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On the Strong Solutions for a Stochastic 2D nonlocal Cahn-Hilliard-Navier-Stokes Model

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Abstract. We study in this article a stochastic version of a well-known diffuse interface model. The model consists of the Navier-Stokes equations for the average velocity, nonlinearly coupled with a nonlocal Cahn-Hilliard equation for the order (phase) parameter. The system describes the evolution of an incompressible isothermal mixture of binary fluids excited by random forces in a two dimensional bounded domain. For a fairly general class of random forces, we prove the existence and uniqueness of a variational solution.

Contents

1. Introduction

In this paper, we study the stochastic version of an evolution system which consists of the Navier-Stokes equations for the fluid velocity u suitably coupled with a nonlocal convective Cahn-Hilliard equation for the order parameter φ on a given smooth bounded domain $\mathcal{O}$ of $\mathbb{R}^2$. The system derives from a diffuse interface model which describes the evolution of an incompressible mixture of two immiscible fluids (see e.g. [**22, 23, 24, 26, 30**] and references therein). We suppose that the temperature variations are negligible and the density is constant and equal to one. Thus u represents an average velocity and φ the relative concentration of one fluid

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(or the difference of two concentrations). The nonlocal Cahn-Hilliard Navier-Stokes system reads as follows

$$\begin{cases} \frac{\partial u}{\partial t} - 2div(\nu(\varphi)Du) + (u.\nabla)u + \nabla\pi = \mu\nabla\varphi + f, \\ div(u) = 0, \\ \frac{\partial\varphi}{\partial t} + u.\nabla\varphi = div(m(\varphi)\nabla\mu), \\ \mu = a\varphi - J * \varphi + F'(\varphi). \end{cases} \tag{1.1}$$

Here ν is the viscosity, π the pressure, f denotes an external force acting on the fluid mixture, $J : \mathbb{R}^2 \to \mathbb{R}$ is a suitable interaction kernel and $J * \varphi$ stands for spatial convolution over $\mathcal{O}$, a is defined as follows $a(x) = \int_{\mathcal{O}} J(x-y)dy$, F is the configuration potential which account for the presence of two phases. The system (1.1) is called nonlocal because of the term J which is averaged over the spatial domain. The local version of the system is obtained by replacing μ equation by $\mu = -\Delta\varphi + F'(\varphi)$. From the mathematical point of view, the nonlocal version is physically more relevant and mathematically challenging too. Indeed the nonlocal Cahn-Hilliard equation which is one of the equations of our system, is widely regarded as a better mathematical representation of the spinodal decomposition phenomenon than the local equation, the latter being a "local approximation" of the nonlocal one (see [**10**]). The model (1.1) is more difficult to handle because of the nonlinear term like the capillarity term (i.e. Korteweg force) $\mu\nabla\varphi$ acting on the fluid. This term can be less regular than the convective term $(u.\nabla)u$ [**9**].

The analysis of the (standard) local Cahn-Hilliard Navier-Stokes system has been investigated in the literature (see [**1, 2, 4, 20, 21, 34, 38**]. In [**4**], the author studied the existence and uniqueness of solutions in dimensions two and three. The solvability of the system (1.1) has been analyzed in [**9, 17, 18, 19**]. In [**9**], the authors proved the existence of a global weak solution for nonlocal Cahn-Hilliard Navier-Stokes system for smooth potentials F of arbitrary polynomial growth. The result was extended in [**19**] for singular potentials. In [**17**], the uniqueness of a weak solution was resolved in two dimensions. The uniqueness of a weak solution remains open in three dimensions. The existence of a unique strong solution in two dimensions is proved in [**18**] and the authors showed that any weak solution regularizes in finite time uniformly with respect to bounded sets of initial data.

However, in order to consider a more realistic model for our problem, it is sensible to consider some kind of noise in the equations. This may reflect, for instance, some environmental effects on the phenomena, some external random forces, etc. To the best of our knowledge, the study of the stochastic version of the system (1.1) has only be analyzed in [**13**]. In [**13**], the authors prove the existence of a global martingale solution for the stochastic nonlocal Cahn-Hilliard Navier-Stokes in dimensions 2 and 3. The proof uses a Faedo-Galerkin approximation scheme, compactness method and the Skorokhod representation theorem.

