.6)

$$\int_e^{\infty}\frac{d\tau}{\tau g^4(\tau^{\frac{1}{2\sigma}})}=2\sigma\int_{e^{\frac{1}{2\sigma}}}^{\infty}\frac{d\tau}{\tau g^4(\tau)}=\infty.$$

Noting the bound

$$\int_0^t\left(1+f(\tau)+\|\Lambda^{\beta}v(\tau)\|_{L^2}^2\right)\,d\tau\le C(t,\,v_0,\,\theta_0),$$

it thus implies

$$X(t)\le C(t,\,v_0,\,\theta_0).$$

Coming back to (3.14), we also have

$$\int_0^t Y(\tau)\,d\tau\le C(t,\,v_0,\,\theta_0).$$

Consequently, we end up with

$$\|\Lambda^{\beta}v(t)\|_{L^2}^2+\int_0^t\|\Lambda^{2\beta}v(\tau)\|_{L^2}^2\,d\tau\le C(t,\,v_0,\,\theta_0),$$

which gives (3.6) and (3.7). This ends the proof of the lemma. □

Next we are ready to establish the following estimates.

LEMMA 3.3. *Assume $v_0\in H^{1+\beta}(\mathbb{R}^n)$ and $\nabla\theta_0\in L^2\cap L^{\infty}(\mathbb{R}^n)$. Let (v,θ) be the corresponding solution of the system (1.5). If $\alpha+\beta=\frac{1}{2}+\frac{n}{4}$ and $\beta>0$, then the following estimate holds*

$$\|\nabla\theta(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^{\infty}}+\|v(t)\|_{H^{1+\beta}}^2+\int_0^t\|v(\tau)\|_{H^{1+2\beta}}^2\,d\tau\le C(t,v_0,\theta_0). \tag{3.15}$$

In particular, we have

$$\int_0^t\left(\|\nabla u(\tau)\|_{L^{\infty}}+\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}\right)d\tau\le C(t,v_0,\theta_0). \tag{3.16}$$

PROOF OF LEMMA 3.3. Applying $\Lambda^{1+\beta}$ to $(1.5)_1$ and taking the inner product with $\Lambda^{1+\beta}v$, it implies

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{1+\beta}v(t)\|_{L^2}^2+\|\Lambda^{1+2\beta}v\|_{L^2}^2=-\int_{\mathbb{R}^n}[\Lambda^{1+\beta},u\cdot\nabla]v\,\Lambda^{1+\beta}v\,dx+\int_{\mathbb{R}^n}\Lambda^{1+\beta}\theta e_n\Lambda^{1+\beta}v\,dx.$$

By the commutator (A.2), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n}[\Lambda^{1+\beta},u\cdot\nabla]v\,\Lambda^{1+\beta}v\,dx\\
&\le C\|[\Lambda^{1+\beta},u\cdot\nabla]v\|_{L^2}\|\Lambda^{1+\beta}v\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{1+\beta}v\|_{L^2}+\|\nabla v\|_{L^{\frac{2n}{n-2\beta}}}\|\Lambda^{1+\beta}u\|_{L^{\frac{n}{\beta}}})\|\Lambda^{1+\beta}v\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}+\|\Lambda^{1+\frac{n}{2}}u\|_{L^2})\|\Lambda^{1+\beta}v\|_{L^2}^2,
\end{aligned}$$

where we have used the Sobolev embedding

$$\dot H^{1+\frac{n}{2}}(\mathbb{R}^n)\hookrightarrow\dot W^{1+\beta,\frac{n}{\beta}}(\mathbb{R}^n),\quad \dot H^{1+\beta}(\mathbb{R}^n)\hookrightarrow\dot W^{1,\frac{2n}{n-2\beta}}(\mathbb{R}^n).$$

Using the Young inequality, we thus arrive at

$$\int_{\mathbb{R}^n}\Lambda^{1+\beta}\theta e_n\Lambda^{1+\beta}v\,dx\le C\|\nabla\theta\|_{L^2}\|\Lambda^{1+2\beta}v\|_{L^2}\le\frac{1}{4}\|\Lambda^{1+2\beta}v\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2.$$

Putting the above estimates together yields

$$\frac{d}{dt}\|\Lambda^{1+\beta}v(t)\|_{L^2}^2+\|\Lambda^{1+2\beta}v\|_{L^2}^2\le C(\|\nabla u\|_{L^\infty}+\|\Lambda^{1+\frac{n}{2}}u\|_{L^2})\|\Lambda^{1+\beta}v\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2.$$

We resort to θ equation to get

$$\partial_t\nabla\theta+(u\cdot\nabla)\nabla\theta=(\nabla u\cdot\nabla)\theta. \tag{3.17}$$

Testing (3.17) by $|\nabla\theta|^{q-2}\nabla\theta$ and using the divergence-free condition

$$\frac{1}{q}\frac{d}{dt}\|\nabla\theta(t)\|_{L^q}^q\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q}^q.$$

It further gives

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^q}\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^q},$$

which also yields

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2.$$

Letting $q\to\infty$, we get

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^\infty}\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty}.$$

Summing up the above estimates, it follows that

$$\frac{d}{dt}\mathcal{Z}(t)+\mathcal{Y}(t)\le C(1+\|\nabla u\|_{L^\infty}+\|\Lambda^{1+\frac{n}{2}}u\|_{L^2})\mathcal{Z}(t),$$

where

$$\mathcal{Z}(t):=\|\Lambda^{1+\beta}v(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^\infty},\quad \mathcal{Y}(t):=\|\Lambda^{1+2\beta}v(t)\|_{L^2}^2.$$

