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A steady model on Navier-Stokes equations with a free surface

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Abstract. We consider the evolution of viscous fluids in a 2D horizontally periodic slab bounded above by a free top surface and below by a fixed flat bottom. The dynamics of the fluid are governed by the incompressible stationary Navier-Stokes equations under the influence of gravity and the effect of surface tension. We develop the existence and uniqueness of solutions in low regularity Sobolev spaces on $[0, T]$ for any $T > 0$. Our methods are mainly based on linear estimates of a geometric formulation of an ε–approximate system.

Contents

1. Introduction

1.1. Formulation of the problem. The stationary Navier-Stokes equations is a classical problem that has been researched by many mathematicians. For a comprehensive introduction of the fixed boundary problems of stationary Navier-Stokes equations, we refer to [**15**] and [**21**]. So we encounter a natural question: what is the behavior of the free boundary Navier-Stokes equation? Especially interesting is determining the effect of the boundary's time evolution. However, if the

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surface is deformable in time, the fluid is hard to stay steady. As far as the authors knowledge, there are still some models concerning steady fluid with a free surface. But in those cases, the surface is assumed to be sufficiently small in order to guarantee that those models are physical, for instance, the Darcy's flow concerned in [**13, 14**]. This is the motivation for us to present the problem.

We consider the viscous incompressible fluid moving in a slab

$$\Omega(t) = \{y = (y_1, y_2) \in \mathbb{T} \times \mathbb{R} | -1 < y_2 < \eta(y_1, t)\}.$$

Here we assume the top free surface is a graph of function of η and denoted by $\Sigma(t) := \{y_2 = \eta(y_1, t)\}$. The flat fixed boundary is denoted by $\Sigma_b := \{y_2 = -1\}$.

For each $t \geq 0$, the fluid is described by its velocity and pressure functions: $(u, p) : \Omega(t) \to \mathbb{R}^2 \times \mathbb{R}$. The stress tensor is determined by $S(p, u) = pI - \mu \mathbb{D}u$, where I is the 2×2 identity matrix, $\mu > 0$ is the coefficient of viscosity and $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ is the symmetric gradient of u. We note that if $\operatorname{div} u = 0$, then $\operatorname{div} S(p, u) = \nabla p - \mu \Delta u$.

The dynamic of the stationary viscous surface waves is governed by the stationary Navier-Stokes equations:

$$\begin{cases} u \cdot \nabla u + \nabla p - \mu \Delta u = 0 & \text{in } \Omega(t), \\ \nabla \cdot u = 0 & \text{in } \Omega(t), \end{cases} \tag{1.1}$$

subjected to the boundary conditions

$$\begin{cases} S(p, u)n = g\eta n - \sigma H n & \text{on } \{y_2 = \eta(y_1, t)\}, \\ u \cdot \nu = 0 & \text{on } \{y_2 = -1\}, \\ (S(p, u)\nu - \beta u) \cdot \tau = 0 & \text{on } \{y_2 = -1\}, \\ \partial_t \eta + u_1 \partial_1 \eta = u_2 & \text{on } \{y_2 = \eta(y_1, t)\}. \end{cases} \tag{1.2}$$

In the above systems (1.1) and (1.2), $g > 0$ is the strength of gravity, $\sigma > 0$ is the coefficient of surface tension, n is the unit-outward-normal of free surface, ν is the unit-outward-normal of bottom, and τ is the unit tangential of bottom. H is the twice mean curvature of free surface given by

$$H = \partial_1 \left(\frac{\partial_1 \eta}{\sqrt{1 + |\partial_1 \eta|^2}} \right).$$

The description of the first and last equations of (1.2), we refer to [**23**]. The velocity of fluid on Σ_b satisfies the Navier-slip condition, which means

$$u \cdot \nu = 0, \quad (S(p, u)\nu - \beta u) \cdot \tau = 0$$

where $\beta > 0$ is the coefficient of friction between fluid and bottom. Sometimes the Dirichlet condition is no-physical, (see the moving contact line [**12**]), so we consider Navier conditions. In addition, we have shifted the pressure by $p = \bar{p} + gy - p_{atm}$, where $\bar{p}$ is the real pressure and p_{atm} is the constant pressure of atmosphere.

This is a free boundary problem, we assume that the initial datum $\eta(t = 0) = \eta_0$ satisfying $1 + \eta_0 > 0$. We also assume that $\int_\Sigma \eta_0 = 0$. It is clear that this condition is conserved in time for the incompressible fluids.

The first pioneer paper on the free boundary of stationary Navier-Stokes should be due to Solonnikov [**17**] for 3D version in a semi-infinite cylinder. Then Solonnikov [**18**] proved the existence and uniqueness of (1.1) in a 3D infinite cylinder in some Hölder spaces. In addition, Solonnikov considered the moving contact lines for stationary flows in [**19**] and two phase stationary flows in [**20**]. As the author's knowledge, there are no any results for the stationary surface waves in 2D domains with Navier-slip boundary conditions. Our aim is to give a proof of the existence and uniqueness of solutions to (1.1) and (1.2). Since our problem has no temporal derivative of velocity u, we could not expect that any methods of dealing with free surface of non-stationary Navier-Stokes equations, i.e., viscous surface waves, (for instance, see [**2, 3, 9, 10, 11, 16, 24**] and references therein), are helpful. So we need to find a new way to prove this quasi-stationary system (1.1).

In previous results, (for instance, see [**9, 10, 11**]), they all assume that the initial data are smooth enough and satisfy many compatible conditions. so our original aim is to study the full Navier-Stokes equations with free boundaries in low regularity (such as $u \in H^2(\Omega)$). Unfortunately, this issue is so complicated, that we have not find a workable way to handle it. So we consider this model (1.1) in low regular spaces first, then we will deal with the full Navier-Stokes equations in the following paper.

1.2. Geometric reformulation. Since the free surface and the change of $\Omega(t)$ will create numerous mathematical difficulties, we straighten the time dependent domain $\Omega(t)$ to a time independent domain $\Omega := \mathbb{T} \times (-1, 0)$. The idea was introduced by J. T. Beale in [**3**, Section 5]. We define $\bar{\eta}$ to be the harmonic extension of η according to

$$\bar{\eta} = \sum_{k\in\mathbb{Z}} e^{2\pi i k x_1} e^{2\pi|k|x_2} \hat{\eta}(k), \tag{1.3}$$

where $\hat{\eta}(k)$ is the coefficient of Fourier series. The harmonic extension $\bar{\eta}$ allows us to introduce the mapping Φ from Ω to $\Omega(t)$ as

$$\Phi : (x_1, x_2) \mapsto (x_1, x_2 + \bar{\eta}(1 + x_2)) = (y_1, y_2). \tag{1.4}$$

If η is sufficiently small in some norms, Φ is a C^1 diffeomorphism. Clearly, Φ maps Ω onto $\Omega(t)$. If we denote $\Sigma = \{x_2 = 0\}$, we have that Φ maps Σ onto $\Sigma(t)$. We also see that Φ keeps the bottom Σ_b.

Then we have its Jacobian matrix $\nabla\Phi$ and the transform matrix $\mathcal{A}$:

$$\nabla\Phi = \begin{pmatrix} 1 & 0 \\ A & J \end{pmatrix}, \quad \mathcal{A} = ((\nabla\Phi)^{-1})^\top = \begin{pmatrix} 1 & -AK \\ 0 & K \end{pmatrix}$$

where

$$A = (1 + x_2)\partial_1\bar{\eta},\ J = 1 + \bar{\eta} + (1 + x_2)\partial_2\bar{\eta},\ K = 1/J. \tag{1.5}$$

In the following, we may assume that Φ is a C^1 diffeomorphism (actually, we will prove this later). Then, we define some transformed operators. The differential operators $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$ and $\Delta_{\mathcal{A}}$ are defined as follows:

$$(\nabla_{\mathcal{A}} f)_i = \mathcal{A}_{ij}\partial_j f,\ \operatorname{div}_{\mathcal{A}} u = \mathcal{A}_{ij}\partial_j u_i,\ \Delta_{\mathcal{A}} f = \operatorname{div}_{\mathcal{A}} \cdot \nabla_{\mathcal{A}} f.$$

The symmetric $\mathcal{A}$-gradient $\mathbb{D}_{\mathcal{A}}$ is defined as $(\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i$. And we write the stress tensor as $S_{\mathcal{A}}(p, u) = pI - \mathbb{D}_{\mathcal{A}} u$, where I is the 2×2 identity matrix. Then we note that $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u$ for vector fields satisfying $\operatorname{div}_{\mathcal{A}} u = 0$. We have also written $\mathcal{N} = (-\partial_1\eta, 1)$ for the nonunit normal to $\{y_2 = \eta(y_1, t)\}$.

Then the original system in this new coordinate becomes

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) + u\cdot\nabla_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(p,u)\mathcal{N} = g\eta\mathcal{N} - \sigma\partial_1^2\eta\mathcal{N} - \sigma\partial_1\mathcal{R}\mathcal{N} & \text{on } \Sigma, \\ u\cdot\nu = 0, \quad (S_{\mathcal{A}}(p,u)\nu - \beta u)\cdot\tau = 0 & \text{on } \Sigma_b, \\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Sigma, \end{cases} \tag{1.6}$$

where $\mathcal{R} = \mathcal{R}(\partial_1\eta)$ is the remainder defined via

$$\mathcal{R}(z) = 3\int_0^z \frac{s(s-z)}{(1+s^2)^{5/2}}\,\mathrm{d}s. \tag{1.7}$$

1.3. Main results. In order to state our result, we need to explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \geq 0$ to be the usual Sobolev spaces. We write norms $\|\partial_t^j u\|_k$ and $\|\partial_t^j p\|_k$ in the space $H^k(\Omega)$, and $\|\partial_t^j \eta\|_k$ in the space $H^k(\Sigma)$. We also need $\mathring{H}^s$, which will be defined in Section 2.

We now define the energy and dissipation used in our main results. The energy $\mathcal{E}(t)$ is defined via

$$\begin{aligned} \mathcal{E}(t) :&= \|u(t)\|_2^2 + \|\partial_t u(t)\|_1^2 + \|p(t)\|_1^2 + \|\partial_t p(t)\|_0^2 + \|\eta(t)\|_{5/2}^2 \\ &\quad + \|\partial_t\eta(t)\|_{3/2}^2 + \sum_{j=0}^1 \|\partial_t^j\eta(t)\|_1^2. \end{aligned} \tag{1.8}$$

The dissipation $\mathcal{D}(t)$ is defined via

$$\begin{aligned} \mathcal{D}(t) :&= \sum_{j=0}^1 \Big(\|\partial_t^j u(t)\|_2^2 + \|\partial_t^j p(t)\|_1^2 + \|\partial_t^j\eta(t)\|_{5/2}^2\Big) \\ &\quad + \sum_{j=0}^2 \Big(\|\partial_t^j u(t)\|_1^2 + \|\partial_t^j u(t)\cdot\tau\|_{H^0(\Sigma_b)}^2\Big) \\ &\quad + \sum_{j=0}^2 \Big(\|\partial_t^j p(t)\|_0^2 + \|\partial_t^j\eta(t)\|_{3/2}^2\Big) + \|\partial_t^3\eta(t)\|_{1/2}^2. \end{aligned} \tag{1.9}$$

Now we state our main results.

THEOREM 1.1. *Assume the initial data satisfy the inclusions* $\eta_0 \in \mathring{H}^{5/2}(\Sigma)$, *and* $\partial_t\eta(0) \in \mathring{H}^{3/2}(\Sigma)$ *and* $\partial_t^2\eta(0) \in \mathring{H}^1(\Sigma)$. *Then there exists* $\gamma_0 > 0$, *such that if*

$$\mathfrak{E}_0 := \|\eta_0\|_{5/2}^2 + \|\partial_t\eta(0)\|_{3/2}^2 + \|\partial_t^2\eta(0)\|_1^2 \leq \gamma_0$$

then there exists a unique solution (u,p,η) *to* (1.6) *on the interval* $[0,T]$ *for* $T>0$ *that achieves the initial data and satisfies*

$$\sup_{0\leq t\leq T}\mathcal{E}(t) + \int_0^T \mathcal{D}(t)\,\mathrm{d}t \leq C\mathfrak{E}_0 \tag{1.10}$$

for a universal constant $C>0$. *Moreover,* Φ *defined by* (1.4) *is a* C^1 *diffeomorphism for each* $t\in[0,T]$.

REMARK 1.2. We emphasize that the small initial data is necessary, which enables our model is physical.

REMARK 1.3. Since Φ is a C^1 diffeomorphism, we can change coordinates from Ω to $\Omega(t)$ to gain solutions of (1.1).

REMARK 1.4. The assumption of flat fixed bottom is not necessary. Actually, if we assume that the bottom is a function $-b(y_1) \in C^\infty(\Sigma)$ with $b(y_1) > 0$, then we modify the flattened mapping Φ via

$$\Phi : (x_1, x_2) \mapsto (x_1, x_2 + \bar{\eta}(1 + x_2/b)) = (y_1, y_2). \tag{1.11}$$

and use the same method in our paper to get the same results as Theorem 1.1.

1.4. Principle difficulties. Our method is ultimately based on the following geometric formulation of the free boundary problem of linear Stokes equations and a fixed point argument. We suppose that η (and hence $\mathcal{A}$, $\mathcal{N}$, etc.) is given and then solve the linear $\mathcal{A}$–Stokes equations for (u, p, ξ):

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(p, u)\mathcal{N} = (g\xi - \sigma\partial_1^2\xi)\mathcal{N} - \sigma\partial_1\mathcal{R}\mathcal{N} + F^3 & \text{on } \Sigma, \\ \partial_t\xi = u \cdot \mathcal{N} & \text{on } \Sigma, \\ u \cdot \nu = 0, \quad (S_{\mathcal{A}}(p, u)\nu - \beta u) \cdot \tau = F^5 & \text{on } \Sigma_b. \end{cases} \tag{1.12}$$

Our procedure is to employ the time-dependent Galerkin method to construct the approximate sequence (u^m, ξ^m) for $m \geq 1$. Unfortunately, in attempting to work directly with (1.12) in energy estimate argument we encounter serious difficulties with estimating a couple key terms. For instance we need to estimate interaction terms of the form

$$\int_\Sigma \mathcal{R}'\partial_1\partial_t\eta\partial_1\partial_t^2\xi^{\varepsilon,m}, \tag{1.13}$$

in (3.43), but the left-hand side of (3.46) has no terms to control $\partial_t^2\xi^{\varepsilon,m}$ for (1.12).

Fortunately, it's possible to bypass this difficulty through a perturbation method, which provides a crucial extra estimate. We consider the following ε–approximate linear $\mathcal{A}$-Stokes instead of (1.12):

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p^\varepsilon, u^\varepsilon) = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u^\varepsilon = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(p^\varepsilon, u^\varepsilon)\mathcal{N} = g(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon) - \sigma\partial_1^2(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon)\mathcal{N} - \sigma\partial_1\mathcal{R}\mathcal{N} + F^3 & \text{on } \Sigma, \\ \partial_t\xi^\varepsilon = u^\varepsilon \cdot \mathcal{N} & \text{on } \Sigma, \\ u^\varepsilon \cdot \nu = 0, \quad (S_{\mathcal{A}}(p^\varepsilon, u^\varepsilon)\nu - \beta u^\varepsilon) \cdot \tau = F^5 & \text{on } \Sigma_b. \end{cases} \tag{1.14}$$

Using the mean curvature term shows then that $\partial_t^2\xi^{\varepsilon,m}$ could be controlled. This allows us to estimate the term (1.13) while retaining the same basic form of the energy-dissipation estimates that the problem (1.12) enjoys. We thus base our analysis on this ε-approximate problem.

For the usual viscous surface waves ([**9, 10, 11**]), the function of free surface is obtained by the transport equation $\partial_t\eta = u \cdot \mathcal{N}$ decoupling with (u, p). However, this method does not work directly for our problem, which could not gains the $H^{5/2}$ regularity for $\partial_t\eta$. Thus we deal with the coupling (u, p, ξ) together. Here ξ stands for some derivatives of η or η itself.

1.5. Notation and terminology. Now, we mention some definitions, notation, and conventions that we will use throughout this paper.

1. Constants. The symbol $C > 0$ will denote a universal constant that only depends on the parameters of the problem and Ω, but does not depend on the data, etc. It is allowed to change from line to line. We will write $C = C(z)$ to indicate that the constant C depends on z. We will write $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$.

2. Norms. We will write H^k for $H^k(\Omega)$ for $k \geq 0$, and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual Sobolev spaces. We will typically write $H^0 = L^2$, though we will also use $L^2([0,T]; H^k)$ (or $L^2([0,T]; H^s(\Sigma))$) to denote the space of temporal square–integrable functions with values in H^k (or $H^s(\Sigma)$). Sometimes we will write $\|\cdot\|_k$ instead of $\|\cdot\|_{H^k(\Omega)}$ or $\|\cdot\|_{H^k(\Sigma)}$. We also will write $\|\cdot\|_{L^2H^k}$ instead of $\|\cdot\|_{L^2([0,T];H^k(\Omega))}$ or $\|\cdot\|_{L^2([0,T];H^k(\Sigma))}$. When we do this it will be clear on which set the norm is evaluated from the context and the argument of the norm.

2. Functional setting

2.1. Function spaces and bilinear forms. We begin with some time independent spaces.

$$\mathring{H}^k(\Omega) = \{p \in H^k(\Omega) | \int_\Omega p = 0\}, \tag{2.1}$$

$$\mathring{H}^s((-\ell,\ell)) = \{\eta \in H^s((-\ell,\ell)) | \int_{-\ell}^{\ell} \eta = 0\}, \tag{2.2}$$

We use $H^k(\Omega)$ and $H^s((-\ell,\ell))$ to denote the usual scalar-valued or vector-valued Sobolev spaces. Then we define vector valued spaces

$$\begin{aligned} &{}_0H^1(\Omega) := \{u \in H^1(\Omega) | u \cdot \nu = 0 \text{ on } \Sigma_b\}, \quad W := \{u \in {}_0H^1(\Omega) | u \cdot e_2 \in H^1(\Sigma)\}, \\ &X := \{u \in W | \operatorname{div} u = 0\}. \end{aligned}$$

Throughout this paper, we usually utilize the following Korn-type inequality.

LEMMA 2.1. *For any $u \in {}_0H^1(\Omega)$, it holds that*

$$\|u\|_1^2 \lesssim \|\mathbb{D}u\|_0^2. \tag{2.3}$$

PROOF. The inequality (2.3) follows easily from the inequality

$$\|u\|_1^2 \lesssim \|\mathbb{D}u\|_0^2 + \|u\|_0^2 \text{ for all } u \in H^1(\Omega), \tag{2.4}$$

and a standard compactness argument. It can also be derived from the Nečas inequality: see for example [**4**, Lemma IV.7.6]. □

We now define the time-dependent spaces. For η is given with $\mathcal{A}$, J, etc determined by η via (1.5), we consider an inner product on L^2 defined by

$$(u,v)_{\mathcal{H}^0} := \int_\Omega u \cdot v J(t) \tag{2.5}$$

corresponding to norm $\|u\|_{\mathcal{H}^0} = \sqrt{(u,u)_{\mathcal{H}^0}}$. Then we denote

$$\mathcal{H}^0(t) = \{u(t) \in L^2(\Omega) | \|u(t)\|_{\mathcal{H}^0} < \infty\}. \tag{2.6}$$

Similarly, we introduce an inner product on ${}_0H^1(\Omega)$ according to

$$(u,v)_{\mathcal{H}^1} := \int_\Omega \mathbb{D}_{\mathcal{A}(t)}u : \mathbb{D}_{\mathcal{A}(t)}vJ(t) \tag{2.7}$$

with corresponding norm $\|u\|_{\mathcal{H}^1} = \sqrt{(u,u)_{\mathcal{H}^1}}$. Then we write

$$\mathcal{H}^1 := \{\|u\|_{\mathcal{H}^1} < \infty\}. \tag{2.8}$$

$$\mathcal{W}(t) := \{u(t) \in \mathcal{H}^1(t) | u \cdot \mathcal{N} \in H^1(\Sigma)\} \tag{2.9}$$

endowed with the norm $\|u\|_{\mathcal{W}} := \|u\|_{\mathcal{H}^1} + \|u \cdot \mathcal{N}\|_{H^1(\Sigma)}$.

$$\mathcal{X}(t) := \{u(t) \in \mathcal{W}(t) | \operatorname{div}_{\mathcal{A}(\mathrm{t})} u = 0\}. \tag{2.10}$$

Finally, we define inner products on scalar-valued or vector-valued spaces $L^2([0,T]; H^1(\Omega))$ for $T > 0$ via

$$(u,v)_{\mathcal{H}^1_T} := \int_0^T (u,v)_{\mathcal{H}^1(t)}\,\mathrm{d}t, \tag{2.11}$$

endowed with norms $\|\cdot\|_{\mathcal{H}^1_T}$ corresponding to spaces $\mathcal{H}^1_T$. Similarly, we might define spaces $\mathcal{W}_T$ and $\mathcal{X}_T$.

LEMMA 2.2. *Suppose that η is given so that*

$$\sup_{0\le t\le T} \|\eta(t)\|_{5/2} \le \gamma_1 \tag{2.12}$$

for a universal constant $\gamma_1 > 0$ and $T > 0$. Then for vector-valued spaces,

$$\frac{1}{\sqrt{2}}\|u\|_k \le \|u\|_{\mathcal{H}^k} \le \sqrt{2}\|u\|_k \tag{2.13}$$

for $k = 0, 1$. Consequently,

$$\frac{1}{\sqrt{2}}\|u\|_{L^2H^k} \le \|u\|_{\mathcal{H}^k_T} \le \sqrt{2}\|u\|_{L^2H^k}. \tag{2.14}$$

PROOF. The proof for vector-valued spaces is similar to [**9**, Lemma 2.1], so we omit the details here. □

Now, we give some useful relations between time-independent spaces and time-dependent spaces. We consider the matrix

$$M := M(t) = K\nabla\Phi = (J\mathcal{A}^\top)^{-1}, \tag{2.15}$$

which induces a linear operator $\mathcal{M}_t : u \mapsto M(t)u$.

PROPOSITION 2.3. *Assume that $\eta \in H^{5/2}(\Sigma)$.*

(1) For each $t \in [0,T]$, $\mathcal{M}_t$ is a bounded isomorphism from $H^k(\Omega)$ to $H^k(\Omega)$ for $k = 0, 1, 2$.

(2) For each $t \in [0,T]$, $\mathcal{M}_t$ is a bounded isomorphism from ${}_0H^1(\Omega)$ to $\mathcal{H}^1(\Omega)$. Moreover,

$$\|Mu\|_{\mathcal{H}^1} \lesssim \sup_{0\le t\le T}(1 + \|\eta\|_{5/2})\|u\|_1 \tag{2.16}$$

(3) Let $u \in H^1(\Omega)$. Then $\operatorname{div} u = p$ if and only if $\operatorname{div}_{\mathcal{A}}(Mu) = Kp$.

PROOF. The proof is similar to [**9**, Proposition 2.5]. We also refer to [**24**] for more information. □

The following proposition is also useful.

PROPOSITION 2.4. *If $u \cdot \nu = 0$ on Σ_b, then $Ru \cdot \nu = 0$ on Σ_b, where $R := \partial_t M M^{-1}$.*

PROOF. Clearly, $K \nabla \Phi^\top \nu = K\nu$ on Σ_b. Hence $Mu \cdot \nu = 0 \Leftrightarrow u \cdot \nu = 0$ on Σ_b, which implies that $M^{-1} u \cdot \nu = 0 \Leftrightarrow u \cdot \nu = 0$ on Σ_b. Then by definition of R,

$$Ru \cdot \nu = \partial_t M M^{-1} u \cdot \nu = -M \partial_t (M^{-1} u) \cdot \nu = 0, \tag{2.17}$$

since $\partial_t(M^{-1}u) \cdot \nu = \partial_t(M^{-1} u \cdot \nu) = 0$. □

At last, we define the bilinear form

$$(\phi, \psi)_{1,\Sigma} := \int_\Sigma g\phi\psi + \sigma \partial_1 \phi \partial_1 \psi \tag{2.18}$$

and $\|\phi\|_{1,\Sigma}^2 := \int_\Sigma g|\phi|^2 + \sigma |\partial_1 \phi|^2$.

2.2. Some useful estimates. We usually use the following two theorems, which have been proved in [**25**], so we omit the details here. The first is to recover the pressure from pressureless weak formulation (for instance, see (3.3)).

THEOREM 2.5. *If $\Lambda_t \in (\mathcal{W}(t))^*$ satisfies $\Lambda_t(v) = 0$ for any $v \in \mathcal{X}(t)$, then there exists a unique $p(t) \in \mathring{\mathcal{H}}^0(t)$ such that*

$$\Lambda_t(v) = (p(t), \operatorname{div}_{\mathcal{A}} w)_{\mathcal{H}^0} \tag{2.19}$$

for all $v \in \mathcal{W}(t)$ and

$$\|p(t)\|_{\mathring{\mathcal{H}}^0} \lesssim (1 + \|\eta(t)\|_{5/2}) \|\Lambda_t\|_{(\mathcal{W})^*}. \tag{2.20}$$

If $\Lambda \in (\mathcal{W}_T)^*$ satisfies $\Lambda(v) = 0$ for any $v \in \mathcal{X}_T$, then there exists a unique $p \in \mathring{\mathcal{H}}^0_T$ such that

$$\Lambda(v) = (p, \operatorname{div}_{\mathcal{A}} w)_{\mathcal{H}^0_T} \tag{2.21}$$

for all $v \in \mathcal{W}_T$ and

$$\|p\|_{\mathring{\mathcal{H}}^0_T} \lesssim (1 + \sup_{0 \le t \le T} \|\eta(t)\|_{5/2}) \|\Lambda\|_{(\mathcal{W}_T)^*}. \tag{2.22}$$

Since $\xi \in H^1$ is not enough to close our *a priori estimates*, we need to utilize the following theorem to enhance the regularity of ξ. In the following, ξ gains more 1/2 derivative.

