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Global solutions of the 3D compressible MHD system in a bounded domain

Jishan Fan, Lulu Jing, Gen Nakamura, and Tong Tang

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Abstract. In this paper we consider the 3D compressible MHD system in a bounded domain. We first prove a regularity criterion and then use it and the bootstrap argument to show the well-posedness of global small solutions. It is important to point out that we do not need the smallness of initial velocity, so we improve the previous results in some senses.

Contents

1. Introduction

Magnetic fields, such as the terrestrial magnetic field and the solar magnetic field, influence many natural and artificial flows. The study of these flows is called magnetohydrodynamics (MHD). In the recent years, there have been lots of studies on MHD by physicists and mathematicians due to its physical importance, complex, rich phenomena and mathematical challenges, such as [**1, 2, 3**]. This paper is

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concerned with the following three-dimensional compressible MHD flows:

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = \operatorname{rot} b \times b, \tag{1.2}$$

$$\partial_t b + \operatorname{rot}(b \times u) - \eta\Delta b = 0, \operatorname{div} b = 0 \ \text{ in } \ \Omega \times (0,\infty), \tag{1.3}$$

$$u = 0, b\cdot n = 0, \operatorname{rot} b \times n = 0 \ \text{ on } \ \partial\Omega \times (0,\infty), \tag{1.4}$$

$$(\rho, u, b)(\cdot, 0) = (\rho_0, u_0, b_0)(\cdot) \ \text{ in } \ \Omega \subset \mathbb{R}^3. \tag{1.5}$$

Here ρ, u and b denote the density, velocity and magnetic field, respectively. $p := a\rho^\gamma$ is the pressure with the constants $a > 0$ and $\gamma > 1$. The constant μ and λ are two viscosity constants satisfying the physical restrictions

$$\mu > 0 \ \text{ and } \ \lambda + \frac{2}{3}\mu \geq 0.$$

η is the resistivity coefficient. Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, n is the unit outward normal vector to the boundary $\partial\Omega$.

We have the well-known vector identities:

$$\operatorname{rot}(b \times u) = u\cdot\nabla b - b\cdot\nabla u + b\operatorname{div} u, \tag{1.6}$$

$$\operatorname{rot} b \times b = \operatorname{div}\left(b \otimes b - \frac{1}{2}|b|^2\mathbb{I}_3\right), \tag{1.7}$$

where $\mathbb{I}_3$ is the identity matrix of order 3.

We will assume the following natural compatibility condition:

$$-\mu\Delta u_0 - (\lambda+\mu)\nabla\operatorname{div} u_0 - \operatorname{rot} b_0 \times b_0 = \sqrt{\rho_0} f, \tag{1.8}$$

for some $f \in L^2(\Omega)$.

Fan and Yu [**4**] show the local well-posedness of strong solutions. Xu and Zhang [**5**] establish the following regularity criterion

$$\rho \in L^\infty(0,T;L^\infty) \ \text{ and } \ u \in L^s(0,T;L^r) \ \text{ with } \ \frac{2}{s} + \frac{3}{r} = 1, 3 < r \leq \infty. \tag{1.9}$$

Zhu and Chen [**6**] show a new regularity criterion:

$$\rho \in L^\infty(0,T;L^{q_1}) \ \text{ and } \ b \in L^\infty(0,T;L^6) \ \text{ for some large } \ q_1 > 3. \tag{1.10}$$

It should be remarked that results of [**5, 6**] are in the whole space $\mathbb{R}^3$.

The aim of this paper is to prove a new regularity criterion

$$7\mu > 9\lambda, \rho \in L^\infty(0,T;L^\infty) \ \text{ and } \ b \in L^{10}(0,T;L^5), \tag{1.11}$$

Then we use it and the bootstrap argument to show the global well-posedness of small strong solutions. We will prove

Theorem 1.1. *Let* $0 \leq \rho_0 \in W^{1,6}, u_0 \in H_0^1 \cap H^2, b_0 \in H^2$ *with* $\operatorname{div} b_0 = 0$ *in* Ω *and* (1.8) *hold true. If* (1.11) *holds true, then*

$$\sup_{0\leq t\leq T}(\|\rho\|_{W^{1,6}} + \|\rho_t\|_{L^6} + \|\sqrt{\rho}u_t\|_{L^2} + \|u\|_{H^2} + \|b\|_{H^2} + \|b_t\|_{L^2}) \leq C,$$

$$\int_0^T (\|u_t\|_{H^1}^2 + \|u\|_{W^{2,6}}^2 + \|b\|_{W^{2,6}}^2 + \|b_t\|_{H^1}^2)\mathrm{d}t \leq C, \tag{1.12}$$

Remark 1.2. Similarly, it is easy to show the regularity criterion (1.9) when Ω is a bounded domain.

REMARK 1.3. Similarly, we can refine (1.10) as

$$\rho \in L^\infty(0,T;L^{q_1}) \ \ \text{and} \ \ b \in L^{10}(0,T;L^5)$$

for some large $q_1 > 3$ when $\Omega := \mathbb{R}^3$. We omit the details here.