Taking advantage of (2.12) and its proof, we have for $\widetilde{s} > 1 + \frac{n}{2}$

$$\|\nabla u\|_{L^\infty} \leq C\|u\|_{L^2} + Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2} + C2^{N(1+\frac{n}{2}-\widetilde{s})}\|\Lambda^{\widetilde{s}}u\|_{L^2}, \tag{3.18}$$

$$\|\Lambda^{1+\frac{n}{2}}u\|_{L^2} \leq C\|u\|_{L^2} + Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2} + C2^{N(1+\frac{n}{2}-\widetilde{s})}\|\Lambda^{\widetilde{s}}u\|_{L^2}. \tag{3.19}$$

Choosing $\widetilde{s} = \rho > 1 + \frac{n}{2}$, we easily get

$$\begin{aligned}\frac{d}{dt}\mathcal{Z}(t) + \mathcal{Y}(t) \leq& C\left(1 + g^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)\mathcal{Z}(t) \\ &+ C2^{N(1+\frac{n}{2}-\rho)}\|\Lambda^\rho u\|_{L^2}\mathcal{Z}(t) \\ \leq& C\left(1 + g^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)\mathcal{Z}(t) + C2^{-\frac{N}{2}}\mathcal{Y}^{\frac{1}{2}}(t)\mathcal{Z}(t) \\ &+ C2^{-\frac{N}{2}}\|u\|_{L^2}\mathcal{Z}(t),\end{aligned}$$

where we have taken $\rho \geq \frac{3+n}{2}$. Taking N as follows

$$2^N \approx e + \mathcal{Z}(t),$$

it directly gives

$$\begin{aligned}\frac{d}{dt}\mathcal{Z}(t) + \mathcal{Y}(t) \leq& C\left(1 + \left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right) g^2\big(e + \mathcal{Z}(t)\big)\sqrt{\ln\big(e + \mathcal{Z}(t)\big)}\big(e + \mathcal{Z}(t)\big) \\ &+ C\mathcal{Y}^{\frac{1}{2}}(t)\big(e + \mathcal{Z}(t)\big)^{\frac{1}{2}} + C\mathcal{Z}(t) \\ \leq& C\left(1 + \left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right) g^2\big(e + \mathcal{Z}(t)\big)\sqrt{\ln\big(e + \mathcal{Z}(t)\big)}\big(e + \mathcal{Z}(t)\big) \\ &+ \frac{1}{2}\mathcal{Y}(t) + C\big(e + \mathcal{Z}(t)\big),\end{aligned}$$

which allows us to conclude

$$\frac{d}{dt}\mathcal{Z}(t) + \mathcal{Y}(t) \leq C\left(1 + \left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right) g^2\big(e + \mathcal{Z}(t)\big)\sqrt{\ln\big(e + \mathcal{Z}(t)\big)}\big(e + \mathcal{Z}(t)\big). \tag{3.20}$$

It is easy to deduce from (3.20) that

$$\int_{e+\mathcal{Z}(0)}^{e+\mathcal{Z}(t)} \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau)} \leq \int_0^t \left(1 + \left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u(\tau)\right\|_{L^2}\right) d\tau \leq C(t,\, u_0,\, \theta_0).$$

Notice that the condition (1.6) yields (2.14), namely

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau)} = \infty,$$

which leads to

$$\mathcal{Z}(t) \leq C(t,\, u_0,\, \theta_0).$$

Noting (3.20), we thus have

$$\int_0^t \mathcal{Y}(\tau)\, d\tau \leq C(t,\, u_0,\, \theta_0).$$

As a result, it entails

$$\|\nabla\theta(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^\infty} + \|v(t)\|_{H^{1+\beta}}^2 + \int_0^t \|v(\tau)\|_{H^{1+2\beta}}^2\, d\tau \le C(t, v_0, \theta_0). \tag{3.21}$$

It just follows from (3.18), (3.19) and (3.21) that

$$\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}} u(\tau)\|_{L^2})\, d\tau \le C(t, v_0, \theta_0). \tag{3.22}$$

This concludes the proof of the lemma. □

THE GLOBAL H^s-ESTIMATE. With the above estimates at our disposal, the global H^s-estimate can be obtained. Applying Λ^s to (1.5), taking the L^2 inner product with $\Lambda^s v$ and $\Lambda^s\theta$ respectively, then adding them up, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\Lambda^{s+\beta} v\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^n} \Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\, dx - \int_{\mathbb{R}^n} \Lambda^s(u\cdot\nabla\theta)\Lambda^s\theta\, dx + \int_{\mathbb{R}^n} \Lambda^s\theta e_n\cdot\Lambda^s v\, dx.
\end{aligned} \tag{3.23}$$

Taking advantage of (A.2) and several interpolation inequalities shows

$$\begin{aligned}
-\int_{\mathbb{R}^n} \Lambda^s\big(u\cdot\nabla v\big)\Lambda^s v\, dx = &-\int_{\mathbb{R}^n} [\Lambda^s, u\cdot\nabla] v\,\Lambda^s v\, dx \\
\le &C\|[\Lambda^s, u\cdot\nabla]v\|_{L^2}\|\Lambda^s v\|_{L^2} \\
\le &C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|\nabla v\|_{L^{\frac{2n}{n-2\beta}}}\|\Lambda^s u\|_{L^{\frac{n}{\beta}}})\|\Lambda^s v\|_{L^2} \\
\le &C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|v\|_{H^{1+\beta}}\|u\|_{H^{s+\frac{n}{2}-\beta}})\|\Lambda^s v\|_{L^2} \\
\le &C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|v\|_{H^{1+\beta}}\|v\|_{H^{s+\beta}})\|\Lambda^s v\|_{L^2} \\
\le &\frac{1}{2}\|\Lambda^{s+\beta} v\|_{L^2}^2 + C(\|v\|_{H^{1+\beta}}^2 + \|\nabla u\|_{L^\infty})(1 + \|\Lambda^s v\|_{L^2}^2).
\end{aligned}$$