The stochastic local Cahn-Hilliard-Navier-Stokes system was analyzed in [**35, 14, 15, 16**]. In [**35**], the author proved the existence and uniqueness of a variational solution in dimension two. The first and the third authors of the present paper in [**15**] proved the existence and uniqueness of a global strong solution for the stochastic 3D globally modified Cahn-Hilliard-Navier-Stokes equations. Using a limiting argument, they also obtained the existence of a global weak martingale solution for the stochastic 3D Cahn-Hilliard Navier-Stokes system. In [**14**], the authors study the stability of weak solutions to a stochastic version of a globally modified Cahn-Hilliard Navier-Stokes model with multiplicative noise. The paper [**16**] concerns the existence of a random attractor for the stochastic 2D Cahn-Hilliard-Navier-Stokes system.

The aim of the present paper is to prove the existence and uniqueness of a strong solution (in the probabilistic sense) for the stochastic nonlocal Cahn-Hilliard Navier-Stokes system in a 2D bounded domain. The model includes an abstract and general form of random external forces dependent eventually on the velocity of the fluid. The proof of the existence combines the Galerkin approximation, the properties of stopping time and the weak convergence in functional analysis. We recall that due to its integro-differential nature and the random external forces, the stochastic nonlocal Cahn-Hilliard Navier-Stokes system is rather difficult to handle compared to the stochastic standard Cahn-Hilliard Navier-Stokes equations (CH-NSE) studied in [**35, 14, 15, 16**].

The plan of the paper goes as follows. In Section 2, we present the stochastic 2D nonlocal Cahn-Hilliard Navier-Stokes system and its mathematical setting. We also introduce the definition of a variational solution of the problem and formulate our main result. In Section 3, We prove the main result. Here we introduce a Galerkin approximation scheme and obtain some priori estimates for the approximating solutions. In Section 4, using the a priori estimates derived in Section 3, we prove that the sequence of approximate solutions converges to the unique variational solution of (2.1).

2. A stochastic nonlocal CH-NSE and its mathematical setting

2.1. Governing equations. Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary $\partial\mathcal{O}$, n the unit outward normal and a final time $T > 0$. We consider the following initial and boundary value problem for the stochastic 2D nonlocal Cahn-Hilliard Navier-Stokes system

$$\begin{cases} u_t - \nu\Delta u + (u.\nabla)u + \nabla\pi = \mu\nabla\varphi + h(t,u) + G(t,u)\dot{W}_t, \text{ in } \mathcal{O}\times(0,T), \\ div(u) = 0, \text{ in } \mathcal{O}\times(0,T), \\ \varphi_t + u.\nabla\varphi = \Delta\mu, \text{ in } \mathcal{O}\times(0,T), \\ \mu = a\varphi - J*\varphi + F'(\varphi), \text{ in } \mathcal{O}\times(0,T), \\ \dfrac{\partial\mu}{\partial n} = 0; u = 0 \text{ on } \partial\mathcal{O}\times(0,T), \\ (u,\varphi)(0) = (u_0,\varphi_0), \text{ in } \mathcal{O}, \end{cases} \tag{2.1}$$

where $u = (u_1, u_2)$, π and ϕ are unknown random fields on $\mathcal{O}\times[0,T]$ representing respectively, the averaged velocity of the fluid, the pressure and the relative concentration of one fluid. (u_0, ϕ_0) is a given initial condition where u_0 is the initial velocity of the fluid and φ_0 the initial concentration of one fluid. The terms $h(t,u)$ and $G(t,u)\dot{W}_t$ represent random external forces depending eventually on u, where $\dot{W}_t$ denotes the time derivative of a cylindrical Wiener process. The abstract and general form of the stochastic term allows to include in the formulation certain random environmental effects as well as the turbulent part of the velocity field (see [**29**] for more details)