THEOREM 1.4. *Let* $0 \le \rho_0 \in W^{1,6}, u_0 \in H_0^1 \cap H^2, b_0 \in H^2$ *with* $\operatorname{div} b_0 = 0$ *in* Ω *and* (1.8) *hold true. If* $7\mu > 9\lambda$ *and* $\|\rho_0\|_{L^\infty} + \|b_0\|_{L^5}$ *is small enough, then* (1.12) *holds true.*

REMARK 1.5. We need not assume that the initial velocity is small.

REMARK 1.6. When the temperature effect is considered, Fan-Yu [**7**], Ducomet-Feireisl [**8**] and Hu-Wang [**9, 10**] proved the existence of global weak solutions. Huang-Li [**11**] (see also [**12**]) show the regularity criterion (1.9).

To prove Theorems 1.4, we will use the following abstract bootstrap argument or continuity argument [**13**, Page 20] (see also [**14, 15**]).

LEMMA 1.7. *(*[**13**]*). Let* $T > 0$. *Assume that two statements* $C(t)$ *and* $H(t)$ *with* $t \in [0,T]$ *satisfy the following conditions:*

(a) *If* $H(t)$ *holds for some* $t \in [0,T]$, *then* $C(t)$ *holds for the same* t;
(b) *If* $C(t)$ *holds for some* $t_0 \in [0,T]$, *then* $H(t)$ *holds for* t *in a neighborhood of* t_0;
(c) *If* $C(t)$ *holds for* $t_m \in [0,T]$ *and* $t_m \to t$, *then* $C(t)$ *holds;*
(d) $C(t)$ *holds for at least one* $t_1 \in [0,T]$.

Then $C(t)$ *holds for all* $t \in [0,T]$.

We now collect several vector identities and the Gauss-Green formula which will be used in the rest of the paper.

LEMMA 1.8. *(*[**16**, Theorem 2.1]*). Let* Ω *be a regular bounded domain in* $\mathbb{R}^3$, $b : \Omega \to \mathbb{R}^3$ *be a sufficiently smooth vector field, and let* $1 < p < \infty$. *Then, the following identity holds.*

$$\begin{aligned} & -\int_\Omega \Delta b \cdot b|b|^{p-2} \mathrm{d}x \\ = & \int_\Omega |b|^{p-2}|\nabla b|^2 \mathrm{d}x + 4\frac{p-2}{p^2}\int_\Omega |\nabla |b|^{\frac{p}{2}}|^2 \mathrm{d}x - \int_{\partial\Omega} |b|^{p-2}(n\cdot\nabla)b\cdot b \mathrm{d}S. \end{aligned} \tag{1.13}$$

Moreover, we have the well-known identity

$$(n\cdot\nabla)b\cdot b = (b\cdot\nabla)b\cdot n + (\operatorname{rot} b \times n)\cdot b. \tag{1.14}$$

LEMMA 1.9. *(*[**17**, Lemma 7.44],[**18**, Corollary 1.7]*). Let a smooth and bounded open set* Ω *be given and let* $1 < p < \infty$. *Then the following inequality holds. There exists a constant* $C > 0$, *such that*

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \tag{1.15}$$

for any $f \in W^{1,p}(\Omega)$.

LEMMA 1.10. *(*[**19**]*). There exists a constant* $C > 0$, *such that*

$$\|f\|_{W^{1,p}(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|\operatorname{div} f\|_{L^p(\Omega)} + \|\operatorname{rot} f\|_{L^p(\Omega)}) \tag{1.16}$$

for any $1 < p < \infty$ *and all* $f \in W^{1,p}(\Omega)$.

When b satisfies $b \cdot n = 0$ on $\partial\Omega$, we will also use the identity

$$(b \cdot \nabla) b \cdot n = -(b \cdot \nabla) n \cdot b \text{ on } \partial\Omega \tag{1.17}$$

for any sufficiently smooth vector field b.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since the local strong solutions to the problem (1.1)-(1.5) was established in [**4**], we only need to show a priori estimates (1.12).

The proof is based on several lemmas.

Lemma 2.1. *We have*

$$\begin{aligned} & \int \left(\rho |u|^2 + |b|^2 + \frac{2a}{\gamma - 1} \rho^\gamma \right) \mathrm{d}x + \\ & 2 \int_0^T \int (\mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + \eta |\operatorname{rot} b|^2) \mathrm{d}x \mathrm{d}t \\ \leq & \int \left(\rho |u_0|^2 + |b_0|^2 + \frac{2a}{\gamma - 1} \rho_0^\gamma \right) \mathrm{d}x. \end{aligned} \tag{2.1}$$

Proof. Testing (1.2) by u and using (1.1), we see that

$$\begin{aligned} & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho |u|^2 \mathrm{d}x + \int (\mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2) \mathrm{d}x \\ = & \int p \operatorname{div} u \mathrm{d}x + \int (\operatorname{rot} b \times b) u \mathrm{d}x. \end{aligned} \tag{2.2}$$

Testing (1.3) by b, we find that

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |b|^2 \mathrm{d}x + \eta \int |\operatorname{rot} b|^2 \mathrm{d}x &= - \int (b \times u) \operatorname{rot} b \mathrm{d}x \\ &= - \int (\operatorname{rot} b \times b) u \mathrm{d}x. \end{aligned} \tag{2.3}$$

Summing up (2.2) and (2.3), and using $p_t + u \cdot \nabla p + \gamma p \operatorname{div} u = 0$, we get

$$\begin{aligned} & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int (\rho |\rho u|^2 + |b|^2) \mathrm{d}x + \int (\mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + \eta |\operatorname{rot} b|^2) \mathrm{d}x \\ = & \int p \operatorname{div} u \mathrm{d}x = -\frac{1}{\gamma - 1} \int p_t \mathrm{d}x, \end{aligned}$$

which gives (2.1).