THEOREM 2.6. *Suppose that $\xi \in \mathring{H}^1(\Sigma)$ satisfying*

$$\begin{aligned} &\frac{\mu}{2}(v, w)_{\mathcal{H}^1} - (q, \operatorname{div}_{\mathcal{A}} w)_{\mathcal{H}^0} + (\xi, w \cdot \mathcal{N})_{1,\Sigma} + \beta \int_{\Sigma_b} (v \cdot \tau)(w \cdot \tau) J \\ &= (F^1, w) - \int_\Sigma \big(\sigma F^6 \cdot \partial_1 (w \cdot \mathcal{N}) + F^4 \cdot w\big) - \int_{\Sigma_b} F^5 (v \cdot \tau) J \end{aligned} \tag{2.23}$$

for each $w \in \mathcal{W}$ and that all the integrals are meaningful. Then $\xi \in H^{3/2}(\Sigma)$ satisfying

$$\|\xi\|_{3/2}^2 \lesssim \|v\|_1^2 + \|q\|_0^2 + \|F^1 - F^4 - F^5\|_{(\mathcal{H}^1)^*}^2 + \|\partial_1 F^6\|_{1/2}^2. \tag{2.24}$$

Finally, we present the elliptic estimates developed in [**25**]. We consider the following elliptic systems

$$
\begin{cases}
\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q,v) = F^1 & \text{in } \Omega,\\
\operatorname{div}_{\mathcal{A}} v = F^2 & \text{in } \Omega,\\
S_{\mathcal{A}}(q,v)\mathcal{N} = g\xi\mathcal{N} - \sigma\partial_1^2\xi\mathcal{N} - \sigma\partial_1 F^6\mathcal{N} + F^3 & \text{on } \Sigma,\\
v\cdot\mathcal{N} = F^4 & \text{on } \Sigma,\\
v\cdot\nu = 0, \quad (S_{\mathcal{A}}(q,v)\nu - \beta v)\cdot\tau = F^5 & \text{on } \Sigma_b.
\end{cases}
\tag{2.25}
$$

Then we have the following theorem.

THEOREM 2.7. *Suppose that* $F^1 \in H^0(\Omega)$, $F^2 \in H^1(\Omega)$, $F^3 \in H^{1/2}(\Sigma)$, $F^4 \in H^{3/2}(\Sigma)$, $F^5 \in H^{1/2}(\Sigma_b)$ *and* $F^6 \in H^{3/2}(\Sigma)$. *Suppose that* $\|\eta\|_{5/2} \le \gamma_0$ *for* γ_0 *sufficiently small . Then there exists a unique triple* $(v,q,\xi) \in H^2(\Omega)\times \mathring{H}^1(\Omega)\times \mathring{H}^{5/2}(\Sigma)$ *solving* (2.25). *Moreover,*

$$
\|v\|_2^2 + \|q\|_1^2 + \|\xi\|_{5/2}^2 \lesssim \|F^1\|_0^2 + \|F^2\|_1^2 + \|F^3\|_{1/2}^2 + \|F^4\|_{3/2}^2 + \|F^5\|_{1/2}^2 + \|F^6\|_{3/2}^2.
\tag{2.26}
$$

3. Linear estimates

Suppose that η as well as $\mathcal{A}$, $\mathcal{N}$, etc. are given. We now consider the following ε-approximate linear stationary equations

$$
\begin{cases}
\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p^\varepsilon,u^\varepsilon) = F^1 & \text{in } \Omega,\\
\operatorname{div}_{\mathcal{A}} u^\varepsilon = 0 & \text{in } \Omega,\\
S_{\mathcal{A}}(p^\varepsilon,u^\varepsilon)\mathcal{N} = g(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon) - \sigma\partial_1^2(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon)\mathcal{N} - \sigma\partial_1\mathcal{R}\mathcal{N} + F^3 & \text{on } \Sigma,\\
\partial_t\xi^\varepsilon = u^\varepsilon\cdot\mathcal{N} & \text{on } \Sigma,\\
u^\varepsilon\cdot\nu = 0, \quad (S_{\mathcal{A}}(p^\varepsilon,u^\varepsilon)\nu - \beta u^\varepsilon)\cdot\tau = F^5 & \text{on } \Sigma_b
\end{cases}
\tag{3.1}
$$

with initial data $\xi^\varepsilon(0) = \eta_0 \in \mathring{H}^{5/2}(\Sigma)$, $\partial_t\xi^\varepsilon(0) = \partial_t\eta(0)$ and $\partial_t^2\xi^\varepsilon(0) = \partial_t^2\eta(0)$. The reason for adding the extra term $\varepsilon\partial_t\xi^\varepsilon$ is to estimate the term $\int_\Sigma \mathcal{R}\partial_1(v\cdot\mathcal{N})$. This method is inspired by [**8**]. To begin our analysis for (3.1), we need to employ two notions of solutions for (3.1): weak and strong.

3.1. Weak solutions.

Our definition for weak solutions is motivated by assuming that there exists a smooth solution $(u^\varepsilon, p^\varepsilon, \xi^\varepsilon)$ of (3.1), then multiply u^ε by vJ for $v \in \mathcal{W}_T$, integrate over Ω by parts, and integrate from 0 to T to see that

$$
\begin{aligned}
&(u^\varepsilon, v)_{\mathcal{H}_T^1} + \beta(u^\varepsilon\cdot\tau, v\cdot\tau)_{L^2H^0(\Sigma_b)} + \int_0^T (\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon, v\cdot\mathcal{N})_{1,\Sigma} - (p^\varepsilon, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}_T^0}\\
&= (F^1, v)_{\mathcal{H}_T^0} - \int_0^T\int_\Sigma \sigma\mathcal{R}\partial_1(v\cdot\mathcal{N}) + F^3\cdot v - \int_0^T\int_{\Sigma_b} F^5(v\cdot\tau)J.
\end{aligned}
\tag{3.2}
$$

Definition 3.1. Suppose that $F^1 - F^3 - F^5 \in (\mathcal{H}^1_T)^*$. Then a triple $(u^\varepsilon, p^\varepsilon, \xi^\varepsilon)$ is a weak solution of (3.1) provided that

(3.3)
$$(u^\varepsilon, v)_{\mathcal{H}^1_T} + \beta(u^\varepsilon\cdot\tau, v\cdot\tau)_{L^2H^0(\Sigma_b)} + \int_0^T (\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon, v\cdot\mathcal{N})_{1,\Sigma} - (p^\varepsilon, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T}$$
$$= \left\langle F^1 - F^3 - F^5, v\right\rangle_{(\mathcal{H}^1_T)^*} + \int_0^T\int_\Sigma \sigma\mathcal{R}\partial_1(v\cdot\mathcal{N})$$

for each $v \in \mathcal{W}_T$, where $\left\langle F^1 - F^3 - F^5, v\right\rangle_{(\mathcal{H}^1_T)^*} = \int_0^T (\int_\Omega F^1\cdot vJ - \int_\Sigma F^3\cdot v - \int_{\Sigma_b} F^5(v\cdot\tau)J)$. In addition, if we choose $v \in \mathcal{X}_T$, then a pair $(u^\varepsilon, \xi^\varepsilon)$ is called a pressureless weak solution of (3.1).

In our following analysis, we only consider the strong solutions of (3.1). That's because weak solutions are only byproducts of the procedure for establishing strong solutions. Now, we only need to study the uniqueness of weak solutions.

Proposition 3.2. *The pressureless weak solutions of* (3.1) *are unique.*

Proof. Suppose that there are two pressureless weak solutions $(u_1^\varepsilon, \xi_1^\varepsilon)$ and $(u_2^\varepsilon, \xi_2^\varepsilon)$ of (3.1). Then for any test function $v \in \mathcal{X}_T$, we employ (3.3) to deduce that

(3.4)
$$(u_1^\varepsilon - u_2^\varepsilon, v)_{\mathcal{H}^1_T} + \beta((u_1^\varepsilon - u_2^\varepsilon)\cdot\tau, v\cdot\tau)_{L^2H^0(\Sigma_b)} + \int_0^T ((\xi_1^\varepsilon - \xi_2^\varepsilon) + \varepsilon\partial_t(\xi_1^\varepsilon - \xi_2^\varepsilon), v\cdot\mathcal{N})_{1,\Sigma} = 0.$$

Especially, if we choose $v = u_1^\varepsilon - u_2^\varepsilon$, (3.4) is reduced to

$$\|u_1^\varepsilon - u_2^\varepsilon\|^2_{\mathcal{H}^1_t} + \beta\int_0^t \|(u_1^\varepsilon - u_2^\varepsilon)\cdot\tau\|^2_{H^0(\Sigma_b)} + \int_0^t \|\xi_1^\varepsilon - \xi_2^\varepsilon\|^2_{1,\Sigma} = 0 \tag{3.5}$$

for any $t \in [0,T]$. Then taking supremum for $t \in [0,T]$, (3.5) implies that $u_1 = u_2$ and $\xi_1 = \xi_2$. □

3.2. Strong solutions. We first introduce an operator D_t via

$$D_t u := \partial_t u - Ru \quad \text{for} \quad R := \partial_t M M^{-1}, \tag{3.6}$$

with $M = K\nabla\Phi$, where K and Φ are defined as in (1.5) and (1.4), respectively. By Lemma B.1 and the definition (2.15) of M, it holds that

$$J\operatorname{div}_{\mathcal{A}} v = J\mathcal{A}_{ij}\partial_j v_i = \partial_j(J\mathcal{A}_{ij}v_i) = \operatorname{div}(J\mathcal{A}^\top v) = \operatorname{div}(M^{-1}v). \tag{3.7}$$

Then we could deduce that

$$J\operatorname{div}_{\mathcal{A}}(D_t v) = J\operatorname{div}_{\mathcal{A}}(M\partial_t(M^{-1}v)) = \operatorname{div}(\partial_t(M^{-1}v)) = \partial_t(J\operatorname{div}_{\mathcal{A}} v), \tag{3.8}$$

which implies that D_t preserves the $\operatorname{div}_{\mathcal{A}}$–free condition. This is the key feature of D_t in our analysis. We now give the definition of strong solutions of (3.1).

Definition 3.3. Assume that $F^1 \in C^0([0,T]; H^0(\Omega))$, $F^3 \in C^0([0,T]; H^{1/2}(\Sigma))$, $F^5 \in L^2([0,T]; H^{1/2}(\Sigma_b))$, and that $\partial_t(F^1 - F^3 - F^5) \in (\mathcal{H}^0_T)^*$. Then a triple (u, p, ξ) is called a strong solution of (3.1) provided that (u, p, ξ) solves (3.1) in the sense that

(3.9)
$$u^\varepsilon \in L^2([0,T]; H^2(\Omega)),\ \partial_t^j u^\varepsilon \in L^2([0,T]; H^1(\Omega)),\ j = 0,1,\ p^\varepsilon \in L^2([0,T]; \mathring{H}^1(\Omega)),$$
$$\xi^\varepsilon \in L^2([0,T]; \mathring{H}^{5/2}(\Sigma)),\ \partial_t\xi^\varepsilon \in L^2([0,T]; H^{3/2}(\Sigma)),\ \partial_t^2\xi^\varepsilon \in L^2([0,T]; H^{1/2}(\Sigma)).$$

Suppose that $\eta_0 \in \mathring{H}^{5/2}(\Sigma)$ and $\partial_t\xi^\varepsilon(0) = u^\varepsilon(0)\cdot\mathcal{N} \in H^{3/2}(\Sigma)$. Then we need to construct the initial data $u^\varepsilon(0)$ and $p^\varepsilon(0)$ of (3.1). We take $t = 0$ in (3.1), then we might use Theorem 2.7 with $F^6 = \mathcal{R}$ to obtain that $u^\varepsilon(0) \in H^2(\Omega)$, $p^\varepsilon(0) \in \mathring{H}^1(\Omega)$ and ξ such that

$$\|u^\varepsilon(0)\|_2^2 + \|p^\varepsilon(0)\|_1^2 \lesssim \|\eta_0\|_{5/2}^2 + \|F^1(0)\|_0^2 + \|F^3(0)\|_{1/2}^2 + \|F^5(0)\|_{1/2}^2. \tag{3.10}$$

We usually use the following lemma. The proof is similar to [**9**, Lemma 2.4], so we omit the details here.

LEMMA 3.4. *Suppose that $u^\varepsilon \in L^2([0,T];\mathcal{H}^1)$ and that $\partial_t u^\varepsilon \in L^2([0,T];\mathcal{H}^1)$. Then $u^\varepsilon \in C^0([0,T];\mathcal{H}^1)$ achieves the initial data $u^\varepsilon(0) \in H^1$, and*

$$\|u^\varepsilon\|_{L^\infty\mathcal{H}^1}^2 \le \|u^\varepsilon(0)\|_1^2 + \|u^\varepsilon\|_{L^2\mathcal{H}^1}^2 + \|\partial_t u^\varepsilon\|_{L^2\mathcal{H}^1}^2. \tag{3.11}$$

Now we state our main theorem for the strong solutions.

THEOREM 3.5. *Suppose that the forcing terms F^1, F^3, and F^5 satisfy the conditions in Definition 3.3, that the initial datum $\eta_0 \in \mathring{H}^{5/2}$. Suppose that $\mathfrak{K}(\eta) \le \gamma_1$ is smaller than γ_0 in Lemma 2.2 and in Theorem 2.7. Then there exists a unique strong solution $(u^\varepsilon, p^\varepsilon, \xi^\varepsilon)$ solving (3.1) such that $(u^\varepsilon, p^\varepsilon, \xi^\varepsilon)$ satisfies (3.9). The solution obeys the estimates*

(3.12)

$$\begin{aligned}
&\|u^\varepsilon\|_{L^2H^1}^2 + \|u^\varepsilon\cdot\tau\|_{L^2H^0(\Sigma_b)}^2 + \|u^\varepsilon\|_{L^2H^2}^2 + \|\partial_t u^\varepsilon\|_{L^2H^1}^2 + \|\partial_t u^\varepsilon\cdot\tau\|_{L^2H^0(\Sigma_b)}^2\\
&\quad + \|p^\varepsilon\|_{L^2H^0}^2 + \|p^\varepsilon\|_{L^2\mathring{H}^1}^2 + \|\partial_t p^\varepsilon\|_{L^2H^0}^2 + \|\xi^\varepsilon\|_{L^\infty H^1}^2 + \|\xi^\varepsilon\|_{L^2H^{3/2}}^2\\
&\quad + \|\xi^\varepsilon\|_{L^2H^{5/2}}^2 + \|\partial_t\xi^\varepsilon\|_{L^\infty H^1}^2 + \|\partial_t\xi^\varepsilon\|_{L^2H^{3/2}}^2\\
&\lesssim (C(\varepsilon)T+1)(\mathfrak{K}(\eta) + \mathfrak{E}_0 + \|F^1(0)\|_0 + \|F^3(0)\|_{1/2}^2 + \|F^3(0)\|_{1/2}^2) + \mathfrak{E}(\eta)\mathfrak{K}(\eta)\\
&\quad + (1+\mathfrak{E}(\eta))(\|F^1\|_{L^2H^0}^2 + \|F^3\|_{L^2H^{1/2}}^2 + \|F^5\|_{L^2H^{1/2}}^2)\\
&\quad + (1+\mathfrak{E}(\eta))\|\partial_t(F^1 - F^3 - F^5)\|_{(\mathcal{H}_T^1)^*}^2.
\end{aligned}$$

Moreover, $(D_t u^\varepsilon, \partial_t p^\varepsilon, \partial_t\xi^\varepsilon)$ satisfies

(3.13)

$$\begin{cases}
-\mu\Delta_{\mathcal{A}} D_t u^\varepsilon + \nabla_{\mathcal{A}}\partial_t p^\varepsilon = D_t F^1 + G^1 & in\ \ \Omega,\\
\mathrm{div}_{\mathcal{A}}(D_t u^\varepsilon) = 0 & in\ \ \Omega,\\
S_{\mathcal{A}}(\partial_t p^\varepsilon, D_t u^\varepsilon)\mathcal{N} = \mathcal{L}(\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon)\mathcal{N} - \sigma\partial_1\partial_t\mathcal{R}\mathcal{N} + \partial_t F^3 + G^3 & on\ \ \Sigma,\\
(S_{\mathcal{A}}(\partial_t p^\varepsilon, D_t u^\varepsilon)\nu - \beta D_t u^\varepsilon)\cdot\tau = \partial_t F^5 + G^5 & on\ \ \Sigma_s,\\
D_t u^\varepsilon\cdot\nu = 0 & on\ \ \Sigma_s,\\
\partial_t^2\xi^\varepsilon = D_t u^\varepsilon\cdot\mathcal{N} & on\ \ \Sigma
\end{cases}$$

in the weak sense of (3.3), where G^1 is defined by

$$G^1 = R^\top\nabla_{\mathcal{A}} p^\varepsilon + \mathrm{div}_{\mathcal{A}}\left(\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon) + \mathbb{D}_{\partial_t\mathcal{A}} u^\varepsilon - R\mathbb{D}_{\mathcal{A}} u^\varepsilon\right), \tag{3.14}$$

and G^4 by

(3.15)

$$G^4 = \mu\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)\mathcal{N} - (p^\varepsilon I - \mu\mathbb{D}_{\mathcal{A}} u^\varepsilon)\partial_t\mathcal{N} + \mu\mathbb{D}_{\partial_t\mathcal{A}} u^\varepsilon\mathcal{N} + \mathcal{L}(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon)\partial_t\mathcal{N} - \sigma\partial_1\mathcal{R}\partial_t\mathcal{N},$$

G^5 by

$$G^5 = (\mu\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)\nu + \mu\mathbb{D}_{\partial_t\mathcal{A}} u^\varepsilon\nu + \beta Ru^\varepsilon)\cdot\tau, \tag{3.16}$$

and

$$\mathcal{L}(\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon) = g(\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon) - \sigma\partial_1^2(\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon).$$

More precisely, (3.13) *holds in the weak sense of*

(3.17)
$$\begin{aligned}
&\frac{\mu}{2}(\partial_t u^\varepsilon, v)_{\mathcal{H}^1} + \beta\int_{\Sigma_b}(\partial_t u^\varepsilon\cdot\tau)(v\cdot\tau)J + (\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon, v\cdot\mathcal{N})_{1,\Sigma}\\
&= (\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon, Rv\cdot\mathcal{N})_{1,\Sigma} - (p^\varepsilon, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0} + \int_\Omega\left[\partial_t F^1\cdot v + \partial_t JKF^1\cdot v\right]J\\
&\quad - \int_\Sigma[\partial_t\mathcal{R}\partial_1(v\cdot\mathcal{N}) + \mathcal{R}\partial_1(v\cdot\partial_t\mathcal{N}) + \partial_t F^3\cdot v] - \int_{\Sigma_b}\left[\partial_t F^5 v + \partial_t JKF^5 v\right]\cdot\tau J\\
&\quad - \int_\Omega\frac{\mu}{2}(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u^\varepsilon : \mathbb{D}_{\partial_t\mathcal{A}}v + \partial_t JK\mathbb{D}_{\mathcal{A}}u^\varepsilon : \mathbb{D}_{\mathcal{A}}v)J\\
&\quad - \int_{\Sigma_b}\beta(u^\varepsilon\cdot\tau)(v\cdot\tau)\partial_t J.
\end{aligned}$$

Proof. We will use the Galerkin method, inspired by [**7**], in the following several steps.

Step 1. Construction of approximate solutions. In order to utilize the Galerkin method, we must first construct a countable basis of $H^2(\Omega)\cap\mathcal{X}(t)$ for each $t\in[0,T]$. Since the requirement $\operatorname{div}_{\mathcal{A}} v = 0$ is time–dependent, any basis of this space must also be time–dependent. For each $t\in[0,T]$, the space $H^2(\Omega)\cap\mathcal{X}(t)$ is separable, so the existence of a countable basis is not an issue. The technical difficulty is that, in order for the basis to be useful in Galerkin method, we must be able to express these time derivatives in terms of finitely many basis elements. Fortunately, it is possible to overcome this difficulty by employing the matrix $M(t)$, defined by (2.15).

Since $H^2(\Omega)\cap V$ is separable, it possess a countable basis $\{w^j\}_{j=1}^\infty$. Note that this basis is not time–dependent. Define $v^j = v^j(t) := M(t)w^j$. According to Proposition 2.3, $v^j(t)\in H^2(\Omega)\cap\mathcal{X}(t)$, and $\{v^j(t)\}_{j=1}^\infty$ is a basis of $H^2(\Omega)\cap\mathcal{X}(t)$ for each $t\in\mathbb{R}^+$. Moreover, we can express $\partial_t v^j(t)$ in terms of $v^j(t)$ as

$$\partial_t v^j(t) = \partial_t M(t)w^j = \partial_t M(t)M^{-1}(t)M(t)w^j = R(t)v^j(t), \tag{3.18}$$

where $R(t)$ is defined by

$$R(t) := \partial_t M(t)M^{-1}(t). \tag{3.19}$$

For any integer $m\ge 1$, we define the finite dimensional space

$$\mathcal{V}_m(t) := \operatorname{span}\{v^1(t),\cdots,v^m(t)\}\subseteq H^2(\Omega)\cap\mathcal{X}(t),$$

and we write

$$\mathcal{P}_t^m : H^2(\Omega)\to\mathcal{V}_m(t) \tag{3.20}$$

for the $H^2(\Omega)$ orthogonal projection onto $\mathcal{V}_m(t)$. Clearly, for each $v\in H^2(\Omega)\cap\mathcal{V}(t)$, we have that $\mathcal{P}_t^m v\to v$ as $m\to\infty$.

For our Galerkin problem, we construct a solution to the pressureless problem as follows. For each $m\ge 1$, we define an approximate solution

$$u^{\varepsilon,m}(t) := d_j^m(t)v^j(t), \text{ with } d_j^m : [0,T]\to\mathbb{R} \text{ for } j=1,\dots,m, \tag{3.21}$$

where as usual we use the Einstein convention of summation of the repeated index j. We similarly define

$$\xi^{\varepsilon,m}(t) = \eta_0 + \int_0^t u^{\varepsilon,m}(s) \cdot \mathcal{N}(s)\,\mathrm{d}s, \tag{3.22}$$

where we understand here that $u^{\varepsilon,m}(\cdot)$ denotes the trace onto Σ.

We want to find the coefficients $d_j^m \in C^0([0,T])$ so that

$$\begin{aligned}&\frac{1}{2}(u^{\varepsilon,m}, v)_{\mathcal{H}^1} + \beta \int_{\Sigma_b} (u^{\varepsilon,m} \cdot \tau)(v \cdot \tau) J + (\xi^{\varepsilon,m} + \varepsilon \partial_t \xi^{\varepsilon,m}, v \cdot \mathcal{N})_{1,\Sigma} \\ &\quad = \int_\Omega F^1 \cdot v J - \int_{\Sigma_b} F^5 (v \cdot \tau) J - \int_\Sigma \left(\sigma \mathcal{R} \partial_1 (v \cdot \mathcal{N}) + F^3 \cdot v\right).\end{aligned} \tag{3.23}$$

It is easy to see that

$$\begin{aligned}&(\xi^{\varepsilon,m}(t) + \varepsilon \partial_t \xi^{\varepsilon,m}(t), v \cdot \mathcal{N}(t))_{1,\Sigma} \\ &= \left(\eta_0 + \int_0^t u^{\varepsilon,m}(s) \cdot \mathcal{N}(s)\,\mathrm{d}s + \varepsilon u^{\varepsilon,m} \cdot \mathcal{N}, v \cdot \mathcal{N}(t)\right)_{1,\Sigma} \\ &= (\eta_0, v \cdot \mathcal{N}(t))_{1,\Sigma} + \int_0^t d_i^m(s)(v^i \cdot \mathcal{N}(s) + \varepsilon d_i^m(t) v^i \cdot \mathcal{N}(t), v \cdot \mathcal{N}(t))_{1,\Sigma}\,\mathrm{d}s.\end{aligned} \tag{3.24}$$

Then (3.23) is equivalent to an equation of d_j^m given by

$$\begin{aligned}&d_j^m \left[\frac{1}{2}(v^j, v^k)_{\mathcal{H}^1} + \beta (v^j \cdot \tau, v^k \cdot \tau)_{H^0(\Sigma_b)} + \varepsilon (v^j \cdot \mathcal{N}, v^k \cdot \mathcal{N})_{1,\Sigma}\right] \\ &\quad + \int_0^t d_j^m(s)(v^j \cdot \mathcal{N}(s), v^k \cdot \mathcal{N}(t))_{1,\Sigma}\,\mathrm{d}s \\ &\quad = -(\eta_0, v^k \cdot \mathcal{N}(t))_{1,\Sigma} + \int_\Omega F^1 \cdot v^k J - \int_{\Sigma_b} F^5 (v^k \cdot \tau) J \\ &\quad - \int_\Sigma \left[\sigma \mathcal{R} \partial_1 (v^k \cdot \mathcal{N}) + F^3 \cdot v^k\right].\end{aligned} \tag{3.25}$$

Since the matrix with j, k entry

$$\left(\frac{1}{2}(v^j, v^k)_{\mathcal{H}^1} + \beta (v^j \cdot \tau, v^k \cdot \tau)_{H^0(\Sigma_b)} + \varepsilon (v^j \cdot \mathcal{N}, v^k \cdot \mathcal{N})_{1,\Sigma}\right)$$

is positively definite, thus (3.25) is equivalent to the form

$$d^m(t) + \int_0^t K(t,s) d^m(s)\,\mathrm{d}s = \mathcal{F}(t), \tag{3.26}$$

where $K(t,s) \in C^0(\{(t,s) | 0 \le s \le t \le T\})$ and $\mathcal{F} \in C^0([0,T])$. Then according to the theory of integral equations (for instance, see [**22**]), there exists a unique vector $d^m \in C^0([0,T])$ solving (3.25). Actually, from the conditions of forces in Definition 3.3, we may differentiate (3.26) once almost everywhere to see that $d^m \in C^{0,1}([0,T])$.