The natural no-flux condition $\frac{\partial \mu}{\partial n} = 0$ implies the conservation of the following quantity

$$\langle \varphi(t) \rangle = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \varphi(x,t) dx,$$

where $|\mathcal{O}|$ stands for the Lebesgue measure of $\mathcal{O}$. More precisely, we have

$$\langle \varphi(t) \rangle = \langle \varphi(0) \rangle, \forall t \geq 0. \tag{2.2}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of φ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle \varphi(t) \rangle = \langle \varphi(0) \rangle = 0, \quad \text{for all} \quad t > 0. \tag{2.3}$$

REMARK 2.1. Adding a stochastic force in the equation for the relative concentration will involve tedious calculations and will increase significantly the size of the paper. For example, we need to apply the Itô formula to the functional (2.14) which will require tedious calculations and probably more assumptions.

2.2. Notations and functional setup. We consider the operator $\mathcal{B} = -\Delta + I$ with homogeneous Neumann boundary conditions. The domain of $\mathcal{B}$ is given by (see [**36**])

$$D(\mathcal{B}) := \left\{ \varphi \in H^2(\mathcal{O}) : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \mathcal{O} \right\}.$$

We introduce

$$H := L^2(\mathcal{O}), \quad U := H^1(\mathcal{O}) \quad \text{and} \quad \mathcal{V} := \left\{ u \in (\mathcal{C}_c^\infty(\mathcal{O}))^2 : \text{div } u = 0 \text{ in } \mathcal{O} \right\}.$$

We denote by G_{div} and V_{div} the closure of $\mathcal{V}$ in $(L^2(\mathcal{O}))^2$ and $(H_0^1(\mathcal{O}))^2$ respectively. We denote by $|.|$ and $(.,.)$ the norm and the scalar product, respectively, on both H and G_{div}. The norms on G_{div} and H are given by

$$|u|^2 := \int_{\mathcal{O}} |u(x)|^2 dx \quad \text{and} \quad |\varphi|^2 := \int_{\mathcal{O}} |\varphi(x)|^2 dx,$$

respectively. The duality between V_{div} and its topological dual V'_{div} is denoted by $\langle .,. \rangle$. It is well-known that V_{div} is endowed with the scalar product $(u,v)_{V_{div}} = (\nabla u, \nabla v) = 2(Du, Dv)$ for all $u, v \in V_{div}$, where $Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$. The norm on V_{div} is given by

$$\|u\|^2 := \int_{\mathcal{O}} |\nabla u(x)|^2 dx = |\nabla u|^2.$$

Let $\mathcal{P} : (L^2(\mathcal{O}))^2 \to G_{div}$ be the Helmholtz-Leray projection from $(L^2(\mathcal{O}))^2$ into G_{div}. We denote by $A = -\mathcal{P}\Delta$ the Stokes operator with domain $D(A) = (H^2(\mathcal{O}))^2 \cap V_{div}$. It is well-known that, see e.g. Constantin [**7**, page 33] or Temam [**36**, page 56], that A is a non-negative self adjoint operator in G_{div}. Moreover, see [**36**, page 57], $V_{div} = D(A^{1/2})$. Furthermore, $A^{-1} : G_{div} \to G_{div}$ is a compact linear operator on G_{div} and $|A\, .|$ is a norm on $D(A)$ that is equivalent to the $H^2(\mathcal{O})$-norm. It is also well-known (see for e.g. [**37**, Chapter I, Section 2.6]) that A possesses a sequence of eigenvalues $\{\lambda_j\}$ with $0 < \lambda_1 \leq \lambda_2 \leq ...$ and $\lambda_j \to \infty$, and a family $w_j \subset D(A)$ of eigenfunctions which is orthonormal in G_{div}. We also recall Poincaré's inequality

$$\lambda_1 |u|^2 \leq |\nabla u|^2, \quad \text{for all} \quad u \in V_{div}.$$

We also define the Hilbert spaces (see [**17**] for more details)