Adopting (A.2), we infer that

$$\begin{aligned}
-\int_{\mathbb{R}^n} \Lambda^s\big(u\cdot\nabla\theta\big)\Lambda^s\theta\, dx = &-\int_{\mathbb{R}^n} [\Lambda^s, u\cdot\nabla]\theta\,\Lambda^s\theta\, dx \\
\le &\ C\|[\Lambda^s, u\cdot\nabla]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2} \\
\le &\ C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s\theta\|_{L^2} \\
\le &\ C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda^s v\|_{L^2})\|\Lambda^s\theta\|_{L^2} \\
\le &\ C(\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).
\end{aligned}$$

Finally, it is easy to get

$$\int_{\mathbb{R}^n} \Lambda^s\theta e_n\cdot\Lambda^s v\, dx \le C(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).$$

Putting all the preceding estimates together, one can get

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\Lambda^{s+\beta} v\|_{L^2}^2 \\
&\le C(1 + \|v\|_{H^{1+\beta}}^2 + \|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})(1 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).
\end{aligned} \tag{3.24}$$

The estimates (3.16) and (3.15) yield the desired global H^s-estimate. This ends the proof of Theorem 1.2. □

4. The proof of Theorem 1.3

Similarly, it suffices to consider the critical case $\alpha = \frac{1}{2} + \frac{n}{4}$. Recalling (3.1), we also have

$$\|v(t)\|_{L^2} \le C(t, v_0, \theta_0),$$

which together with the relation $v = u + \mathcal{L}^2 u$ leads to

$$\|u(t)\|_{L^2} + \left\| \frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)} u(t) \right\|_{L^2} \le C(t, v_0, \theta_0).$$

Recalling (3.23), it reads

$$\frac{1}{2}\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2) = -\int_{\mathbb{R}^n} \Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^n} \Lambda^s(u\cdot\nabla\theta)\Lambda^s\theta\,dx + \int_{\mathbb{R}^n} \Lambda^s\theta e_n\cdot\Lambda^s v\,dx.$$

Taking p as

$$\frac{1}{2} - \frac{s-1}{n} < \frac{1}{p} < \frac{1}{2}$$

and using (A.2) as well as (A.4), we now conclude

$$\begin{aligned}
&-\int_{\mathbb{R}^n} \Lambda^s\big(u\cdot\nabla v\big)\Lambda^s v\,dx \\
&\le C\|[\Lambda^s, u\cdot\nabla]v\|_{L^2}\|\Lambda^s v\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|\nabla v\|_{L^p}\|\Lambda^s u\|_{L^{\frac{2p}{p-2}}})\|\Lambda^s v\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|\nabla v\|_{L^p}\|\Lambda^{s+\frac{n}{p}} u\|_{L^2})\|\Lambda^s v\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|v\|_{L^2}^{1-\lambda}\|\Lambda^s v\|_{L^2}^{\lambda}\|\Lambda^{1+\frac{n}{2}} u\|_{L^2}^{\lambda}\|\Lambda^{s+1+\frac{n}{2}} u\|_{L^2}^{1-\lambda})\|\Lambda^s v\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + \|v\|_{L^2}^{1-\lambda}\|\Lambda^s v\|_{L^2}^{\lambda}\|\Lambda^{1+\frac{n}{2}} u\|_{L^2}^{\lambda}\|\Lambda^s v\|_{L^2}^{1-\lambda})\|\Lambda^s v\|_{L^2} \\
&\le C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}} u\|_{L^2})\|\Lambda^s v\|_{L^2}^2,
\end{aligned}$$

where λ is given by

$$\lambda = \frac{\frac{1}{2} + \frac{1}{n} - \frac{1}{p}}{\frac{s}{n}} \in \Big(\frac{1}{s}, 1\Big).$$

By the same argument, it also gives

$$-\int_{\mathbb{R}^n} \Lambda^s\big(u\cdot\nabla\theta\big)\Lambda^s\theta\,dx \le C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}} u\|_{L^2})\|\Lambda^s\theta\|_{L^2}^2.$$

The Young inequality yields

$$\int_{\mathbb{R}^n} \Lambda^s\theta e_n\cdot\Lambda^s v\,dx \le C(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).$$

Putting together all the above estimates yields

$$\frac{d}{dt}(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2) \le C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}} u\|_{L^2})(\|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).$$

Making use of (3.18) and (3.19), we have

$$\|\nabla u\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}} u\|_{L^2} \le C\|u\|_{L^2} + Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)} u\right\|_{L^2}$$

$$
\begin{aligned}
&+ C2^{N(1+\frac{n}{2}-s)}\|\Lambda^s u\|_{L^2}\\
\leq & C\|u\|_{L^2} + Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\\
&+ C2^{N(1+\frac{n}{2}-s)}\|\Lambda^s v\|_{L^2}.
\end{aligned}
$$

Now we denote

$$A(t) := \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2,$$

then it gives

$$\frac{d}{dt}A(t) \leq C\left(1 + g^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2} + 2^{N(1+\frac{n}{2}-s)}\|\Lambda^s v\|_{L^2}\right)A(t).$$