Step 2. Energy estimates for $u^{\varepsilon,m}$ and $\xi^{\varepsilon,m}$. We choose the test function $v = u^{\varepsilon,m}$ in (3.23), then Hölder inequality implies that

$$\begin{aligned}&\frac{d}{dt}\|\xi^{\varepsilon,m}\|_{1,\Sigma}^2 + \frac{1}{2}\|u^{\varepsilon,m}\|_{\mathcal{H}^1}^2 + \beta\|u^{\varepsilon,m}\cdot\tau\|_{L^2(\Sigma_b)}\\ &\lesssim \|F^1\|_{\mathcal{H}^0}\|u^{\varepsilon,m}\|_{\mathcal{H}^1} + \|F^5\|_{1/2}\|u^{\varepsilon,m}\cdot\tau\|_{L^2(\Sigma_b)}\\ &\quad + \|\mathcal{R}\|_{1/2}\|u^{\varepsilon,m}\cdot\mathcal{N}\|_{1/2} + \|F^3\|_{1/2}\|u^{\varepsilon,m}\|_{H^{1/2}(\Sigma)}.\end{aligned} \tag{3.27}$$

Then employing Cauchy-Schwarz inequality, Lemma 2.2, and trace theory, we have that

$$\begin{aligned}&\frac{d}{dt}\|\xi^{\varepsilon,m}\|_{1,\Sigma}^2 + \frac{1}{2}\|u^{\varepsilon,m}\|_{\mathcal{H}^1}^2 + \beta\|u^{\varepsilon,m}\cdot\tau\|_{L^2(\Sigma_b)}\\ &\lesssim \|F^1\|_{\mathcal{H}^0}^2 + \|F^3\|_{1/2}^2 + \|F^5\|_{1/2}^2 + \|\eta\|_{5/2}^2.\end{aligned} \tag{3.28}$$

Then integrating from 0 to $t < T$ and taking sup-norm on $[0,T]$ reveal that

$$\begin{aligned}&\sup_{0\le t\le T}\|\xi^{\varepsilon,m}(t)\|_{1,\Sigma}^2 + \|u^{\varepsilon,m}\|_{\mathcal{H}_T^1}^2 + \|u^{\varepsilon,m}\cdot\tau\|_{L^2H^0(\Sigma_b)}\\ &\lesssim \|\eta_0\|_{5/2}^2 + \|F^1\|_{\mathcal{H}_T^0}^2 + \|F^3\|_{L^2H^{1/2}}^2 + \|F^5\|_{L^2H^{1/2}}^2 + \mathfrak{K}(\eta).\end{aligned} \tag{3.29}$$

Step 3. Estimates for $u^{\varepsilon,m}(0)\cdot\mathcal{N}(0)$ and $u^{\varepsilon,m}(0)$. In order to use Lemma 3.4, we need to obtain the initial data $u^{\varepsilon,m}(0)$. Taking $t = 0$ in (3.1), we have that

$$\begin{aligned}&\frac{1}{2}(u^\varepsilon(0), v)_{\mathcal{H}^1} + \beta\int_{\Sigma_b}(u^\varepsilon(0)\cdot\tau)(v\cdot\tau)J(0) + (\eta_0 + \varepsilon\partial_t\xi^\varepsilon(0), v\cdot\mathcal{N}(0))_{1,\Sigma}\\ &= \int_\Omega F^1(0)\cdot vJ(0) - \int_{\Sigma_b}F^5(0)(v\cdot\tau)J(0) - \int_\Sigma \sigma\mathcal{R}\partial_1(v\cdot\mathcal{N}(0)) + F^3(0)\cdot v.\end{aligned} \tag{3.30}$$

Hence the Lax-Milgram theorem guarantees that there exists a unique $u^\varepsilon(0) \in \mathcal{W}$ and $u^\varepsilon(0)\cdot\tau \in H^0(\Sigma_b)$. Then we might use the elliptic theory to obtain that $u^\varepsilon(0) \in H^2(\Omega)$ enjoys

$$\|u^\varepsilon(0)\|_2^2 \lesssim \|F^1(0)\|_0^2 + \|F^3(0)\|_{1/2}^2 + \|F^5\|_{1/2}^2 + \|\eta_0\|_{5/2}^2. \tag{3.31}$$

Then we supplement the condition $u^{\varepsilon,m}(0) = \mathcal{P}^m u^\varepsilon(0)$, whence

$$\|u^{\varepsilon,m}(0)\|_2^2 \lesssim \|u^\varepsilon(0)\|_2^2 \lesssim \|F^1(0)\|_0^2 + \|F^3(0)\|_{1/2}^2 + \|F^5(0)\|_{1/2}^2 + \|\eta_0\|_{5/2}^2, \tag{3.32}$$

and

$$\begin{aligned}\|u^{\varepsilon,m}(0)\cdot\mathcal{N}(0)\|_{H^{3/2}(\Sigma)}^2 &\lesssim (1 + \|\eta_0\|_{5/2}^2)\|u^{\varepsilon,m}(0)\|_2^2\\ &\lesssim \|F^1(0)\|_0^2 + \|F^3(0)\|_{1/2}^2 + \|F^5(0)\|_{1/2}^2 + \|\eta_0\|_{5/2}^2.\end{aligned} \tag{3.33}$$

Step 4. Energy estimates for $\partial_t u^{\varepsilon,m}$ and $\partial_t\xi^{\varepsilon,m}$.

Suppose that $w = a_i^m v^i \in \mathcal{V}_m$ for $a_i^m \in C^{0,1}([0,T])$. Then it is easy to see that $\partial_t w - R(t)w \in \mathcal{V}_m$. We now take this w as a test function in (3.23), differentiate temporally and subtract the resulting equation from the equation (3.23) with test function $\partial_t w - R(t)w$. Thus terms of $\partial_t w$ are cancelled and we have the equality

$$
\begin{aligned}
&\frac{1}{2}(\partial_t u^{\varepsilon,m}, w)_{\mathcal{H}^1} + (\partial_t \xi^{\varepsilon,m} + \varepsilon\partial_t^2\xi^{\varepsilon,m}, w\cdot\mathcal{N})_{1,\Sigma} + \beta\int_{\Sigma_b}(\partial_t u^{\varepsilon,m}\cdot\tau)(w\cdot\tau)J \\
&= -\frac{1}{2}\int_\Omega (\mathbb{D}_{\partial_t\mathcal{A}}u^m : \mathbb{D}_\mathcal{A} w + \mathbb{D}_\mathcal{A} u^{\varepsilon,m} : \mathbb{D}_{\partial_t\mathcal{A}} w + \mathbb{D}_\mathcal{A} u^{\varepsilon,m} : \mathbb{D}_\mathcal{A} w\partial_t JK)\,J \\
&\quad - \frac{1}{2}(u^{\varepsilon,m}, Rw)_{\mathcal{H}^1} - \beta\int_{\Sigma_b}((u^{\varepsilon,m}\cdot\tau)(Rw\cdot\tau) + (u^{\varepsilon,m}\cdot\tau)(w\cdot\tau)\partial_t JK)\,J \\
&\quad + \int_\Omega \left(\partial_t F^1\cdot w + F^1\cdot Rw + F^1\cdot w\partial_t JK\right) J \\
&\quad - \int_{\Sigma_b}\left(\partial_t F^5(\tau\cdot w) + F^5(\tau\cdot Rw) + F^5(\tau\cdot w)\partial_t JK\right) J \\
&\quad - \int_\Sigma \sigma\partial_t\mathcal{R}\partial_1(w\cdot\mathcal{N}) + \partial_t F^3\cdot w + F^3\cdot Rw,
\end{aligned}
\tag{3.34}
$$

where we have used the fact that $Rw\cdot\mathcal{N} = -w\cdot\partial_t\mathcal{N}$ on Σ. Then we choose $w = \partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}$ as the test function. Since $\partial_t u^{\varepsilon,m}\cdot\mathcal{N} - Ru^{\varepsilon,m}\cdot\mathcal{N} = \partial_t(u^{\varepsilon,m}\cdot\mathcal{N}) = \partial_t^2\xi^{\varepsilon,m}$ on Σ, we write that

$$
(\partial_t\xi^{\varepsilon,m} + \varepsilon\partial_t^2\xi^{\varepsilon,m}, (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})\cdot\mathcal{N})_{1,\Sigma} = \frac{d}{dt}\frac{1}{2}\|\partial_t\xi^{\varepsilon,m}\|_{1,\Sigma}^2 + \varepsilon\|\partial_t^2\xi^{\varepsilon,m}\|_{1,\Sigma}^2.
$$

Then the equation (3.34) is reduced to

$$
\frac{d}{dt}\frac{1}{2}\|\partial_t\xi^{\varepsilon,m}\|_{1,\Sigma}^2 + \varepsilon\|\partial_t^2\xi^{\varepsilon,m}\|_{1,\Sigma}^2 + \frac{1}{2}\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} + \beta\int_{\Sigma_b}|\partial_t u^{\varepsilon,m}\cdot\tau|^2 J := I + II + III,
\tag{3.35}
$$

where

$$
\begin{aligned}
I :=& \frac{1}{2}(\partial_t u^{\varepsilon,m}, Ru^{\varepsilon,m})_{\mathcal{H}^1} - \frac{1}{2}(u^{\varepsilon,m}, R(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}))_{\mathcal{H}^1} \\
&- \frac{1}{2}\int_\Omega (\mathbb{D}_{\partial_t\mathcal{A}}u^{\varepsilon,m} : \mathbb{D}_\mathcal{A}(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}))J \\
&+ \frac{1}{2}\int_\Omega \mathbb{D}_\mathcal{A} u^{\varepsilon,m} : \mathbb{D}_{\partial_t\mathcal{A}}(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})J \\
&+ \frac{1}{2}\int_\Omega \mathbb{D}_\mathcal{A} u^{\varepsilon,m} : \mathbb{D}_\mathcal{A}(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})\partial_t J,
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
II :=& \beta\int_{\Sigma_b}(\partial_t u^{\varepsilon,m}\cdot\tau)(Ru^{\varepsilon,m}\cdot\tau)J - \beta\int_{\Sigma_b}((u^{\varepsilon,m}\cdot\tau)(R(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})\cdot\tau)J \\
&+ \beta\int_{\Sigma_b}(u^{\varepsilon,m}\cdot\tau)((\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})\cdot\tau)\partial_t JK)J,
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
III := \int_\Omega \Big(&\partial_t F^1 \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}) + F^1 \cdot R(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}) \\
&+ F^1 \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})\partial_t JK \Big) J \\
&- \int_{\Sigma_b} \Big(\partial_t F^5 (\tau \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})) + F^5 (\tau \cdot R(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})) \\
&+ F^5 (\tau \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}))\partial_t JK \Big) J \\
&- \int_\Sigma \sigma \partial_t \mathcal{R} \partial_1 ((\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}) \cdot \mathcal{N}) + \partial_t F^3 \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}) \\
&- \int_\Sigma F^3 \cdot R(\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}).
\end{aligned} \tag{3.38}
$$

Then we estimate these three terms as follows. First, we employ Hölder inequality, Lemma 2.2, Sobolev embedding theory, usual trace theory and Cauchy inequality to deduce that

$$
\begin{aligned}
I \lesssim\; &\|\nabla R\|_{L^4(\Omega)} \|\mathcal{A}\|_{L^\infty(\Omega)} (1 + \|\partial_t J\|_{L^\infty(\Omega)}) (\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} \|u^{\varepsilon,m}\|_{L^4} \\
&+ \|u^{\varepsilon,m}\|_{\mathcal{H}^1} \|\partial_t u^{\varepsilon,m}\|_{L^4}) + \|R\|^2_{L^\infty(\Omega)} \|u^{\varepsilon,m}\|^2_{\mathcal{H}^1} \\
&+ (\|R\|_{L^\infty(\Omega)} + \|\partial_t \mathcal{A}\|_{L^\infty(\Omega)} + \|\partial_t J\|_{L^\infty(\Omega)}) \|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} \|u^{\varepsilon,m}\|_{\mathcal{H}^1} \\
&+ \|\nabla R\|_{L^4(\Omega)} \|\mathcal{A}\|_{L^\infty(\Omega)} (\|R\|_{L^\infty(\Omega)} + \|\partial_t \mathcal{A}\|_{L^\infty(\Omega)}) \|u^{\varepsilon,m}\|_{\mathcal{H}^1} \|u^{\varepsilon,m}\|_{L^4} \\
\lesssim\; &\|\partial_t \eta\|_{5/2} \|u^{\varepsilon,m}\|_{\mathcal{H}^1} \|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} + \|\partial_t \eta\|^2_{5/2} \|u^{\varepsilon,m}\|^2_{\mathcal{H}^1}.
\end{aligned} \tag{3.39}
$$

Then we use the fact that $R^\top \tau = K\tau$ on Σ_b, Hölder inequality and Sobolev embedding theorems to deduce that

$$
\begin{aligned}
II &\lesssim (1 + \|\partial_t J\|_{L^\infty(\Sigma_b)}) \|\partial_t u^{\varepsilon,m} \cdot \tau\|_{H^0(\Sigma_b)} \|u^{\varepsilon,m} \cdot \tau\|_{H^0(\Sigma_b)} + \|u^{\varepsilon,m} \cdot \tau\|^2_{H^0(\Sigma_b)} \\
&\lesssim (1 + |\partial_t \eta\|_{5/2}) \|\partial_t u^{\varepsilon,m} \cdot \tau\|_{H^0(\Sigma_b)} \|u^{\varepsilon,m} \cdot \tau\|_{H^0(\Sigma_b)} + \|u^{\varepsilon,m} \cdot \tau\|^2_{H^0(\Sigma_b)}.
\end{aligned} \tag{3.40}
$$

Now, we estimate III more carefully use the same method as I and II.

$$
\begin{aligned}
&\int_\Omega \partial_t F^1 (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}) J - \int_\Sigma \partial_t F^3 \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m}) \\
&\quad - \int_{\Sigma_b} \partial_t F^5 (\tau \cdot (\partial_t u^{\varepsilon,m} - Ru^{\varepsilon,m})) J \\
&\lesssim \|\partial_t (F^1 - F^3 - F^5)\|_{(\mathcal{H}^1)^*} (\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} + \|Ru^{\varepsilon,m}\|_{\mathcal{H}^1}) \\
&\lesssim \|\partial_t (F^1 - F^3 - F^5)\|_{(\mathcal{H}^1)^*} (\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} + \|\nabla R\|_{L^4(\Omega)} \|u^{\varepsilon,m}\|_{L^4} \\
&\quad + \|R\|_{L^\infty(\Omega)} \|u^{\varepsilon,m}\|_{\mathcal{H}^1}) \\
&\lesssim \|\partial_t (F^1 - F^3 - F^5\|_{(\mathcal{H}^1)^*} (\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1} + \|\partial_t \eta\|_{5/2} \|u^{\varepsilon,m}\|_{\mathcal{H}^1}).
\end{aligned} \tag{3.41}
$$

For terms relating to force F^1, F^3 and F^5, we estimate them as follows.

$$\begin{aligned}&\int_\Omega F^1\cdot R(\partial_t u^{\varepsilon,m}-Ru^{\varepsilon,m})J+F^1\cdot(\partial_t u^{\varepsilon,m}-Ru^{\varepsilon,m})\partial_t J\\&\lesssim \|F^1\|_{\mathcal{H}^0}((\|R\|_{L^4(\Omega)}+\|\partial_t J\|_{L^4(\Omega)})\|\partial_t u^{\varepsilon,m}\|_{L^4}+\|R\|_{L^6(\Omega)}(\|R\|_{L^6(\Omega)}\\&\quad+\|\partial_t J\|_{L^6(\Omega)})\|u^{\varepsilon,m}\|_{L^6})\\&\lesssim \|F^1\|_{\mathcal{H}^0}(\|\partial_t\eta\|_{3/2}\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1}+\|\partial_t\eta\|_{3/2}^2\|u^{\varepsilon,m}\|_{\mathcal{H}^1}).\end{aligned}\tag{3.42}$$

$$\begin{aligned}&\int_{\Sigma_b}(F^5(\tau\cdot R(\partial_t u^{\varepsilon,m}-Ru^{\varepsilon,m}))+F^5(\tau\cdot(\partial_t u^{\varepsilon,m}-Ru^{\varepsilon,m}))\partial_t JK)J\\&\lesssim(\|F^5\|_{1/2}+\|F^5\|_{L^4}\|\partial_t J\|_{L^4(\Sigma_b)})(\|\partial_t u^{\varepsilon,m}\|_1+\|u^{\varepsilon,m}\|_1)\\&\lesssim(\|F^5\|_{1/2}+\|F^5\|_{1/2}\|\partial_t\eta\|_{3/2})(\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1}+\|u^{\varepsilon,m}\|_{\mathcal{H}^1}).\end{aligned}\tag{3.43}$$

$$\begin{aligned}&\int_\Sigma F^4\cdot R(\partial_t u^{\varepsilon,m}-Ru^{\varepsilon,m})\\&\lesssim\|F^4\|_{1/2}\|R\|_{H^1(\Omega)}(\|\partial_t u^{\varepsilon,m}\|_1+\|R\|_{H^1(\Omega)}\|u^{\varepsilon,m}\|_1)\\&\lesssim\|\partial_t\eta\|_{3/2}\|F^4\|_{1/2})(\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1}+\|\partial_t\eta\|_{5/2}\|u^{\varepsilon,m}\|_{\mathcal{H}^1}).\end{aligned}\tag{3.44}$$

Then we bound

$$\begin{aligned}&\int_\Sigma\partial_t\mathcal{R}\partial_1((\partial_t u^{\varepsilon,m}-Ru^{\varepsilon,m})\cdot\mathcal{N})=\int_\Sigma\mathcal{R}'\partial_1\partial_t\eta\partial_1\partial_t^2\xi^{\varepsilon,m}\\&\lesssim\|\partial_1\eta\|_{L^4}\|\partial_1\partial_t\eta\|_{L^4}\|\partial_1\partial_t^2\xi^{\varepsilon,m}\|_{L^2}\lesssim\|\eta\|_{3/2}\|\partial_t\eta\|_{3/2}\|\partial_t^2\xi^{\varepsilon,m}\|_{1,\Sigma}.\end{aligned}\tag{3.45}$$

Then, (3.35) coupled with (3.39)–(3.45) might be reduced by Cauchy-Schwarz inequality to

$$\begin{aligned}&\frac{d}{dt}\frac12\|\partial_t\xi^{\varepsilon,m}\|_{1,\Sigma}^2+\frac12\varepsilon\|\partial_t^2\xi^{\varepsilon,m}\|_{1,\Sigma}^2+\frac14\|\partial_t u^{\varepsilon,m}\|_{\mathcal{H}^1}^2+\frac12\beta\int_{\Sigma_b}|\partial_t u^{\varepsilon,m}\cdot\tau|^2J\\&\lesssim C(\varepsilon)\|\eta\|_{3/2}^2\|\partial_t\eta\|_{3/2}^2+\|\partial_t(F^1-F^3-F^5)\|_{(\mathcal{H}^1)^*}^2\\&\quad+\|\partial_t\eta\|_{3/2}^2(\|F^1\|_{\mathcal{H}^1}^2+\|F^3\|_{1/2}^2+\|F^5\|_{1/2}^2)+\|\partial_t\eta\|_{5/2}^2\|u^{\varepsilon,m}\|_{\mathcal{H}^1}^2.\end{aligned}\tag{3.46}$$

Then integrating over $[0,t]$ for $t<T$ and taking sup-norm on $[0,T]$ imply that

$$\begin{aligned}&\sup_{0\le t\le T}\|\partial_t\xi^{\varepsilon,m}(t)\|_1^2+\|\partial_t u^{\varepsilon,m}\|_{L^2\mathcal{H}^1}^2+\|\partial_t u^{\varepsilon,m}\cdot\tau\|_{L^2H^0(\Sigma_b)}^2\\&\lesssim\|\partial_t\xi^{\varepsilon,m}(0)\|_1^2+C(\varepsilon)T\mathfrak{E}(\eta)+\|\partial_t(F^1-F^3-F^5)\|_{(\mathcal{H}^1_T)^*}^2\\&\quad+\mathfrak{E}(\eta)(\|F^1\|_{\mathcal{H}^1_T}^2+\|F^3\|_{L^2H^{1/2}}^2+\|F^5\|_{L^2H^{1/2}}^2)+\mathfrak{D}(\eta)\|u^{\varepsilon,m}\|_{L^\infty\mathcal{H}^1}^2.\end{aligned}\tag{3.47}$$

Then choosing $\gamma_1>0$ sufficiently small, using Lemma 3.4 and estimates for initial data $u^{\varepsilon,m}(0)$ and $\partial_t\xi^{\varepsilon,m}(0)$, we have that

$$\begin{aligned}&\sup_{0\le t\le T}\|\partial_t\xi^{\varepsilon,m}(t)\|_1^2+\|\partial_t u^{\varepsilon,m}\|_{L^2\mathcal{H}^1}^2+\|\partial_t u^{\varepsilon,m}\cdot\tau\|_{L^2H^0(\Sigma_b)}^2\\&\lesssim C(\varepsilon)T\mathfrak{E}(\eta)\mathfrak{D}(\eta)+\mathfrak{K}(\eta)(\|F^1\|_{L^2H^0}^2+\|F^3\|_{L^2H^{1/2}}^2+\|F^5\|_{L^2H^{1/2}}^2)\\&\quad+\|\partial_t(F^1-F^3-F^5)\|_{(\mathcal{H}^1_T)^*}^2+(\mathfrak{E}_0(\eta)+\mathfrak{D}(\eta)).\end{aligned}\tag{3.48}$$

Step 5. Estimates for $\xi^{\varepsilon,m}$. We utilize Theorem 2.6 for (3.23) to obtain that

$$\|\xi^{\varepsilon,m}+\varepsilon\partial_t\xi^{\varepsilon,m}\|_{3/2}\lesssim\|u^{\varepsilon,m}\|_1+\|F^1-F^3-F^5\|_{(\mathcal{H}^1)^*}+\|\mathcal{R}\|_{1/2}.\tag{3.49}$$

We set $\zeta = \xi^{\varepsilon,m} + \varepsilon\partial_t\xi^{\varepsilon,m}$. Then we might represent $\xi^{\varepsilon,m}$ in terms of ζ as

$$\xi^{\varepsilon,m} = \eta_0 e^{-\frac{t}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^t e^{-\frac{t-s}{\varepsilon}}\zeta(s)\,\mathrm{d}s. \tag{3.50}$$

Then we use the Fubini's theorem to obtain that

$$\begin{aligned}
\|\xi^{\varepsilon,m}\|^2_{L^2H^{3/2}} &\le \|\eta_0\|_{3/2}\left(\int_0^T e^{-\frac{2}{t}\varepsilon}\,\mathrm{d}t\right) + \left\|\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}\|\zeta\|_{3/2}(s)\,\mathrm{d}s\right\|^2_{L^2}\\
&\le C(\varepsilon)T\|\eta_0\|^2_{3/2} + \left\|\left(\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}\|\zeta\|^2_{3/2}(s)\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}\,\mathrm{d}s\right)^{1/2}\right\|^2_{L^2}\\
&\lesssim C(\varepsilon)T\|\eta_0\|^2_{3/2} + \|\xi^{\varepsilon,m} + \varepsilon\partial_t\xi^{\varepsilon,m}\|^2_{L^2H^{3/2}},
\end{aligned} \tag{3.51}$$

which coupling (3.49) gives the bound

$$\begin{aligned}
\|\xi^{\varepsilon,m}\|^2_{L^2H^{3/2}} \lesssim{}& C(\varepsilon)T\|\eta_0\|^2_{3/2} + \|u^{\varepsilon,m}\|^2_{L^2H^1}\\
&+\|F^1 - F^3 - F^5\|^2_{(\mathcal{H}^1_T)^*} + \|\mathcal{R}\|^2_{L^2H^{1/2}}.
\end{aligned} \tag{3.52}$$

Similarly, from (3.34), we have that

$$\begin{aligned}
&\|\partial_t\xi^{\varepsilon,m}\|^2_{L^2H^{3/2}} \lesssim C(\varepsilon)T\varepsilon^2\|\partial_t\xi^{\varepsilon,m}(0)\|^2_{3/2} + \|\partial_t\xi^{\varepsilon,m} + \varepsilon\partial_t^2\xi^{\varepsilon,m}\|^2_{L^2H^{3/2}}\\
&\lesssim C(\varepsilon)T(\|\xi^{\varepsilon,m}(0)\|^2_{3/2} + \|\xi^{\varepsilon,m}(0) + \varepsilon\partial_t\xi^{\varepsilon,m}(0)\|^2_{3/2}) + \|\partial_t\xi^{\varepsilon,m} + \varepsilon\partial_t^2\xi^{\varepsilon,m}\|^2_{L^2H^{3/2}}\\
&\lesssim C(\varepsilon)T(\mathfrak{E}_0(\eta) + \|F^1(0)\|^2_0 + \|F^3(0)\|^2_{1/2} + \|F^5(0)\|^2_{1/2})\\
&\quad + \|\partial_t u^{\varepsilon,m}\|^2_{L^2H^1} + \|\partial_t(F^1 - F^3 - F^5)\|^2_{(\mathcal{H}^1_T)^*} + \mathfrak{K}(\eta).
\end{aligned} \tag{3.53}$$

Step 6. Passing to the limit. From the estimates of (3.29), (3.48) and (3.49) and Lemma 2.2, we know that $\{u^{\varepsilon,m}\}$ and $\{\partial_t u^{\varepsilon,m}\}$ is uniformly bounded in $L^2([0,T];H^1(\Omega)) \cap L^2([0,T];H^0(\Sigma_b))$, and that $\{\xi^{\varepsilon,m}\}$ and $\{\partial_t\xi^{\varepsilon,m}\}$ are uniformly bounded in $L^\infty([0,T];H^1(\Sigma)) \cap L^2([0,T];H^{3/2}(\Sigma))$. Up to an extraction of a subsequence, we have that

$$u^{\varepsilon,m} \rightharpoonup u^\varepsilon \text{ weakly in } L^2([0,T];H^1(\Omega)) \cap L^2([0,T];H^0(\Sigma_b)),$$

$$\partial_t u^{\varepsilon,m} \rightharpoonup \partial_t u^\varepsilon \text{ weakly in } L^2([0,T];H^1(\Omega)) \cap L^2([0,T];H^0(\Sigma_b)),$$

$$\xi^{\varepsilon,m} \rightharpoonup \xi^\varepsilon \text{ weakly-}* \text{ in } L^\infty([0,T];H^1(\Sigma)),$$

$$\partial_t\xi^{\varepsilon,m} \rightharpoonup \partial_t\xi^\varepsilon \text{ weakly-}* \text{ in } L^\infty([0,T];H^1(\Sigma)),$$

and

$$\xi^{\varepsilon,m} \rightharpoonup \xi^\varepsilon \text{ weakly in } L^2([0,T];H^{3/2}(\Sigma)),$$

$$\partial_t\xi^{\varepsilon,m} \rightharpoonup \partial_t\xi^\varepsilon \text{ weakly in } L^2([0,T];H^{3/2}(\Sigma)).$$

The lower semicontinuity of norms implies that

$$\|u^{\varepsilon,m}\|^2_{L^2H^1} + \|u^{\varepsilon,m}\cdot\tau\|^2_{L^2H^0(\Sigma_b)} + \|\partial_t u^{\varepsilon,m}\|^2_{L^2H^1} + \|\partial_t u^{\varepsilon,m}\cdot\tau\|^2_{L^2H^0(\Sigma_b)}$$

and

$$\|\xi^{\varepsilon,m}\|^2_{L^\infty H^1} + \|\partial_t\xi^{\varepsilon,m}\|^2_{L^\infty H^1} + \|\xi^{\varepsilon,m}\|_{L^2H^{3/2}} + \|\partial_t\xi^{\varepsilon,m}\|_{L^2H^{3/2}}$$

are bounded.

