

Least energy sign-changing solution to a fractional p -Laplacian problem involving singularities

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ABSTRACT. In this paper we study the existence of a least energy sign-changing solution to a nonlocal elliptic PDE involving singularities by using the Nehari manifold method, the constraint variational method and Brouwer degree theory.

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1. Introduction and Main results

In this paper we consider the following fractional p -Laplacian problem involving singularities and power nonlinearities.

$$(P) \quad \begin{cases} (-\Delta_p)^\alpha u = \lambda g(u) + f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$, $p \in (1, +\infty)$, $\alpha \in (0, 1)$, $N > p\alpha$, $0 < \delta < 1$, $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $g : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^+$ is continuous, nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, +\infty)$ which satisfies some growth conditions. The fractional p -Laplacian operator, $(-\Delta)_p^\alpha$ is defined as,

$$(-\Delta_p)^\alpha u(x) = C_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+p\alpha}} dy, \quad x \in \mathbb{R}^N,$$

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where $C_{N,\alpha}$ is a positive, normalizing constant. One of the classical topics in the analysis of PDEs is the study of existence and multiplicity of nonnegative solutions for the p -Laplacian and the fractional p -Laplacian operator ($1 < p < \infty$) as well as for the Laplacian and the fractional Laplacian operator ($p = 2$), involving concave-convex nonlinearity and singularity-power nonlinearity. The literature is rich for problems involving $(-\Delta_p)$, $1 < p < \infty$ and singularity-power nonlinearity. Some of the references are [16, 20, 21, 23] and the references therein. The methods used by the authors in these references are mainly variational techniques. In the recent past there has been considerable interest in studying the following general fractional p -Laplacian problem involving singularity

$$(1.1) \quad \begin{aligned} (-\Delta_p)^\alpha u &= \frac{\lambda a(x)}{u^\delta} + Mf(x, u) \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 \text{ in } \Omega, \end{aligned}$$

where $N > p\alpha$, $M \geq 0$, $a : \Omega \rightarrow \mathbb{R}$ is a nonnegative bounded function. Ghanmi & Saoudi [19] guaranteed the existence of multiple weak solutions to the problem (1.1), for $0 < \delta < 1$ and $1 < p - 1 < q \leq p_s^*$ by using the Nehari manifold method. Recently, multiplicity and Hölder regularity of solutions to the problem (1.1) has been studied by Saoudi et al. [31]. On the other hand, for $p = 2$, the problems of the type (1.1), have been investigated by many researchers. For references see [28, 30, 31] and the references therein.

The existence of a sign-changing solution of nonlinear elliptic PDEs with power nonlinearities has been studied extensively for the p -Laplacian operator ($1 < p < \infty$) as well as the fractional p -Laplacian operator ($1 < p < \infty$). We refer the reader to see [2, 6, 8, 9, 10, 13, 14, 25, 38, 40] and the references therein. Consider the nonlocal problem

$$(1.2) \quad \begin{aligned} (-\Delta)_p^\alpha u &= f(x, u) \text{ in } \Omega, \\ u &= 0 \text{ on } \mathbb{R}^N \setminus \partial\Omega. \end{aligned}$$

For $p = 2$, the authors in [13] have studied the problem (1.2), where the fractional Laplacian operator is defined through spectral decomposition to obtain the sign-changing solution. The method of harmonic extension was introduced by Caffarelli and Silvestre [10] to transform the nonlocal problem in Ω to a local problem in the half cylinder $\Omega \times (0, +\infty)$, by using an equivalent definition of the fraction Laplacian operator [9]. For $p \in (1, \infty)$, the problem studied by Chang et al. [14], where the authors have guaranteed the existence of a least energy sign-changing solutions by using Nehari manifold method. A few more refernces where the existence of sign-changing solution has been guaranteed can be found in the table after Theorem 1.4. Recently, the study of the nonlocal problems with singularity has drawn interest to many researchers. For recent studies on nonlocal PDEs involving singularities, we refer [15, 16, 20, 21, 23, 31] and the references therein.

The main goal of this article is to obtain a sign-changing solutions to the nonlocal problem (P) involving singularity. For $p \neq 2$, the harmonic extension method can not be applied on an equivalent definition of $(-\Delta)_p^\alpha$. However, Sire and Valdinoci [35] gave an approach where a nonlocal fractional reaction equation can be handled by considering a problem with extended variable by the results due to [10] in which one could interpret the operator on the boundary as a nonlocal quasilinear operator.

On a similar note, we can not have the decomposition $\Phi(u) = \Phi(u^+) + \Phi(u^-)$ for $u = u^+ + u^-$, where Φ is the corresponding energy functional to the problem (P). Therefore, by using the method as in [13], one can not guarantee the existence of a sign-changing solution. We will apply the Nehari manifold method combining with a constrained variational method and Brouwer degree theory to obtain a least energy sign-changing solution.

We first recall some preliminary results on the fractional Sobolev space [1, 17]. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $\alpha \in (0, 1)$. We denote the fractional Sobolev space by $W^{\alpha,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{\alpha,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\alpha p}} dx dy \right)^{\frac{1}{p}}.$$

We set, $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$ then the space $(X, \|\cdot\|_X)$ is defined by

$$X = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is measurable} \mid u|_{\Omega} \in L^p(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+p\alpha}{p}}} \in L^p(Q) \right\}$$

equipped with the Gagliardo norm

$$\|u\|_X = \|u\|_p + \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+p\alpha}} dx dy \right)^{\frac{1}{p}}.$$

Here $\|u\|_p$ refers to the L^p -norm of u . We now define the space

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equipped with the norm

$$\|u\| = \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+p\alpha}} dx dy \right)^{\frac{1}{p}}.$$

The best Sobolev constant is defined as

$$(1.3) \quad S = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+p\alpha}} dx dy}{\left(\int_{\Omega} |u|^{p_{\alpha}^*} dx \right)^{\frac{p}{p_{\alpha}^*}}}$$

For $p > 1$, the space X_0 is a uniformly convex Banach space [33, 34] and the embedding $X_0 \hookrightarrow L^q(\Omega)$ is compact for $q \in [1, p_{\alpha}^*]$ and is continuous for $q \in [1, p_{\alpha}^*]$, where p_{α}^* is the Sobolev conjugate of p , defined as $p_{\alpha}^* = \frac{Np}{N-p\alpha}$.

Henceforth, we have the following assumptions on f and g .

(f₁): $f \in C(\bar{\Omega} \times \mathbb{R})$, $\lim_{|u| \rightarrow 0} \frac{f(x,u)}{|u|^{p-2}u} = 0$, uniformly in x ;

(f₂): there exist constants $C_0 > 0$ and $q \in (p, p_{\alpha}^*)$ with $p_{\alpha}^* = \frac{pN}{N-p\alpha}$ such that

$$|f(x, u)| \leq C_0(1 + |u|^{q-1}), \quad \forall u \in \mathbb{R}, \forall x \in \Omega;$$

(f₃): there exist $\mu > p$ and $M_0 > 0$ such that $f(x, u)u \geq \mu F(x, u) > 0$ for $|u| \geq M_0$, uniformly in x , where $F(x, u) \doteq \int_0^u f(x, \tau) d\tau$;

(f₄): $\lim_{|u| \rightarrow +\infty} \frac{f(x,u)}{|u|^{p-2}u} = +\infty$ uniformly in x ;

(f₅): $\frac{f(x,u)}{|u|^{p-2}u}$ is strictly increasing on $(0, +\infty)$ and strictly decreasing on $(-\infty, 0)$, uniformly in x .