$$V_0 := \{\varphi \in U : \langle \varphi \rangle = 0\}, V_0' := \{\varphi \in U' : \langle \varphi \rangle = 0\} \text{ and } L^2_{(0)}(\mathcal{O}) := \{\varphi \in H : \langle \varphi \rangle = 0\},$$

where U' and $\langle\varphi\rangle$ denote the dual space of U and the average of φ over $\mathcal{O}$, respectively.
Following [**28**, Proposition 1.24] we can define a self-adjoint operator $\mathcal{A} : U \to U'$, $\mathcal{A} \in \mathcal{L}(U, U')$ by

$$\langle \mathcal{A}u, w\rangle = \int_{\mathcal{O}} \nabla u.\nabla w \; dx, \quad \text{for all} \;\; u, w \in U.$$

We recall that $\mathcal{A}$ maps U onto $V_0^{'}$ and the restriction $\mathcal{A}_N$ of $\mathcal{A}$ to V_0 maps V_0 onto $V_0^{'}$ isomorphically. Further, we denote by $\mathcal{A}_N^{-1} : V_0^{'} \to V_0$ the inverse map defined by

$$\mathcal{A}\mathcal{A}_N^{-1} f = f, \quad \text{for all} \;\; f \in V_0^{'} \;\; \text{and} \;\; \mathcal{A}_N^{-1}\mathcal{A}v = v, \quad \text{for all} \;\; v \in V_0.$$

We know that for every $f \in V_0^{'}$, $\mathcal{A}_N^{-1} f$ is the unique solution with zero mean value of the Neumann problem

$$\begin{cases} -\Delta\varphi = f, & \text{in} \;\; \mathcal{O} \\ \frac{\partial\varphi}{\partial\eta} = 0 & \text{in} \;\; \partial\mathcal{O}. \end{cases}$$

Since $\mathcal{A}_N^{-1}$ is a positive self-adjoint operator, fractional powers of it are well defined. Hereafter, we will denote by

$$\|\varphi\|_{V_0'} = |\mathcal{A}_N^{-1/2}\varphi|_{L^2} := |\mathcal{A}_N^{-1/2}\varphi|$$

the norm on the space $V_0^{'}$ with associated inner product given by

$$(\varphi, \psi)_{V_0'} = \left(\mathcal{A}_N^{-1/2}\varphi, \mathcal{A}_N^{-1/2}\psi\right)_{L^2} := \left(\mathcal{A}_N^{-1/2}\varphi, \mathcal{A}_N^{-1/2}\psi\right) \tag{2.4}$$

for φ, ψ belonging to $V_0^{'}$. If $\varphi \in V_0^{'}$ and $\psi \in H$, then

$$(\varphi, \psi)_{V_0'} = \left(\mathcal{A}_N^{-1}\varphi, \psi\right)_{L^2} := \left(\mathcal{A}_N^{-1}\varphi, \psi\right). \tag{2.5}$$

In addition, we have

$$\left\langle \mathcal{A}\varphi, \mathcal{A}_N^{-1} f\right\rangle = \langle\varphi, f\rangle, \quad \forall\varphi \in V_0, \quad \forall f \in V_0', \tag{2.6}$$

$$\left\langle f, \mathcal{A}_N^{-1} g\right\rangle = \left\langle g, \mathcal{A}_N^{-1} f\right\rangle = \int_{\mathcal{O}} \nabla(\mathcal{A}_N^{-1} f).\nabla(\mathcal{A}_N^{-1} g) dx, \quad \forall f, g \in V_0'. \tag{2.7}$$

In particular for all $f \in V_0'$, we have

$$\left|\mathcal{A}_N^{-1/2} f\right| = \left|\nabla\mathcal{A}_N^{-1} f\right|.$$

We introduce the trilinear form b which appears in the weak formulation of the Navier-Stokes equations:

$$b(u, v, w) = \int_{\mathcal{O}} (u.\nabla)v.w \;\; dx, \quad \forall u, v, w \in V_{div}.$$