□

It follows from [**20**, (4.21)] that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho |u|^5 \mathrm{d}x + \frac{C_0}{4} \int |u|^3 |\nabla u|^2 \mathrm{d}x \leq C \int \rho |u|^5 \mathrm{d}x + C \left(\int |b|^5 \mathrm{d}x \right)^2 + C,$$

which gives

$$\int \rho |u|^5 + \int_0^T \int |u|^3 |\nabla u|^2 \mathrm{d}x \mathrm{d}t \leq C, \tag{2.4}$$

and thus

$$\|u\|_{L^5(0,T;L^{15})} \leq C. \tag{2.5}$$

It follows from (2.1) and (2.4) that

$$\int_0^T\int |u|^2|\nabla u|^2\mathrm{d}x\mathrm{d}t \le \int_0^T\int\left(\frac{2}{3}|u|^3+\frac{1}{3}\right)|\nabla u|^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.6}$$

Testing (1.3) by $|b|^4 b$, using (1.13), (1.14), (1.15), (1.17) and (2.5), we have

$$\begin{aligned}
&\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^6}^6+\int |b|^4|\nabla b|^2\mathrm{d}x+\frac{4}{9}\int|\nabla|b|^3|^2\mathrm{d}x\\
= &-\int_{\partial\Omega}|b|^4(b\cdot\nabla)n\cdot b dS-\int(b\times u)\mathrm{rot}\,(|b|^4 b)\mathrm{d}x\\
\le & \ C\int_{\partial\Omega}|b|^6 dS-\int(b\times u)(|b|^4\mathrm{rot}\,b+D|b|^4\times b)\mathrm{d}x\\
\le & \ C\int_{\partial\Omega}a^2 dS+\frac{1}{4}\int|b|^4|\nabla b|^2\mathrm{d}x+C\|u\|_{L^p}^2\|a\|_{L^{\frac{2p}{p-2}}}^2 \quad (a:=|b|^3, p:=15)\\
\le & \ C\|a\|_{L^2(\Omega)}\|a\|_{H^1(\Omega)}+\frac{1}{4}\int|b|^4|\nabla b|^2\mathrm{d}x+C\|u\|_{L^p}\|a\|_{L^2}^{2(1-\theta)}\|a\|_{H^1}^{2\theta} \quad \left(\theta:=\frac{3}{p}\right)\\
\le & \ \frac{2}{9}\int|\nabla|b|^3|^2\mathrm{d}x+\frac{1}{4}\int|b|^4|\nabla b|^2\mathrm{d}x+C\|a\|_{L^2}^2+C\|u\|_{L^p}^{\frac{2}{1-\theta}}\|a\|_{L^2}^2,
\end{aligned}$$

which implies

$$\int|b|^6\mathrm{d}x+\int_0^T\int|b|^4|\nabla b|^2\mathrm{d}x\mathrm{d}t\le C. \tag{2.7}$$

Similarly to (2.6), we observe at

$$\int_0^T\int|b|^2|\mathrm{rot}\,b|^2\mathrm{d}x\mathrm{d}t\le C\int_0^T\int|b|^2|\nabla b|^2\mathrm{d}x\mathrm{d}t\le C. \tag{2.8}$$

Following the argument by [**21**], we let $v:=\mathcal{L}^{-1}\nabla p(\rho)$ be the solution of the Lamé system:

$$\begin{cases}\mathcal{L}v:=\mu\Delta v+(\lambda+\mu)\nabla\mathrm{div}\,v=\nabla p(\rho),\\ v=0 \ \text{ on } \ \partial\Omega.\end{cases} \tag{2.9}$$

Then it follows from [**21**] Proposition 2.1 that

$$\|\nabla v\|_{L^q}\le C\|p(\rho)\|_{L^q}\le C \ \text{ for any } \ 1<q<\infty. \tag{2.10}$$

Denote $w:=u-v$, then w satisfies

$$\begin{cases}\rho w_t-\mathcal{L}w=\rho F+\mathrm{rot}\,b\times b,\\ w=0 \ \text{ on } \ \partial\Omega\times(0,\infty),\\ w(\cdot,0)=u_0-v_0 \ \text{ in } \ \Omega,\end{cases} \tag{2.11}$$

where

$$\begin{aligned}
F: &= -u\cdot\nabla u-\mathcal{L}^{-1}\nabla p_t\\
&= -u\cdot\nabla u+\mathcal{L}^{-1}\nabla\mathrm{div}\,(pu)-\mathcal{L}^{-1}\nabla((p-p'(\rho)\rho)\mathrm{div}\,u).
\end{aligned}$$