Step 7. The strong solutions. We may pass to the limit for (3.23) as $m \to \infty$ to obtain that

$$\begin{aligned}&\frac{\mu}{2}(u^\varepsilon, v)_{\mathcal{H}^1} + \beta \int_{\Sigma_b} (u^\varepsilon \cdot \tau)(v \cdot \tau) J + (\xi^\varepsilon + \varepsilon \partial_t \xi^\varepsilon, v \cdot \mathcal{N})_{1,\Sigma} \\ &= (F^1, v)_{\mathcal{H}^0} - \int_\Sigma \left(\sigma \mathcal{R} \partial_1 (v \cdot \mathcal{N}) + F^3 \cdot v\right) - \int_{\Sigma_b} F^5 (v \cdot \tau) J\end{aligned} \tag{3.54}$$

for each $v \in \mathcal{V}(t)$. Then we might use Theorem 2.5 to introduce the pressure. Define the linear functional $\Lambda \in (\mathcal{W})^*$ by setting $\Lambda(v)$ to be the difference of the left-hand side and the right-hand side of (3.54) for each $v \in \mathcal{W}$. Then Theorem 2.5 ensures a unique $p^\varepsilon(t) \in \mathring{H}^0(\Omega)$ such that $(p^\varepsilon, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \Lambda(v)$ for each $v \in \mathcal{W}$. Moreover,

$$\|p^\varepsilon(t)\|_0^2 \lesssim \|u^\varepsilon\|_1^2 + \|F^1\|_0^2 + \|F^3\|_{1/2}^2 + \|F^5\|_{1/2}^2. \tag{3.55}$$

Hence

$$\begin{aligned}&\frac{\mu}{2}(u^\varepsilon, v)_{\mathcal{H}^1} + \beta \int_{\Sigma_b} (u^\varepsilon \cdot \tau)(v \cdot \tau) J - (p^\varepsilon, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} + (\xi^\varepsilon + \varepsilon \partial_t \xi^\varepsilon, v \cdot \mathcal{N})_{1,\Sigma} \\ &= (F^1, v)_{\mathcal{H}^0} - \int_\Sigma \left(\sigma \mathcal{R} \partial_1 (v \cdot \mathcal{N}) + F^3 \cdot v\right) - \int_{\Sigma_b} F^5 (v \cdot \tau) J\end{aligned} \tag{3.56}$$

for each $v \in \mathcal{W}(t)$, which shows that $(u^\varepsilon(t), p^\varepsilon(t), \xi(t) + \varepsilon \partial_t \xi(t))$ is the unique weak solution of the elliptic system (2.25). Then we employ the elliptic theory in Theorem 2.7 to deduce that

$$\begin{aligned}&\|u^\varepsilon(t)\|_2^2 + \|p^\varepsilon(t)\|_1^2 + \|\xi^\varepsilon(t) + \varepsilon \partial_t \xi^\varepsilon(t)\|_{5/2}^2 \\ &\lesssim \|F^1(t)\|_0^2 + \|F^3(t)\|_{1/2}^2 + \|F^5(t)\|_{1/2}^2 + \|\partial_t \xi^\varepsilon(t)\|_{3/2}^2 + \|\partial_1 \mathcal{R}\|_{1/2}^2.\end{aligned} \tag{3.57}$$

Then from the integral representation of ξ^ε as

$$\xi^\varepsilon = \eta_0 e^{-\frac{t}{\varepsilon}} + \frac{1}{\varepsilon} \int_0^t e^{-\frac{t-s}{\varepsilon}} (\xi^\varepsilon + \varepsilon \partial_t \xi^\varepsilon)(s) \mathrm{d}s,$$

we have that

$$\|\xi^\varepsilon\|_{L^2 H^{5/2}}^2 \lesssim C(\varepsilon) T \|\eta_0\|_{5/2}^2 + \|\xi^\varepsilon + \varepsilon \partial_t \xi^\varepsilon\|_{L^2 H^{5/2}}^2 \tag{3.58}$$

Finally, along with (3.53), we have that

$$\begin{aligned}&\|u^\varepsilon\|_{L^2 H^2}^2 + \|p^\varepsilon\|_{L^2 H^1}^2 + \|\xi^\varepsilon\|_{L^2 H^{5/2}}^2 \\ &\lesssim C(\varepsilon) T \|\eta_0\|_{5/2}^2 + \|F^1\|_{L^2 H^0}^2 + \|F^3\|_{L^2 H^{1/2}}^2 + \|F^5\|_{L^2 H^{1/2}}^2 + \|\partial_t \xi^\varepsilon\|_{L^2 H^{3/2}}^2 + \mathfrak{K}(\eta) \\ &\lesssim C(\varepsilon) T (\mathfrak{E}_0(\eta) + \|F^1(0)\|_0^2 + \|F^3(0)\|_{1/2}^2 + \|F^5(0)\|_{1/2}^2) \\ &\quad + \|\partial_t u^{\varepsilon,m}\|_{L^2 H^1}^2 + \|\partial_t (F^1 - F^3 - F^5)\|_{(\mathcal{H}_T^1)^*}^2 + \mathfrak{K}(\eta).\end{aligned} \tag{3.59}$$

Step 8. The weak solution for $D_t u^\varepsilon$ and $\partial_t p^\varepsilon$.

Now we seek to use (3.34) to determine the PDE satisfied by $D_t u^\varepsilon$ and $\partial_t p^\varepsilon$. We may pass to the limit $m \to \infty$, and use (3.34) with the test function v replaced

by Rv to derive that
(3.60)

$$\begin{aligned}
&\frac{\mu}{2}(\partial_t u^\varepsilon, v)_{\mathcal{H}^1} + \beta \int_{\Sigma_b} (\partial_t u^\varepsilon \cdot \tau)(v\cdot\tau)J + (\partial_t \xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon, v\cdot\mathcal{N})_{1,\Sigma} \\
&= (\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon, Rv\cdot\mathcal{N})_{1,\Sigma} - (p^\varepsilon, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0} + \int_\Omega \left[\partial_t F^1 \cdot v + \partial_t JKF^1\cdot v\right] J \\
&\quad - \int_\Sigma [\sigma\partial_t\mathcal{R}\partial_1(v\cdot\mathcal{N}) + \sigma\mathcal{R}\partial_1(v\cdot\partial_t\mathcal{N}) + \partial_t F^3\cdot v] \\
&\quad - \int_{\Sigma_b} \left[\partial_t F^5 v + \partial_t JKF^5 v\right]\cdot\tau J - \int_{\Sigma_b} \beta(u^\varepsilon\cdot\tau)(v\cdot\tau)\partial_t J \\
&\quad - \int_\Omega \frac{\mu}{2}(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u^\varepsilon : \mathbb{D}_{\partial_t\mathcal{A}}v + \partial_t JK\mathbb{D}_{\mathcal{A}}u^\varepsilon : \mathbb{D}_{\mathcal{A}}v)J.
\end{aligned}$$

According to the Lemma B.1, we know that $-R^\top\mathcal{N} = \partial_t\mathcal{N}$ on Σ. Then integrating by parts, we have that

$$-(p^\varepsilon, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0} = (R^\top\nabla_{\mathcal{A}}p^\varepsilon, v)_{\mathcal{H}^0} + \langle p^\varepsilon\partial_t\mathcal{N}, v\rangle_{-1/2}, \tag{3.61}$$

where we have used the Proposition 2.4 to cancel the term on boundary of solid wall. Then the definition of R and integration by parts yield that

$$\begin{aligned}
&- \int_\Omega \frac{\mu}{2}(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u^\varepsilon : \mathbb{D}_{\partial_t\mathcal{A}}v + \partial_t JK\mathbb{D}_{\mathcal{A}}u^\varepsilon : \mathbb{D}_{\mathcal{A}}v)J \\
&= - \int_\Omega \mu(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon - R\mathbb{D}_{\mathcal{A}}u^\varepsilon) : \nabla_{\mathcal{A}}vJ \\
&= (\operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon - R\mathbb{D}_{\mathcal{A}}u^\varepsilon), v)_{\mathcal{H}^0} - \langle \mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\mathcal{N} + \mathbb{D}_{\mathcal{A}}u^\varepsilon\partial_t\mathcal{N}, v\rangle_{-1/2}.
\end{aligned} \tag{3.62}$$

Similarly, we have that

$$\begin{aligned}
&- \int_{\Sigma_b} \beta(u^\varepsilon\cdot\tau)(v\cdot\tau)\partial_t J \\
&= \int_{\Sigma_b} \mu\mathbb{D}_{\mathcal{A}}u^\varepsilon\nu\cdot v\partial_t J \\
&= \int_{\Sigma_b} \mu\mathbb{D}_{\mathcal{A}}u^\varepsilon\nu\cdot v\partial_t JKJ + \mu R\mathbb{D}_{\mathcal{A}}u^\varepsilon\nu\cdot vJ - \mu R\mathbb{D}_{\mathcal{A}}u^\varepsilon\nu\cdot vJ \\
&= \int_{\Sigma_b} \mu\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\nu\cdot vJ - \mu R\mathbb{D}_{\mathcal{A}}u^\varepsilon\nu\cdot vJ \\
&= \int_{\Sigma_b} \mu\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\nu\cdot vJ + \beta Ru^\varepsilon\cdot vJ.
\end{aligned} \tag{3.63}$$

Combining the above equalities (3.60)–(3.63),
(3.64)

$$\begin{aligned}&\frac{\mu}{2}(\partial_t u^\varepsilon, v)_{\mathcal{H}^1} + \beta\int_{\Sigma_b}(\partial_t u^\varepsilon\cdot\tau)(v\cdot\tau)J + (\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon, v\cdot\mathcal{N})_{1,\Sigma}\\
&= \int_\Omega\left[\operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon - R\mathbb{D}_{\mathcal{A}}u^\varepsilon) + R^\top\nabla_{\mathcal{A}}p^\varepsilon\right]\cdot vJ + \int_\Omega(\partial_t F^1 + \partial_t JKF^1)\cdot vJ\\
&\quad + \int_{\Sigma_b}(\mu\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\nu + \beta Ru^\varepsilon)\cdot\tau(\tau\cdot v)J - \int_{\Sigma_b}(\partial_t F^5 + \partial_t JKF^5)(v\cdot\tau)J\\
&\quad - \int_\Sigma \partial_t\sigma\mathcal{R}\partial_1(v\cdot\mathcal{N}) + \mathcal{R}\partial_1(v\cdot\partial_t\mathcal{N}) + F^3\cdot v + (\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\mathcal{N} + \mathbb{D}_{\mathcal{A}}u^\varepsilon\partial_t\mathcal{N})\cdot v\\
&\quad + \int_\Sigma\left(-p^\varepsilon + g(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon) - \sigma\partial_1\left(\mathcal{R} + \partial_1(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon)\right)\right)\partial_t\mathcal{N}\cdot v,\end{aligned}$$

where we have used the integration by parts for the term $(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon, Rv\cdot\mathcal{N})_{1,\Sigma}$. Then there exists a unique $\partial_t p^\varepsilon \in \mathring{H}^0$ by employing Theorem 2.5, such that
(3.65)

$$\begin{aligned}&\frac{\mu}{2}(\partial_t u^\varepsilon, v)_{\mathcal{H}^1} + \beta\int_{\Sigma_b}(\partial_t u^\varepsilon\cdot\tau)(v\cdot\tau)J - (\partial_t p^\varepsilon, \operatorname{div}_{\mathcal{A}} v)_0 + (\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon, v\cdot\mathcal{N})_{1,\Sigma}\\
&= \int_\Omega\left[\operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon - R\mathbb{D}_{\mathcal{A}}u^\varepsilon) + R^\top\nabla_{\mathcal{A}}p^\varepsilon\right]\cdot vJ + \int_\Omega(\partial_t F^1 + \partial_t JKF^1)\cdot vJ\\
&\quad + \int_{\Sigma_b}(\mu\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\nu + \beta Ru^\varepsilon)\cdot\tau(\tau\cdot v)J - \int_{\Sigma_b}(\partial_t F^5 + \partial_t JKF^5)(v\cdot\tau)J\\
&\quad - \int_\Sigma \partial_t\sigma\mathcal{R}\partial_1(v\cdot\mathcal{N}) + \mathcal{R}\partial_1(v\cdot\partial_t\mathcal{N}) + F^3\cdot v + (\mathbb{D}_{\partial_t\mathcal{A}}u^\varepsilon\mathcal{N} + \mathbb{D}_{\mathcal{A}}u^\varepsilon\partial_t\mathcal{N})\cdot v\\
&\quad + \int_\Sigma\left(-p^\varepsilon + g(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon) - \sigma\partial_1\left(\mathcal{R} + \partial_1(\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon)\right)\right)\partial_t\mathcal{N}\cdot v,\end{aligned}$$

and

$$\begin{aligned}\|\partial_t p^\varepsilon\|_0^2 &\lesssim \|\partial_t u^\varepsilon\|_1^2 + \|\eta\|_{3/2}^2\|\partial_t\eta\|_{3/2}^2(\|u^\varepsilon\|_2^2 + \|p^\varepsilon\|_1^2 + \|\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon\|_{5/2}^2\\
&\quad + \|\eta\|_{5/2}^2 + \|\eta\|_{3/2}^2 + 1 + \|F^1\|_0^2 + \|F^3\|_{1/2}^2 + \|F^5\|_{1/2}^2\\
&\quad + \|\partial_t(F^1 - F^3 - F^5)\|_{(\mathcal{H}^1)^*}^2).\end{aligned} \tag{3.66}$$

Thus integrating temporally from 0 to T reveals that
(3.67)

$$\begin{aligned}&\|\partial_t u^\varepsilon\|_{L^2H^1} + \|\partial_t p^\varepsilon\|_{L^2H^0}^2\\
&\lesssim (C(\varepsilon)T + 1)(\mathfrak{K}(\eta) + \mathfrak{E}_0 + \|F^1(0)\|_0 + \|F^3(0)\|_{1/2}^2 + \|F^3(0)\|_{1/2}^2) + \mathfrak{E}(\eta)\mathfrak{K}(\eta)\\
&\quad + (1 + \mathfrak{E}(\eta))(\|F^1\|_{L^2H^0}^2 + \|F^3\|_{L^2H^{1/2}}^2 + \|F^5\|_{L^2H^{1/2}}^2)\\
&\quad + (1 + \mathfrak{E}(\eta))\|\partial_t(F^1 - F^3 - F^5)\|_{(\mathcal{H}_T^1)^*}^2.\end{aligned}$$

□

3.3. Higher order regularity. Now we define notions as follows:

$$\begin{aligned}\mathfrak{E}^\varepsilon(u^\varepsilon, p^\varepsilon, \xi^\varepsilon) :&= \|u^\varepsilon\|_{L^\infty H^2}^2 + \|\partial_t u^\varepsilon\|_{L^\infty H^1}^2 + \|p^\varepsilon\|_{L^\infty H^1}^2 + \|\partial_t p^\varepsilon\|_{L^\infty H^0}^2\\
&\quad + \|\xi^\varepsilon\|_{L^\infty H^{5/2}}^2 + \|\partial_t\xi^\varepsilon\|_{L^\infty H^{3/2}}^2 + \sum_{j=0}^2\|\partial_t^j\xi^\varepsilon\|_{L^\infty H^1}^2,\end{aligned} \tag{3.68}$$

$$\begin{aligned}\mathfrak{D}^\varepsilon(u^\varepsilon,p^\varepsilon,\xi^\varepsilon):=&\sum_{j=0}^{1}\Big(\|\partial_t^j u^\varepsilon\|_{L^2H^2}^2+\|\partial_t^j p^\varepsilon\|_{L^2H^1}^2+\|\partial_t^j\xi^\varepsilon\|_{L^2H^{5/2}}^2\Big)\\&+\sum_{j=0}^{2}\Big(\|\partial_t^j u^\varepsilon\|_{L^2H^1}^2+\|\partial_t^j u^\varepsilon\cdot\tau\|_{L^2H^0(\Sigma_b)}^2+\|\partial_t^j p^\varepsilon\|_{L^2H^0}^2\Big)\\&+\sum_{j=0}^{2}\|\partial_t^j\xi^\varepsilon\|_{L^2H^{3/2}}^2+\|\partial_t^3\xi^\varepsilon\|_{L^2H^{1/2}}^2.\end{aligned}\tag{3.69}$$

In order to state our higher regularity results for the problem (3.1), we must be able to define the forcing terms and initial data for the problem that results from temporally differentiating (3.1) one time. First, we define some mappings. Given F^6, v, q, $\tilde{\xi}$, we define the vector fields $\mathfrak{G}^1$ in Ω, $\mathfrak{G}^3$ on Σ and $\mathfrak{G}^4$ on Σ_b by

$$\begin{aligned}\mathfrak{G}^1(v,q)&=R^\top\nabla_{\mathcal{A}}q+\operatorname{div}_{\mathcal{A}}\left(\mathbb{D}_{\mathcal{A}}(Rv)+\mathbb{D}_{\partial_t\mathcal{A}}v-R\mathbb{D}_{\mathcal{A}}v\right),\\\mathfrak{G}^3(v,q,\tilde{\xi})&=\mu\mathbb{D}_{\mathcal{A}}(Rv)\mathcal{N}-(qI-\mu\mathbb{D}_{\mathcal{A}}v)\partial_t\mathcal{N}+\mu\mathbb{D}_{\partial_t\mathcal{A}}v)\mathcal{N}+\mathcal{L}(\tilde{\xi})\partial_t\mathcal{N}\\&\quad-\sigma\partial_1F^6\partial_t\mathcal{N},\\\mathfrak{G}^5(v)&=(\mu\mathbb{D}_{\mathcal{A}}(Rv)\nu+\mu\mathbb{D}_{\partial_t\mathcal{A}}v\nu+\beta Rv)\cdot\tau.\end{aligned}\tag{3.70}$$

These mappings allow us to define the forcing terms as follows. We write $F^{1,0}=F^1$, $F^{3,0}=F^3$ and $F^{5,0}=F^5$. Then we write

$$F^{1,1}:=D_tF^{1,0}+\mathfrak{G}^1(u^\varepsilon,p^\varepsilon),\quad F^{3,1}:=\partial_tF^{3,0}+\mathfrak{G}^3(u^\varepsilon,p^\varepsilon,\xi^\varepsilon+\varepsilon\partial_t\xi^\varepsilon),\quad F^{5,1}:=\partial_tF^{5,0}+\mathfrak{G}^5(u^\varepsilon).\tag{3.71}$$

When $F^6=\mathcal{R}$, u^ε, p^ε and ξ^ε are sufficiently regular for the following to make sense, we define the vectors

$$F^{1,2}:=\mathfrak{G}^1(D_tu^\varepsilon,\partial_tp^\varepsilon)+D_tF^{1,1},\ F^{3,2}:=\mathfrak{G}^3(D_tu^\varepsilon,\partial_tp^\varepsilon,\partial_t\xi^\varepsilon+\varepsilon\partial_t^2\xi^\varepsilon)+\partial_tF^{3,1},\ F^{5,2}:=\mathfrak{G}^5(D_tu^\varepsilon)+\partial_tF^{5,1}.\tag{3.72}$$

In order to deduce the higher regularity, we need to control the forcing terms $F^{i,j}$. Before that, we need the following useful lemma.

Lemma 3.6. *Suppose that the right-hand side of the following estimates are finite. Then we have the inclusions* $u^\varepsilon\in C^0([0,T];H^2(\Omega))$, $p^\varepsilon\in C^0([0,T];\mathring{H}^1(\Omega))$, $\xi^\varepsilon\in C^0([0,T];H^{5/2}(\Sigma))$, *as well as the estimates*

$$\|u^\varepsilon\|_{L^\infty H^2}^2\lesssim\|u^\varepsilon(0)\|_{3/2}^2+\|u^\varepsilon\|_{L^2H^2}^2+\|\partial_tu^\varepsilon\|_{L^2H^2}^2,\tag{3.73}$$

$$\|p^\varepsilon\|_{L^\infty\mathring{H}^1}^2\lesssim\|p^\varepsilon(0)\|_1^2+\|p^\varepsilon\|_{L^2\mathring{H}^1}^2+\|\partial_tp^\varepsilon\|_{L^2\mathring{H}^1}^2,\tag{3.74}$$

$$\|\xi^\varepsilon\|_{L^\infty H^{5/2}}^2\lesssim\|\eta_0\|_{H^{5/2}}^2+\|\xi^\varepsilon\|_{L^2H^{5/2}}^2+\|\partial_t\xi^\varepsilon\|_{L^2H^{5/2}}^2.\tag{3.75}$$

$$\|\xi^\varepsilon+\varepsilon\partial_t\xi^\varepsilon\|_{L^\infty H^{5/2}}^2\lesssim\|\eta_0\|_{5/2}^2+\|u^\varepsilon(0)\|_2^2+\|\xi^\varepsilon+\varepsilon\partial_t\xi^\varepsilon\|_{L^2H^{5/2}}^2+\|\partial_t\xi^\varepsilon+\varepsilon\partial_t^2\xi^\varepsilon\|_{L^2H^{5/2}}^2.\tag{3.76}$$

Proof. First, (3.73) and (3.74) are obtained by a computation similar to that of Lemma 3.4. (3.75) can be obtained after employing the extension theory on Sobolev spaces, and then using the restriction theory on Sobolev spaces. From the third equation of (3.1), we know that

$$\|(\xi^\varepsilon+\varepsilon\partial_t\xi^\varepsilon)(0)\|_{5/2}^2\lesssim\|\eta_0\|_{5/2}^2+\|u^\varepsilon(0)\|_2^2,$$

which together with (3.75) imply (3.76). □

Now, we need to estimate the forcing terms of $F^{i,j}$.

LEMMA 3.7. *The following estimates hold whenever the right hand side are finite.*

(3.77)
$$\|F^{1,1}\|^2_{L^2H^0} \lesssim \|\partial_t F^1\|_{L^2H^0} + \mathfrak{K}(\eta)(\|F^1(0)\|^2_0 + \|u^\varepsilon(0)\|^2_0 + \|p^\varepsilon(0)\|^2_1 + \|F^1\|_{L^2H^0} + \|u^\varepsilon\|^2_{L^2H^1} + \|\partial_t u^\varepsilon\|^2_{L^2H^1} + \|u^\varepsilon\|^2_{L^2H^2} + \|\partial_t u^\varepsilon\|^2_{L^2H^2} + \|p^\varepsilon\|^2_{L^2\mathring{H}^1} + \|\partial_t p^\varepsilon\|^2_{L^2\mathring{H}^1}),$$

$$\begin{aligned}\|F^{3,1}\|^2_{L^2H^{1/2}} \lesssim \|\partial_t F^3\|_{L^2H^{1/2}} + \mathfrak{K}(\eta)\Big(&\|F^3(0)\|^2_{1/2} + \|\eta_0\|^2_{5/2} \\ &+ \|u^\varepsilon(0)\|^2_2 + \|p^\varepsilon(0)\|^2_1 + \|p^\varepsilon\|^2_{L^2H^1} + \|u^\varepsilon\|^2_{L^2H^2} + \|\partial_t p^\varepsilon\|^2_{L^2H^1} \\ &+ \|\xi^\varepsilon + \epsilon\partial_t\xi^\varepsilon\|^2_{L^2H^{5/2}} + \|\partial_t u^\varepsilon\|^2_{L^2H^2} + \|\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon\|^2_{L^2H^{5/2}}\Big),\end{aligned} \tag{3.78}$$

$$\begin{aligned}\|F^{5,1}\|^2_{L^2H^{1/2}} \lesssim \|\partial_t F^5\|_{L^2H^{1/2}} + \mathfrak{K}(\eta)\Bigg(&\|F^5(0)\|^2_{1/2} + \|u^\varepsilon(0)\|^2_0 + \|u^\varepsilon\|^2_{L^2H^2} \\ &+ \|\partial_t u^\varepsilon\|^2_{L^2H^2} + \|u^\varepsilon\|^2_{L^2H^1}\Bigg).\end{aligned} \tag{3.79}$$

PROOF. The estimates follow from simple but lengthy computations. For this reason, we only give a sketch of proving these estimates.