The associated bilinear operator B from $V_{div} \times V_{div}$ into $V_{div}^{'}$ is defined by

$$\langle B(u, v), w\rangle := b(u, v, w), \quad \forall u, v, w \in V_{div}.$$

Hereafter, we set $B(u, u) := B(u)$, for all $u \in V_{div}$. We recall that we have

$$b(u, v, w) = -b(u, w, v), \quad \forall u, v, w \in V_{div}, \tag{2.8}$$

and the following estimate holds in dimension two

$$|b(u, v, w)| \leq c|u|^{1/2}|\nabla u|^{1/2}|\nabla v||w|^{1/2}|\nabla w|^{1/2}, \forall u, v, w \in V_{div}. \tag{2.9}$$

For more properties of the operator B, we refer the reader to [**37**, Lemma II.1.3] or [**20, 21**].
We mention that $|\mathcal{A}_N^{1/2}\varphi|^2 = (\mathcal{A}_N\varphi,\varphi) = |\nabla\varphi|^2$, for all $\varphi \in D(\mathcal{A}_N)$ and hence $|\mathcal{A}_N^{1/2}\varphi| = |\nabla\varphi|$, which also holds, by density, for all $\varphi \in D(\mathcal{A}_N^{1/2}) = V_0$, see ([**17**], p. 13).
We set

$$\mathbb{Y} = G_{div} \times H. \tag{2.10}$$

The space $\mathbb{Y}$ is a complete metric space with respect to the norm

$$|(v,\varphi)|_{\mathbb{Y}}^2 = |v|^2 + |\varphi|^2. \tag{2.11}$$

We define the Hilbert space $\mathbb{V}$ by

$$\mathbb{V} = V_{div} \times V_0 = V_{div} \times D(\mathcal{A}_N^{1/2}), \tag{2.12}$$

endowed with the scalar product whose associated norm is

$$\|(v,\varphi)\|_{\mathbb{V}}^2 = \|v\|^2 + |\nabla\varphi|^2. \tag{2.13}$$

Hereafter, for any $\varphi \in H$ and $u \in V_{div}$, we set

$$\begin{aligned} &\mathcal{E}(\varphi) = \frac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)|\varphi(x)-\varphi(y)|^2 dxdy + \int_{\mathcal{O}} F(\varphi(x))dx,\\ &\mathcal{E}_{tot}(u,\varphi) = |u|^2 + 2\mathcal{E}(\varphi). \end{aligned} \tag{2.14}$$

2.3. The cylindrical Wiener process. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ an increasing and right continuous family of sub σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Let $(\beta_t^k : t \geq 0, k = 1,2,...)$ be a sequence of mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on this space, and suppose that K is a given separable Hilbert space, and $(e_k : k \in \mathbb{N}\cap[1,\infty))$ an orthonormal basis of K. We denote by $\{W(t); t \geq 0\}$ the cylindrical Wiener process on K defined formally as

$$W(t) = \sum_{k=1}^{\infty} \beta_t^k e_k, \quad t \geq 0. \tag{2.15}$$

It is well known that this series does not converge in K, but rather in any Hilbert space $\tilde{K}$ such that $K \subset \tilde{K}$, and the injection of K into $\tilde{K}$ is Hilbert-Schmidt (see, e.g. [**12**], Proposition 4.11).

Let $T > 0$ be given. For any separable Banach space X, we denote by

$$M^2_{\mathcal{F}_t}(0,T;X)$$

the space of all processes $\varphi \in L^2(\Omega\times(0,T), d\mathbb{P}\times dt; X)$ that are $\mathcal{F}_t$-progressively measurable. The space $M^2_{\mathcal{F}_t}(0,T;X)$ is a Hilbert subspace of

$$L^2(\Omega\times(0,T), d\mathbb{P}\times dt; X),$$

see [**8**].