Testing $(2.11)_1$ by w_t, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int(\mu|\nabla w|^2+(\lambda+\mu)(\operatorname{div} w)^2)\mathrm{d}x+\int\rho|w_t|^2\mathrm{d}x\\
\leq\ & \|\sqrt{\rho}F\|_{L^2}^2-2\frac{\mathrm{d}}{\mathrm{d}t}\int\left(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\right):\nabla w\mathrm{d}x+C\int|b||b_t||\nabla w|\mathrm{d}x\\
=\ & :I_1+I_2+I_3.
\end{aligned}\tag{2.12}$$

For I_1, we have

$$\begin{aligned}
I_1\ \lesssim\ & \|\sqrt{\rho}u\cdot\nabla u\|_{L^2}^2+\|\sqrt{\rho}\mathcal{L}^{-1}\nabla\operatorname{div}(pu)\|_{L^2}^2+\|\sqrt{\rho}\mathcal{L}^{-1}\nabla((p-p'(\rho)\rho)\operatorname{div} u)\|_{L^2}^2\\
=\ & :\sum_{j=1}^{3}I_{1j}.
\end{aligned}\tag{2.13}$$

For I_{11}, we have

$$I_{11}\lesssim\|\sqrt{\rho}\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}^2\lesssim\|u\cdot\nabla u\|_{L^2}^2. \tag{2.14}$$

For I_{12} and I_{13}, by [**21**] Proposition 2.1, and (2.1), and Sobolev's inequality, we have

$$I_{12}\ \lesssim\ \|pu\|_{L^2}^2\lesssim\|\sqrt{\rho}u\|_{L^2}^2\leq C, \tag{2.15}$$

$$\begin{aligned}
I_{13}\ \lesssim\ & \|\sqrt{\rho}\|_{L^3}^2\|\mathcal{L}^{-1}\nabla((p-p'(\rho)\rho)\operatorname{div} u)\|_{L^6}^2\\
\lesssim\ & \|\nabla\mathcal{L}^{-1}\nabla((p-p'(\rho)\rho)\operatorname{div} u)\|_{L^2}^2\lesssim\|(p-p'(\rho)\rho)\operatorname{div} u\|_{L^2}^2\\
\lesssim\ & \|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^2\leq C\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{2.16}$$

Putting (2.14), (2.15) and (2.16) into (2.13), we obtain

$$I_1\leq C\|u\cdot\nabla u\|_{L^2}^2+C+C\|\nabla u\|_{L^2}^2. \tag{2.17}$$

For I_3, we have

$$\begin{aligned}
I_3\ \leq\ & C\|b\|_{L^6}\|b_t\|_{L^2}\|\nabla w\|_{L^3}\leq C\|b_t\|_{L^2}\|\nabla w\|_{L^3}\\
\leq\ & \frac{1}{8}\|b_t\|_{L^2}^2+C\|\nabla w\|_{L^3}^2\leq\frac{1}{8}\|b_t\|_{L^2}^2+C\|\nabla w\|_{L^2}\|\nabla w\|_{L^6}\\
\leq\ & \frac{1}{8}\|b_t\|_{L^2}^2+C\|\nabla w\|_{L^2}^2+\epsilon\|\nabla w\|_{L^6}^2.
\end{aligned}\tag{2.18}$$

Again by [**21**], Proposition 2.1 and (2.8), we have

$$\|\nabla^2 w\|_{L^2}\lesssim\|\sqrt{\rho}w_t\|_{L^2}+\|\sqrt{\rho}F\|_{L^2}+\|\operatorname{rot} b\times b\|_{L^2}. \tag{2.19}$$

Substituting (2.17) and (2.18) into (2.12), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int(\mu|\nabla w|^2+(\lambda+\mu)(\operatorname{div} w)^2)\mathrm{d}x+\frac{1}{2}\int\rho|w_t|^2\mathrm{d}x\\
\leq\ & -2\frac{\mathrm{d}}{\mathrm{d}t}\int\left(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\right):\nabla w\mathrm{d}x+C\|\nabla w\|_{L^2}^2\\
& +\frac{1}{8}\|b_t\|_{L^2}^2+C+C\|u\cdot\nabla u\|_{L^2}^2+C\|\operatorname{rot} b\times b\|_{L^2}^2.
\end{aligned}\tag{2.20}$$

On the other hand, testing (1.3) by b_t and using (2.1), we deduce that

$$\begin{aligned} & \frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\, b|^2 \mathrm{d}x + \int |b_t|^2 \mathrm{d}x \\ \leq\ & C(\|u\|_{L^6}\|\nabla b\|_{L^3} + \|b\|_{L^6}\|\nabla u\|_{L^3})\|b_t\|_{L^2} \\ \leq\ & C(\|\nabla u\|_{L^6}\|\nabla b\|_{L^2}^{\frac12}\|\nabla b\|_{L^6}^{\frac12} + \|\nabla w\|_{L^3} + \|\nabla v\|_{L^3})\|b_t\|_{L^2} \\ \leq\ & \epsilon\|b_t\|_{L^2}^2 + C\|u\|_{L^6}^4\|\nabla b\|_{L^2}^2 + \epsilon\|\nabla b\|_{L^6}^2 + C + C\|\nabla w\|_{L^2}\|\nabla w\|_{L^6} \\ \leq\ & \epsilon\|b_t\|_{L^2}^2 + C\|u\|_{L^6}^4\|\nabla b\|_{L^2}^2 + \epsilon\|\nabla b\|_{L^6}^2 + C + C\|\nabla w\|_{L^2}^2 + \epsilon\|\nabla w\|_{L^6}^2. \end{aligned} \tag{2.21}$$