According to the definition of $F^{1,1}$, $F^{4,1}$ and $F^{5,1}$ in (3.71), we use Leibniz rule to rewrite $F^{i,1}$ as a sum of products for two terms. One term is a product of various derivatives of $\bar\eta$, and the other is linear for derivatives of u^ε, p^ε and ξ^ε. Then for a.e. $t \in [0,T]$, we estimate these resulting products using the usual Sobolev embedding theorems and Lemma 3.4. Then the resulting inequalities after integrating over $[0,T]$ reveals

$$\begin{aligned}\|F^{1,1}\|^2_{L^2H^0} \lesssim \|\partial_t F^1\|^2_{L^2H^0} + P(\mathfrak{E}(\eta))\mathfrak{D}(\eta)(&\|F^1(0)\|^2_0 + \|F^1\|^2_{L^2H^0} \\ &+ \|u^\varepsilon(0)\|^2_0 + \|p^\varepsilon(0)\|^2_1 + \|u\|^2_{L^2H^1} + \|\partial_t u\|^2_{L^2H^1} \\ &+ \|u\|^2_{L^2H^2} + \|\partial_t u\|^2_{L^2H^2} + \|p\|^2_{L^2\mathring{H}^1} + \|\partial_t p\|^2_{L^2\mathring{H}^1}),\end{aligned} \tag{3.80}$$

where $P(\cdot)$ is a polynomial. Since $\mathfrak{K}(\eta) \le 1$, we know that $P(\mathfrak{E}(\eta))\mathfrak{D}(\eta) \lesssim \mathfrak{K}(\eta)$. Thus we have the bounds for (3.77). Similarly, we have the bounds for (3.78) and (3.79), and (3.78) also needs (3.76). □

Now, we give some estimates for the difference between $\partial_t u^\varepsilon$ and $D_t u^\varepsilon$. The proof is similar as that of Lemma 3.7, so we omit it here.

LEMMA 3.8.

$$\|\partial_t u^\varepsilon - D_t u^\varepsilon\|^2_{L^2H^2} \lesssim \mathfrak{D}(\eta)(\|u^\varepsilon(0)\|^2_2 + \|u^\varepsilon\|^2_{L^2H^2} + \|\partial_t u^\varepsilon\|^2_{L^2H^2}), \tag{3.81}$$

$$\|\partial_t u^\varepsilon - D_t u^\varepsilon\|^2_{L^2H^1} + \|\partial_t u^\varepsilon - D_t u^\varepsilon\|^2_{L^2H^0(\Sigma_b)} \lesssim \mathfrak{E}(\eta)(\|u^\varepsilon\|^2_{L^2H^1} + \|u^\varepsilon\|^2_{L^2H^2}), \tag{3.82}$$

and

(3.83)
$$\|\partial_t^2 u^\varepsilon - \partial_t D_t u^\varepsilon\|^2_{L^2H^1} + \|\partial_t^2 u^\varepsilon - \partial_t D_t u^\varepsilon\|^2_{L^2H^0(\Sigma_b)} \lesssim \mathfrak{K}(\eta)(\|u^\varepsilon\|^2_{L^2H^2} + \|\partial_t u^\varepsilon\|^2_{L^2H^2}).$$

LEMMA 3.9. *It holds that*

$$\begin{aligned} \|F^{1,1} - F^{3,1} - F^{5,1}\|^2_{(\mathcal{H}^1_T)^*} &\lesssim \|\partial_t F^1\|^2_{L^2H^0} + \|\partial_t F^3\|^2_{L^2H^{1/2}} + \|\partial_t F^5\|^2_{L^2H^{1/2}} \\ &+ \mathfrak{E}(\eta)(\|F^1\|^2_{L^2L^2} + \|p^\varepsilon\|^2_{L^2\mathring{H}^1} + \|u^\varepsilon\|^2_{L^2H^2} + \|\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon\|^2_{L^2H^{5/2}}), \end{aligned} \tag{3.84}$$

and

$$\begin{aligned} &\|\partial_t(F^{1,1} - F^{3,1} - F^{5,1})\|^2_{(\mathcal{H}^1_T)^*} \lesssim \|\partial_t^2(F^1 - F^3 - F^5)\|_{(\mathcal{H}^1_T)^*} \\ &\quad + \mathfrak{K}(\eta)(\|F^1(0)\|^2_0 + \|F^1\|_{\mathcal{H}^1_T} + \|\partial_t F^1\|_{\mathcal{H}^1_T} + \|u^\varepsilon(0)\|^2_2 + \|p^\varepsilon(0)\|^2_1 \\ &\quad + \|\eta_0\|^2_{5/2} + \|u^\varepsilon\|^2_{L^2H^1} + \|\partial_t u^\varepsilon\|^2_{L^2H^1} + \|u^\varepsilon\|^2_{L^2H^2} + \|\partial_t u^\varepsilon\|^2_{L^2H^2} \\ &\quad + \|p^\varepsilon\|^2_{L^2\mathring{H}^1} + \|\partial_t p\|^2_{L^2\mathring{H}^1} + \|\xi^\varepsilon + \varepsilon\partial_t\xi^\varepsilon\|^2_{L^2H^{5/2}} + \|\partial_t\xi^\varepsilon + \varepsilon\partial_t^2\xi^\varepsilon\|^2_{L^2H^{5/2}}). \end{aligned} \tag{3.85}$$

Then $F^{1,1} - F^{3,1} - F^{5,1} \in C([0,T];(\mathcal{H}^1)^*)$. *Moreover,*

$$\|(F^{1,1} - F^{3,1} - F^{5,1})(0)\|^2_{(\mathcal{H}^1)^*} \lesssim \mathfrak{E}_0 + \|F^1(0)\|^2_0 + \|F^3(0)\|^2_{1/2} + \|F^5(0)\|^2_{1/2}. \tag{3.86}$$

PROOF. Since the proof of the first two inequalities are similar, we only give the proof of second inequality. From the notation in Definition 3.1, we have that

$$\begin{aligned} &\langle \partial_t(F^{1,1} - F^{4,1} - F^{5,1}), v\rangle_{(\mathcal{H}^1_T)^*} \\ &= \int_0^T \left(\int_\Omega \partial_t(D_tF^1)\cdot vJ - \int_\Sigma \partial_t^2F^3\cdot v - \int_{\Sigma_b}\partial_t^2F^5(v\cdot\tau)J\right) \\ &\quad + \int_0^T\left(\int_\Omega \partial_t\mathfrak{G}^1\cdot vJ - \int_\Sigma\partial_t\mathfrak{G}^3\cdot v - \int_{\Sigma_b}\partial_t\mathfrak{G}^5(v\cdot\tau)J\right) \end{aligned} \tag{3.87}$$

for each $v \in \mathcal{X}$. Then we use an integration by parts to compute

$$\begin{aligned} \int_\Omega \operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon))vJ = -\frac{1}{2}\int_\Omega \mathbb{D}_{\mathcal{A}}(Ru^\varepsilon) : \mathbb{D}_{\mathcal{A}}vJ + \int_\Sigma \mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)\mathcal{N}\cdot v \\ + \int_{\Sigma_b}\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)\nu\cdot\tau(v\cdot\tau)J, \end{aligned} \tag{3.88}$$

which reduces (3.87) to the following equality:

$$\begin{aligned} &\langle \partial_t(F^{1,1} - F^{4,1} - F^{5,1}), v\rangle_{(\mathcal{H}^1_T)^*} \\ &= \int_0^T \left(\int_\Omega \partial_t(D_tF^1)\cdot vJ - \int_\Sigma \partial_t^2F^3\cdot v - \int_{\Sigma_b}\partial_t^2F^5(v\cdot\tau)J\right) \\ &\quad - \frac{1}{2}\int_0^T\int_\Omega \partial_t(\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)) : \mathbb{D}_{\mathcal{A}}vJ \\ &\quad + \int_0^T\int_\Omega\left[\partial_t(\mathfrak{G}^1 - \mu\operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon))) + \mu\operatorname{div}_{\partial_t\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon))\right]\cdot vJ \\ &\quad - \int_0^T\int_\Sigma \partial_t(\mathfrak{G}^3 - \mu\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)\mathcal{N})\cdot v - \int_0^T\int_{\Sigma_b}\partial_t(\mathfrak{G}^5 - \mu\mathbb{D}_{\mathcal{A}}(Ru^\varepsilon)\nu\cdot\tau)(v\cdot\tau)J. \end{aligned} \tag{3.89}$$

Then we use Hölder's inequality and the same computation in Lemma 3.7 to derive the resulting bounds. □

For our higher order estimates, we need to construct $D_tu^\varepsilon(0)$ and $\partial_tp^\varepsilon(0)$. Suppose that $\partial_t\xi^\varepsilon(0) = \partial_t^2\eta(0) \in \mathring{H}^1(\Sigma)$ is given. We take temporal derivative to

(3.1) and take $t=0$ to obtain that

$$
\begin{cases}
-\mu\Delta_{\mathcal{A}(0)}(D_tu^\varepsilon(0))+\nabla_{\mathcal{A}(0)}\partial_tp^\varepsilon(0)=F^{1,1}(0) & \text{in} \quad \Omega,\\
\operatorname{div}_{\mathcal{A}(0)}(D_tu^\varepsilon(0))=0 & \text{in} \quad \Omega,\\
S_{\mathcal{A}(0)}(\partial_tp^\varepsilon(0),D_tu^\varepsilon(0))\mathcal{N}(0)=\mathcal{L}(\partial_t\xi^\varepsilon(0)+\varepsilon\partial_t^2\xi^\varepsilon(0))\mathcal{N}(0) & \\
\qquad -\sigma\partial_1\partial_t\mathcal{R}(0)\mathcal{N}(0)+F^{3,1}(0) & \text{on} \quad \Sigma,\\
(S_{\mathcal{A}(0)}(\partial_tp^\varepsilon(0),D_tu^\varepsilon(0))\nu-\beta(D_tu^\varepsilon)(0))\cdot\tau=F^5(0) & \text{on} \quad \Sigma_b,\\
D_tu^\varepsilon(0)\cdot\nu=0 & \text{on} \quad \Sigma_b,\\
\partial_t^2\xi^\varepsilon(0)=D_tu^\varepsilon(0)\cdot\mathcal{N}(0) & \text{on} \quad \Sigma,
\end{cases}
\tag{3.90}
$$

Then as Definition 3.1, we have the following pressureless weak formulation

$$
\begin{aligned}
&(D_tu^\varepsilon(0),v)_{\mathcal{H}^1}+\beta(D_tu^\varepsilon(0)\cdot\tau,J(0)v\cdot\tau)_{H^0(\Sigma_b)}+(\partial_t\xi^\varepsilon(0)+\varepsilon\partial_t^2\xi^\varepsilon(0),v\cdot\mathcal{N})_{1,\Sigma}\\
&=(F^{1,1}(0),v)_{\mathcal{H}^0}-\int_\Sigma\sigma\partial_t\mathcal{R}\partial_1(v\cdot\mathcal{N}(0))+F^{3,1}(0)\cdot v-\int_{\Sigma_b}F^{5,1}(0)(v\cdot\tau)J(0).
\end{aligned}
\tag{3.91}
$$

Then Lax-Milgram Theorem guarantees the existence of $D_tu^\varepsilon(0)\in H^1(\Omega)$, which along with Lemma 3.9 and the estimates for $u^\varepsilon(0)$ in (3.10) implies that

$$
\|\partial_tu^\varepsilon(0)\|_1^2\lesssim\|D_tu^\varepsilon(0)\|_1^2+\|R(0)u^\varepsilon(0)\|_1^2\lesssim\mathfrak{E}_0+\|F^1(0)\|_0^2+\|F^3(0)\|_{1/2}^2+\|F^5(0)\|_{1/2}^2.
\tag{3.92}
$$

Then Theorem 2.5 recovers the pressure $\partial_tp^\varepsilon(0)\in L^2(\Omega)$ satisfying

$$
\|\partial_tp^\varepsilon(0)\|_0^2\lesssim\mathfrak{E}_0+\|F^1(0)\|_0^2+\|F^3(0)\|_{1/2}^2+\|F^5(0)\|_{1/2}^2.
\tag{3.93}
$$

THEOREM 3.10. *Suppose that $\eta_0\in\mathring{H}^{5/2}(\Sigma)$ and $\partial_t^2\eta(0)\in\mathring{H}^1(\Sigma)$, that $\mathfrak{K}(\eta)\le\gamma_1$ is sufficiently small satisfying the assumption in Lemma 2.2 and* [**25**, Theorem 3.3]. *Let $u^\varepsilon(0)\in H^2(\Omega)$, $D_tu^\varepsilon(0)\in H^1(\Omega)$, $p^\varepsilon\in\mathring{H}^1(\Omega)$, $\partial_tp^\varepsilon(0)\in\mathring{H}^0(\Omega)$, $\partial_t\xi^\varepsilon(0)\in H^{3/2}(\Sigma)$ and $\partial_t^2\xi^\varepsilon(0)\in H^1(\Sigma)$ constructed as above, all be determined in terms of η_0 and $\partial_t^2\eta(0)$. Then for each $0<\varepsilon\le1$, there exists $T_\varepsilon>0$ such that for $0<T\le T_\varepsilon$, then there exists a unique strong solution $(u^\varepsilon,p^\varepsilon,\xi^\varepsilon)$ to (3.1) on $[0,T]$. The pair $(D_t^ju^\varepsilon,\partial_t^jp^\varepsilon,\partial_t^j\xi^\varepsilon)$ satisfies*

$$
\begin{cases}
-\mu\Delta_{\mathcal{A}}(D_t^ju^\varepsilon)+\nabla_{\mathcal{A}}\partial_t^jp^\varepsilon=F^{1,j} & \textit{in} \quad \Omega,\\
\operatorname{div}_{\mathcal{A}}(D_t^ju^\varepsilon)=0 & \textit{in} \quad \Omega,\\
S_{\mathcal{A}}(\partial_t^jp^\varepsilon,D_t^ju^\varepsilon)\mathcal{N}=\mathcal{L}(\partial_t^j\xi^\varepsilon+\varepsilon\partial_t^{j+1}\xi^\varepsilon)\mathcal{N}-\sigma\partial_1\partial_t^j\mathcal{R}\mathcal{N}+F^{3,j} & \textit{on} \quad \Sigma,\\
(S_{\mathcal{A}}(\partial_t^jp^\varepsilon,D_t^ju^\varepsilon)\nu-\beta(D_t^ju^\varepsilon))\cdot\tau=F^{5,j} & \textit{on} \quad \Sigma_b,\\
D_t^ju^\varepsilon\cdot\nu=0 & \textit{on} \quad \Sigma_b,\\
\partial_t^{j+1}\xi^\varepsilon=D_t^ju^\varepsilon\cdot\mathcal{N} & \textit{on} \quad \Sigma,
\end{cases}
\tag{3.94}
$$

in the strong sense with initial data $(D_t^ju(0),\partial_t^jp(0),\partial_t^j\xi(0))$ for $j=0,1$ and in the weak sense for $j=2$, where

$$
\mathcal{L}(\partial_t^j\xi^\varepsilon+\varepsilon\partial_t^{j+1}\xi^\varepsilon)=g(\partial_t^j\xi^\varepsilon+\varepsilon\partial_t^{j+1}\xi^\varepsilon)-\sigma\partial_1^2(\partial_t^j\xi^\varepsilon+\varepsilon\partial_t^{j+1}\xi^\varepsilon).
$$

Moreover, the solution satisfies the estimate

(3.95)

$$\mathfrak{K}(u^\varepsilon,p^\varepsilon,\xi^\varepsilon) \le C_0(C(\varepsilon)T+1)(\mathfrak{K}(\eta)+\mathfrak{E}_0+\sum_{j=0}^{1}(\|\partial_t^j F^1(0)\|_0+\|\partial_t^j F^3(0)\|_{1/2}^2 + \|\partial_t^j F^5(0)\|_{1/2}^2)) + C_0(1+\mathfrak{E}(\eta))\sum_{j=0}^{1}(\|\partial_t^j F^1\|_{L^2H^0}^2+\|\partial_t^j F^3\|_{L^2H^{1/2}}^2 + \|\partial_t^j F^5\|_{L^2H^{1/2}}^2) + C_0(1+\mathfrak{E}(\eta))\|\partial_t^2(F^1-F^3-F^5)\|_{(\mathcal{H}_T^1)^*}^2.$$

where C_0 is a positive constant independent of ε, and $\mathfrak{K}(u^\varepsilon,p^\varepsilon,\xi^\varepsilon)=\mathfrak{E}(u^\varepsilon,p^\varepsilon,\xi^\varepsilon)+\mathfrak{D}(u^\varepsilon,p^\varepsilon,\xi^\varepsilon)$.

Proof. Step 1 – Following Theorem 3.5. For the case $j=0$, Theorem 3.5 guarantees the existence of $(u^\varepsilon,p^\varepsilon,\xi^\varepsilon)$ such that $(D_t^j u^\varepsilon,\partial_t^j p^\varepsilon,\partial_t^j\xi^\varepsilon)$ is a unique solution of (3.94) in the strong sense when $j=0$ and in the weak sense when $j=1$. For $j=1$, the assumption of Theorem 3.5 are satisfied according to Lemma 3.7, Lemma 3.9. Then according to Theorem 3.5 and the elliptic estimate for $\xi^\varepsilon+\varepsilon\partial_t\xi^\varepsilon$, we have that $(D_tu^\varepsilon,\partial_tp^\varepsilon,\partial_t\xi^\varepsilon)$ is a unique strong solution of (3.94), and $(D_t^2u^\varepsilon,\partial_t^2p^\varepsilon,\partial_t^2\xi^\varepsilon)$ is a unique weak solution of (3.94). Moreover,

(3.96)

$$\begin{aligned}&\|D_tu^\varepsilon\|_{L^2H^1}^2+\|D_tu^\varepsilon\|_{L^2H^0(\Sigma_b)}^2+\|D_tu^\varepsilon\|_{L^2H^2}^2+\|\partial_tD_tu^\varepsilon\|_{L^2H^1}^2\\&\quad+\|\partial_tD_tu^\varepsilon\|_{L^2H^0(\Sigma_b)}^2+\|\partial_tp^\varepsilon\|_{L^2H^0}^2+\|\partial_tp^\varepsilon\|_{L^2\mathring{H}^1}^2+\|\partial_t^2p^\varepsilon\|_{L^2H^0}^2+\|\partial_t\xi^\varepsilon\|_{L^\infty H^1}^2\\&\quad+\|\partial_t^2\xi^\varepsilon\|_{L^2H^{3/2}}^2+\|\partial_t\xi\|_{L^2H^{5/2}}^2+\|\partial_t^2\xi^\varepsilon\|_{L^\infty H^1}^2+\|\partial_t\xi^\varepsilon\|_{L^2H^{3/2}}^2\\&\lesssim(C(\varepsilon)T+1)(\mathfrak{K}(\eta)+\mathfrak{E}_0+\|F^{1,1}(0)\|_0+\|F^{3,1}(0)\|_{1/2}^2+\|F^{5,1}(0)\|_{1/2}^2)\\&\quad+(1+\mathfrak{E}(\eta))(\|F^{1,1}\|_{L^2H^0}^2+\|F^{3,1}\|_{L^2H^{1/2}}^2+\|F^{5,1}\|_{L^2H^{1/2}}^2)\\&\quad+(1+\mathfrak{E}(\eta))\|\partial_t(F^{1,1}-F^{3,1}-F^{5,1})\|_{(\mathcal{H}_T^1)^*}^2.\end{aligned}$$

Since $\mathfrak{K}(\eta)$ is sufficiently small, then Lemma 3.7–3.9 can reduce the above estimate to

(3.97)

$$\begin{aligned}&\|\partial_tu^\varepsilon\|_{L^2H^1}^2+\|\partial_tu^\varepsilon\|_{L^2H^0(\Sigma_b)}^2+\|\partial_tu^\varepsilon\|_{L^2H^2}^2+\|\partial_t^2u^\varepsilon\|_{L^2H^1}^2+\|\partial_t^2u^\varepsilon\|_{L^2H^0(\Sigma_b)}^2\\&\quad+\|\partial_tp^\varepsilon\|_{L^2H^0}^2+\|\partial_tp^\varepsilon\|_{L^2\mathring{H}^1}^2+\|\partial_t^2p^\varepsilon\|_{L^2H^0}^2+\|\partial_t\xi^\varepsilon\|_{L^\infty H^1}^2+\|\partial_t^2\xi^\varepsilon\|_{L^2H^{3/2}}^2\\&\quad+\|\partial_t\xi^\varepsilon\|_{L^2H^{5/2}}^2+\|\partial_t^2\xi^\varepsilon\|_{L^\infty H^1}^2+\|\partial_t\xi^\varepsilon\|_{L^2H^{3/2}}^2\\&\lesssim(C(\varepsilon)T+1)(\mathfrak{K}(\eta)+\mathfrak{E}_0+\sum_{j=0}^{1}(\|\partial_t^jF^1(0)\|_0+\|\partial_t^jF^3(0)\|_{1/2}^2+\|\partial_t^jF^5(0)\|_{1/2}^2))\\&\quad+(1+\mathfrak{E}(\eta))\sum_{j=0}^{1}(\|\partial_t^jF^1\|_{L^2H^0}^2+\|\partial_t^jF^3\|_{L^2H^{1/2}}^2+\|\partial_t^jF^5\|_{L^2H^{1/2}}^2)\\&\quad+(1+\mathfrak{E}(\eta))\|\partial_t^2(F^1-F^3-F^5)\|_{(\mathcal{H}_T^1)^*}^2.\end{aligned}$$

We can directly estimate $\partial_t^3\xi^\varepsilon$ by employing [**5**, Lemma 6] with $p=t=2, s>1/2, \theta=1/(2s)\in(0,1)$, and $r=2/(1-\theta)=4s/(2s-1)$, together with Sobolev embedding theory

$$\begin{aligned}\|\partial_t^3\xi^\varepsilon\|_{H^{1/2}}^2 &= \|\partial_t D_t u^\varepsilon \cdot \mathcal{N} + D_t u^\varepsilon \cdot \partial_t \mathcal{N}\|_{H^{1/2}}^2 \\ &\lesssim \|\partial_t D_t u^\varepsilon\|_1^2 \|\eta\|_{5/2}^2 + \|D_t u^\varepsilon\|_1^2 \|\partial_t \eta\|_{5/2}^2.\end{aligned} \tag{3.98}$$

Then from (3.97), and Lemma 3.8, we see that

$$\|\partial_t^3\xi^\varepsilon\|_{L^2H^{1/2}}^2 \lesssim \mathfrak{K}(\eta)(\|u^\varepsilon(0)\|_0^2 + \|u^\varepsilon\|_{L^2H^1}^2 + \|u^\varepsilon\|_{L^2H^2}^2 + \|\partial_t u^\varepsilon\|_{L^2H^2}^2) \tag{3.99}$$

Thus(3.97) – (3.98) imply
(3.100)

$$\begin{aligned}\mathfrak{D}(u^\varepsilon, p^\varepsilon, \xi^\varepsilon) \le C_0(C(\varepsilon)T + 1)(\mathfrak{K}(\eta) + \mathfrak{E}_0 + \sum_{j=0}^{1}(\|\partial_t^j F^1(0)\|_0 + \|\partial_t^j F^3(0)\|_{1/2}^2 \\ + \|\partial_t^j F^5(0)\|_{1/2}^2)) + C_0(1+\mathfrak{E}(\eta))\sum_{j=0}^{1}(\|\partial_t^j F^1\|_{L^2H^0}^2 + \|\partial_t^j F^3\|_{L^2H^{1/2}}^2 \\ + \|\partial_t^j F^5\|_{L^2H^{1/2}}^2) + C_0(1+\mathfrak{E}(\eta))\|\partial_t^2(F^1 - F^3 - F^5)\|_{(\mathcal{H}_T^1)^*}^2.\end{aligned}$$

Step 2 – Other terms in $\mathfrak{E}$. Arguing as in Lemma 3.4, we may directly derive the bounds

$$\begin{aligned}\|\partial_t u^\varepsilon\|_{L^\infty H^1}^2 &\lesssim \|\partial_t u^\varepsilon(0)\|_{H^1}^2 + \|\partial_t u^\varepsilon\|_{L^2H^1}^2 + \|\partial_t^2 u^\varepsilon\|_{L^2H^1}^2, \\ \|\partial_t p^\varepsilon\|_{L^\infty H^0}^2 &\lesssim \|\partial_t p^\varepsilon(0)\|_{H^1}^2 + \|\partial_t p^\varepsilon\|_{L^2H^1}^2 + \|\partial_t^2 p^\varepsilon\|_{L^2H^1}^2, \\ \|\partial_t \xi^\varepsilon\|_{L^\infty H^{3/2}}^2 &\lesssim \|\partial_t \xi^\varepsilon(0)\|_{H^{3/2}}^2 + \|\partial_t \xi^\varepsilon\|_{L^2H^{3/2}}^2 + \|\partial_t^2 \xi^\varepsilon\|_{L^2H^{3/2}}^2,\end{aligned}$$

which together with Lemma 3.4, Lemma 3.6 a imply that

$$\mathfrak{E}(u^\varepsilon, p^\varepsilon, \xi^\varepsilon) \le \mathfrak{D}(u^\varepsilon, p^\varepsilon, \xi^\varepsilon). \tag{3.101}$$

Then (3.100) and (3.101) imply the conclusion (3.95). □

4. Local well-posedness for the full nonlinear equation

We now consider the local well-posedness of the full problem (1.6). We first construct an approximate solution $(u^\varepsilon, p^\varepsilon, \eta^\varepsilon)$ for (1.6) and for each $\varepsilon > 0$. Then our plan is to let $\varepsilon \to 0$ to obtain the solution of (1.6).