(g₁): $g : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^+$ continuous on $\mathbb{R} \setminus \{0\}$, g is nondecreasing on $(-\infty, 0)$ and g is nonincreasing on $(0, +\infty)$,

(g₂): $c_1 \leq \liminf_{t \rightarrow 0^+} g(t)t^\delta \leq \limsup_{t \rightarrow 0^+} g(t)t^\delta = c_2$ for some $c_1, c_2 > 0$ and $d_1 \leq \liminf_{t \rightarrow 0^-} g(t)|t|^\delta \leq \limsup_{t \rightarrow 0^-} g(t)|t|^\delta = d_2$ for some $d_1, d_2 > 0$.

REMARK 1.1. (1) By (f₅) it follows that

$$t^2 f'(t) - (p - 1)f(t)t > 0, \forall |t| > 0.$$

(2) From (g₂), g is singular at the origin and $\lim_{t \rightarrow 0^\pm} g(t) = \infty$.

We now define a weak solution to the problem defined in (P).

DEFINITION 1.2. A function $u \in X_0$ is a weak solution to the problem (P), if

$$\int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+p\alpha}} dx dy - \lambda \int_\Omega g(u)\phi - \int_\Omega f(x, u)\phi = 0$$

for each $\phi \in X_0$. The corresponding Euler-Lagrange energy functional is

$$I_\lambda(u) = \frac{1}{p} \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+p\alpha}} dx dy - \lambda \int_\Omega G_\lambda(u) dx - \int_\Omega F_\lambda(x, u) dx.$$

It is easy to observe that I_λ is not C^1 due the presence of the singular term in it but I_λ is continuous and Gâteaux differentiable (see Corollary 6.3 of [31]). Therefore, we can not apply the Nehari manifold method corresponding to the functional I_λ . Hence, we will establish the existence of a sign-changing solution to the problem (P) by obtaining a critical point to a C^1 cutoff functional. We define,

$$\Lambda = \inf\{\lambda > 0 : \text{The problem (P) has no weak solution}\}.$$

We now state the existence of a unique solution due to [11] to the following problem.

$$\begin{aligned} (-\Delta_p)^s w &= \lambda g(w) \text{ in } \Omega, \\ w &> 0 \text{ in } \Omega, \\ (1.4) \quad w &= 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

LEMMA 1.3. Assume $0 < \delta < 1$ and $\lambda > 0$. Then the problem (1.4) has a unique solution, $\underline{u}_\lambda \in W_0^{s,p}(\Omega)$, such that for every $K \subset\subset \Omega$, $\text{ess. inf}_K \underline{u}_\lambda > 0$.

Define,

$$\bar{g}(t) = \begin{cases} g(t), & \text{if } |t| > \underline{u}_\lambda \\ g(\underline{u}_\lambda), & \text{if } |t| \leq \underline{u}_\lambda \end{cases}$$

and

$$\bar{f}(x, t) = \begin{cases} f(x, t), & \text{if } |t| > \underline{u}_\lambda \\ f(x, \underline{u}_\lambda), & \text{if } |t| \leq \underline{u}_\lambda, \end{cases}$$

where \underline{u}_λ is the solution to (1.4). Let $G(s) = \int_0^s \bar{g}(t) dt$ and $F(x, s) = \int_0^s \bar{f}(x, t) dt$. We now define the energy functional $\Phi : X_0 \rightarrow \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p} \|u\|^p - \lambda \int_\Omega G(u) dx - \int_\Omega F(x, u) dx.$$

Under the assumptions (f₁) – (f₅) and (g₁) – (g₂), the functional Φ is C^1 on X_0 (see Lemma 6.4 in [31]) and weakly lower semicontinuous by a standard arguments. Define

$$\begin{aligned} \zeta(u) &= \langle \Phi'(u), u \rangle_{X_0^*, X_0} = \|u\|^p - \lambda \int_\Omega g(u) u dx - \int_\Omega f(x, u) u dx, \forall u \in X_0, \\ \mathcal{N} &= \{u \in X_0 \setminus \{0\} : \zeta(u) = 0\}, \end{aligned}$$

where X_0^* is the dual space of X_0 . For simplicity, we will denote $\langle \cdot, \cdot \rangle_{X_0^*, X_0}$ by $\langle \cdot, \cdot \rangle$. Clearly, every nontrivial solutions of (P) belongs to \mathcal{N} . We now define the set of sign-changing solutions of (P) as

$$\mathcal{M} = \{u \in X_0 : u^\pm \neq 0, \langle \Phi'(u), u^+ \rangle = \langle \Phi'(u), u^- \rangle = 0\},$$

where $u^+(x) \doteq \max\{u(x), 0\}$, $u^-(x) \doteq \min\{u(x), 0\}$. We set $m_\alpha \doteq \inf_{u \in \mathcal{M}} \Phi(u)$ and $c_\alpha \doteq \inf_{u \in \mathcal{N}} \Phi(u)$. The main result proved in this article is the following.

THEOREM 1.4. *Suppose that the assumptions $(f_1) - (f_5)$ and $(g_1) - (g_2)$ holds. Then there exists a $\Lambda > 0$, such that for $\lambda \in (0, \Lambda)$, the problem (P) admits one sign-changing solution $u^* \in X_0$ and $\Phi(u^*) = m_\alpha$.*

REMARK 1.5. To the best of our knowledge, the literature pertaining to an elliptic problem involving singularity and a power nonlinearity is unavailable, even in the problems involving local operator. Therefore it is important to mention here that our contribution in the form of Theorem 1.4 is a new addition to the literature. Before we proceed further, we would like the readers to have a look at the following tabulated list of references which led to the development of this work.

S No.	Reference	$(-\Delta_p)^\alpha$	singularity	non linearity
1	[26]	$\alpha = 1, p = 2$	Yes	No
2	[12], [5], [32]	$\alpha = 1, p = 2$	No	Yes
3	[18], [13], [36]	$0 < \alpha < 1, p = 2$	No	Yes
4	[14]	$0 < \alpha < 1, 1 < p < \infty$	No	Yes
5	[4], [7]	$\alpha = 1, 1 < p < \infty$	No	Yes

The paper is organized as follows. In Section 2, we present some useful notations and give some preliminary results. In Section 3, we apply the method of Nehari manifold to prove Theorem 1.4. Throughout the paper, we always denote by C_1, C_2, \dots positive constants (possibly different in different places) and let $|\cdot|_p$ denote the usual $L^p(\Omega)$ norm for all $p \in [1, +\infty]$.

2. Important Lemmas

We begin this section with the following Lemma.

LEMMA 2.1. *Assume $0 < \delta < 1 < q < p_s^* - 1$. Then $0 < \Lambda < \infty$.*

PROOF. This result can be proved by working on similar lines as of [31]. We consider the functions \bar{f} , \bar{g} and the functional Φ which is defined in the previous section to obtain

$$(2.1) \quad \Phi(u) \geq \frac{1}{p} \|u\|^p - \lambda c_1 \|u\|^{1-\gamma} - c_2 \|u\|^{q+1}$$

where, c_1, c_2 are constants. Fix such an r , so that the other negative terms in (2.1) may be made arbitrarily small by choosing λ small enough. One must also make sure that the chosen λ is sufficiently small so that the term

$$\frac{1}{p} \int_{\{x: |u(x)| \leq \underline{u}_\lambda\}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy + \int_{\{x: |u(x)| \leq \underline{u}_\lambda\}} (\lambda G(u) + F(x, u)) dx$$

is also small compared to

$$\frac{1}{p} \int_{\{x:|u(x)| \geq \underline{u}_\lambda\}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy + \int_{\{x:|u(x)| \geq \underline{u}_\lambda\}} (\lambda G(u) + F(x, u)) dx.$$

This is possible since \underline{u}_λ becomes smaller with $\lambda > 0$ becoming small. Thus we have a pair of (λ, r) such that

$$\min_{u \in \partial B_r} \{\Phi(u)\} > 0.$$

Now for $\phi \in X_0$ such that $\phi \geq 0$, $\phi \neq 0$ and $t > 1$ we have

$$(2.2) \quad \begin{aligned} \Phi(t\phi) &= \frac{|t|^p}{p} \|\phi\|^p - \frac{t^{1-\gamma}}{1-\gamma} \int_{\Omega} \phi^{1-\gamma} dx \\ &\quad - \frac{t^{q+1}}{q+1} \int_{\Omega} \phi^{q+1} dx, \end{aligned}$$

Thus $\Phi(t\phi) \rightarrow -\infty$ as $t \rightarrow \infty$, since $1 - \gamma < 1 < p < q + 1$. Therefore, we can conclude that $\inf_{\|u\|_{X_0} \leq r} \Phi(u) = c < 0$. The proof follows *verbatim* thereafter as in Lemma 3.5 of [31]. □

The following Lemma due to [3] will be useful in the proof of Theorem 1.4.