Choosing N such that

$$2^{N(s-1-\frac{n}{2})} \approx \sqrt{e + A(t)},$$

it implies

$$
\begin{aligned}
\frac{d}{dt}A(t) \leq & C\left(1 + g^2\left[\left(e + A(t)\right)^{\frac{1}{2s-2-n}}\right]\sqrt{\ln\left(e + A(t)\right)}\left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)A(t)\\
\leq & C\left(1 + \left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)g^2\left[\left(e + A(t)\right)^{\frac{1}{2s-2-n}}\right]\sqrt{\ln\left(e + A(t)\right)}\left(e + A(t)\right).
\end{aligned}
$$

Thanks to

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau^{\frac{1}{2s-2-n}})} = \sqrt{2s-2-n}\int_{e^{\frac{1}{2s-2-n}}}^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau)} = \infty$$

and the fact

$$\int_0^t \left(1 + \left\|\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u(\tau)\right\|_{L^2}\right)d\tau \leq C(t, v_0, \theta_0),$$

we may deduce

$$A(t) \leq C(t, v_0, \theta_0).$$

This is nothing but the desired global H^s-bound of Theorem 1.3. This ends the proof of Theorem 1.3.

Appendix A. Besov spaces

This appendix includes several parts. It recalls the Littlewood-Paley theory, introduces the Besov spaces, provides Bernstein inequalities as well as several facts used in the proof of our main result. We start with the Littlewood-Paley theory. We choose some smooth radial non increasing function χ with values in $[0,1]$ such that $\chi \in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B} := \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$ and and with value 1 on $\{\xi \in \mathbb{R}^n, |\xi| \leq \frac{3}{4}\}$, then we set $\varphi(\xi) = \chi\big(\frac{\xi}{2}\big) - \chi(\xi)$. One easily verifies that $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C} := \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and satisfies

$$\chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $\widetilde{h} = \mathcal{F}^{-1}(\chi)$, then we introduce the dyadic blocks Δ_j of our decomposition by setting

$$\Delta_j u = 0, \quad j \leq -2; \qquad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^n}\widetilde{h}(y)u(x-y)\,dy;$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u = \chi(2^{-j}D)u = \sum_{-1\le k\le j-1} \Delta_k u = 2^{jn}\int_{\mathbb{R}^n} \widetilde{h}(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}.$$

The nonhomogeneous Besov spaces are defined through the dyadic decomposition.

DEFINITION A.1. Let $s \in \mathbb{R}, (p,r) \in [1,+\infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$B^s_{p,r} = \{f \in S'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \Big(\sum_{j\ge -1} 2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & r<\infty,\\ \sup_{j\ge -1} 2^{js}\|\Delta_j f\|_{L^p}, & r=\infty. \end{cases}$$

We now introduce the Bernstein's inequalities.

LEMMA A.1 (see [**2**]). *Assume $1 \le a \le b \le \infty$. If the integer $j \ge -1$, then it holds*

$$\|\Lambda^k \Delta_j f\|_{L^b} \le C_1\, 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|\Delta_j f\|_{L^a}, \quad k \ge 0.$$

If the integer $j \ge 0$, then we have

$$C_2\, 2^{jk}\|\Delta_j f\|_{L^b} \le \|\Lambda^k \Delta_j f\|_{L^b} \le C_3\, 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|\Delta_j f\|_{L^a}, \quad k \in \mathbb{R},$$

where C_1, C_2 and C_3 are constants depending on k, a and b only.

We recall the following commutator estimates (see [**26**, Lemma 2.6]).

LEMMA A.2. *Let f be a divergence-free vector field and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p \in [2,\infty)$, $p_1, p_2 \in [2,\infty]$, $r \in [1,\infty]$ as well as $s \in (-1, 1-\delta)$ for $\delta \in (0,2)$, then it holds*

$$\|[\Lambda^\delta, f\cdot\nabla]g\|_{B^s_{p,r}} \le C(p,r,\delta,s)\big(\|\nabla f\|_{L^{p_1}}\|g\|_{B^{s+\delta}_{p_2,r}} + \|f\|_{L^2}\|g\|_{L^2}\big). \tag{A.1}$$

We also need the Kato-Ponce type commutator estimate [**15**] in that following.

LEMMA A.3. *Let $s > 0$. Let $p, p_1, p_3 \in (1,\infty)$ and $p_2, p_4 \in [1,\infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then there exists some constants C such that

$$\|[\Lambda^s, f]g\|_{L^p} \le C\left(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|\Lambda^{s-1}g\|_{L^{p_3}}\|\nabla f\|_{L^{p_4}}\right). \tag{A.2}$$

In what follows, we also need the following variant version of (A.2) (its proof is the same one as for (A.2))

$$\|[\Lambda^{s-1}\partial_{x_i}, f]g\|_{L^r} \le C\left(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right). \tag{A.3}$$

The following lemma is the fractional version of Gagliardo-Nirenberg inequality (see [**10**] for example).