We write $L^2(\Omega; C([0,T];X))$ to denote the space of all continuous and $\mathcal{F}_t$-progressively measurable X-valued processes $\{\varphi_t; t \in [0,T]\}$, satisfying

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\varphi_t\|_X^2\right) < \infty.$$

For another separable Hilbert space $\tilde{H}$, with scalar product $(,.,)_{\tilde{H}}$, we denote by $L_2(K,\tilde{H})$ the separable Hilbert space of Hilbert-Schmidt operators from K into $\tilde{H}$, and by $((.,.))_{L_2(K,\tilde{H})}$ and $\|.\|_{L_2(K,\tilde{H})}$ the scalar product and norm in $L_2(K,\tilde{H})$, where for R and S in $L_2(K,\tilde{H})$,

$$(R,S))_{L^2(K,\tilde{H})} = \sum_{j=1}^{\infty}(Re_j, Se_j)_{\tilde{H}}.$$

For any process $\psi \in M^2_{\mathcal{F}_t}(0,T;L_2(K,\tilde{H}))$, one can define the stochastic integral of ψ with respect to the cylindrical Wiener process W_t, denoted

$$\int_0^t \psi(s)dW(s), 0 \le t \le T,$$

as the unique continuous $\tilde{H}$-valued $\mathcal{F}_t$-martingale, such that for all $g \in \tilde{H}$

$$\left(\int_0^t \psi(s)dW(s), g\right)_{\tilde{H}} = \sum_{j=1}^{\infty}\int_0^t (\psi(s)e_j, g)_{\tilde{H}}\, d\beta_s^j, \quad t \in [0,T],$$

where the integral with respect to β_s^j is the Itô integral. The above series converges in $L^2(\Omega;\mathcal{C}([0,T];\mathbb{R}))$. See [**12**] for the properties of the stochastic integral defined in this way. In particular, if $\psi \in M^2_{\mathcal{F}_t}(0,T;L_2(K,\tilde{H}))$ and $g \in L^2(\Omega;L^\infty(0,T;\tilde{H}))$ is $\mathcal{F}_t$-progressively measurable, then the series

$$\sum_{j=1}^{\infty}\int_0^t (\psi(s)e_j, g(s))d\beta_s^j,\ t \in [0,T],$$

converges in $L^1(\Omega;\mathcal{C}([0,T];\mathbb{R}))$, and defines a real-valued continuous $\mathcal{F}_t$-martingale. We will use the notation

$$\int_0^t (\psi(s)dW_s, g(s)) = \sum_{j=1}^{\infty}\int_0^t (\psi(s)e_j, g(s))_{\tilde{H}} d\beta_s^j,\ t \in [0,T].$$

In order to solve our problem (2.1), we make precise some assumptions.

2.4. Assumptions.

(H_1) We assume that h and G are measurable Lipschitz and sublinear mappings from $\Omega\times(0,T)\times G_{div}$ into V'_{div} and from $\Omega\times(0,T)\times G_{div}$ into $L_2(K,G_{div})$ respectively. More precisely, for all v_1, $v_2 \in V_{div}$, $h(.,v_1)$ and $G(.,v_1)$ are $\mathcal{F}_t$-progressively measurable, and $d\mathbb{P}\times dt$-a.e. in $\Omega\times(0,T)$

$$\|h(t,v_1) - h(t,v_2)\|_{V'_{div}} \le l_h|v_1 - v_2|,$$
$$h(t,0) \in M^2_{\mathcal{F}_t}(0,T;V'_{div}),$$
$$\|G(t,v_1) - G(t,v_2)\|_{L_2(K,G_{div})} \le l_g|v_1 - v_2|,$$
$$G(t,0) \in M^2_{\mathcal{F}_t}\left(0,T;L_2(K,G_{div})\right),$$

where $l_h > 0, l_g >$ are some constants.
As usual by writing $G(t,v_1)$ we mean the map $\omega \mapsto G(\omega,t,v_1)$. Analogously for h.
For the initial data u_0 and φ_0 we assume that

$$(u,\varphi)(0) = (u_0,\varphi_0) \in L^2\left(\Omega,\mathcal{F}_0,\mathbb{P};G_{div}\times H\right).$$

As in [**9**] (see also [**3**]) the assumptions on the kernel J, the potential F are the following:

(H$_2$) $J \in W^{1,1}(\mathbb{R}^2;\mathbb{R})$, $J(x) = J(-x)$ and $a(x) = \int_{\mathcal{O}} J(x-y)dy \geq 0$ a.e, in $\mathcal{O}$.