LEMMA 2.2. *Let $\xi, \eta \in \mathbb{R}^N$. Then we have*

$$(2.3) \quad \begin{aligned} (i) \text{ For } 2 < p < \infty, \text{ there exists } d'_1, d'_2 > 0 \text{ such that, for all } \xi, \eta \in \mathbb{R}^N, \\ &(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq d'_1 |\xi - \eta|^p, \\ &||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq d'_2 (|\xi| + |\eta|)^{p-2} |\xi - \eta| \end{aligned}$$

$$(2.4) \quad \begin{aligned} (ii) \text{ For } 1 < p \leq 2, \text{ there exist } d_3, d_4 > 0 \text{ such that, for all } \xi, \eta \in \mathbb{R}^N, \\ &(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq d_3 \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}, \\ &||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq d_4 |\xi - \eta|^{p-1} \end{aligned}$$

We will now prove a comparison principle for the fractional p -Laplacian operator involving singularities. It must be noted that the proof of the comparison principle here may not work when general singular nonlinearities are considered. It will be interesting to establish (or de-establish) the result for a general g . However, if the singular nonlinearity g is continuous and nonincreasing in $\mathbb{R} \setminus \{0\}$ together with (g_2) , then the following comparison principle holds true.

LEMMA 2.3 (Weak Comparison Principle). *Let $u, v \in X_0$ be such that $v = u = 0$ in $\mathbb{R}^N \setminus \Omega$. Suppose $(-\Delta_p)^s v - \lambda g(v) \geq (-\Delta_p)^s u - \lambda g(u)$ weakly on Ω . Then we have $v \geq u$ in \mathbb{R}^N .*

PROOF. Since $(-\Delta_p)^s v - \lambda g(v) \geq (-\Delta_p)^s u - \lambda g(u)$ weakly with $u = v = 0$ in $\mathbb{R}^N \setminus \Omega$, we have

$$(2.5) \quad \langle (-\Delta_p)^s v, \phi \rangle - \int_{\Omega} \lambda g(v) \phi dx \geq \langle (-\Delta_p)^s u, \phi \rangle - \int_{\Omega} \lambda g(u) \phi dx, \quad \forall \phi \geq 0 \in X_0.$$

In particular choose $\phi = (u - v)^+$. To this choice, (2.5) looks as follows.

$$(2.6) \quad \langle (-\Delta_p)^s v - (-\Delta_p)^s u, (u - v)^+ \rangle - \int_{\Omega} \lambda (u - v)^+ (g(v) - g(u)) dx \geq 0.$$

Let $\psi = u - v$. The identity

$$(2.7) \quad |b|^{p-2}b - |a|^{p-2}a = (p-1)(b-a) \int_0^1 |a+t(b-a)|^{p-2} dt$$

with $a = v(x) - v(y)$, $b = u(x) - u(y)$ gives

$$(2.8) \quad \begin{aligned} & |u(x) - u(y)|^{p-2}(u(x) - u(y)) - |v(x) - v(y)|^{p-2}(u(x) - u(y)) \\ &= (p-1)\{(u(y) - v(y)) - (u(x) - v(x))\}Q(x, y) \end{aligned}$$

where

$$(2.9) \quad Q(x, y) = \int_0^1 |(u(x) - u(y)) + t((v(x) - v(y)) - (u(x) - u(y)))|^{p-2} dt.$$

We choose the test function $\phi = (u - v)^+$. We express,

$$\psi = u - v = (u - v)^+ - (u - v)^-$$

to further obtain

$$(2.10) \quad [\psi(y) - \psi(x)][\phi(x) - \phi(y)] = -(\psi^+(x) - \psi^+(y))^2.$$

The equation (2.10) implies

$$(2.11) \quad \begin{aligned} 0 &\geq \langle (-\Delta_p)^s v - (-\Delta_p)^s u, (v - u)^+ \rangle \\ &= -(p-1) \frac{Q(x, y)}{|x - y|^{N+sp}} (\psi^+(x) - \psi^+(y))^2 \\ &\geq 0. \end{aligned}$$

This leads to the conclusion that the Lebesgue measure of Ω^+ , i.e., $|\Omega^+| = 0$. In other words $v \geq u$ a.e. in Ω . \square

3. Proof of Theorem 1.4

In this section we will prove the existence of a sign-changing solution for (P) by obtaining a minimizer of the energy functional Φ over

$$\mathcal{M} = \{u \in X_0 : u^\pm \neq 0, \langle \Phi'(u), u^+ \rangle = \langle \Phi'(u), u^- \rangle = 0\}.$$

Further we will verify that the obtained minimizer is a sign-changing solution to (P). Since it is difficult to show that $\mathcal{M} \neq \emptyset$, we will prove that $\mathcal{M} \neq \emptyset$ by using the parametric method. We will prove that, if $u \in X_0$ with $u^\pm \neq 0$ then there exists a unique pair $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $su^+ + tu^- \in \mathcal{M}$. Finally to conclude that the minimizer of the constrained problem is a sign-changing solution, we use the quantitative deformation lemma (see Lemma 2.3 of [41]) and the Brouwer degree theory.

LEMMA 3.1. *Let the assumptions of Theorem 1.4 holds, then there exist $\lambda_1 > 0$ and $\mu_1, \mu_2 > 0$ such that*

- (i): $\|u^\pm\| \geq \mu_1, \forall u \in \mathcal{M}$;
- (ii): $\int_\Omega |u^\pm|^q dx \geq \mu_2, \forall u \in \mathcal{M}$.