4.1. Initial data. For simplicity, we will typically drop ε in the notation and denote the unknown as (u, p, η) instead of $(u^\varepsilon, p^\varepsilon, \eta^\varepsilon)$. Suppose that $\eta_0 \in H^{5/2}$ satisfying $\|\eta_0\|_{5/2}^2 < \gamma_4$ for $\gamma_4 > 0$ sufficiently small, then we consider the elliptic system

$$\begin{cases}\operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(p(0), u(0)) = -u(0)\cdot\nabla_{\mathcal{A}(0)}u(0) & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}(0)} u(0) = 0 & \text{in } \Omega, \\ \mu\mathbb{D}_{\mathcal{A}(0)}u(0)\mathcal{N}(0)\cdot\mathcal{T}(0) = 0, \quad u(0)\cdot\mathcal{N}(0) = \partial_t\eta(0) & \text{on } \Sigma, \\ (S_{\mathcal{A}(0)}(p(0), u(0))\cdot\nu - \beta u(0))\cdot\tau = 0, \quad u(0)\cdot\nu = 0 & \text{on } \Sigma_b,\end{cases} \tag{4.1}$$

where $\mathcal{T}$ is the tangential of top surface. Now we might use Banach's fixed point theorem to prove the following theorem.

THEOREM 4.1. *Suppose that $\eta_0 \in \mathring{H}^{5/2}(\Sigma)$ and $\partial_t\eta(0) \in \mathring{H}^{3/2}(\Sigma)$, as well as $\mathcal{A}(0)$, $\mathcal{N}(0)$, etc, in terms of η_0. The system (4.1) has a solution $(u(0), p(0)) \in (H^2(\Omega) \cap {}_0H^1(\Omega)) \times (H^1(\Omega) \cap \mathring{H}^0(\Omega))$ satisfying*

$$\|u(0)\|_2^2 + \|u(0)\|_1^2 + \|p(0)\|_1^2 + \|p(0)\|_0^2 \lesssim \|\eta_0\|_{5/2}^2 + \|\partial_t\eta(0)\|_{3/2}^2.$$

PROOF. Let

$$X_\Lambda = \{(u,p) \in {}_0H^1(\Omega) \cap \mathring{H}^0(\Omega) | \|u\|_{H^2}^2 + \|p\|_{H^1}^2 + \|u\|_1^2 + \|p\|_0^2 \le \Lambda\},$$

with the metric

$$d((u,p),(v,q)) = \|u-v\|_1 + \|p-q\|_0.$$

Given $u(0), p(0) \in X_\Lambda$, then $f = -u(0)\cdot\nabla_{\mathcal{A}(0)}u(0) \in L^2(\Omega)$. Now, we define the linear operator $L : H^2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$

Then we consider the following linear system

$$\begin{cases} \operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(q,v) = f & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}(0)} v = 0 & \text{in } \Omega, \\ \mu\mathbb{D}_{\mathcal{A}(0)} v\mathcal{N}(0)\cdot\mathcal{T}(0) = 0, \quad v\cdot\mathcal{N}(0) = \partial_t\eta(0) & \text{on } \Sigma, \\ (S_{\mathcal{A}(0)}(q,v)\cdot\nu - \beta v)\cdot\tau = 0, \quad v\cdot\nu = 0 & \text{on } \Sigma_b, \end{cases} \tag{4.2}$$

which, by employing [**25**, Theorem 3.3], has a unique strong solution $(q,v) \in X_\Lambda$ satisfying

$$\|v\|_2^2 + \|q\|_1^2 \lesssim \|f\|_0^2 + \|h\|_{1/2}^2 \le C(\|u(0)\|_1^4 + \|\eta_0\|_{5/2}^2 + \|\partial_t\eta(0)\|_{3/2}^2). \tag{4.3}$$

Thus for small $\Lambda > 0$ and restricting $\gamma_4 > 0$, we have that $(v,q) \in X_\Lambda$.

Define then operator $A : X_\Lambda \to X_\Lambda$ with $(v,q) = A(u(0),p(0))$. Suppose that for $i = 1,2$

$$\begin{cases} \operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(q^i,v^i) = f^i & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}(0)} v^i = 0 & \text{in } \Omega, \\ \mu\mathbb{D}_{\mathcal{A}(0)} v^i\mathcal{N}(0)\cdot\mathcal{T}(0) = 0, \quad v^i\cdot\mathcal{N}(0) = \partial_t\eta(0) & \text{on } \Sigma, \\ (S_{\mathcal{A}(0)}(q^i,v^i)\cdot\nu - \beta v^i)\cdot\tau = 0, \quad v^i\cdot\nu = 0 & \text{on } \Sigma_b, \end{cases} \tag{4.4}$$

where $f^i = -u^i(0)\cdot\nabla_{\mathcal{A}}u^i(0)$. Then by energy estimates, we have that

$$\|v^1 - v^2\|_1^2 + \beta\|(v^1-v^2)\cdot\tau\|_{H^0(\Sigma_b)}^2 \le C(\|u^1(0)\|_1^2 + \|u^2(0)\|_1^2)(\|u^1(0) - u^2(0)\|_1^2), \tag{4.5}$$

which along with Theorem 2.5 and small $\Lambda > 0$ implies that

$$\|v^1 - v^2\|_1^2 + \|q^1 - q^2\|_0^2 \le \frac{1}{2}(\|u^1(0) - u^2(0)\|_1^2), \tag{4.6}$$

whence $A : X_\Lambda \to X_\Lambda$ is a contraction. Then Banach's fixed point theorem guarantees that there exists a unique solution (u(0),p(0)) satisfying

$$\|u(0)\|_2^2 + \|p(0)\|_1^2 \lesssim \|\eta_0\|_{5/2}^2 + \|\partial_t\eta(0)\|_{3/2}^2. \tag{4.7}$$

□

In order to estimate $\partial_t u(0)$, for $\partial_t^2\eta(0) \in \mathring{H}^1(\Sigma)$ given, we differentiate (1.1) and (1.2) temporally, and then take $t = 0$ to have the following linear system

(4.8)
$$\begin{cases} \operatorname{div}_{\mathcal{A}_0} S_{\mathcal{A}_0}(\partial_t p(0), D_t u(0)) + (D_t u(0)) \cdot \nabla_{\mathcal{A}_0} u_0 + u_0 \cdot \nabla_{\mathcal{A}_0}(D_t u(0)) \\ \qquad = \tilde{F}^1(0) & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}_0} D_t u(0) = 0 & \text{in} \quad \Omega, \\ S_{\mathcal{A}_0}(\partial_t p(0), D_t u(0))\mathcal{N}(0) = g\partial_t\eta(0)\mathcal{N}(0) - \sigma\partial_1^2\partial_t\eta(0)\mathcal{N}(0) \\ \qquad + \tilde{F}^6(0)\mathcal{N}(0) + \tilde{F}^3(0) & \text{on} \quad \Sigma, \\ D_t u(0) \cdot \mathcal{N}(0) = \partial_t^2\eta(0) & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}_0}(\partial_t p(0), D_t u(0))\nu - \beta D_t u(0)) \cdot \tau = \tilde{F}^5, \quad D_t u(0) \cdot \nu = 0 & \text{on} \quad \Sigma_b, \end{cases}$$

where

$$\begin{aligned} \tilde{F}^1(0) &= -\operatorname*{div}_{\partial_t\mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0, u_0) + \mu \operatorname*{div}_{\mathcal{A}(0)} \mathbb{D}_{\partial_t\mathcal{A}(0)} u_0 + \mu \operatorname*{div}_{\mathcal{A}(0)} \mathbb{D}_{\mathcal{A}(0)}(R(0)u_0) \\ &\quad - (R(0)u_0) \cdot \nabla_{\mathcal{A}(0)} u_0 - u_0 \cdot \nabla_{\partial_t\mathcal{A}(0)} u_0, \\ \tilde{F}^4(0) &= \mu\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0)\mathcal{N}(0) + \mu\mathbb{D}_{\partial_t\mathcal{A}(0)} u_0\mathcal{N}(0) \\ &\quad + \left[g\eta_0 - \sigma\partial_1\left(\partial_1\eta_0 + \partial_1\mathcal{R}\right)\right]\partial_t\mathcal{N}(0), \\ \tilde{F}^5(0) &= \mu\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0)\nu \cdot \tau + \mu\mathbb{D}_{\partial_t\mathcal{A}(0)} u_0\nu \cdot \tau + \beta R(0)u_0 \cdot \tau, \\ \tilde{F}^6(0) &= \mathcal{R}'\partial_1\partial_t\eta(0). \end{aligned}$$

Then we have the pressureless weak formulation

(4.9)
$$\begin{aligned} B[D_t u(0), w] &:= \frac{\mu}{2}(D_t u(0), w)_{\mathcal{H}^1} + \beta\int_{\Sigma_b}(D_t u(0) \cdot \tau, w \cdot \tau)J(0) \\ &\quad + (D_t u(0) \cdot \mathcal{N}(0), w \cdot \mathcal{N}(0))_{1,\Sigma} \\ &\quad + \int_\Omega \left[(D_t u(0)) \cdot \nabla_{\mathcal{A}(0)} u_0 + u_0 \cdot \nabla_{\mathcal{A}(0)}(D_t u(0))\right] \cdot wJ(0) \\ &= (\partial_t^2\eta(0), w \cdot \mathcal{N}(0))_{1,\Sigma} - (\partial_t\eta(0), w \cdot \mathcal{N}(0))_{1,\Sigma} - \int_\Sigma \mathcal{R}'\partial_1\partial_t\eta(0)\partial_1(w \cdot \mathcal{N}(0)) \\ &\quad - \int_\Sigma \left[g\eta_0 - \sigma\partial_1\left(\partial_1\eta_0 + \mathcal{R}(\partial_1\eta_0)\right)\right]\partial_t\mathcal{N}(0) \cdot w - \int_{\Sigma_b}\beta(R(0)u_0 \cdot \tau)(w \cdot \tau)J(0) \\ &\quad - \int_\Omega \left(\operatorname*{div}_{\partial_t\mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0, u_0) \cdot w + \frac{\mu}{2}\mathbb{D}_{\partial_t\mathcal{A}(0)} u_0 : \mathbb{D}_{\mathcal{A}(0)} w\right) J(0) \\ &\quad - \int_\Omega \frac{\mu}{2}\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0) : \mathbb{D}_{\mathcal{A}(0)} wJ(0). \end{aligned}$$

Then Lax-Millgram theorem guarantees that there exists a unique $D_t u(0) \in \mathcal{X}(0)$, such that (4.9) holds for each $w \in \mathcal{X}(0)$. Moreover,

$$\|D_t u(0)\|_1^2 \lesssim \|\eta_0\|_{5/2}^2 + \|\partial_t\eta(0)\|_{3/2}^2 + \|\partial_t^2\eta(0)\|_1^2. \tag{4.10}$$

Now from Theorem 2.5, we may recover $\partial_t p(0) \in \mathring{H}^0(\Omega)$ such that

$$\|\partial_t p(0)\|_0^2 \lesssim \|\eta_0\|_{5/2}^2 + \|\partial_t\eta(0)\|_{3/2}^2 + \|\partial_t^2\eta(0)\|_1^2. \tag{4.11}$$

4.2. Existence of approximate solutions. We now construct a sequence of approximate solutions $(u^\varepsilon, p^\varepsilon, \eta^\varepsilon)$ for each $0 < \varepsilon \le 1$. For simplicity, we will typically drop ε in the notation and denote the unknown as (u, p, η) instead of $(u^\varepsilon, p^\varepsilon, \eta^\varepsilon)$.

Now we consider the ε–perturbation problem of the original system (1.6) as

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = -\mu\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = -u\cdot\nabla_{\mathcal{A}} u & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(p,u)\mathcal{N} = g(\eta+\varepsilon\eta_t)\mathcal{N} - \sigma\partial_1\left(\partial_1\eta + \varepsilon\partial_1\eta_t\right)\mathcal{N} - \sigma\partial_1\mathcal{R}\mathcal{N} & \text{on } \Sigma, \\ (S_{\mathcal{A}}(p,u)\nu - \beta u)\cdot\tau = 0, \quad u\cdot\nu = 0 & \text{on } \Sigma_b, \\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Sigma. \end{cases} \tag{4.12}$$

where $\mathcal{A}$, $\mathcal{N}$ are in terms of η^ε and the initial data are $\eta(x_1,0) = \eta_0(x_1)$, $\partial_t\eta(x_1,0)$ and $\partial_t^2\eta(x_1,0)$.

Our strategy is to work in a metric space that requires high regularity estimates to hold but that is endowed with a low-regularity metric. First we will find a complete metric space, endowed with a weak choice of a metric, compatible with the linear estimates in Theorem 3.10. Then we will prove that the fixed point on this metric space gives a solution to (4.12).

We now define the desired metric space.

Definition 4.2. Suppose that $T > 0$. For $\kappa \in (0,\infty)$ we define the space

$$S(T,\kappa) = \Big\{(u,p,\eta) \in L^2H^1 \times L^2\mathring{H}^0 \times L^\infty H^{5/2} \Big| \mathfrak{K}(u,p,\eta)^{1/2} \le \kappa \text{ and } (u,p,\eta) \text{ achieve the initial data as Section 4.1}\Big\}. \tag{4.13}$$

We endow this space with the metric

$$d((u,p,\eta),(v,q,\xi)) = \|u-v\|_{L^2H^1} + \|p-q\|_{L^2\mathring{H}^0} + \|\eta-\xi\|_{L^\infty H^{5/2}}, \tag{4.14}$$

where here the temporal norm is evaluated on the set $[0,T]$.

It is obvious that $S(T,\kappa)$ is a complete metric space, so that we could admit the contraction mapping principle. Then we employ the metric space $S(T,\kappa)$ and a contraction mapping argument to produce a solution to (4.12).

Theorem 4.3. *There exists a constant $C > 0$ such that for each $0 < \varepsilon \le \min\{1, 1/(8C)\}$ there exists a unique solution $(u^\varepsilon, p^\varepsilon, \eta^\varepsilon)$ to (4.12) belong to the metric space $S(T_\varepsilon, \kappa)$, where $T_\varepsilon > 0$ and $\sigma > 0$ are sufficiently small.*

Proof. Throughout the proof $P(\cdot)$ denotes a polynomial such that $P(0) = 0$, which is allowed to be changed from line to line.

Step 1 – The metric space. Suppose that $\mathfrak{K}(\eta) \le \gamma_0$ is sufficiently small. For $F^1 = -u\cdot\nabla_{\mathcal{A}} u$, $F^3 = 0$ and $F^5 = 0$ in Theorem 3.10, let $T_\varepsilon > 0$ and the initial data small enough such that $C_0(C(\varepsilon)T+1)(\mathfrak{K}(\eta) + \mathfrak{E}_0 + \sum_{j=0}^1(\|\partial_t^j F^1(0)\|_0 + \|\partial_t^j F^3(0)\|_{1/2}^2 + \|\partial_t^j F^5(0)\|_{1/2}^2)) < \gamma_0/2$. Then we take γ_0 small enough such that

$$C_0\mathfrak{E}(\eta)\mathfrak{K}(\eta) + C_0(1+\mathfrak{E}(\eta))\sum_{j=0}^1(\|\partial_t^j F^1\|_{L^2H^0}^2 + \|\partial_t^j F^3\|_{L^2H^{1/2}}^2 + \|\partial_t^j F^5\|_{L^2H^{1/2}}^2) + C_0(1+\mathfrak{E}(\eta))\|\partial_t^2(F^1 - F^3 - F^5)\|_{(\mathcal{H}_T^1)^*}^2 < \gamma_0/2.$$

Then we take $\kappa \le {\gamma_0}^{1/2}$. For every $(u,p,\eta) \in S(T_\varepsilon,\kappa)$, let $(\tilde{u},\tilde{p},\tilde{\eta})$ be the unique solution of the linear problem of

(4.15)
$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(\tilde{p},\tilde{u}) = -\mu\Delta_{\mathcal{A}}\tilde{u} + \nabla_{\mathcal{A}}\tilde{p} = -u\cdot\nabla_{\mathcal{A}}u & \text{in } \Omega,\\ \operatorname{div}_{\mathcal{A}}\tilde{u} = 0 & \text{in } \Omega,\\ S_{\mathcal{A}}(\tilde{p},\tilde{u})\mathcal{N} = g(\tilde{\eta}+\varepsilon\partial_t\tilde{\eta})\mathcal{N} - \sigma\partial_1\left(\partial_1\tilde{\eta}+\varepsilon\partial_1\partial_t\tilde{\eta}\right)\mathcal{N} - \sigma\partial_1\mathcal{R}\mathcal{N} & \text{on } \Sigma,\\ (S_{\mathcal{A}}(\tilde{p},\tilde{u})\nu - \beta\tilde{u})\cdot\tau = 0, \quad \tilde{u}\cdot\nu = 0 & \text{on } \Sigma_b,\\ \partial_t\tilde{\eta} = \tilde{u}\cdot\mathcal{N} & \text{on } \Sigma, \end{cases}$$

where $\mathcal{A}$ and $\mathcal{N}$ are in terms of η, and the initial data $\tilde{\eta}(0) = \eta_0$, $\partial_t\tilde{\eta}(0) = \partial_t\eta(0)$ and $\partial_t^2\tilde{\eta}(0) = \partial_t^2\eta(0)$. By Theorem 3.10 we have the estimate

$$\mathfrak{K}(\tilde{u},\tilde{p},\tilde{\eta}) \le \kappa^2, \tag{4.16}$$

which implies that

$$(\tilde{u},\tilde{p},\tilde{\eta}) \in S(T_\varepsilon,\kappa). \tag{4.17}$$

Step 2 – Contraction. Define $A : (u,p,\eta) = (\tilde{u},\tilde{p},\tilde{\eta})$. Now we prove that $A : S(T_\varepsilon,\kappa) \to S(T_\varepsilon,\kappa)$ is a strict contraction mapping with the metric in the Definition 4.2. Choose $(u^i,p^i,\eta^i) \in S(T_\varepsilon,\kappa)$, and define $A(u^i,p^i,\eta^i) = (\tilde{u}^i,\tilde{p}^i,\tilde{\eta}^i)$ as above, $i = 1,2$. For simplicity, we will abuse notation and denote $u = u^1 - u^2$, $p = p^1 - p^2$, $\eta = \eta^1 - \eta^2$ and the same for $\tilde{u},\tilde{p},\tilde{\eta}$. From the difference of equation for $(\tilde{u}^i,\tilde{p}^i,\tilde{\eta}^i)$, $i = 1,2$, we know that

$$\begin{cases} \underset{\mathcal{A}^1}{\operatorname{div}} S_{\mathcal{A}^1}(\tilde{p},\tilde{u}) = \mu\underset{\mathcal{A}^1}{\operatorname{div}}(\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2) + R^1 & \text{in } \Omega,\\ \underset{\mathcal{A}^1}{\operatorname{div}}\tilde{u} = R^2 & \text{in } \Omega,\\ S_{\mathcal{A}^1}(\tilde{p},\tilde{u})\mathcal{N}^1 = \mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\mathcal{N}^1 + g(\tilde{\eta}+\varepsilon\partial_t\tilde{\eta})\mathcal{N}^1 & \\ \qquad -\sigma\partial_1\left(\partial_1\tilde{\eta}+\varepsilon\partial_1\partial_t\tilde{\eta}\right)\mathcal{N}^1 - \sigma\partial_1\mathcal{R}\mathcal{N}^1 + R^3 & \text{on } \Sigma,\\ (S_{\mathcal{A}^1}(\tilde{p},\tilde{u})\nu - \beta\tilde{u})\cdot\tau = \mu\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2\nu\cdot\tau, \quad \tilde{u}\cdot\nu = 0 & \text{on } \Sigma_b,\\ \partial_t\tilde{\eta} = \tilde{u}\cdot\mathcal{N}^1 + R^5 & \text{on } \Sigma, \end{cases} \tag{4.18}$$

where R^1, R^2, R^3, R^4, R^5 are defined by

$$\begin{aligned} R^1 &= \mu\underset{(\mathcal{A}^1-\mathcal{A}^2)}{\operatorname{div}}(\mathbb{D}_{\mathcal{A}^2}\tilde{u}^2) - \nabla_{(\mathcal{A}^1-\mathcal{A}^2)}\tilde{p}^2,\\ R^2 &= -\underset{(\mathcal{A}^1-\mathcal{A}^2)}{\operatorname{div}}\tilde{u}^2,\\ R^3 &= -\tilde{p}^2(\mathcal{N}^1-\mathcal{N}^2) + \mathbb{D}_{\mathcal{A}^2}\tilde{u}^2(\mathcal{N}^1-\mathcal{N}^2) + g(\tilde{\eta}^2+\varepsilon\partial_1\partial_t\tilde{\eta}^2)(\mathcal{N}^1-\mathcal{N}^2)\\ &\quad - \sigma\partial_1\left(\partial_1\tilde{\eta}^2+\varepsilon\partial_1\partial_t\tilde{\eta}^2\right)(\mathcal{N}^1-\mathcal{N}^2) - \sigma\partial_1\mathcal{R}^2(\mathcal{N}^1-\mathcal{N}^2),\\ R^5 &= \tilde{u}^2\cdot(\mathcal{N}^1-\mathcal{N}^2), \end{aligned}$$

and $\mathcal{A}^i$, $\mathcal{N}^i$, $\mathcal{R}^i = \mathcal{R}(\partial_1\eta^i)$ are in terms of η^i, $i = 1,2$. Here $\mathcal{R} = \mathcal{R}^1 - \mathcal{R}^2$.