LEMMA A.4. *Let* $1 < p, q, r < \infty$, $0 \le \theta \le 1$ *and* $s, s_1, s_2 \in \mathbb{R}$, *then the following fractional Gagliardo-Nirenberg inequality*

$$\|\Lambda^s u\|_{L^p(\mathbb{R}^n)} \le C\|\Lambda^{s_1}u\|^{1-\theta}_{L^q(\mathbb{R}^n)}\|\Lambda^{s_2}u\|^{\theta}_{L^r(\mathbb{R}^n)}, \tag{A.4}$$

holds if and only if

$$\frac{1}{p}-\frac{s}{n}=(1-\theta)(\frac{1}{q}-\frac{s_1}{n})+\theta(\frac{1}{r}-\frac{s_2}{n}),\quad s\le(1-\theta)s_1+\theta s_2.$$

The proof of (2.12) Invoking the high-low frequency technique, we arrive at

$$\|\nabla u\|_{L^\infty}\le\|\Delta_{-1}\nabla u\|_{L^\infty}+\sum_{l=0}^{N-1}\|\Delta_l\nabla u\|_{L^\infty}+\sum_{l=N}^{\infty}\|\Delta_l\nabla u\|_{L^\infty}.$$

One easily obtains by the Bernstein lemma

$$\|\Delta_{-1}\nabla u\|_{L^\infty}\le C\|u\|_{L^2}$$

and

$$\sum_{l=N}^{\infty}\|\Delta_l\nabla u\|_{L^\infty}\le C\sum_{l=N}^{\infty}2^{l(1+\frac{n}{2}-\widetilde{s})}\|\Delta_l\Lambda^{\widetilde{s}}u\|_{L^2}\le C2^{N(1+\frac{n}{2}-\widetilde{s})}\|\Lambda^{\widetilde{s}}u\|_{L^2},$$

where $\widetilde{s} > 1+\frac{n}{2}$. According to the Bernstein lemma and the assumptions of the function g, the middle term can be handled as follows

$$\begin{aligned}\sum_{l=0}^{N-1}\|\Delta_l\nabla u\|_{L^\infty}\le{}& C\sum_{l=0}^{N-1}2^{l(1+\frac{n}{2})}\|\Delta_l u\|_{L^2}\\
\le{}& C\sum_{l=0}^{N-1}\Big\|\Delta_l\Big(g^2(\Lambda)\frac{\Lambda^{1+\frac{n}{2}}}{g^2(\Lambda)}u\Big)\Big\|_{L^2}\\
\le{}& C\sum_{l=0}^{N-1}\Big\|\varphi(2^{-l}\xi)g^2(|\xi|)\frac{|\xi|^{1+\frac{n}{2}}}{g^2(|\xi|)}\widehat{u}(\xi)\Big\|_{L^2}\\
\le{}& C\sum_{l=0}^{N-1}g^2(2^l)\Big\|\frac{|\xi|^{1+\frac{n}{2}}}{g^2(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|_{L^2}\\
\le{}& C\Big(\sum_{l=0}^{N-1}g^4(2^l)\Big)^{\frac{1}{2}}\left(\sum_{l=0}^{N-1}\Big\|\frac{|\xi|^{1+\frac{n}{2}}}{g^2(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|^2_{L^2}\right)^{\frac{1}{2}}\\
\le{}& Cg^2(2^N)\Big(\sum_{l=0}^{N-1}1\Big)^{\frac{1}{2}}\|\mathcal{L}^2u\|_{L^2}\\
\le{}& Cg^2(2^N)\sqrt{N}\|\mathcal{L}^2u\|_{L^2}.\end{aligned}$$

Summarizing the above three estimates implies

$$\|\nabla u\|_{L^\infty}\le C\|u\|_{L^2}+Cg^2(2^N)\sqrt{N}\|\mathcal{L}^2u\|_{L^2}+C2^{N(1+\frac{n}{2}-\widetilde{s})}\|\Lambda^{\widetilde{s}}u\|_{L^2},$$

which is nothing but the desired inequality (2.12).

Proof of (2.14) On one hand, if $g(\tau)\geq (\ln\tau)^{\frac14}$, then one has

$$\infty=\int_e^\infty \frac{d\tau}{\tau g^4(\tau)}\leq \int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}g^2(\tau)}.$$

This yields

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}g^2(\tau)}=\infty.$$

On the other hand, if $g(\tau)\leq (\ln\tau)^{\frac14}$, then we have

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}g^2(\tau)}\geq \int_e^\infty \frac{d\tau}{\tau\ln\tau}=\infty,$$

which immediately leads to

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}g^2(\tau)}=\infty.$$

Combining the above two cases, the desired (2.14) follows directly.

Appendix B. The proof of Theorem 1.5

The proof of Theorem 1.5 can be proved via that of Theorem 1.2. We only consider the critical case $\alpha+\beta=\frac12+\frac n4$. To begin with, the basic energy estimate reads as follows.

Lemma B.1. *Assume $v_0\in L^2(\mathbb{R}^n)$ and $\theta_0\in L^p(\mathbb{R}^n)$ for some $p\in[2,\infty]$. Then the corresponding solution (v,θ) of the system (1.9) admits the following bounds for any $t>0$*

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2+\int_0^t\|\mathcal{L}v(\tau)\|_{L^2}^2\,d\tau\leq C(t,v_0,\theta_0),\\ &\|\theta(t)\|_{L^p}\leq\|\theta_0\|_{L^p}.\end{aligned}$$

Next we would like to show the key estimate.