PROOF. We have $\langle \Phi'(u), u^\pm \rangle = 0$ for every $u \in \mathcal{M}$. Therefore,

$$\lambda \int_\Omega g(u)u^\pm dx + \int_\Omega f(x, u)u^\pm dx = \lambda \int_\Omega g(u^\pm)u^\pm dx + \int_\Omega f(x, u^\pm)u^\pm dx.$$

By a simple computation one can obtain

$$\langle \Phi'(u), u^+ \rangle = \langle \Phi'(u^+), u^+ \rangle + 2C_1^+(u),$$

where

$$C_1^+(u) = \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x) - u^-(y)|^{p-1} u^+(x)}{|x - y|^{N+p\alpha}} dx dy - \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x)|^p}{|x - y|^{N+p\alpha}} dx dy > 0.$$

Therefore, $\langle \Phi'(u^+), u^+ \rangle < 0$ and hence it follows that

$$\|u^+\|^p < \lambda \int_{\Omega} g(u^+) u^+ dx + \int_{\Omega} f(x, u^+) u^+ dx.$$

Similarly, we obtain

$$\|u^-\|^p < \lambda \int_{\Omega} g(u^-) u^- dx + \int_{\Omega} f(x, u^-) u^- dx.$$

Now by the assumptions $(f_1) - (f_2)$, we have for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$(3.1) \quad f(x, \tau) \tau \leq \epsilon |\tau|^p + C_\epsilon |\tau|^q, \forall x \in \bar{\Omega}, \forall \tau \in \mathbb{R}.$$

Therefore, by the Sobolev inequality and the growth condition of g , there exists $C_1, C_2 > 0$ such that

$$(3.2) \quad \|u^\pm\|^p \leq \epsilon C_1 \|u^\pm\|^p + C_\epsilon C_1 \|u^\pm\|^q + C_2 \lambda \|u^\pm\|^{1-\delta}.$$

We now choose, $\lambda > 0$ (say, λ_1) very small such that

$$(3.3) \quad C_2 \lambda_1 \|u^\pm\|^{1-\delta} \leq \frac{1}{4} \|u^\pm\|^p.$$

Since $q \in (p, p_\alpha^*)$ then by using (3.2), (3.3) and for $\epsilon = \frac{1}{2C_1}$, one can see (i) holds. Again, by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} \mu_1^p &\leq \|u^\pm\|^p \\ &\leq \epsilon C_1 \|u^\pm\|^p + C_\epsilon \|u^\pm\|_q^q + C_2 \lambda \|u^\pm\|^{1-\delta} \\ &\leq \epsilon C_1 \|u^\pm\|^p + C_\epsilon \|u^\pm\|_q^q + \frac{1}{4} \|u^\pm\|^p. \end{aligned}$$

Thus for $\epsilon = \frac{1}{2C_1}$, we can obtain that

$$\|u^\pm\|_q^q \geq \frac{\mu_1^p}{4C_\epsilon} = \mu_2.$$

This completes the proof. \square

LEMMA 3.2. *Let $u \in X_0$ be such that $u^\pm \neq 0$. Then there exists a unique pair $(t_u, s_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $t_u u^+ + s_u u^- \in \mathcal{M}$.*

PROOF. For every $t, s > 0$, let us define g_1 and g_2 as

$$\begin{aligned}
g_1(t, s) &= \langle \Phi'(tu^+ + su^-), tu^+ \rangle \\
&= \int_{\Omega^+} \int_{\Omega^+} \frac{|tu^+(x) - tu^+(y)|^p}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^+} \int_{\Omega^c} \frac{|tu^+(x)|^p}{|x - y|^{N+p\alpha}} dx dy \\
&\quad + \int_{\Omega^c} \int_{\Omega^+} \frac{|tu^+(y)|^p}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^+} \int_{\Omega^-} \frac{|tu^+(x) - su^-(y)|^{p-1} tu^+(x)}{|x - y|^{N+p\alpha}} dx dy \\
&\quad + \int_{\Omega^-} \int_{\Omega^+} \frac{|su^-(x) - tu^+(y)|^{p-1} tu^+(y)}{|x - y|^{N+p\alpha}} dx dy - \lambda \int_{\Omega} g(tu^+) tu^+ dx \\
&\quad - \int_{\Omega} f(x, tu^+) tu^+ dx
\end{aligned}$$

and

$$\begin{aligned}
g_2(t, s) &= \langle \Phi'(tu^+ + su^-), su^- \rangle \\
&= \int_{\Omega^+} \int_{\Omega^-} \frac{|tu^+(x) - su^-(y)|^{p-1} (-su^-(y))}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^c} \int_{\Omega^-} \frac{|-su^-(y)|^p}{|x - y|^{N+p\alpha}} dx dy \\
&\quad + \int_{\Omega^-} \int_{\Omega^+} \frac{|su^-(x) - tu^+(y)|^{p-1} (-su^-(x))}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^-} \int_{\Omega^c} \frac{|su^-(x)|^p}{|x - y|^{N+p\alpha}} dx dy \\
&\quad + \int_{\Omega^-} \int_{\Omega^-} \frac{|su^-(x) - su^-(y)|^p}{|x - y|^{N+p\alpha}} dx dy - \lambda \int_{\Omega} g(su^-) su^- dx \\
&\quad - \int_{\Omega} f(x, su^-) su^- dx.
\end{aligned}$$

Now by using (f_4) , we have for any $C_1 > 0$ there exists $C_2 > 0$ such that

$$(3.4) \quad f(x, \tau)\tau \geq C_1 |\tau|^p - C_2, \quad \forall x \in \overline{\Omega}, \forall \tau \in \mathbb{R}.$$

Therefore, by using $q \in (p, p_\alpha^*)$, (3.1), (3.4) and Lemma 3.1, there exist $r_1 > 0$, $\lambda > 0$ small enough and $R_1 > 0$ large enough such that

$$(3.5) \quad g_1(t, t) > 0, \quad g_2(t, t) > 0, \quad \forall t \in (0, r_1),$$

$$(3.6) \quad g_1(t, t) < 0, \quad g_2(t, t) < 0, \quad \forall t \in (R_1, +\infty).$$

Observe that, for a fixed $t > 0$, $g_1(t, s)$ is increasing in s on $(0, +\infty)$ and for a fixed $s > 0$, $g_2(t, s)$ is increasing in t on $(0, +\infty)$. Therefore, by using (3.5) and (3.6) there exist $\lambda > 0$, $r > 0$ and $R > 0$ with $r < R$ such that

$$(3.7) \quad g_1(r, s) > 0, \quad g_1(R, s) < 0, \quad \forall s \in (r, R],$$

$$(3.8) \quad g_2(t, r) > 0, \quad g_2(t, R) < 0, \quad \forall t \in (r, R].$$

Now by applying the Miranda's theorem [27], $g_1(t_u, s_u) = g_2(t_u, s_u) = 0$, for some $t_u, s_u \in [r, R]$. This implies that $t_u u^+ + s_u u^- \in \mathcal{M}$.

We now prove the uniqueness of the pair (t_u, s_u) . Assume that there exists (t_1, s_1) and (t_2, s_2) such that $t_i u^+ + s_i u^- \in \mathcal{M}$, $i = 1, 2$. We prove the uniqueness by dividing into two cases.

Case 1. Let $u \in \mathcal{M}$.

Without loss of generality, we assume that $(t_1, s_1) = (1, 1)$ and $t_2 \leq s_2$. Now, for $u \in X_0$, we define

$$\begin{aligned} A^+(u) &= \int_{\Omega^+} \int_{\Omega^+} \frac{|u^+(x) - u^+(y)|^p}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^+} \int_{\Omega^c} \frac{|u^+(x)|^p}{|x - y|^{N+p\alpha}} dx dy \\ &\quad + \int_{\Omega^c} \int_{\Omega^+} \frac{|u^+(y)|^p}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x) - u^-(y)|^{p-1} u^+(x)}{|x - y|^{N+p\alpha}} dx dy \\ &\quad + \int_{\Omega^-} \int_{\Omega^+} \frac{|u^-(x) - u^+(y)|^{p-1} u^+(y)}{|x - y|^{N+p\alpha}} dx dy \end{aligned}$$

and

$$\begin{aligned} A^-(u) &= \int_{\Omega^-} \int_{\Omega^-} \frac{|u^-(x) - u^-(y)|^p}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^-} \int_{\Omega^c} \frac{|u^-(x)|^p}{|x - y|^{N+p\alpha}} dx dy \\ &\quad + \int_{\Omega^c} \int_{\Omega^-} \frac{|-u^-(y)|^p}{|x - y|^{N+p\alpha}} dx dy + \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x) - u^-(y)|^{p-1} (-u^-(y))}{|x - y|^{N+p\alpha}} dx dy \\ &\quad + \int_{\Omega^-} \int_{\Omega^+} \frac{|u^-(x) - u^+(y)|^{p-1} (-u^-(x))}{|x - y|^{N+p\alpha}} dx dy. \end{aligned}$$