On the other hand, using H^2-theory of elliptic equations, we get

$$\begin{aligned} \|b\|_{H^2} &\lesssim \|b_t\|_{L^2} + \|\mathrm{rot}\,(b\times u)\|_{L^2} \\ &\lesssim \|b_t\|_{L^2} + \|b\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^6}\|\nabla b\|_{L^3} \\ &\lesssim \|b_t\|_{L^2} + \|\nabla w\|_{L^3} + 1 + \|u\|_{L^6}^2\|\nabla b\|_{L^2} + \epsilon\|b\|_{H^2}, \end{aligned}$$

whence

$$\|b\|_{H^2} \lesssim \|b_t\|_{L^2} + \|\nabla w\|_{L^2} + \epsilon\|\nabla w\|_{L^6} + 1 + C\|u\|_{L^6}^2\|\nabla b\|_{L^2}. \tag{2.22}$$

Putting (2.22) into (2.21), we have

$$\begin{aligned} & \eta\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\, b|^2\mathrm{d}x + \int |b_t|^2\mathrm{d}x \\ \leq\ & C\|u\|_{L^6}^4\|\nabla b\|_{L^2}^2 + C + C\|\nabla w\|_{L^2}^2 + \frac18\|\sqrt{\rho}w_t\|_{L^2}^2 \\ & +C\|u\cdot\nabla u\|_{L^2}^2 + C\|\mathrm{rot}\, b\times b\|_{L^2}^2. \end{aligned} \tag{2.23}$$

Summing up (2.20) and (2.21) and integrating over $(0,t)$, we have

$$\begin{aligned} & \int (\mu|\nabla w|^2 + (\lambda+\mu)(\mathrm{div}\, w)^2 + \eta|\mathrm{rot}\, b|^2)\mathrm{d}x + \int_0^t\int(\rho|w_t|^2 + |b_t|^2)\mathrm{d}x\mathrm{d}s \\ \leq\ & C\|b\|_{L^4}^2\|\nabla w\|_{L^2} + C + C\int_0^t \|u\|_{L^6}^4\|\nabla b\|_{L^2}^2 ds \\ \leq\ & \frac{\mu}{2}\|\nabla w\|_{L^2}^2 + C + C\int_0^t \|u\|_{L^6}^4\|\nabla b\|_{L^2}^2 ds. \end{aligned} \tag{2.24}$$

This proves

Lemma 2.2. *We have*

$$\int(|\nabla w|^2 + |\nabla b|^2)\mathrm{d}x + \int_0^T\int(\rho|w_t|^2 + |\nabla^2 w|^2 + |b_t|^2 + |\nabla^2 b|^2)\mathrm{d}x\mathrm{d}t \leq C. \tag{2.25}$$

□

Using $u = w + v$, we easily observe that

$$\|\nabla u\|_{L^\infty(0,T;L^2)} \leq C, \|\nabla u\|_{L^2(0,T;L^6)} \leq C. \tag{2.26}$$

Lemma 2.3. *We have*

$$\int(\rho|\dot u|^2 + |b_t|^2)\mathrm{d}x + \int_0^T\int(|\nabla\dot u|^2 + |\nabla b_t|^2)\mathrm{d}x\mathrm{d}t \leq C, \tag{2.27}$$

where $\dot f$ is the material derivative:

$$\dot f := f_t + u\cdot\nabla f.$$

Proof. By the definition of material derivative, we can rewrite (1.2) as follows.

$$\rho\dot{u} + \nabla p = \mathcal{L}u + \operatorname{rot} b \times b. \tag{2.28}$$

Applying ∂_t to (2.28) and using (1.1), we have

$$\begin{aligned} &\rho\dot{u}_t + \rho u\cdot\nabla\dot{u} + \nabla p_t - (\operatorname{rot} b\times b)_t \\ = \; &\mathcal{L}\dot{u} - \mathcal{L}(u\cdot\nabla u) + \operatorname{div}[\mathcal{L}u\otimes u - \nabla p\otimes u + (\operatorname{rot} b\times b)\otimes u]. \end{aligned} \tag{2.29}$$