Lemma B.2. *Assume $\mathcal{L}v_0\in L^2(\mathbb{R}^n)$. Let (v,θ) be the corresponding solution of the system (1.9). If $\alpha+\beta=\frac12+\frac n4$ and $\beta>1$, then the following estimate holds*

$$\|\mathcal{L}v(t)\|_{L^2}^2+\int_0^t\|\mathcal{L}^2v(\tau)\|_{L^2}^2\,d\tau\leq C(t,v_0,\theta_0),$$

or

$$\left\|\frac{\Lambda^{2\alpha+\beta}}{g^2(\Lambda)}u(t)\right\|_{L^2}^2+\int_0^t\left\|\frac{\Lambda^{1+\frac n2}}{g^2(\Lambda)}u(\tau)\right\|_{L^2}^2\,d\tau\leq C(t,v_0,\theta_0).$$

Proof of Lemma B.2. Multiplying the first equation of (1.9) by $\mathcal{L}^2v$, we get

$$\frac12\frac{d}{dt}\|\mathcal{L}v(t)\|_{L^2}^2+\|\mathcal{L}^2v\|_{L^2}^2=-\int_{\mathbb{R}^n}(u\cdot\nabla v)\mathcal{L}^2v\,dx+\int_{\mathbb{R}^n}\theta e_n\mathcal{L}^2v\,dx.$$

It is easy to check for $\beta>1$ that

$$\begin{aligned}\left|-\int_{\mathbb{R}^n}(u\cdot\nabla v)\mathcal{L}^2v\,dx\right|\leq&C\|u\|_{L^{\frac{n}{\beta-1}}}\|\nabla v\|_{L^{\frac{2n}{n+2-2\beta}}}\|\mathcal{L}^2v\|_{L^2}\\ \leq&C\|\Lambda^{\frac{n+2-2\beta}{2}}u\|_{L^2}\|\Lambda^\beta v\|_{L^2}\|\mathcal{L}^2v\|_{L^2}\end{aligned}$$

$$\leq C\|\Lambda^{\beta}v\|_{L^2}^2\|\mathcal{L}^2 v\|_{L^2},$$

where we have applied the fact $v = u + \Lambda^{2\alpha}u$. The Young inequality leads to

$$\Big|\int_{\mathbb{R}^n}\theta e_n\mathcal{L}^2 v\,dx\Big| \leq \|\theta\|_{L^2}\|\mathcal{L}^2 v\|_{L^2} \leq \frac{1}{16}\|\mathcal{L}^2 v\|_{L^2}^2 + C\|\theta\|_{L^2}^2.$$

Collecting the above estimates yields

$$\frac{d}{dt}\|\mathcal{L}v(t)\|_{L^2}^2 + \|\mathcal{L}^2 v\|_{L^2}^2 \leq C\|\Lambda^{\beta}v\|_{L^2}^2\|\mathcal{L}^2 v\|_{L^2} + C\|\theta\|_{L^2}^2. \tag{B.1}$$

Now making use of the same argument used in proving (2.3), we have for any $\lambda_1 \in (0,\, 2\alpha+\beta)$ and any $\lambda_2 \in (0,\, 2\alpha+2\beta)$ that

$$\|\mathcal{L}v\|_{L^2}^2 \geq C_1\|\Lambda^{\lambda_1}u\|_{L^2}^2 - C_2\|u\|_{L^2}^2, \tag{B.2}$$

$$\|\mathcal{L}^2 v\|_{L^2}^2 \geq C_3\|\Lambda^{\lambda_2}u\|_{L^2}^2 - C_4\|u\|_{L^2}^2. \tag{B.3}$$

It follows from the high-low frequency technique that

$$\|\Lambda^{\beta}v\|_{L^2} \leq \|S_N\Lambda^{\beta}v\|_{L^2} + \sum_{j\geq N}\|\Delta_j\Lambda^{\beta}v\|_{L^2}.$$

According to Plancherel's theorem and Sobolev embedding, we have

$$\begin{aligned}
\|S_N\Lambda^{\beta}v\|_{L^2} =&\|\chi(2^{-N}\xi)|\xi|^{\beta}\widehat{v}(\xi)\|_{L^2}\\
=&C\Big\|\chi(2^{-N}\xi)g(|\xi|)\frac{|\xi|^{\beta}}{g(|\xi|)}\widehat{v}(\xi)\Big\|_{L^2}\\
\leq&Cg(2^N)\|\mathcal{L}v\|_{L^2}.
\end{aligned}$$

By Lemma A.1 and the fact $v = u + \Lambda^{2\alpha}u$, one gets

$$\begin{aligned}
\sum_{j\geq N}\|\Delta_j\Lambda^{\beta}v\|_{L^2} \leq& C\sum_{j\geq N}2^{-\frac{1}{4}j}\|\Delta_j\Lambda^{\beta+\frac{1}{4}}u\|_{L^2} + C\sum_{j\geq N}2^{-\frac{1}{4}j}\|\Delta_j\Lambda^{2\alpha+\beta+\frac{1}{4}}u\|_{L^2}\\
\leq& C\sum_{j\geq N}2^{-\frac{1}{4}j}2^{-2\alpha j}\|\Delta_j\Lambda^{2\alpha+\beta+\frac{1}{4}}u\|_{L^2}\\
&+ C\sum_{j\geq N}2^{-\frac{1}{4}j}\|\Delta_j\Lambda^{2\alpha+\beta+\frac{1}{4}}u\|_{L^2}\\
\leq& C\sum_{j\geq N}2^{-\frac{1}{4}j}\|\Delta_j\Lambda^{2\alpha+\beta+\frac{1}{4}}u\|_{L^2}\\
\leq& C2^{-\frac{N}{4}}\|\Lambda^{\frac{1+8\alpha+4\beta}{4}}u\|_{L^2}\\
\leq& C2^{-\frac{N}{4}}\|\Lambda^{\lambda_1}u\|_{L^2}^{1-\varrho}\|\Lambda^{\lambda_2}u\|_{L^2}^{\varrho},
\end{aligned}$$

where

$$\varrho = \frac{1+8\alpha+4\beta-4\lambda_1}{4\lambda_2-4\lambda_1},\quad 0<\lambda_1<2\alpha+\beta,\quad \frac{1+8\alpha+4\beta}{4}<\lambda_2<2\alpha+2\beta.$$