Since $u \in \mathcal{M}$, therefore by using $\langle \Phi'(u), u^+ \rangle = \langle \Phi'(u), u^- \rangle = 0$, we get

$$(3.9) \quad A^+(u) = \int_{\Omega} f(x, u^+) u^+ dx + \lambda \int_{\Omega} g(u^+) u^+ dx,$$

$$(3.10) \quad A^-(u) = \int_{\Omega} f(x, u^-) u^- dx + \lambda \int_{\Omega} g(u^-) u^- dx.$$

Again by using $\langle \Phi'(t_2 u^+ + s_2 u^-), t_2 u^+ \rangle = 0 = \langle \Phi'(t_2 u^+ + s_2 u^-), s_2 u^- \rangle$, we have

$$\begin{aligned} (3.11) \quad &t_2^p (A^+(u) + B_1^+(u) + B_2^+(u)) \\ &= \int_{\Omega} f(x, t_2 u^+) t_2 u^+ dx + \lambda \int_{\Omega} g(t_2 u^+) t_2 u^+ dx \end{aligned}$$

and

$$\begin{aligned} (3.12) \quad &s_2^p (A^-(u) + B_1^-(u) + B_2^-(u)) \\ &= \int_{\Omega} f(x, s_2 u^-) s_2 u^- dx + \lambda \int_{\Omega} g(s_2 u^-) s_2 u^- dx, \end{aligned}$$

where

$$\begin{aligned} B_1^+(u) &= \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x) - \frac{s_2}{t_2} u^-(y)|^{p-1} u^+(x)}{|x - y|^{N+p\alpha}} dx dy \\ &\quad - \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x) - u^-(y)|^{p-1} u^+(x)}{|x - y|^{N+p\alpha}} dx dy, \\ B_2^+(u) &= \int_{\Omega^-} \int_{\Omega^+} \frac{|\frac{s_2}{t_2} u^-(x) - u^+(y)|^{p-1} u^+(y)}{|x - y|^{N+p\alpha}} dx dy \\ &\quad - \int_{\Omega^-} \int_{\Omega^+} \frac{|u^-(x) - u^+(y)|^{p-1} u^+(y)}{|x - y|^{N+p\alpha}} dx dy, \end{aligned}$$

$$\begin{aligned}
B_1^-(u) &= \int_{\Omega^+} \int_{\Omega^-} \frac{|\frac{t_2}{s_2}u^+(x) - u^-(y)|^{p-1}(-u^-(y))}{|x-y|^{N+p\alpha}} dx dy \\
&\quad - \int_{\Omega^+} \int_{\Omega^-} \frac{|u^+(x) - u^-(y)|^{p-1}(-u^-(y))}{|x-y|^{N+p\alpha}} dx dy, \\
B_2^-(u) &= \int_{\Omega^-} \int_{\Omega^+} \frac{|u^-(x) - \frac{t_2}{s_2}u^+(y)|^{p-1}(-u^-(x))}{|x-y|^{N+p\alpha}} dx dy \\
&\quad - \int_{\Omega^-} \int_{\Omega^+} \frac{|u^-(x) - u^+(y)|^{p-1}(-u^-(x))}{|x-y|^{N+p\alpha}} dx dy.
\end{aligned}$$

Furthermore, $t_2 \leq s_2$ implies that $B_1^+(u), B_2^+(u) \geq 0$. Hence, from (3.9) and (3.11) we have

$$\begin{aligned}
(3.13) \quad & \int_{\Omega} \left(\frac{f(x, t_2 u^+)}{|t_2 u^+|^{p-2} t_2 u^+} - \frac{f(x, u^+)}{|u^+|^{p-2} u^+} \right) |u^+|^p \\
& \quad + \lambda \int_{\Omega} \left(\frac{g(t_2 u^+)}{|t_2 u^+|^{p-2} t_2 u^+} - \frac{g(u^+)}{|u^+|^{p-2} u^+} \right) |u^+|^p \\
& = \int_{\Omega} I_1 + \int_{\Omega} I_2 \\
& \geq 0,
\end{aligned}$$

where

$$I_1 = \left(\frac{f(x, t_2 u^+)}{|t_2 u^+|^{p-2} t_2 u^+} - \frac{f(x, u^+)}{|u^+|^{p-2} u^+} \right) |u^+|^p$$

and

$$I_2 = \lambda \left(\frac{g(t_2 u^+)}{|t_2 u^+|^{p-2} t_2 u^+} - \frac{g(u^+)}{|u^+|^{p-2} u^+} \right) |u^+|^p.$$

Claim. $t_2 \geq 1$.

To prove our claim, we consider the following four possibilities.

- I.** When $I_1 > 0, I_2 > 0$: Now, $I_1 > 0$ implies that $t_2 \geq 1$ by (f_5) . Again, $I_2 > 0$ implies $t_2 \leq 1$ by using (g_1) - (g_2) . Therefore, on combining both the cases, we get $t_2 = 1$.
- II.** When $I_1 > 0, I_2 < 0$: Since, $I_2 < 0$, we have $\int_{\Omega} I_1 + \int_{\Omega} I_2 \leq \int_{\Omega} I_1$. Therefore, by (f_5) , $I_1 > 0$ implies $t_2 \geq 1$ and similarly, $I_2 < 0$ implies $t_2 \geq 1$ by (g_1) - (g_2) . Thus $t_2 \geq 1$.
- III.** When $I_1 < 0, I_2 > 0$: Since, $I_1 < 0$, we may choose, $\lambda > 0$ small enough such that $\int_{\Omega} I_1 + \int_{\Omega} I_2 \leq 0$, which is a contradiction to (3.13).
- IV.** When $I_1 < 0, I_2 < 0$: In this case, both $I_1 < 0$ and $I_2 < 0$, yield $\int_{\Omega} I_1 + \int_{\Omega} I_2 \leq 0$, which is a contradiction to (3.13).

Therefore, we can conclude that $t_2 \geq 1$. Again, from $B_1^-(u), B_2^-(u) \leq 0$, we have by using (3.10) and (3.12) that

$$\begin{aligned}
0 &\geq \int_{\Omega} \left[\frac{f(x, s_2 u^-)}{|s_2 u^-|^{p-2} s_2 u^-} - \frac{f(x, u^-)}{|u^-|^{p-2} u^-} \right] |u^-|^p dx \\
&\quad + \lambda \int_{\Omega} \left[\frac{g(s_2 u^-)}{|s_2 u^-|^{p-2} s_2 u^-} - \frac{g(u^-)}{|u^-|^{p-2} u^-} \right] |u^-|^p dx.
\end{aligned}$$

By proceeding as in the above proof of the claim together with (f_5) , (g_1) - (g_2) , one can prove that $s_2 \leq 1$. Hence, $t_2 = s_2 = 1$.

Case 2. Let $u \notin \mathcal{M}$.

Let $v_1 = t_1 u^+ + s_1 u^-$ and $v_2 = t_2 u^+ + s_2 u^-$. Again, by using the above arguments as in the proof of the claim in Case 1, it is easy to prove that $\frac{t_2}{t_1} = \frac{s_2}{s_1} = 1$. Hence, $(t_1, s_1) = (t_2, s_2)$. This completes the proof. \square

LEMMA 3.3. *Assume that (f_1) - (f_5) and (g_1) - (g_2) holds. Then there exists $u \in \mathcal{M}$ such that $\Phi(u) = m_\alpha$, where $m_\alpha = \inf_{u \in \mathcal{M}} \Phi(u)$.*

PROOF. Clearly, by the above Lemma 3.2, we have $\mathcal{M} \neq \emptyset$. Consider a minimizing sequence $\{u_n\} \subset \mathcal{M}$ such that $\Phi(u_n) \rightarrow m_\alpha$ as $n \rightarrow \infty$.