We now have the pressureless weak formulation of (4.18) as

$$\begin{aligned} &\frac{\mu}{2}\int_\Omega \mathbb{D}_{\mathcal{A}^1}\tilde{u} : \mathbb{D}_{\mathcal{A}^1}wJ^1 + \beta\int_{\Sigma_b} J^1(\tilde{u}\cdot\tau)(w\cdot\tau) + (\tilde{\eta}+\varepsilon\partial_t\tilde{\eta}, w\cdot\mathcal{N}^1)_{1,\Sigma}\\ &= \mu\int_\Omega \mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2 : \mathbb{D}_{\mathcal{A}^1}wJ^1 + \int_\Omega R^1\cdot wJ^1 - \int_\Sigma \sigma\mathcal{R}\partial_1(w\cdot\mathcal{N}^1) + R^3\cdot w, \end{aligned} \tag{4.19}$$

for each $w \in \mathcal{X}(t)$. Then according to Theorem 2.5, there exists a unique $\tilde{p} \in \mathring{H}^0(\Omega)$ such that

(4.20)

$$\frac{\mu}{2}\int_\Omega \mathbb{D}_{\mathcal{A}^1}\tilde{u} : \mathbb{D}_{\mathcal{A}^1} w J^1 + \beta\int_{\Sigma_b} J^1(\tilde{u}\cdot\tau)(w\cdot\tau) - (\tilde{p}, \operatorname*{div}_{\mathcal{A}^1} w)_0 + (\tilde{\eta}+\varepsilon\partial_t\tilde{\eta}, w\cdot\mathcal{N}^1)_{1,\Sigma}$$
$$= \mu\int_\Omega \mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2 : \mathbb{D}_{\mathcal{A}^1} w J^1 + \int_\Omega R^1\cdot w J^1 - \int_\Sigma \sigma\mathcal{R}\partial_1(w\cdot\mathcal{N}^1) + R^3\cdot w,$$

for each $w \in \mathcal{W}(t)$. Moreover,

$$\|\tilde{p}\|_0 \lesssim \|\tilde{u}\|_1 + \|\eta\|_{3/2}\left((\|\eta^1\|_{3/2}+\|\eta^2\|_{3/2})\|\tilde{u}^2\|_2 + \|\tilde{p}^2\|_1\right). \tag{4.21}$$

Multiplying the first equation of (4.18) by $\tilde{u}J^1$ and integrating by parts reveals that

$$\begin{aligned}
&\partial_t\left(\int_\Sigma \frac{g}{2}|\tilde{\eta}|^2 + \frac{\sigma}{2}|\partial_1\tilde{\eta}|^2\right) + \varepsilon\int_\Sigma g|\partial_t\tilde{\eta}|^2 + \sigma|\partial_1\partial_t\tilde{\eta}|^2 + \frac{\mu}{2}\int_\Omega |\mathbb{D}_{\mathcal{A}^1}\tilde{u}|^2 J^1\\
&= \frac{\mu}{2}\int_\Omega \mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}\tilde{u}^2 : \mathbb{D}_{\mathcal{A}^1}\tilde{u}J^1 + \int_\Omega R^1\cdot\tilde{u}J^1 + \tilde{p}R^2J^1 - \int_{\Sigma_b} J^1(\tilde{u}\cdot\tau)R^4\\
&\quad - \int_\Sigma \sigma\mathcal{R}\partial_1(\tilde{u}\cdot\mathcal{N}^1) + R^3\cdot\tilde{u} - g(\tilde{\eta}+\varepsilon\partial_t\tilde{\eta})R^5 - \sigma\partial_1(\tilde{\eta}+\varepsilon\partial_t\tilde{\eta})\partial_1 R^5.
\end{aligned} \tag{4.22}$$

We will now estimate the terms in right-hand side of (4.22). First:

$$\begin{aligned}
&\int_\Omega R^1\cdot\tilde{u}J^1 + \tilde{p}R^2J^1\\
&= \int_\Omega \left(\operatorname*{div}_{(\mathcal{A}^1-\mathcal{A}^2)}(\mathbb{D}_{\mathcal{A}^2}\tilde{u}^2) - \nabla_{(\mathcal{A}^1-\mathcal{A}^2)}\tilde{p}^2\right)\cdot\tilde{u}J^1 + \tilde{p}\operatorname*{div}_{(\mathcal{A}^1-\mathcal{A}^2)}\tilde{u}^2J^1\\
&\lesssim \int_\Omega |\nabla\bar{\eta}|\left(|\nabla^2\bar{\eta}^2||\nabla\tilde{u}^2| + |\nabla\bar{\eta}^2||\nabla^2\tilde{u}^2| + |\nabla\tilde{p}^2|\right)|\tilde{u}| + |\tilde{p}||\nabla\bar{\eta}||\nabla\tilde{u}^2|\\
&\lesssim \left(\|\tilde{u}\|_1\|\eta^2\|_{5/2} + \|\tilde{p}\|_0\right)\|\eta\|_{3/2}\|\eta^1\|_{5/2}\|\tilde{u}^2\|_2,
\end{aligned}$$

Now we consider the integrals on Σ. We know that $\mathcal{N}^1-\mathcal{N}^2 = (-\partial_1\eta, 0)$ and $\partial_1\mathcal{R}^2 = (\mathcal{R})'\partial_1^2\eta^2$, where $|\mathcal{R}'| \lesssim |\partial_1\eta^2|$. Then we use Hölder inequality, Sobolev inequality and trace theory to derive that

$$\begin{aligned}
\int_\Sigma R^3\cdot\tilde{u} = &\int_\Sigma \big[-\tilde{p}^2(\mathcal{N}^1-\mathcal{N}^2) + \mathbb{D}_{\mathcal{A}^2}\tilde{u}^2(\mathcal{N}^1-\mathcal{N}^2) + g(\tilde{\eta}^2+\varepsilon\partial_1\partial_t\tilde{\eta}^2)(\mathcal{N}^1-\mathcal{N}^2)\\
&- \sigma\partial_1\left(\partial_1\tilde{\eta}^2+\varepsilon\partial_1\partial_t\tilde{\eta}^2\right)(\mathcal{N}^1-\mathcal{N}^2) - \sigma\partial_1\mathcal{R}^2(\mathcal{N}^1-\mathcal{N}^2)\big]\cdot\tilde{u}\\
\lesssim &\Bigg(\|\tilde{p}^2\|_0\|\eta\|_{3/2} + \|\eta^2\|_{3/2}\|\tilde{u}^2\|_{W^2_\delta}\|\eta\|_{3/2} + \|\tilde{\eta}^2+\epsilon\partial_t\tilde{\eta}^2\|_{5/2}\|\eta\|_{3/2}\\
&+ \|\eta^2\|_{3/2}^2\|\eta\|_{3/2} + \|\eta^2\|_{3/2}\|\eta^2\|_{5/2}\|\eta\|_{3/2}\Bigg)\|\tilde{u}\|_1
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Sigma \sigma\mathcal{R}\partial_1(\tilde{u}\cdot\mathcal{N}^1) - g(\tilde{\eta}+\varepsilon\partial_t\tilde{\eta})R^5 - \sigma\partial_1(\tilde{\eta}+\epsilon\partial_t\tilde{\eta})\partial_1R^5\\
&\lesssim \|\eta\|_{3/2}\left(\|\eta\|_{3/2}+\|\eta^2\|_{3/2}\right)\|\partial_t\tilde{\eta}\|_1 + \|\tilde{u}\|_1\|\tilde{u}^2\|_1\|\eta\|_{3/2}\|\eta^1\|_{3/2}\\
&\quad + \|\tilde{\eta}^2+\varepsilon\partial_t\tilde{\eta}^2\|_{5/2}\|\tilde{u}\|_1\|\eta\|_{3/2} + \|\eta^2\|_{3/2}\|\eta\|_{3/2} + \|\eta^2\|_{3/2}^2\|\eta\|_{3/2}\\
&\quad + \|\tilde{\eta}+\varepsilon\partial_t\tilde{\eta}\|_1\left(\|\eta\|_{3/2}\|\tilde{u}^2\|_2 + \|\tilde{u}^2\|_1\|\eta\|_{5/2}\right),
\end{aligned}$$

Then the Cauchy-Schwarz inequality, Sobolev embedding theorem and Gronwall's inequality imply that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\tilde\eta\|_1^2+\varepsilon\int_0^T\|\partial_t\tilde\eta\|_1^2+\int_0^T\|\tilde u\|_1^2\\
&\lesssim\int_0^T\|\eta\|_{5/2}^2\Big((\|\eta^1\|_{5/2}^2+\|\eta^2\|_{5/2}^2)\|\tilde u^2\|_2^2+\|\eta^1\|_{3/2}^2+\|\eta^2\|_{3/2}^2+\|\tilde p^2\|_0^2\Big)\\
&\quad+\int_0^T\|\tilde\eta^2+\varepsilon\partial_t\tilde\eta^2\|_{5/2}^2\|\eta\|_{3/2}^2+\|\eta\|_{5/2}^2\|\tilde u^2\|_2^2.
\end{aligned}\tag{4.23}$$

From the weak formulation (4.19) and Theorem 2.6, we have that

$$\begin{aligned}
\|\tilde\eta+\varepsilon\partial_t\tilde\eta\|_{3/2}^2&\lesssim\|\tilde u\|_1^2+\|\eta\|_{3/2}^2(\|\eta^1\|_{5/2}^2\|\tilde u^2\|_2^2+\|\tilde p^2\|_1^2\\
&\quad+\|\tilde\eta^2+\varepsilon\partial_t\tilde\eta^2\|_{5/2}^2)+\|F^3\|_{1/2}^2\\
&\lesssim\|\tilde u\|_1^2+\|\eta\|_{3/2}^2(\|\eta^1\|_{5/2}^2\|\tilde u^2\|_2^2+\|\tilde p^2\|_1^2+\|\tilde\eta^2+\varepsilon\partial_t\tilde\eta^2\|_{5/2}^2)\\
&\quad+\|\eta\|_{3/2}^2(\|\eta\|_{3/2}^2+\|\tilde u^2\|_2^2).
\end{aligned}\tag{4.24}$$

Since

$$\tilde\eta=\frac{1}{\varepsilon}\int_0^t e^{-\frac{t-s}{\varepsilon}}(\tilde\eta+\varepsilon\partial_t\tilde\eta),\tag{4.25}$$

thus

$$\partial_t\tilde\eta=\frac{1}{\varepsilon}(\tilde\eta+\varepsilon\partial_t\tilde\eta)-\frac{1}{\varepsilon^2}\int_0^t e^{-\frac{t-s}{\varepsilon}}(\tilde\eta+\varepsilon\partial_t\tilde\eta),$$

then we have

$$\begin{aligned}
\|\partial_t\tilde\eta\|_{L^2H^{3/2}}^2&\le\frac{1}{\varepsilon^2}\|\tilde\eta+\varepsilon\partial_t\tilde\eta\|_{L^2H^{3/2}}^2+\int_0^T\left(\frac{1}{\varepsilon^2}\int_0^t e^{-\frac{t-s}{\varepsilon}}\|\tilde\eta+\varepsilon\partial_t\tilde\eta\|_{3/2}\right)^2\\
&\lesssim C(\varepsilon)\|\tilde\eta+\varepsilon\partial_t\tilde\eta\|_{L^2H^{3/2}}^2\lesssim C(\varepsilon)P(\kappa)T\|\eta\|_{L^\infty H^{5/2}}^2.
\end{aligned}\tag{4.26}$$

From Theorem 2.7, we have that

$$\begin{aligned}
&\|\tilde u\|_2^2+\|\tilde p\|_1^2\\
&\lesssim\|-\mu\,\mathrm{div}_{\mathcal A^1}(\mathbb D_{\mathcal A^1-\mathcal A^2}\tilde u^2)+R^1\|_0^2+\|R^2\|_1^2+\|\partial_t\tilde\eta-R^5\|_{3/2}^2\\
&\quad+\|\mu\mathbb D_{\mathcal A^1-\mathcal A^2}\tilde u^2\mathcal N^1+R^3\|_{1/2}^2+\|\mu\mathbb D_{\mathcal A^1-\mathcal A^2}\tilde u^2\nu\cdot\tau\|_{1/2}^2\\
&\lesssim\|\eta\|_{5/2}^2\left(\Big(1+\|\eta^1\|_{5/2}^2+\|\eta^2\|_{5/2}^2\Big)\|\tilde u^2\|_2^2+\|\tilde p^2\|_1^2\right)+\|\partial_t\tilde\eta\|_{3/2}^2\\
&\quad+\|\eta\|_{5/2}^2\Big(\|\tilde\eta^2+\varepsilon\partial_t\tilde\eta^2\|_{5/2}^2+\|\eta^1\|_{5/2}^2+\|\eta^2\|_{5/2}^2\Big),
\end{aligned}\tag{4.27}$$

then the Theorem 2.7 with $F^6=\mathcal R$ implies

$$\begin{aligned}
&\|\tilde\eta+\varepsilon\partial_t\tilde\eta\|_{5/2}^2\\
&\lesssim\|\tilde u\|_2^2+\|\tilde p\|_1^2+\|\partial_1\mathcal R\|_{1/2}^2+\|\mu\mathbb D_{\mathcal A^1-\mathcal A^2}\tilde u^2\mathcal N^1+R^3\|_{1/2}^2\\
&\lesssim\|\eta\|_{5/2}^2\left(\Big(1+\|\eta^1\|_{5/2}^2+\|\eta^2\|_{5/2}^2\Big)\|\tilde u^2\|_2^2+\|\tilde p^2\|_1^2\right)+\|\partial_t\tilde\eta\|_{3/2}^2\\
&\quad+\|\eta\|_{5/2}^2\Big(\|\tilde\eta^2+\varepsilon\partial_t\tilde\eta^2\|_{5/2}^2+\|\eta^1\|_{5/2}^2+\|\eta^2\|_{5/2}^2\Big).
\end{aligned}\tag{4.28}$$

Combining (4.23)–(4.26), the Cauchy-Schwarz inequality, Sobolev embeddings, and the linear estimates of Theorem 3.5 and 3.10 then show that

$$\begin{aligned}\|\tilde{\eta}\|^2_{L^\infty H^{5/2}} &\le C(\varepsilon)T\|\tilde{\eta}+\varepsilon\partial_t\tilde{\eta}\|^2_{L^2H^{5/2}} \\ &\lesssim C(\varepsilon)TP(\|\eta^1\|^2_{L^\infty H^{5/2}}, \|\eta^2\|^2_{L^\infty H^{5/2}}, \|\tilde{u}^2\|^2_{L^2H^2}, \|\tilde{p}^2\|^2_{L^2H^1})\|\eta\|^2_{L^\infty H^{5/2}} \\ &\lesssim C(\varepsilon)TP(\kappa)\|\eta\|^2_{L^\infty H^{5/2}},\end{aligned} \tag{4.29}$$

where the first inequality is obtained by (4.25) and the second inequality used the fact that $\|\tilde{\eta}^2+\varepsilon\partial_t\tilde{\eta}^2\|^2_{L^2H^{5/2}} \le \kappa$ which is included in the proof of Theorem 3.5.

Then (4.21) and (4.23) imply that

$$\|\tilde{u}\|^2_{L^2H^1}+\|\tilde{p}\|^2_{L^2\mathring{H}^0} \lesssim P(\kappa)\|\eta\|^2_{L^\infty H^{5/2}}. \tag{4.30}$$

Hence (4.29) and (4.30) reveals that

$$\|\tilde{u}\|^2_{L^2H^1}+\|\tilde{p}\|^2_{L^2\mathring{H}^0}+\|\tilde{\eta}\|^2_{L^\infty H^{5/2}} \le (C(\varepsilon)T+C)P(\kappa)\|\eta\|^2_{L^\infty H^{5/2}}, \tag{4.31}$$

where C is a universal constant independent of ε.

We may restrict σ such that $CP(\sigma)\le 1/8$. For each $0<\varepsilon\le 1/(8CP(\sigma))$, we choose $T'>0$ such that $C(\varepsilon)T'P(\sigma)\le 1/8$. This implies

$$\begin{aligned}d(A(u^1,p^1,\eta^1), A(u^2,p^2,\eta^2)) &= d((\tilde{u}^1,\tilde{p}^1,\tilde{\eta}^1),(\tilde{u}^2,\tilde{p}^2,\tilde{\eta}^2)) \\ &\le \frac{1}{2}d((u^1,p^1,\eta^1),(u^2,p^2,\eta^2)).\end{aligned} \tag{4.32}$$

Finally we see that A is a strict contraction on $S(T_\varepsilon,\kappa)$. Since the metric space $S(T_\varepsilon,\kappa)$ is complete, the contraction mapping principle reveals the existence of a unique $(u,p,\eta)\in S(T_\varepsilon,\kappa)$ such that $A(u,p,\eta)=(\tilde{u},\tilde{p},\tilde{\eta})=(u,p,\eta)$. □

4.3. Existence of solutions. In this section, we consider the solution of original problem (1.6). For $(u^\varepsilon,p^\varepsilon,\eta^\varepsilon)$ solving (4.12) obtained in Theorem 4.3, we want to send $\varepsilon\to 0$ to get a uniform $T>0$ independent of ϵ, so we need some uniform estimates. Fortunately, this could be easily done due to [**25**, Theorem 4.12]. In [**25**], the author gives the a priori estimate for (1.6), and the ε terms in (4.12) has no effect on a priori estimate. So the details of proof of the following theorem is the same as [**25**, Theorem 4.12] using a standard continuity argumet, in which $\mathcal{E}^\varepsilon(t)$ and $\mathcal{D}^\varepsilon(t)$ denotes the modified energy and dissipation with (u,p,η) replaced by $(u^\varepsilon,p^\varepsilon,\eta^\varepsilon)$ in $\mathcal{E}(t)$ and $\mathcal{D}(t)$ respectively.

Theorem 4.4. *There exists a universal constant C and for any large $T>0$ independent of ε such that for each $\varepsilon>0$ sufficiently small,*

$$\sup_{0\le t\le T}\mathcal{E}^\varepsilon(t)+\int_0^T\mathcal{D}^\varepsilon(t)\,\mathrm{d}t \le C. \tag{4.33}$$

Then we present the main result of this section.

Theorem 4.5. *There exists a solution (u,p,η) solving the equation* (1.6).

Proof. According to the energy estimate in Theorem 4.4, there exists a sequence ε_k tending to zero and a pair (u,p,η)

$$\begin{cases}(u^{\varepsilon_k},p^{\varepsilon_k},\eta^{\varepsilon_k}) \overset{*}{\rightharpoonup} (u,p,\eta) & \text{weakly-}* \text{ in the spaces of energy as (3.68)},\\ (u^{\varepsilon_k},p^{\varepsilon_k},\eta^{\varepsilon_k}) \rightharpoonup (u,p,\eta) & \text{weakly in the spaces of dissipation as (3.69)}.\end{cases} \tag{4.34}$$

Choose a function $w \in \mathcal{W}$, then from the weak formulation, we deduce that

$$\begin{aligned}
&\int_0^T \int_\Sigma g\eta^{\varepsilon_k}(w\cdot \mathcal{N}^{\varepsilon_k}) + \sigma\partial_1\eta^{\varepsilon_k}\partial_1(w\cdot\mathcal{N}^{\varepsilon_k}) \\
&\quad + \varepsilon_k \int_0^T \int_\Sigma g\partial_t\eta^{\varepsilon_k}(w\cdot \mathcal{N}^{\varepsilon_k}) + \sigma\partial_1\partial_t\eta^{\varepsilon_k}\partial_1(w\cdot\mathcal{N}^{\varepsilon_k}) \\
&\quad + \int_0^T \int_\Omega \frac{\mu}{2}\mathbb{D}_{\mathcal{A}^{\varepsilon_k}} u^{\varepsilon_k} : \mathbb{D}_{\mathcal{A}^{\varepsilon_k}} w J^{\varepsilon_k} + \int_0^T \int_{\Sigma_b} \beta(u^{\varepsilon_k}\cdot\tau)(w\cdot\tau)J^{\varepsilon_k} \\
&\quad - \int_0^T \int_\Omega p^{\varepsilon_k} \operatorname{div}_{\mathcal{A}^{\varepsilon_k}} w J^{\varepsilon_k} \\
&= -\sigma\int_0^T \int_\Sigma \mathcal{R}(\partial_1\eta^{\varepsilon_k})\partial_1(w\cdot\mathcal{N}^{\varepsilon_k}) - \int_\Omega (u^{\varepsilon_k}\cdot\nabla_{\mathcal{A}^{\varepsilon_k}} u^{\varepsilon_k})\cdot wJ^{\varepsilon_k}.
\end{aligned}$$

Passing the limit $\varepsilon_k \to 0$, the convergence (4.34) reveals that

$$\begin{aligned}
&\int_0^T \int_\Sigma g\eta(w\cdot\mathcal{N}) + \sigma\partial_1\eta\partial_1(w\cdot\mathcal{N}) + \int_0^T\int_\Omega \frac{\mu}{2}\mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} wJ \\
&\qquad + \int_0^T \int_{\Sigma_b} \beta(u\cdot\tau)(w\cdot\tau)J - \int_0^T\int_\Omega p \operatorname{div}_{\mathcal{A}} wJ \\
&\qquad = -\sigma\int_0^T\int_\Sigma \mathcal{R}(\partial_1\eta)\partial_1(w\cdot\mathcal{N}) - \int_\Omega (u\cdot\nabla_{\mathcal{A}} u)\cdot wJ.
\end{aligned}$$

Thus the limit (u, p, η) is a weak solution of (1.6). Then integrating by parts,
(4.35)

$$\begin{aligned}
&\int_0^T \int_\Sigma g\eta(w\cdot\mathcal{N}) - \sigma\partial_1\left(\partial_1\eta + \mathcal{R}(\partial_1\eta)\right) w\cdot\mathcal{N} - \int_0^T\int_\Omega \mu(\Delta_{\mathcal{A}} u)wJ \\
&+ \int_0^T\int_\Sigma \mu\mathbb{D}_{\mathcal{A}} u\mathcal{N}\cdot w + \int_0^T\int_{\Sigma_b} \mu\mathbb{D}_{\mathcal{A}} u\nu\cdot w + \beta(u\cdot\tau)(w\cdot\tau)J + \int_0^T\int_\Omega \nabla_{\mathcal{A}} p\cdot wJ \\
&\qquad - \int_0^T\int_\Sigma p\mathcal{N}\cdot w + \int_\Omega (u\cdot\nabla_{\mathcal{A}} u)\cdot wJ = 0,
\end{aligned}$$

we know that (u, p, η) satisfy the boundary condition of (1.6). In addition, we could use the same argument as Section 4.1 to construct the initial data of (4.12) and then let $\varepsilon \to 0$ to obtain that the limits are consistent with results in Section 4.1. Thus (u, p, η) is a strong solution of (1.6) because of its regularity. □

4.4. Uniqueness. We refer to velocities as u^j, pressures as p^j, surface functions as η^j, for $j = 1, 2$.

THEOREM 4.6. *Let u^1, u^2, p^1, p^2 and η^1, η^2 satisfy*
(4.36)

$$\sup_{0\le t\le T} \{\mathcal{E}(u^1,p^1,\eta^1), \mathcal{E}(u^2,p^2,\eta^2)\} < \delta, \quad \textit{and} \quad \int_0^T \{\mathcal{D}(u^1,p^1,\eta^1), \mathcal{D}(u^2,p^2,\eta^2)\} < \delta,$$

with $T > 0$. Suppose that for $j = 1, 2$,

$$\begin{cases} -\mu\Delta_{\mathcal{A}^j} u^j + \nabla_{\mathcal{A}^j} p^j = -u^j \cdot \nabla_{\mathcal{A}^j} u^j & in \quad \Omega, \\ \operatorname{div}_{\mathcal{A}^j} u^j = 0 & in \quad \Omega, \\ S_{\mathcal{A}^j}(p^j, u^j)\mathcal{N}^j = g\eta^j \mathcal{N}^j - \sigma\partial_1\left(\partial_1\eta^j + \mathcal{R}^j\right)\mathcal{N}^j & on \quad \Sigma, \\ \left(S_{\mathcal{A}^j}(p^j, u^j)\nu - \beta u^j\right)\cdot\tau = 0, \quad u^j \cdot \nu = 0 & on \quad \Sigma_b, \\ \partial_t \eta^j = u^j \cdot \mathcal{N}^j & on \quad \Sigma. \end{cases} \tag{4.37}$$

where $\mathcal{A}^j$, $\mathcal{N}^j$, $\mathcal{R}^j$ are determined by η^j as usual. Suppose that $\partial_t^k \eta^1(0) = \partial_t^k \eta^2(0)$ for $k = 0, 1$.

Then there exist $\delta_1 > 0$ such that if $0 < \delta \le \delta_1$, then

$$u^1 = u^2, \quad p^1 = p^2, \quad \eta^1 = \eta^2 \tag{4.38}$$

on $[0, T]$.

Proof. First, we define $v = u^1 - u^2$, $q = p^1 - p^2$, $\theta = \eta^1 - \eta^2$ and derive the PDEs satisfied by v, q, θ. We still use $\mathcal{R}$ to denote $\mathcal{R} = \mathcal{R}^1 - \mathcal{R}^2$.