Testing (2.29) by $\dot{u}$ and using the fact $\dot{u}=0$ on $\partial\Omega$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\dot{u}|^2\mathrm{d}x + \int(\mu|\nabla\dot{u}|^2 + (\lambda+\mu)|\operatorname{div}\dot{u}|^2)\mathrm{d}x \\ = \; &\int((p(\rho))_t\operatorname{div}\dot{u} + u\otimes\nabla(p(\rho)) : \nabla\dot{u})\mathrm{d}x \\ &+\mu\int(\operatorname{div}(\Delta u\otimes u) - \Delta(u\cdot\nabla u))\cdot\dot{u}\mathrm{d}x \\ &+(\lambda+\mu)\int(\operatorname{div}(\nabla\operatorname{div} u\otimes u) - \nabla\operatorname{div}(u\cdot\nabla u))\cdot\dot{u}\mathrm{d}x \\ &-\int(u\otimes(\operatorname{rot} b\times b)) : \nabla\dot{u}\mathrm{d}x \\ &-\int(b_t\otimes b + b\otimes b_t - b\cdot b_t\mathbb{I}_3) : \nabla\dot{u}\mathrm{d}x =: \sum_{i=1}^5 J_i. \end{aligned} \tag{2.30}$$

By (1.1), we have

$$\begin{aligned} J_1 &= \int(-\operatorname{div}(p(\rho)u)\operatorname{div}\dot{u} - (p'(\rho)\rho - p(\rho))\operatorname{div} u\operatorname{div}\dot{u} + u\otimes\nabla(p(\rho)) : \nabla\dot{u})\mathrm{d}x \\ &= \int(p(\rho)u\cdot\nabla\operatorname{div}\dot{u} + (p(\rho) - p'(\rho)\rho)\operatorname{div} u\operatorname{div}\dot{u} + p(\rho)(\nabla u)^t : \nabla\dot{u} \\ &\quad -p(\rho)u\cdot\nabla\operatorname{div}\dot{u})\mathrm{d}x \\ &= \int((p(\rho) - p'(\rho)\rho)\operatorname{div} u\operatorname{div}\dot{u}\mathrm{d}x + p(\rho)(\nabla u)^t : \nabla\dot{u})\mathrm{d}x \lesssim \|\nabla u\|_{L^2}\|\nabla\dot{u}\|_{L^2}. \end{aligned}$$

By the product rule, we can see

$$\operatorname{div}(\Delta u\otimes u) - \Delta(u\cdot\nabla u) = \nabla_k(\operatorname{div} u\nabla_k u) - \nabla_k(\nabla_k u^j\nabla_j u) - \nabla_j(\nabla_k u^j\nabla_k u),$$

so that by integration by parts, we have

$$J_2 = \mu\int(\nabla_k(\operatorname{div} u\nabla_k u) - \nabla_k(\nabla_k u^j\nabla_j u) - \nabla_j(\nabla_k u^j\nabla_k u))\cdot\dot{u}\mathrm{d}x \lesssim \|\nabla\dot{u}\|_{L^2}\|\nabla u\|_{L^4}^2.$$

Similarly, since

$$\operatorname{div}(\nabla\operatorname{div} u\otimes u) - \nabla\operatorname{div}(u\cdot\nabla u) = \nabla_k(\nabla_j u^j\nabla_i u^i) - \nabla_k(\nabla_j u^i\nabla_i u^j) - \nabla_i(\nabla_k u^i\nabla_j u^j),$$

we have

$$\begin{aligned} J_3 = (\lambda+\mu)\int(\nabla_k(\nabla_j u^j\nabla_i u^i) - \nabla_k(\nabla_j u^i\nabla_i u^j) - \nabla_i(\nabla_k u^i\nabla_j u^j))\dot{u}^k\mathrm{d}x \\ \lesssim \|\nabla\dot{u}\|_{L^2}\|\nabla u\|_{L^4}^2. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} J_4 &\lesssim \|\nabla\dot{u}\|_{L^2}\|\operatorname{rot} b\|_{L^2}\|b\|_{L^\infty} \lesssim \|\nabla\dot{u}\|_{L^2}\|b\|_{L^\infty}, \\ J_5 &\lesssim \int|\nabla\dot{u}||b_t||b|\mathrm{d}x \lesssim \|\nabla\dot{u}\|_{L^2}\|b_t\|_{L^2}\|b\|_{L^\infty}. \end{aligned}$$

Putting all these estimates into (2.30), using Young's inequality and Sobolev's inequality, Lemma 2.2 and (2.26), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\dot u|^2\mathrm{d}x+\int(\mu|\nabla\dot u|^2+(\lambda+\mu)(\operatorname{div}\dot u)^2)\mathrm{d}x\\
\lesssim\ & \|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}+\|\nabla\dot u\|_{L^2}\|\nabla u\|_{L^4}^2+\|\nabla\dot u\|_{L^2}\|b\|_{L^\infty}+\|\nabla\dot u\|_{L^2}\|b_t\|_{L^2}\|b\|_{L^\infty}\\
\le\ & \frac{\mu}{2}\|\nabla\dot u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C+C\|b\|_{L^\infty}^2\|b_t\|_{L^2}^2+C\|\nabla u\|_{L^4}^4+C\|b\|_{L^\infty}^2. \qquad (2.31)
\end{aligned}$$

By the definition of w, we have

$$\mathcal{L}w=\rho\dot u+\operatorname{rot} b\times b. \tag{2.32}$$

By H^2-estimates of (2.32), we deduce that

$$\|\nabla^2 w\|_{L^2}\lesssim\|\rho\dot u\|_{L^2}+\|\operatorname{rot} b\times b\|_{L^2}\lesssim\|\sqrt{\rho}\dot u\|_{L^2}+\|\operatorname{rot} b\times b\|_{L^2}. \tag{2.33}$$