Claim: The sequence $\{u_n\}$ is uniformly bounded in X_0 .

Proof. We will prove it by contradiction. Let us assume that $\lim_{n \rightarrow \infty} \|u_n\| \rightarrow \infty$. We set $w_n = \frac{u_n}{\|u_n\|}$. Clearly, $\|z_n\| = 1$ and upto a subsequence, there exists $w_0 \in X_0$ such that

- (i) $w_n \rightarrow w_0$ in X_0 ,
- (ii) $w_n \rightarrow w_0$ in $L^r(\Omega)$ for all $r \in [1, p_\alpha^*)$ and
- (iii) $w_n(x) \rightarrow w_0(x)$ almost everywhere in Ω .

We further claim that $w_0 = 0$. Suppose not, define $\Omega_1 = \{x \in \Omega : w_0(x) \neq 0\}$, then by (f_4) and Fatou's lemma, we get

$$\begin{aligned} \frac{1}{p} - \frac{m_\alpha + o(1)}{\|u_n\|^p} &= \frac{1}{p} - \frac{\Phi(u_n)}{\|u_n\|^p} \\ &= \int_{\Omega} \frac{F(x, u_n)}{u_n^p} w_n^p dx + \int_{\Omega} \frac{G(u_n)}{u_n^p} w_n^p dx \\ &\geq \int_{\Omega_1} \frac{F(x, u_n)}{u_n^p} w_n^p dx + \int_{\Omega_1} \frac{G(u_n)}{u_n^p} w_n^p dx \\ &\geq \int_{\Omega_1} \frac{F(x, u_n)}{u_n^p} w_n^p dx \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

This is a contradiction. Thus, $w_0 \equiv 0$. Therefore, $\{u_n\}$ is uniformly bounded in X_0 . Then there exists $u^* \in X_0$ such that

$$(3.14) \quad u_n^\pm \rightharpoonup (u^*)^\pm \text{ in } X_0,$$

$$(3.15) \quad u_n^\pm \rightarrow (u^*)^\pm \text{ in } L^r(\Omega) \text{ for } r \in [1, p_\alpha^*) \text{ and}$$

$$(3.16) \quad u_n(x) \rightarrow u^*(x) \text{ a.e. } x \in \Omega.$$

From Lemma 3.1, we have $(u^*)^\pm \neq 0$. In addition, under the assumptions (f_1) - (f_2) and (g_1) - (g_2) , by using the compact embedding $X_0 \hookrightarrow L^r(\Omega)$ for $r \in [1, p_\alpha^*)$ and by applying some standard arguments (see [41]), we get that

$$(3.17) \quad \lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n^\pm) u_n^\pm dx = \int_{\Omega} f(x, (u^*)^\pm) (u^*)^\pm dx,$$

$$(3.18) \quad \lim_{n \rightarrow \infty} \int_{\Omega} F(x, u_n^\pm) dx = \int_{\Omega} F(x, (u^*)^\pm) dx,$$

$$(3.19) \quad \lim_{n \rightarrow \infty} \int_{\Omega} g(u_n^\pm) u_n^\pm dx = \int_{\Omega} g((u^*)^\pm) (u^*)^\pm dx \text{ and}$$

$$(3.20) \quad \lim_{n \rightarrow \infty} \int_{\Omega} G(u_n^\pm) dx = \int_{\Omega} G((u^*)^\pm) dx.$$

From Lemma 3.2, we can conclude that there exists a pair (t^*, s^*) with $t^*, s^* > 0$ such that $t^*(u^*)^+ + s^*(u^*)^- \in \mathcal{M}$. This implies

$$(3.21) \quad \begin{aligned} & (t^*)^p [A^+(u^*) + B_1^+(u^*) + B_2^+(u^*)] \\ &= \int_{\Omega} f(x, t^*(u^*)^+) t^*(u^*)^+ dx + \int_{\Omega} g(t^*(u^*)^+) t^*(u^*)^+ dx \text{ and} \end{aligned}$$

$$(3.22) \quad \begin{aligned} & (s^*)^p [A^-(u^*) + B_1^-(u^*) + B_2^-(u^*)] \\ &= \int_{\Omega} f(x, s^*(u^*)^-) s^*(u^*)^- dx + \int_{\Omega} g(s^*(u^*)^-) s^*(u^*)^- dx. \end{aligned}$$

We now prove that $t^*, s^* \leq 1$. Since the minimizing sequence $\{u_n\} \subset \mathcal{M}$, we get $\langle \Phi'(u_n), u_n^{\pm} \rangle = 0$ which implies that

$$(3.23) \quad A^{\pm}(u_n) = \int_{\Omega} f(x, u_n^{\pm}) u_n^{\pm} dx + \int_{\Omega} g(u_n^{\pm}) u_n^{\pm} dx.$$

Therefore, by using the above inequalities (3.14)-(3.21) and Fatou's lemma, we obtain

$$(3.24) \quad A^{\pm}(u^*) \leq \int_{\Omega} f(x, (u^*)^{\pm}) (u^*)^{\pm} dx + \int_{\Omega} g((u^*)^{\pm}) (u^*)^{\pm} dx.$$

Furthermore, without loss of generality, we assume that $t^* \leq s^*$. Again from (3.22) and (3.24) and the fact $B_1^-(u^*), B_2^-(u^*) \leq 0$, we get

$$(3.25) \quad \begin{aligned} 0 &\leq \int_{\Omega} \left[\frac{f(x, (u^*)^-)}{|(u^*)^-|^{p-2} (u^*)^-} - \frac{f(x, s^*(u^*)^-)}{|s^*(u^*)^-|^{p-2} s^*(u^*)^-} \right] |(u^*)^-|^p dx \\ &+ \lambda \int_{\Omega} \left[\frac{g((u^*)^-)}{|(u^*)^-|^{p-2} (u^*)^-} - \frac{g(s^*(u^*)^-)}{|s^*(u^*)^-|^{p-2} s^*(u^*)^-} \right] |u^+|^p dx. \end{aligned}$$

Consider the four possibilities of the Claim as in the proof of Case 1 in Lemma 3.2. Proceeding on similar arguments as in the proof of Case 1 in Lemma 3.2 and by using (3.25), one can easily obtain $s^* \leq 1$. Therefore, we have $0 < t^* \leq s^* \leq 1$. Let us define, $\mathcal{H}(x, \tau) = f(x, \tau)\tau - pF(x, \tau)$ and $\mathcal{H}_1(\tau) = g(\tau)\tau - pG(\tau)$. Then, from (f_5) we have $\mathcal{H}(x, \tau)$ is increasing with respect to τ on $(0, +\infty)$, decreasing with respect to τ on $(-\infty, 0)$ and $\mathcal{H}(x, \tau) \geq 0$. Again by (g_1) - (g_2) , we have $\mathcal{H}_1(\tau) \leq 0$ for $\tau \in \mathbb{R} \setminus \{0\}$. Therefore, by the definition of Φ and Fatou's lemma, we get