Step 1 – PDEs and energy for differences. Subtracting equations in (4.37) with $j = 2$ from the same equations with $j = 1$, we can write the resulting equations in terms of v, q, θ as

$$\begin{cases} \operatorname{div}_{\mathcal{A}^1} S_{\mathcal{A}^1}(q, v) = \mu \operatorname{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\right) + H^1 & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}^1} v = H^2 & \text{in} \quad \Omega, \\ S_{\mathcal{A}^1}(q, v)\mathcal{N}^1 = \mu\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\mathcal{N}^1 + g\theta\mathcal{N}^1 - \sigma\partial_1^2\theta\mathcal{N}^1 - \sigma\partial_1\mathcal{R}\mathcal{N}^1 + H^3 & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}^1}(q, v)\nu - \beta v)\cdot\tau = \mu\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\nu\cdot\tau, \quad v\cdot\nu = 0 & \text{on} \quad \Sigma_b, \\ \partial_t\theta = v\cdot\mathcal{N}^1 + H^5 & \text{on} \quad \Sigma, \end{cases} \tag{4.39}$$

where H^1, H^2, H^3, H^4, H^5 are defined by

$$\begin{aligned} H^1 &= \mu \operatorname{div}_{(\mathcal{A}^1-\mathcal{A}^2)}(\mathbb{D}_{\mathcal{A}^2}u^2) - \nabla_{(\mathcal{A}^1-\mathcal{A}^2)}p^2 - u^1\cdot\nabla_{\mathcal{A}^1}u^1 + u^2\cdot\nabla_{\mathcal{A}^2}u^2 - v\cdot\nabla_{\mathcal{A}^1}u^1 \\ &\quad - u^2\cdot\nabla_{\mathcal{A}^1-\mathcal{A}^2}u^2 - u^2\cdot\nabla_{\mathcal{A}^2}v, \\ H^2 &= -\operatorname{div}_{(\mathcal{A}^1-\mathcal{A}^2)} u^2, \\ H^3 &= -p^2(\mathcal{N}^1-\mathcal{N}^2) + \mathbb{D}_{\mathcal{A}^1}u^2(\mathcal{N}^1-\mathcal{N}^2) - \mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}u^2\mathcal{N}^2 + g\eta^2(\mathcal{N}^1-\mathcal{N}^2) \\ &\quad - \sigma\partial_1^2\eta^2(\mathcal{N}^1-\mathcal{N}^2) - \sigma\partial_1\mathcal{R}^2(\mathcal{N}^1-\mathcal{N}^2), \\ H^5 &= u^2\cdot(\mathcal{N}^1-\mathcal{N}^2). \end{aligned}$$

The solutions are sufficiently regular for us to differentiate (4.39) in time, which results in the equations

(4.40)
$$\begin{cases} \operatorname{div}_{\mathcal{A}^1} S_{\mathcal{A}^1}(\partial_t q, \partial_t v) = \mu \operatorname{div}_{\mathcal{A}^1} \left(\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2\right) + \tilde{H}^1 & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}^1} \partial_t v = \tilde{H}^2 & \text{in} \quad \Omega, \\ S_{\mathcal{A}^1}(\partial_t q, \partial_t v)\mathcal{N}^1 = \mu \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 \mathcal{N}^1 + g \partial_t \theta \mathcal{N}^1 - \sigma \partial_1 \left(\partial_1 \partial_t \theta\right) \mathcal{N}^1 & \\ \qquad - \sigma \partial_1 \partial_t \mathcal{R}^1 \mathcal{N}^1 + \tilde{H}^3 & \text{on} \quad \Sigma, \\ (S_{\mathcal{A}^1}(\partial_t q, \partial_t v)\nu - \beta \partial_t v) \cdot \tau = \mu \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 \nu \cdot \tau + \tilde{H}^4 & \text{on} \quad \Sigma_b, \\ \partial_t v \cdot \nu = 0 & \text{on} \quad \Sigma_b, \\ \partial_t^2 \theta = \partial_t v \cdot \mathcal{N}^1 + \tilde{H}^5 & \text{on} \quad \Sigma, \end{cases}$$

where

$$\begin{aligned} \tilde{H}^1 &= \partial_t H^1 + \operatorname{div}_{\partial_t \mathcal{A}^1} (\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} u^2) + \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t u^2) + \operatorname{div}_{\partial_t \mathcal{A}^1} (\mathbb{D}_{\mathcal{A}^1} v) \\ &\quad + \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{\partial_t \mathcal{A}^1} v) - \nabla_{\partial_t \mathcal{A}^1} q, \\ \tilde{H}^2 &= \partial_t H^2 - \operatorname{div}_{\partial_t \mathcal{A}^1} v, \\ \tilde{H}^3 &= \partial_t H^3 + \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t u^2 \mathcal{N}^1 + \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} u^2 \partial_t \mathcal{N}^1 - S_{\mathcal{A}^1}(q, v) \partial_t \mathcal{N}^1 + \mathbb{D}_{\partial_t \mathcal{A}^1} v \mathcal{N}^1 \\ &\quad + g \theta \partial_t \mathcal{N}^1 - \sigma \partial_1^2 \theta \partial_t \mathcal{N}^1, \\ \tilde{H}^4 &= \mu \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t u^2 \nu \cdot \tau + \mathbb{D}_{\partial_t \mathcal{A}^1} v \nu \cdot \tau, \\ \tilde{H}^5 &= \partial_t H^5 + v \cdot \partial_t \mathcal{N}^1. \end{aligned}$$

Now we multiply (4.40) by $J^1 \partial_t u^1$, integrate over Ω and integrate by parts to deduce that

(4.41)
$$\begin{aligned} &\partial_t \left(\int_\Sigma \frac{g}{2} |\partial_t \theta|^2 + \frac{\sigma}{2} |\partial_1 \partial_t \theta|^2 \right) + \frac{\mu}{2} \int_\Omega |\mathbb{D}_{\mathcal{A}^1} \partial_t v|^2 J^1 + \beta \int_{\Sigma_b} J^1 |\partial_t v \cdot \tau|^2 \\ &= \int_\Omega \mu \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2) \cdot \partial_t v J^1 + \tilde{H}^1 \cdot \partial_t v J^1 + \partial_t q \tilde{H}^2 J^1 - \int_{\Sigma_b} J^1 (\partial_t v \cdot \tau) \tilde{H}^4 \\ &\quad - \int_\Sigma \sigma \partial_t F^3 \partial_1 (\partial_t v \cdot \mathcal{N}^1) + (\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 \mathcal{N}^1 + \tilde{H}^3) \cdot \partial_t v - g \partial_t \theta \tilde{H}^5 \\ &\quad + \int_\Sigma \sigma \partial_1 \partial_t \theta \partial_1 \tilde{H}^5 - \int_{\Sigma_b} J^1 (\partial_t v \cdot \tau) \mu \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 \nu \cdot \tau. \end{aligned}$$

Another integration by parts reveals that

$$\begin{aligned} \int_\Omega \mu \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2) \cdot \partial_t v J^1 &= -\frac{\mu}{2} \int_\Omega J^1 \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 : \mathbb{D}_{\mathcal{A}^1} \partial_t v \\ &\quad + \int_\Sigma \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 \mathcal{N}^1 \cdot \partial_t v + \int_{\Sigma_b} \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} u^2 \nu \cdot \partial_t v J^1. \end{aligned} \tag{4.42}$$

We combine (4.41) and (4.42), and then integrate in time from 0 to $t < T$ to derive that

$$\begin{aligned}
&\int_\Sigma \frac{g}{2}|\partial_t\theta|^2 + \frac{\sigma}{2}|\partial_1\partial_t\theta|^2 + \frac{\mu}{2}\int_0^t\int_\Omega |\mathbb{D}_{\mathcal{A}^1}\partial_t v|^2 J^1 + \beta\int_0^t\int_{\Sigma_b} J^1|\partial_t v\cdot\tau|^2 \\
&= -\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2 : \mathbb{D}_{\mathcal{A}^1}\partial_t v + \int_0^t\int_\Omega \tilde{H}^1\cdot\partial_t v J^1 + \partial_t q\tilde{H}^2 J^1 \\
&\quad - \int_0^t\int_\Sigma \sigma\partial_t F^3\partial_1(\partial_t v\cdot\mathcal{N}^1) + \tilde{H}^3\cdot\partial_t v - g\partial_t\theta\tilde{H}^5 - \sigma\partial_1\partial_t\theta\partial_1\tilde{H}^5 \\
&\quad - \int_0^t\int_{\Sigma_b} J^1(\partial_t v\cdot\tau)\tilde{H}^4.
\end{aligned} \tag{4.43}$$

Step 2 – Estimate of pressure. In order to handle the term related to $\partial_t q$, we multiply (4.40) by $J^1 w$, integrate over Ω and integrate by parts to deduce that

$$\begin{aligned}
&\frac{\mu}{2}\int_\Omega \mathbb{D}_{\mathcal{A}^1}\partial_t v : \mathbb{D}_{\mathcal{A}^1} w J^1 + \beta\int_{\Sigma_b}(\partial_t v\cdot\tau)(w\cdot\tau) + (\partial_t\theta, w\cdot\mathcal{N}^1)_{1,\Sigma} \\
&= \int_\Omega (\mu \operatorname{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\right) + \tilde{H}^1)\cdot w J^1 \\
&\quad - \int_{\Sigma_b} J^1(w\cdot\tau)(\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\nu\cdot\tau + \tilde{H}^4) \\
&\quad - \int_\Sigma \sigma\partial_t(F^{3,1} - F^{3,2})\partial_1(w\cdot\mathcal{N}^1) + (\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\mathcal{N}^1 + \tilde{H}^3)\cdot w,
\end{aligned} \tag{4.44}$$

for each $w \in \mathcal{X}(t)$ and a.e. $t \in [0,T]$. Then $\partial_t q \in \mathring{H}^0(\Omega)$ might be recovered from Theorem 2.5 such that

$$\begin{aligned}
&\frac{\mu}{2}\int_\Omega \mathbb{D}_{\mathcal{A}^1} v : \mathbb{D}_{\mathcal{A}^1} w J^1 + \beta\int_{\Sigma_b}(v\cdot\tau)(w\cdot\tau) - (\partial_t q, \operatorname{div}_{\mathcal{A}^1} w)_0 + (\partial_t\theta, w\cdot\mathcal{N}^1)_{1,\Sigma} \\
&= \int_\Omega (\mu \operatorname{div}_{\mathcal{A}^1}\left(\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\right) + \tilde{H}^1)\cdot w J^1 \\
&\quad - \int_{\Sigma_b} J^1(w\cdot\tau)(\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\nu\cdot\tau + \tilde{H}^4) \\
&\quad - \int_{-\ell}^{\ell} \sigma\partial_t(\mathcal{R}^1 - \mathcal{R}^2)\partial_1(w\cdot\mathcal{N}^1) + (\mu\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2\mathcal{N}^1 + \tilde{H}^3)\cdot w,
\end{aligned} \tag{4.45}$$

for each $w \in \mathcal{W}(t)$ and a.e. $t \in [0,T]$. Moreover,

$$\|\partial_t q\|^2_{L^2\mathring{H}^0} \lesssim \|\partial_t v\|^2_{L^2H^1} + P(\sqrt{\delta})(\|\partial_t\theta\|^2_{L^2\mathring{H}^{3/2}} + \|\theta\|^2_{L^2H^{5/2}} + \|q\|^2_{L^2\mathring{H}^1} + \|v\|^2_{L^2H^2}), \tag{4.46}$$

where the temporal L^2 norm is computed on $[0,T]$, and $P(\cdot)$ is a polynomial which would be allowed to change from line to line.

Step 3 – Estimates of the forcing terms. To handle the term $\partial_t(\mathcal{R}^1-\mathcal{R}^2)$, we rewrite it as

$$
\begin{aligned}
&\int_\Sigma \sigma\partial_t(\mathcal{R}^1-\mathcal{R}^2)\partial_1(\partial_t v\cdot\mathcal{N}^1) = \int_\Sigma \sigma[(\mathcal{R}^1)'\partial_1\partial_t\theta+(\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2]\partial_1(\partial_t^2\theta-\tilde{H}^5)\\
&\quad=\frac{d}{dt}\left(\int_\Sigma(\mathcal{R}^1)'\frac{|\partial_1\partial_t\theta|^2}{2}-(\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)-\int_\Sigma(\mathcal{R}^1)''\partial_1\partial_t\eta^1\frac{|\partial_1\partial_t\theta|^2}{2}\\
&\qquad-\int_\Sigma|\partial_1\partial_t\theta|^2(\mathcal{R}^1)''\partial_1\partial_t\eta^2+(\mathcal{R}^1-\mathcal{R}^2)''(\partial_1\partial_t\eta^2)^2\partial_1\partial_t\theta\\
&\qquad+(\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t^2\eta^2\partial_1\partial_t\theta-(\mathcal{R}^1-\mathcal{R}^2)'\tilde{H}^5,
\end{aligned}
\tag{4.47}
$$

Then we rewrite (4.41) as

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t\theta\|_{1,\Sigma}^2+\int_\Sigma(\mathcal{R}^1)'\frac{|\partial_1\partial_t\theta|^2}{2}-(\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)\\
&\quad+\frac{\mu}{2}\int_\Omega|\mathbb{D}_{\mathcal{A}^1}\partial_t v|^2J^1+\beta\int_{\Sigma_b}J^1|\partial_t v\cdot\tau|^2\\
&=-\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2:\mathbb{D}_{\mathcal{A}^1}\partial_t v+\int_\Omega\tilde{H}^1\cdot\partial_t vJ^1+\partial_t q\tilde{H}^2J^1\\
&\quad-\int_{\Sigma_b}J^1(\partial_t v\cdot\tau)\tilde{H}^4-\int_\Sigma|\partial_1\partial_t\theta|^2(\mathcal{R}^1)''\partial_1\partial_t\eta^2\\
&\quad-\int_\Sigma(\mathcal{R}^1-\mathcal{R}^2)''(\partial_1\partial_t\eta^2)^2\partial_1\partial_t\theta+(\mathcal{R}^1-\mathcal{R}^2)'(\partial_1\partial_t^2\eta^2\partial_1\partial_t\theta-\tilde{H}^5)\\
&\quad-\int_\Sigma(\mathcal{R}^1)''\partial_1\partial_t\eta^1\frac{|\partial_1\partial_t\theta|^2}{2}-\int_\Sigma\left(\tilde{H}^3\cdot\partial_t v-g\partial_t\theta\tilde{H}^5-\sigma\partial_1\partial_t\theta\partial_1\tilde{H}^5\right).
\end{aligned}
\tag{4.48}
$$

We now estimate the terms on the right hand side of (4.43).

$$\frac{\mu}{2}\int_\Omega J^1\mathbb{D}_{(\partial_t\mathcal{A}^1-\partial_t\mathcal{A}^2)}u^2:\mathbb{D}_{\mathcal{A}^1}\partial_t v\lesssim P(\sqrt{\delta})\|\partial_t v\|_1\|\partial_t\theta\|_{3/2}. \tag{4.49}$$

$$\int_\Omega\tilde{H}^1\cdot\partial_t vJ^1\lesssim P(\sqrt{\delta})\|\partial_t v\|_1(\|\partial_t\theta\|_{3/2}+\|\theta\|_{5/2}+\|q\|_1+\|v\|_2). \tag{4.50}$$

$$\int_{\Sigma_b}J^1(\partial_t v\cdot\tau)\tilde{H}^4\lesssim P(\sqrt{\delta})\|\partial_t v\|_1(\|\theta\|_{5/2}+\|\partial_t\theta\|_{3/2}). \tag{4.51}$$

$$\int_\Omega\partial_t q\tilde{H}^2J^1\lesssim P(\sqrt{\delta})\|\partial_t q\|_0(\|\partial_t\theta\|_{3/2}+\|\theta\|_{5/2}+\|v\|_2). \tag{4.52}$$

By the direct computation for derivatives of (1.7), we may employ the Sobolev embedding theory to derive that

$$
\begin{aligned}
&-\int_\Sigma|\partial_1\partial_t\theta|^2(\mathcal{R}^1)''\partial_1\partial_t\eta^2+(\mathcal{R}^1-\mathcal{R}^2)''(\partial_1\partial_t\eta^2)^2\partial_1\partial_t\theta\\
&-\int_\Sigma(\mathcal{R}^1-\mathcal{R}^2)'(\partial_1\partial_t^2\eta^2\partial_1\partial_t\theta-\tilde{H}^5)+(\mathcal{R}^1)''\partial_1\partial_t\eta^1\frac{|\partial_1\partial_t\theta|^2}{2}\\
&\qquad\lesssim P(\sqrt{\delta})(\|\partial_t\theta\|_{3/2}^2+\|\theta\|_{5/2}^2+\|\theta\|_{5/2}+\|v\|_2),
\end{aligned}
\tag{4.53}
$$

and

$$\int_\Sigma (\mathcal{R}^1)' \frac{|\partial_1\partial_t\theta|^2}{2} - (\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2\partial_1\partial_t\theta \lesssim P(\sqrt{\delta})(\|\partial_t\theta\|_{3/2}^2 + \|\theta\|_{5/2}). \tag{4.54}$$

$$\int_\Sigma \sigma\tilde{H}^3\cdot\partial_t v \lesssim P(\sqrt{\delta})\|\partial_t v\|_1(\|\theta\|_{5/2}+\|\partial_t\theta\|_{3/2}+\|q\|_1+\|v\|_2). \tag{4.55}$$

After integrating by parts,

$$\begin{aligned}\int_\Sigma -g\partial_t\theta\tilde{H}^5 - \sigma\partial_1\partial_t\theta\partial_1\tilde{H}^5 \lesssim P(\sqrt{\delta})\left[\|\partial_t\theta\|_1(\|\theta\|_{5/2}+\|\partial_t\theta\|_1+\|v\|_2)+\|\partial_t\theta\|_{3/2}^2\right]\\ +\|\partial_t\eta^1\|_{5/2}\|\partial_t\theta\|_{3/2}\|v\|_1.\end{aligned} \tag{4.56}$$

Then combining all the above estimates (4.49)–(4.56), we can derive that

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t\theta\|_{1,\Sigma}^2 + \int_\Sigma (\mathcal{R}^1)'\frac{|\partial_1\partial_t\theta|^2}{2} - (\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right) + \|\partial_t v\|_1^2\\ &\le CP(\sqrt{\delta})\left(\|\partial_t\theta\|_{1,\Sigma}^2 + \int_\Sigma (\mathcal{R}^1)'\frac{|\partial_1\partial_t\theta|^2}{2} - (\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2\partial_1\partial_t\theta\right)\\ &\quad + CP(\sqrt{\delta})\|\theta\|_{5/2}^2 + \|\partial_t\theta\|_{3/2}^2 + \|q\|_1^2 + \|v\|_2^2.\end{aligned} \tag{4.57}$$

Since

$$\begin{aligned}\sup_{0\le t\le T}\int_\Sigma (\mathcal{R}^1)'\frac{|\partial_1\partial_t\theta|^2}{2} - (\mathcal{R}^1-\mathcal{R}^2)'\partial_1\partial_t\eta^2\partial_1\partial_t\theta\\ \lesssim P(\sqrt{\delta})(\|\partial_t\theta\|_{L^\infty\mathring{H}^1}^2 + \|\theta\|_{L^\infty\mathring{H}^{3/2}}^2)\\ \lesssim P(\sqrt{\delta})(\|\partial_t\theta\|_{L^\infty\mathring{H}^1}^2 + \|\partial_t\theta\|_{L^2\mathring{H}^{3/2}}^2 + \|\theta\|_{L^2H^{5/2}}^2),\end{aligned}$$

Gronwall's lemma together with the smallness of δ implies that

$$\begin{aligned}\|\partial_t\theta\|_{L^\infty\mathring{H}^1}^2 + \|\partial_t v\|_{L^2H^1}^2 &\lesssim e^{CP(\sqrt{\delta})T_1}CP(\sqrt{\delta})(\|\theta\|_{L^2H^{5/2}}^2 + \|\partial_t\theta\|_{L^2\mathring{H}^{3/2}}^2\\ &\qquad + \|\partial_t q\|_{L^2\mathring{H}^0}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|v\|_{L^2H^2}^2)\\ &\lesssim e^{CP(\sqrt{\delta})T_1}CP(\sqrt{\delta})(\|\theta\|_{L^2H^{5/2}}^2 + \|\partial_t\theta\|_{L^2\mathring{H}^{3/2}}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|v\|_{L^2H^2}^2),\end{aligned} \tag{4.58}$$

where the temporal L^∞ and L^2 norms are computed over $[0,T]$. We assume that δ_1 is sufficiently small for $e^{CP(\sqrt{\delta})T} \le e^{CP(\sqrt{\delta_1})T_1} \le 2$. Then we deduce the bound

$$\|\partial_t\theta\|_{L^\infty\mathring{H}^1}^2 + \|\partial_t v\|_{L^2H^1}^2 \lesssim P(\sqrt{\delta})(\|\theta\|_{L^2H^{5/2}}^2 + \|\partial_t\theta\|_{L^2\mathring{H}^{3/2}}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|v\|_{L^2H^2}^2). \tag{4.59}$$

Since $\partial_t\theta \in \mathring{H}^1(\Sigma)$ and (4.44), with δ sufficient small, Theorem 2.6 reveals that

$$\|\partial_t\theta\|_{L^2\mathring{H}^{3/2}}^2 \lesssim P(\sqrt{\delta})(\|\theta\|_{L^2H^{5/2}}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|v\|_{L^2H^2}^2). \tag{4.60}$$

Step 4 – Elliptic estimates for v, q and θ. In order to close our estimates, we must be able to estimate v, q and θ. The elliptic estimates imply that

$$\begin{aligned}\|v\|_{H^2}^2 + \|q\|_{\mathring{H}^1}^2 + \|\theta\|_{H^{5/2}}^2 &\lesssim \|\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}u^2) + H^1\|_{H^0}^2 + \|H^2\|_{H^1}^2\\ &\quad + \|\partial_t\theta - H^5\|_{H^{3/2}}^2 + \|H^3\|_{H^{1/2}}^2 + \|\mathbb{D}_{\mathcal{A}^1-\mathcal{A}^2}u^2\nu\cdot\tau\|_{H^{1/2}}^2\\ &\quad + \|\partial_1(\mathcal{R}^1-\mathcal{R}^2)\|_{H^{1/2}}^2.\end{aligned} \tag{4.61}$$

Then after integrating temporally from 0 to T, we have that

$$\begin{aligned}\|v\|_{L^2H^2}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|\theta\|_{L^2H^{5/2}}^2 &\lesssim P(\sqrt{\delta})\|\theta\|_{L^2H^{5/2}}^2 + \|\partial_t\theta\|_{L^2H^{3/2}}^2 \\ &\leq CP(\sqrt{\delta})(\|\theta\|_{L^2H^{5/2}}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|v\|_{L^2H^2}^2),\end{aligned} \tag{4.62}$$

where $P(0) = 0$. Since δ is sufficiently small, we might restrict δ_1 such that $CP(\sqrt{\delta}) < 1$. Thus

$$\|v\|_{L^2H^2}^2 + \|q\|_{L^2\mathring{H}^1}^2 + \|\theta\|_{L^2H^{5/2}}^2 = 0. \tag{4.63}$$

□

4.5. Continuity of energy $\mathcal{E}(t)$. We have previously established the boundedness of the map $t \mapsto \mathcal{E}(t)$, but we will now show that actually this map is continuous. A simple variant of Lemma 3.6 establishes the continuity of $t \mapsto \|u(t)\|_{H^2(\Omega)}^2 + \|p(t)\|_{\mathring{H}^1(\Omega)}^2 + \|\eta(t)\|_{H^{5/2}(\Sigma)}^2$ and so to establish the continuity of $t \mapsto \mathcal{E}(t)$ it suffices to prove that $t \mapsto \|\partial_t u\|_{H^1}^2 + \|\partial_t p\|_{\mathring{H}^0}^2 + \|\partial_t\eta\|_{H^{3/2}}^2 + \|\partial_t^2\eta\|_{\mathring{H}^1}^2$ is continuous. Due to the dissipation estimates we have the inclusions

$$\begin{aligned}&\partial_t u \in L^2([0,T]; H^1(\Omega)) \text{ and } \partial_t^2 u \in L^2([0,T]; H^1(\Omega)),\\ &\partial_t p \in L^2([0,T]; H^0(\Omega)) \text{ and } \partial_t^2 u \in L^2([0,T]; H^0(\Omega)),\\ &\partial_t \eta \in L^2([0,T]; H^{3/2}(\Sigma)) \text{ and } \partial_t^2 \eta \in L^2([0,T]; H^{3/2}(\Sigma)),\end{aligned} \tag{4.64}$$

and so the Sobolev embeddings (in the time variable) guarantee that

$$\partial_t u \in C^0([0,T]; H^1(\Omega)), \partial_t p \in C^0([0,T]; H^0(\Omega)), \partial_t\eta \in C^0([0,T]; H^{3/2}(\Sigma)) \tag{4.65}$$

and that $t \mapsto \|\partial_t u\|_{H^1}^2 + \|\partial_t p\|_{\mathring{H}^0}^2 + \|\partial_t\eta\|_{H^{3/2}}^2$ is continuous. The dissipation estimate also tell us that $\partial_t^2\eta \in L^2([0,T]; H^{3/2}(\Sigma))$ and that $\partial_t^3\eta \in L^2([0,T]; H^{1/2}(\Sigma))$. From this and, for instance, [**9**, Lemma A.4] we find that $\partial_t^2\eta \in C^0([0,T]; H^1(\Sigma))$ and that the map $t \mapsto \|\partial_t^2\eta(t)\|_1^2$ is continuous. Consequently, $t \mapsto \mathcal{E}(t)$ is continuous.

4.6. Diffeomorphism of Φ. From the definition of J and restrict theory in Sobolev spaces, we can derive that

$$\|J\|_{L^\infty} \geq 1 - C(\|\bar{\eta}\|_{L^\infty} + \|\partial_2\bar{\eta}\|_{L^\infty}) \geq 1 - C\|\eta\|_{5/2}.$$

The smallness of $\mathfrak{K}(\eta)$ sufficiently guarantees that Φ, defined in (1.4), is a C^1 diffeomorphism for each $t \in [0,T]$. For more details, one can see [**11**] in $3D$ domains.

Appendix A. Boundedness for $\mathcal{R}$

The following lemma concerning the estimates of $\mathcal{R}$ could be derived directly from the computation for (1.7), so we omit the proof here.

Lemma A.1. *The mapping $\mathcal{R} \in C^\infty(\mathbb{R}^2)$ defined by (1.7) satisfies*

$$\sup_{z\in\mathbb{R}}\left(\left|\frac{1}{z^3}\int_0^z \mathcal{R}(s)\,\mathrm{d}s\right| + \left|\frac{\mathcal{R}(z)}{z^2}\right| + \left|\frac{\mathcal{R}'(z)}{z}\right| + |\mathcal{R}''(z)| + |\mathcal{R}'''(z)|\right) < \infty.$$

In addition, $|\mathcal{R}'(z)| < \infty$.

Appendix B. Properties involving $\mathcal{A}$

We now record some useful properties involving $\mathcal{A}$.

LEMMA B.1. *The following identities hold.*

(1) $\partial_j(J\mathcal{A}_{ij}) = 0$ *for* $j = 1, 2$ *and each* $i = 1, 2$.
(2) $J\mathcal{A}e_2 = \mathcal{N}$ *on* Σ,
(3) $R^\top \mathcal{N} = -\partial_t \mathcal{N}$ *on* Σ, *where* R *is defined by* $R = \partial_t M M^{-1}$.

PROOF. The first equality comes from [**9**, Lemma A.3]. On Σ,

$$J\mathcal{A}e_2 = \begin{pmatrix} J & -A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -A \\ 1 \end{pmatrix} = \begin{pmatrix} -\partial_1 \eta \\ 1 \end{pmatrix} = \mathcal{N}.$$

It is easily to compute that $R^\top = J\partial_t K I_{2\times 2} - \partial_t \mathcal{A}\mathcal{A}^{-1}$. Since $J\mathcal{A}e_2 = \mathcal{N}$,

$$\begin{aligned} R^\top \mathcal{N} &= (J\partial_t K - \partial_t \mathcal{A}\mathcal{A}^{-1}) J\mathcal{A}e_2 \\ &= (-K\partial_t J - \partial_t \mathcal{A}\mathcal{A}^{-1}) J\mathcal{A}e_2 \\ &= (-\partial_t J\mathcal{A} - J\partial_t \mathcal{A})e_2 = -\partial_t(J\mathcal{A}e_2) = -\partial_t \mathcal{N}. \end{aligned}$$

□

LEMMA B.2. *Let* $\mathcal{A}$, $\mathcal{N}$, J, K *be defined as* (1.5). *Then there exists a universal constant* $0 < \gamma < 1$ *such that if* $\|\eta\|_{5/2}^2 < \gamma$, *then*

$$\|J\|_{L^\infty(\Omega)} + \|\mathcal{A}\|_{L^\infty(\Omega)} \lesssim 1,$$

and

$$\|\mathcal{N}\|_{L^\infty(\Sigma)} + \|K\|_{L^\infty(\Sigma)} \lesssim 1.$$

This proof is very simply by utilizing Sobolev embedding theorems, so we omit the details here. For more information, we refer to [**9**].

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