By interpolation inequality, (2.10) and (2.26), we obtain

$$\begin{aligned}
\|\nabla u\|_{L^4}^4 &\lesssim \|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3\lesssim\|\nabla u\|_{L^6}\|\nabla u\|_{L^6}^2\\
&\lesssim \|\nabla u\|_{L^6}(\|\nabla w\|_{L^6}^2+\|\nabla v\|_{L^6}^2)\lesssim\|\nabla u\|_{L^6}(\|\nabla^2 w\|_{L^2}^2+1)\\
&\lesssim \|\nabla u\|_{L^6}(\|\sqrt{\rho}\dot u\|_{L^2}^2+1+\|\nabla b\|_{L^3}^2)\\
&\lesssim \|\nabla u\|_{L^6}\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla u\|_{L^6}+\|\nabla u\|_{L^6}^2+\|\nabla b\|_{L^3}^4\\
&\lesssim \|\nabla u\|_{L^6}\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla u\|_{L^6}+\|\nabla u\|_{L^6}^2+\|b\|_{H^2}^2. \qquad (2.34)
\end{aligned}$$

Applying ∂_t to (1.3), testing by b_t and using (2.26), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|b_t|^2\mathrm{d}x+\int|\operatorname{rot} b_t|^2\mathrm{d}x=-\int(b_t\times u+b\times u_t)\operatorname{rot} b_t\mathrm{d}x\\
\le\ & (\|u\|_{L^6}\|b_t\|_{L^3}+\|b\|_{L^3}\|\dot u\|_{L^6}+\|b\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^6})\|\operatorname{rot} b_t\|_{L^2}\\
\le\ & \frac{1}{2}\|\operatorname{rot} b_t\|_{L^2}^2+C\|u\|_{L^6}^4\|b_t\|_{L^2}^2+C_0\|\nabla\dot u\|_{L^2}^2+C\|\nabla u\|_{L^6}^2. \qquad (2.35)
\end{aligned}$$

Doing (2.31)+ϵ_0(2.35) and using (2.34) and using the Gronwall inequality, we arrive at (2.27).

□

It is easy to verify that

$$\begin{aligned}
\|b\|_{H^2} &\lesssim \|b_t\|_{L^2}+\|\operatorname{rot}(b\times u)\|_{L^2}\\
&\le C+C\|u\|_{L^6}\|\nabla b\|_{L^3}+C\|b\|_{L^\infty}\|\nabla u\|_{L^2}\\
&\le C+C\|\nabla b\|_{L^3}+C\|b\|_{L^\infty}\\
&\le C+C\|\nabla b\|_{L^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}},
\end{aligned}$$

and therefore,

$$\|b\|_{H^2}\le C. \tag{2.36}$$

□

It follows from (2.27) and (2.33) that

$$\|\nabla^2 w\|_{L^\infty(0,T;L^2)}\le C. \tag{2.37}$$

Using $W^{2,p}$-theory of (2.32), we get

$$\|w\|_{L^2(0,T;W^{2,p})}\lesssim\|\rho\dot u+\operatorname{rot} b\times b\|_{L^2(0,T;L^p)}\le C. \tag{2.38}$$

Following the same argument of [**21**, Section 5], we have

$$\|\rho\|_{L^\infty(0,T;W^{1,6})} \le C. \tag{2.39}$$

By [**21**, Proposition 2.1], (2.27) and (2.39), we obtain

$$\|\nabla^2 u\|_{L^2} \lesssim \|\rho\dot{u}\|_{L^2} + \|\nabla p\|_{L^2} + \|\mathrm{rot}\, b \times b\|_{L^2} \le C. \tag{2.40}$$

Similarly, we have

$$\|\nabla^2 u\|_{L^6} \lesssim \|\rho\dot{u}\|_{L^6} + \|\nabla p\|_{L^6} + \|\mathrm{rot}\, b \times b\|_{L^6} \lesssim \|\rho\dot{u}\|_{L^6} + 1,$$

which leads to

$$\|\nabla^2 u\|_{L^2(0,T;L^6)} \le C. \tag{2.41}$$

It is easy to show that

$$\begin{aligned} \int \rho|u_t|^2 \mathrm{d}x &\lesssim \int \rho|\dot{u}|^2 \mathrm{d}x + \int \rho|u\cdot\nabla u|^2 \mathrm{d}x \\ &\lesssim 1 + \|\rho\|_{L^\infty}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \le C. \end{aligned} \tag{2.42}$$

$$\begin{aligned} \|\rho_t\|_{L^6} &\lesssim \|u\cdot\nabla\rho\|_{L^6} + \|\rho \mathrm{div}\, u\|_{L^6} \\ &\lesssim \|u\|_{L^\infty}\|\nabla\rho\|_{L^6} + \|\rho\|_{L^\infty}\|\mathrm{div}\, u\|_{L^6} \le C. \end{aligned} \tag{2.43}$$

$$\begin{aligned} \|\nabla u_t\|_{L^2} &\lesssim \|\nabla\dot{u}\|_{L^2} + \|\nabla(u\cdot\nabla u)\|_{L^2} \\ &\lesssim \|\nabla\dot{u}\|_{L^2} + \|u\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^4}^2 \\ &\lesssim \|\nabla\dot{u}\|_{L^2} + 1, \end{aligned}$$

which implies

$$\|\nabla u_t\|_{L^2(0,T;L^2)} \le C. \tag{2.44}$$

Now it is easy to show that

$$\|b\|_{L^2(0,T;W^{2,6})} \le C. \tag{2.45}$$

This completes the proof.