$$\begin{aligned} m_{\alpha} &\leq \Phi(t^*(u^*)^+ + s^*(u^*)^-) \\ &= \Phi(t^*(u^*)^+ + s^*(u^*)^-) - \frac{1}{p} \langle \Phi'(t^*(u^*)^+ + s^*(u^*)^-), t^*(u^*)^+ + s^*(u^*)^- \rangle \\ &= \frac{1}{p} \int_{\Omega} \mathcal{H}(x, t^*(u^*)^+ + s^*(u^*)^-) dx + \frac{1}{p} \int_{\Omega} \mathcal{H}_1(t^*(u^*)^+ + s^*(u^*)^-) dx \\ &\leq \frac{1}{p} \int_{\Omega} \mathcal{H}(x, t^*(u^*)^+ + s^*(u^*)^-) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p} \left[\int_{\Omega^+} \mathcal{H}(x, t^*(u^*)^+) dx + \int_{\Omega^-} \mathcal{H}(x, s^*(u^*)^-) dx \right] \\
&\leq \frac{1}{p} \left[\int_{\Omega^+} \mathcal{H}(x, (u^*)^+) dx + \int_{\Omega^-} \mathcal{H}(x, (u^*)^-) dx \right] \\
&\leq \liminf_{n \rightarrow \infty} \frac{1}{p} \int_{\Omega} \mathcal{H}(x, u_n) dx \\
&= \lim_{n \rightarrow \infty} \left[\Phi(u_n) - \frac{1}{p} \langle \Phi'(u_n), u_n \rangle \right] \\
&= m_\alpha.
\end{aligned}$$

Thus, we conclude that $t^* = s^* = 1$ and hence $\Phi(u^*) = m_\alpha$. This completes the proof. \square

LEMMA 3.4. *Let $u \in \mathcal{M}$. Then for every $t, s \geq 0$ with $(t, s) \neq (1, 1)$, we have*

$$\Phi(u) > \Phi(tu^+ + su^-).$$

PROOF. For each $u \in X_0$ such that $u^\pm \neq 0$, let us define

$$I_u : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$$

such that

$$I_u(t, s) = \Phi(tu^+ + su^-), \quad \forall t, s \geq 0.$$

Observe that from (f_4) , we get

$$\lim_{|(t,s)| \rightarrow \infty} I_u(t, s) = -\infty.$$

Therefore, I_u admits a global maximum at some $(t_0, s_0) \in [0, \infty) \times [0, \infty)$. We now prove that $t_0 > 0, s_0 > 0$ by showing that the other three possibilities can not hold, which are as follow.

- (i) $t_0 = s_0 = 0$,
- (ii) $t_0 > 0, s_0 = 0$ and
- (iii) $t_0 = 0, s_0 > 0$.

Let $s_0 = 0$. Since I_u has a global maximum at (t_0, s_0) , then $\Phi(t_0 u^+) \geq \Phi(tu^+)$ for every $t > 0$. Therefore, we have $\langle \Phi'(t_0 u^+), t_0 u^+ \rangle = 0$, which implies

$$(3.26) \quad t_0^p \|u^+\|^p = \int_{\Omega} f(x, t_0 u^+) t_0 u^+ dx + \int_{\Omega} g(t_0 u^+) t_0 u^+ dx.$$

Again, since $u \in \mathcal{M}$, we get $\langle \Phi'(u^+), u^+ \rangle < 0$, i.e.

$$\|u^+\|^p < \int_{\Omega} f(x, u^+) u^+ dx + \int_{\Omega} g(u^+) u^+ dx.$$

Now by using this inequality and (3.26), we get

$$\begin{aligned}
0 &< \int_{\Omega} \left[\frac{f(x, u^+)}{|u^+|^{p-2} u^+} - \frac{f(x, t_0 u^+)}{|t_0 u^+|^{p-2} t_0 u^+} \right] |u^+|^p dx \\
&\quad + \int_{\Omega} \left[\frac{g(u^+)}{|u^+|^{p-2} u^+} - \frac{g(t_0 u^+)}{|t_0 u^+|^{p-2} t_0 u^+} \right] |u^+|^p dx.
\end{aligned}$$

Again, by repeating similar arguments as in the Claim of Case 1 of the Lemma 3.2 together with (g_1) - (g_2) and (f_5) , we get $t_0 \leq 1$. Furthermore, $\mathcal{H}(x, \tau) \geq 0, \forall x \in \bar{\Omega}$

and $\tau \in \mathbb{R}$ and $\mathcal{H}_1(\tau) \leq 0, \forall \tau \in \mathbb{R} \setminus \{0\}$. In addition, $\mathcal{H}(x, \tau)$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$ with respect to τ . Therefore, we have

$$\begin{aligned}
 I_u(t_0, 0) &= \Phi(t_0 u^+) \\
 &= \Phi(t_0 u^+) - \frac{1}{p} \langle \Phi'(t_0 u^+), t_0 u^+ \rangle \\
 &= \frac{1}{p} \int_{\Omega} \mathcal{H}(x, t_0 u^+) dx + \frac{1}{p} \int_{\Omega} \mathcal{H}_1(t_0 u^+) dx \\
 &\leq \frac{1}{p} \int_{\Omega} \mathcal{H}(x, t_0 u^+) dx \\
 &\leq \frac{1}{p} \int_{\Omega^+} \mathcal{H}(x, u^+) dx \\
 &< \frac{1}{p} \left[\int_{\Omega^+} \mathcal{H}(x, u^+) dx + \int_{\Omega^-} \mathcal{H}(x, u^-) dx \right] \\
 &= \Phi(u) - \frac{1}{p} \langle \Phi'(u), u \rangle \\
 &= \Phi(u) = I_u(1, 1).
 \end{aligned}$$

This contradicts that I_u has a global maximum at (t_0, s_0) . Hence, $s_0 > 0$. Similarly, we can prove that $t_0 > 0$. Finally, Lemma 3.2 guarantees that $(1, 1)$ is the unique critical point of I_u in $(0, \infty) \times (0, \infty)$. This readily implies that, if $t_0, s_0 \in (0, 1]$ such that $(t_0, s_0) \neq (1, 1)$, then we have

$$I_u(t_0, s_0) < I_u(1, 1).$$

This completes the proof. \square

The following lemma concludes the existence of a critical point of Φ , which is a least energy solution to our problem (P).

LEMMA 3.5. *Let there exists $u^* \in \mathcal{M}$ be such that $\Phi(u^*) = m_{\alpha}$. Then u^* is a critical point of Φ , i.e. $\Phi'(u^*) = 0$.*

PROOF. We will prove by method of contradiction. Let $\Phi'(u^*) \neq 0$, then there exist $\rho_1, \mu_1 > 0$ such that

$$\|\Phi'(u)\| \geq \rho_1, \quad \forall B_{3\mu_1}(u^*),$$

where $B_{3\mu_1}(u^*) = \{u \in X_0 : \|u - u^*\| \leq 3\mu_1\}$ a closed ball of radius $3\mu_1$ in X_0 centered at u^* . Now, $u^* \in \mathcal{M}$ implies that $(u^*)^{\pm} \neq 0$, then we can choose a sufficiently small $\mu_1 > 0$ such that $u^{\pm} \neq 0$ for all $u \in B_{3\mu_1}(u^*)$. For sufficiently small $\delta_1 \in (0, \frac{1}{2})$, let us define $D = (1 - \delta_1, 1 + \delta_1) \times (1 - \delta_1, 1 + \delta_1)$ such that $t(u^*)^+ + s(u^*)^- \in B_{3\mu_1}(u^*)$ for all $(t, s) \in \overline{D}$. From Lemma 3.4, one can say

$$(3.27) \quad \tilde{m}_{\alpha} = \max_{(t,s) \in \partial D} \Phi(t(u^*)^+ + s(u^*)^-) < m_{\alpha}.$$

Choose, $\epsilon_1 = \min\{\frac{m_{\alpha} - \tilde{m}_{\alpha}}{2}, \frac{\rho_1 \mu_1}{8}\}$. Therefore, from the quantitative deformation lemma (see Lemma 2.3 of [41]), it follows that there exists a continuous map $\eta : \mathbb{R} \times X_0 \rightarrow X_0$ such that

- (i) $\eta(1, u) = u$ if $u \notin \Phi^{-1}[m_{\alpha} - 2\epsilon_1, m_{\alpha} + 2\epsilon_1] \cap B_{2\mu_1}(u^*)$,
- (ii) $\eta(1, \Phi^{m_{\alpha} + \epsilon_1} \cap B_{\mu_1}(u^*)) \subset \Phi^{m_{\alpha} - \epsilon_1}$ and
- (iii) $\Phi(\eta(1, u)) \leq \Phi(u), \forall u \in X_0$.