Thus, it is obvious to check that

$$\|\Lambda^{\beta}v\|_{L^2} \leq Cg(2^N)\|\mathcal{L}v\|_{L^2} + C2^{-\frac{N}{4}}\|\Lambda^{\lambda_1}u\|_{L^2}^{1-\varrho}\|\Lambda^{\lambda_2}u\|_{L^2}^{\varrho}.$$

Consequently, one has by using (B.2) and (B.3)

$$\begin{aligned}
&C\|\Lambda^{\beta}v\|_{L^2}^2\|\mathcal{L}^2 v\|_{L^2}\\
&\leq C\Big(g^2(2^N)\|\mathcal{L}v\|_{L^2}^2 + 2^{-\frac{N}{2}}\|\Lambda^{\lambda_1}u\|_{L^2}^{2(1-\varrho)}\|\Lambda^{\lambda_2}u\|_{L^2}^{2\varrho}\Big)\|\mathcal{L}^2 v\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\Big(g^2(2^N)\|\mathcal{L}v\|_{L^2}^2+2^{-\frac{N}{2}}(\|u\|_{L^2}+\|\mathcal{L}v\|_{L^2})^{2(1-\varrho)}(\|u\|_{L^2}+\|\mathcal{L}^2v\|_{L^2})^{2\varrho}\Big)\|\mathcal{L}^2v\|_{L^2}\\
&=Cg^2(2^N)\|\mathcal{L}v\|_{L^2}^2\|\mathcal{L}^2v\|_{L^2}+C2^{-\frac{N}{2}}\|\mathcal{L}v\|_{L^2}^{2-2\varrho}\|\mathcal{L}^2v\|_{L^2}^{1+2\varrho}+C2^{-\frac{N}{2}}\|u\|_{L^2}^2\|\mathcal{L}^2v\|_{L^2}\\
&\quad+C2^{-\frac{N}{2}}\|u\|_{L^2}^{2\varrho}\|\mathcal{L}v\|_{L^2}^{2-2\varrho}\|\mathcal{L}^2v\|_{L^2}+C2^{-\frac{N}{2}}\|u\|_{L^2}^{2-2\varrho}\|\mathcal{L}^2v\|_{L^2}^{1+2\varrho}\\
&\leq\frac{1}{2}\|\mathcal{L}^2v\|_{L^2}^2+Cg^4(2^N)\|\mathcal{L}v\|_{L^2}^4+C2^{-\frac{N}{1-2\varrho}}\|\mathcal{L}v\|_{L^2}^{\frac{4(1-\varrho)}{1-2\varrho}}+C2^{-N}\|u\|_{L^2}^{4\varrho}\|\mathcal{L}v\|_{L^2}^{4(1-\varrho)}\\
&\quad+C\|u\|_{L^2}^2+C\|u\|_{L^2}^{\frac{4(1-\varrho)}{1-2\varrho}}.
\end{aligned}
$$

This along with (B.1) yields

$$
\begin{aligned}
\frac{d}{dt}\|\mathcal{L}v(t)\|_{L^2}^2+\|\mathcal{L}^2v\|_{L^2}^2\leq&Cg^4(2^N)\|\mathcal{L}v\|_{L^2}^4+C2^{-\frac{N}{1-2\varrho}}\|\mathcal{L}v\|_{L^2}^{\frac{4(1-\varrho)}{1-2\varrho}}\\
&+C2^{-N}\|u\|_{L^2}^{4\varrho}\|\mathcal{L}v\|_{L^2}^{4(1-\varrho)}+C\|u\|_{L^2}^2+C\|u\|_{L^2}^{\frac{4(1-\varrho)}{1-2\varrho}}\\
&+C\|\theta\|_{L^2}^2,
\end{aligned}\tag{B.4}
$$

where $\varrho<\frac{1}{2}$. Now we denote

$$
X(t):=\|\mathcal{L}v(t)\|_{L^2}^2,\qquad Y(t):=\|\mathcal{L}^2v(t)\|_{L^2}^2,
$$
$$
h(t):=\|u(t)\|_{L^2}^2+\|u(t)\|_{L^2}^{\frac{4(1-\varrho)}{1-2\varrho}}+\|\theta(t)\|_{L^2}^2
$$

and take N as

$$
2^N\approx e+X(t).
$$

By tedious computations, it follows from (B.4) that

$$
\frac{d}{dt}X(t)+Y(t)\leq C(1+\|\mathcal{L}v\|_{L^2}^2)(e+X(t))g^4\big(e+X(t)\big)+Ch(t).\tag{B.5}
$$

Thanks to

$$
g^4\big(e+X(t)\big)\big(e+X(t)\big)\geq 1,
$$

we deduce from (B.5) that

$$
\int_{e+X(0)}^{e+X(t)}\frac{d\tau}{\tau g^4(\tau)}\leq C\int_0^t\big(1+\|\mathcal{L}v(\tau)\|_{L^2}^2+h(\tau)\big)\,d\tau.
$$

Keeping in mind the condition (1.10), namely,

$$
\int_e^\infty\frac{d\tau}{\tau g^4(\tau)}=\infty
$$

and the bound

$$
\int_0^t\big(1+\|\mathcal{L}v(\tau)\|_{L^2}^2+h(\tau)\big)\,d\tau\leq C(t,\,v_0,\,\theta_0),
$$

it implies

$$
X(t)\leq C(t,\,v_0,\,\theta_0).
$$

Thanks to (3.14), one also obtains

$$
\int_0^t Y(\tau)\,d\tau\leq C(t,\,v_0,\,\theta_0).
$$

So we finally discover that

$$
\|\mathcal{L}v(t)\|_{L^2}^2+\int_0^t\|\mathcal{L}^2v(\tau)\|_{L^2}^2\,d\tau\leq C(t,\,v_0,\,\theta_0).
$$

This completes the proof of the lemma. □

Finally, we will derive the following lemma inspired by the proof of Lemma 3.3.