□

3. Proof of Theorem 1.4

We will use the bootstrap argument to prove Theorem 1.4.

Let $\delta > 0$ be a fixed number, say,

$$2\|\rho_0\|_{L^\infty} + 2\|b_0\|_{L^5} \le \delta. \tag{3.1}$$

Denote by $H(t)$ the statement that, for $t \in [0,T]$,

$$\|\rho\|_{L^\infty(\Omega\times[0,t])} + \|b\|_{L^\infty(0,t;L^5(\Omega))} \le \delta \tag{3.2}$$

and $C(t)$ the statement that

$$\|\rho\|_{L^\infty(\Omega\times[0,t])} + \|b\|_{L^\infty(0,t;L^5(\Omega))} \le \frac{\delta}{2}. \tag{3.3}$$

The conditions (b)-(d) in Lemma 1.7 are clearly true and it remains to verify (a) under the conditions that $7\mu > 9\lambda$ and $\|\rho_0\|_{L^\infty} + \|b_0\|_{L^5}$ is sufficiently small. Once this is verified, then the bootstrap argument would imply that $C(t)$, or (3.3) actually holds for any $t \in [0,T]$ and thus (1.12) holds true.

Now we assume that (3.2) holds true for some $t \in [0,T]$. By Theorem 1.1, we have

$$\|u\|_{L^2(0,t;W^{2,6})} \le C_1. \tag{3.4}$$

Testing (1.1) by ρ^{q-1} $(q > 2)$ and using (3.4), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^q} \leq \left(1-\frac{1}{q}\right)\|\mathrm{div}\, u\|_{L^\infty}\|\rho\|_{L^q} \leq C_0\|u\|_{W^{2,6}}\|\rho\|_{L^q},$$

which leads to

$$\|\rho\|_{L^q} \leq \|\rho_0\|_{L^q}\exp(C_0\sqrt{T}C_1).$$

Taking $q \to +\infty$ and letting

$$\|\rho_0\|_{L^\infty} \ \text{ be small},$$

we arrive at

$$\|\rho\|_{L^\infty(\Omega\times[0,t])} \leq \|\rho_0\|_{L^\infty}\exp(C_0\sqrt{T}C_1) \leq \frac{\delta}{4}. \tag{3.5}$$

Testing (1.3) by $|b|^{p-2}b$ $(p := 5)$ and using (1.6), (1.13), (1.14), (1.15) and (1.17), we have

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^p}^p + \eta\int |b|^{p-2}|\nabla b|^2\mathrm{d}x + 4\frac{p-2}{p^2}\eta\int |\nabla|b|^{\frac{p}{2}}|^2\mathrm{d}x\\
=\ & -\int(u\cdot\nabla b - b\cdot\nabla u + b\,\mathrm{div}\, u)|b|^{p-2}b\mathrm{d}x - \eta\int_{\partial\Omega}|b|^{p-2}(b\cdot\nabla)n\cdot b\mathrm{d}S\\
=\ & \frac{1}{p}\int |b|^p\mathrm{div}\, u\mathrm{d}x + \int(b\cdot\nabla u)|b|^{p-2}b\mathrm{d}x - \int |b|^p\mathrm{div}\, u\mathrm{d}x\\
& -\eta\int_{\partial\Omega}|b|^{p-2}(b\cdot\nabla)n\cdot b\mathrm{d}S\\
\leq\ & C\|\nabla u\|_{L^\infty}\|b\|_{L^p}^p + C\eta\int_{\partial\Omega}|b|^p\mathrm{d}S\\
\leq\ & C\|\nabla u\|_{L^\infty}\|b\|_{L^p}^p + \frac{p-2}{p^2}\eta\int|\nabla|b|^{\frac{p}{2}}|^2\mathrm{d}x + C\eta\|b\|_{L^p}^p,
\end{aligned}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^p} \leq C(\|\nabla u\|_{L^\infty}+1)\|b\|_{L^p}, \tag{3.6}$$

which implies

$$\begin{aligned}
\|b\|_{L^5} &\leq C\|b_0\|_{L^5}\exp\left(t + \int_0^t \|u\|_{W^{2,6}}\mathrm{d}s\right)\\
&\leq C\|b_0\|_{L^5} \leq \frac{\delta}{4}.
\end{aligned} \tag{3.7}$$

This completes the proof.

□

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Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
Email address: fanjishan@njfu.edu.cn

Center for Information, Nanjing Forestry University, Nanjing 210037, P.R.China
Email address: jinglulu@njfu.edu.cn

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
Email address: nakamuragenn@gmail.com

Department of Mathematics, College of Sciences, Hohai University, Nanjing 210098, China
Email address: tt0507010156@126.com