We define, $\sigma(t, s) = \eta(1, t(u^*)^+ + s(u^*)^-), \forall (t, s) \in \overline{D}$. Thus from the Lemma 3.4 together with (ii)-(iii) of the deformation lemma, we get

$$(3.28) \quad \max_{(t,s) \in \overline{D}} \Phi(\eta(1, t(u^*)^+ + s(u^*)^-)) < m_\alpha,$$

which implies that $\{\sigma(t, s) : (t, s) \in \overline{D}\} \cap \mathcal{M} = \emptyset$. Again by the following argument, we will prove that $\{\sigma(t, s) : (t, s) \in \overline{D}\} \cap \mathcal{M} \neq \emptyset$ to arrive at a contradiction. Now for $(t, s) \in \overline{D}$, we define

$$\begin{aligned} J_1(t, s) &= (\langle \Phi'(t(u^*)^+ + s(u^*)^-), (u^*)^+ \rangle, \langle \Phi'(t(u^*)^+ + s(u^*)^-), (u^*)^- \rangle), \\ J_2(t, s) &= \left(\frac{1}{t} \langle \Phi'(\sigma(t, s)), \sigma^+(t, s) \rangle, \frac{1}{s} \langle \Phi'(\sigma(t, s)), \sigma^-(t, s) \rangle \right). \end{aligned}$$

Since $f, g \in C^1$, the functional J_1 is C^1 . Therefore, from $\langle \Phi'(u^*), (u^*)^\pm \rangle = 0$ we get

$$\begin{aligned} & \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} (u^*(x) - u^*(y)) ((u^*)^+(x) - (u^*)^+(y))}{|x - y|^{N+p\alpha}} dx dy \\ &= \int_\Omega f(x, (u^*)^+) (u^*)^+ dx + \int_\Omega g((u^*)^+) (u^*)^+ dx \end{aligned}$$

and

$$\begin{aligned} & \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} (u^*(x) - u^*(y)) ((u^*)^-(x) - (u^*)^-(y))}{|x - y|^{N+p\alpha}} dx dy \\ &= \int_\Omega f(x, (u^*)^-) (u^*)^- dx + \int_\Omega g((u^*)^-) (u^*)^- dx. \end{aligned}$$

From (g_1) - (g_2) and (f_5) , we have $\mathcal{H}'(x, \tau)\tau = f'(x, \tau)\tau^2 - (p-1)f(x, \tau)\tau > 0$ for all $\tau \in \mathbb{R} \setminus \{0\}$. We denote

$$\begin{aligned} \alpha_1 &= \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} |(u^*)^+(x) - (u^*)^+(y)|^2}{|x - y|^{N+p\alpha}} dx dy, \\ \alpha_2 &= \int_\Omega f'_u(x, (u^*)^+) |(u^*)^+|^2 dx + \int_\Omega g'_u((u^*)^+) |(u^*)^+|^2 dx, \\ \alpha_3 &= \int_\Omega f(x, (u^*)^+) (u^*)^+ dx + \int_\Omega g((u^*)^+) (u^*)^+ dx, \\ \beta_1 &= \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} |(u^*)^-(x) - (u^*)^-(y)|^2}{|x - y|^{N+p\alpha}} dx dy, \\ \beta_2 &= \int_\Omega f'_u(x, (u^*)^-) |(u^*)^-|^2 dx + \int_\Omega g'_u((u^*)^-) |(u^*)^-|^2 dx, \\ \beta_3 &= \int_\Omega f(x, (u^*)^-) (u^*)^- dx + \int_\Omega g((u^*)^-) (u^*)^- dx, \\ \gamma_1 &= \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} ((u^*)^-(x) - (u^*)^-(y)) ((u^*)^+(x) - (u^*)^+(y))}{|x - y|^{N+p\alpha}} dx dy \text{ and} \\ \gamma_2 &= \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} ((u^*)^+(x) - (u^*)^+(y)) ((u^*)^-(x) - (u^*)^-(y))}{|x - y|^{N+p\alpha}} dx dy. \end{aligned}$$

It is easy to observe that

$$\begin{aligned} \alpha_1 &> 0, \quad \alpha_2 > (p-1)\alpha_3 > 0, \\ \beta_1 &> 0, \quad \beta_2 > (p-1)\beta_3 > 0, \\ \gamma_1 &= \int_Q \frac{|u^*(x) - u^*(y)|^{p-2} (-(u^*)^-(x)(u^*)^+(y) - (u^*)^-(y)(u^*)^+(x))}{|x-y|^{N+p\alpha}} dx dy \\ &= \gamma_2 > 0 \text{ and} \\ \alpha_3 &= \alpha_1 + \gamma_1, \quad \beta_3 = \beta_1 + \gamma_2. \end{aligned}$$

Hence, we get

$$\begin{aligned} \det(J'_1(1, 1)) &= \langle \Phi''(u^*)(u^*)^+, (u^*)^+ \rangle \cdot \langle \Phi''(u^*)(u^*)^-, (u^*)^- \rangle \\ &\quad - \langle \Phi''(u^*)(u^*)^+, (u^*)^- \rangle \cdot \langle \Phi''(u^*)(u^*)^-, (u^*)^+ \rangle \\ &= [(p-1)\alpha_1 - \alpha_2] \cdot [(p-1)\beta_1 - \beta_2] - (p-1)^2 \gamma_1 \cdot \gamma_2 \\ &> (p-1)^2 \gamma_1 \cdot \gamma_2 - (p-1)^2 \gamma_1 \cdot \gamma_2 \\ &= 0. \end{aligned}$$

Therefore, by using the Brouwer degree theory, we get $\deg(J_1, D, 0) = 1$. Again, from (3.28), we have $\sigma(t, s) = t(u^*)^+ + s(u^*)^-$, $\forall (t, s) \in \partial D$. Hence,

$$\deg(J_2, D, 0) = \deg(J_1, D, 0) = 1.$$

Therefore, there exists $(t_0, s_0) \in D$ such that $J_2(t_0, s_0) = 0$. On using the conditions (i)-(ii) in the deformation lemma, one can obtain that

$$u_0 = \sigma(t_0, s_0) = \eta(1, t_0(u^*)^+ + s_0(u^*)^-) \in B_{3\mu_1}(u^*).$$

Therefore, we can say $\langle \Phi'(u_0), u_0^+ \rangle = \langle \Phi'(u_0), u_0^- \rangle = 0$ such that $u_0^\pm \neq 0$, that is, $u_0 \in \{\eta(t, s) : (t, s) \in \bar{D}\} \cap \mathcal{M}$. Hence, we have a contradiction. Thus we conclude that u^* is a critical point of Φ and a least energy sign-changing solution to the problem corresponding to Φ . Finally, since the critical points of Φ are also critical points of I_λ , we have u^* is a critical point of I_λ . Hence u^* is a sign-changing solution to the problem (P). \square

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