LEMMA B.3. *Assume $v_0 \in H^{1+\beta}(\mathbb{R}^n)$ and $\nabla\theta_0 \in L^2 \cap L^\infty(\mathbb{R}^n)$. Let (v, θ) be the corresponding solution of the system (1.9). If $\alpha + \beta = \frac{1}{2} + \frac{n}{4}$ and $\beta > 1$, then the following estimate holds*

$$\begin{aligned}\|\nabla\theta(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^\infty} + \|v(t)\|_{H^{1+\beta-\epsilon}}^2 + \int_0^t (\|\mathcal{L}v(\tau)\|_{H^{1+\beta-\epsilon}}^2 + \|v(\tau)\|_{H^{1+2\beta-2\epsilon}}^2)\, d\tau \\ \le C(t, v_0, \theta_0),\end{aligned} \tag{B.6}$$

where $0 < \epsilon \ll 1$ (can be arbitrarily small). In particular, there holds

$$\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}} u(\tau)\|_{L^2})\, d\tau \le C(t, v_0, \theta_0). \tag{B.7}$$

PROOF OF LEMMA B.3. It is sufficient to modify the proof of Lemma 3.3. Applying $\Lambda^{1+\beta-\epsilon}$ to $(1.2)_1$ and multiplying it by $\Lambda^{1+\beta-\epsilon}v$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^{1+\beta-\epsilon}v(t)\|_{L^2}^2 + \|\Lambda^{1+\beta-\epsilon}\mathcal{L}v\|_{L^2}^2 = &-\int_{\mathbb{R}^n} [\Lambda^{1+\beta-\epsilon}, u\cdot\nabla]v\, \Lambda^{1+\beta-\epsilon}v\, dx \\ &+ \int_{\mathbb{R}^n} \Lambda^{1+\beta-\epsilon}\theta e_n \Lambda^{1+\beta-\epsilon}v\, dx.\end{aligned}$$

According to the proof of (2.3), it ensures that

$$\|\Lambda^{1+\beta-\epsilon}\mathcal{L}v\|_{L^2}^2 \ge C_7\|\Lambda^{1+2\beta-2\epsilon}v\|_{L^2}^2 - C_8\|\Lambda^{1+\beta-\epsilon}v\|_{L^2}^2. \tag{B.8}$$

The commutator (A.2) allows us to show

$$\begin{aligned}&-\int_{\mathbb{R}^n} [\Lambda^{1+\beta-\epsilon}, u\cdot\nabla]v\, \Lambda^{1+\beta-\epsilon}v\, dx \\ &\le C\|[\Lambda^{1+\beta-\epsilon}, u\cdot\nabla]v\|_{L^2}\|\Lambda^{1+\beta-\epsilon}v\|_{L^2} \\ &\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{1+\beta-\epsilon}v\|_{L^2} + \|\nabla v\|_{L^{\frac{2n}{n+2\epsilon-2\beta}}}\|\Lambda^{1+\beta-\epsilon}u\|_{L^{\frac{n}{\beta-\epsilon}}})\|\Lambda^{1+\beta-\epsilon}v\|_{L^2} \\ &\le C(\|\nabla u\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}}u\|_{L^2})\|\Lambda^{1+\beta-\epsilon}v\|_{L^2}^2.\end{aligned}$$

In addition, owing to the Young inequality and (B.8), we also claim that

$$\begin{aligned}\int_{\mathbb{R}^n} \Lambda^{1+\beta-\epsilon}\theta e_n \Lambda^{1+\beta-\epsilon}v\, dx \le& C\|\nabla\theta\|_{L^2}\|\Lambda^{1+2\beta-2\epsilon}v\|_{L^2} \\ \le& C\|\nabla\theta\|_{L^2}(\|\Lambda^{1+\beta-\epsilon}\mathcal{L}v\|_{L^2} + \|\Lambda^{1+\beta-\epsilon}v\|_{L^2}) \\ \le& \frac{1}{16}\|\Lambda^{1+\beta-\epsilon}\mathcal{L}v\|_{L^2}^2 + C(\|\Lambda^{1+\beta-\epsilon}v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).\end{aligned}$$

Putting the above estimates together yields

$$\begin{aligned}&\frac{d}{dt}\|\Lambda^{1+\beta-\epsilon}v(t)\|_{L^2}^2 + \|\Lambda^{1+\beta-\epsilon}\mathcal{L}v\|_{L^2}^2 + \|\Lambda^{1+2\beta-2\epsilon}v\|_{L^2}^2 \\ &\le C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{1+\frac{n}{2}}u\|_{L^2})(\|\Lambda^{1+\beta-\epsilon}v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).\end{aligned}$$

The remainder proof is the same as that of Lemma 3.3. We thus omit the details. Therefore, we conclude the proof of Lemma 3.3. □

The global H^s-estimate. It follows from (3.24) that

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)+\|\mathcal{L}\Lambda^s v\|_{L^2}^2\\&\le C(1+\|v\|_{H^{1+\beta}}^2+\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})(1+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2).\end{aligned}$$

Therefore, the desired global H^s-estimate is an easy consequence of the Gronwall inequality and (B.6) as well as (B.7). This completes the proof of Theorem 1.5. □

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