(H$_3$) We assume that $F \in C^2(\mathbb{R})$ and there exists $c_0 > 0$ such that $F''(s) + a(x) \geq c_0$, $\forall s \in \mathbb{R}$, a.e., $x \in \mathcal{O}$.

(H$_4$) Moreover, there exist c_1, $c_2 > 0$ and $\kappa > 0$ such that $F''(s) + a(x) \geq c_1|s|^{2\kappa} - c_2$, $\forall s \in \mathbb{R}$, a.e., $x \in \mathcal{O}$.

(H$_5$) There exist $c_3 > 0$, $c_4 \geq 0$ and $r \in (1, 2]$ such that

$$|F'(s)|^r \leq c_3|F(s)| + c_4, \quad \text{for all} \quad s \in \mathbb{R}.$$

REMARK 2.2. (H$_4$) implies the existence of $c_7 > 0$ and $c_8 > 0$ such that

$$F(s) \geq c_7|s|^{2+2\kappa} - c_8, \quad \text{for all} \quad s \in \mathbb{R}. \tag{2.16}$$

Using the above notations, we can rewrite (2.1) as follows:

$$\begin{cases} \dfrac{du}{dt} + \nu Au + B(u) + \mu\nabla\varphi = h(t,u) + G(t,u)\dot{W}_t \quad \text{in} \quad V_{div}^{'}, \\ \dfrac{d\varphi}{dt} + (u.\nabla)\varphi = \Delta\mu, \quad \text{in} \quad V_0', \\ \mu = a\varphi - J * \varphi + F'(\varphi), \\ (u,\varphi)(0) = (u_0,\varphi_0), \end{cases} \tag{2.17}$$

or equivalently

$$\begin{cases} u(t) + \int_0^t (\nu Au(s) + B(u(s)))ds = u_0 + \int_0^t \mu(s)\nabla\varphi(s)ds \\ \qquad + \int_0^t h(s,u(s))ds + \int_0^t G(s,u(s))dW(s) \text{ in } V_{div}^{'}, \\ \dfrac{d\varphi}{dt} + (u.\nabla)\varphi = \Delta\mu, \quad \text{in} \quad V_0', \\ \mu = a\varphi - J * \varphi + F'(\varphi), \end{cases} \tag{2.18}$$

$\mathbb{P}$-a.s., and for all $t \in [0,T]$.

REMARK 2.3. One (formally) has

$$\mu\nabla\varphi = (a\varphi - J * \varphi + F'(\varphi))\nabla\varphi = \nabla\left(F(\varphi) + a\frac{\varphi^2}{2}\right) - \frac{\varphi^2}{2}\nabla a - (J * \varphi)\nabla\varphi.$$

In this paper c will stand for a nonnegative constant depending possibly only on J, f, $\mathcal{O}$, ν, λ_1, c_0 and T. The value of c may vary even within the same line.

DEFINITION 2.4. A variational solution to problem (2.1) is a process (u,φ) such that

$$u \in M^2_{\mathcal{F}_t}(0,T;V_{div}) \cap L^2(\Omega; L^\infty(0,T;G_{div})),$$
$$\varphi \in M^2_{\mathcal{F}_t}(0,T;V_0) \cap L^{2\kappa+2}(\Omega; L^\infty(0,T;L^{2\kappa+2}(\mathcal{O}))),$$

(u,φ) is strongly continuous with values in $G_{div} \times H$ and such that (2.18) is satisfied in $\mathbb{V}^{'}$, $\mathbb{P}$-a.s., for all $t \in [0,T]$.

2.5. Statement of the main result. Now we state the main result of this work.

THEOREM 2.5. *Suppose that the hypotheses $(H_1) - (H_5)$ hold and that*

$$h(t,0) \in L^4(\Omega; L^2(0,T;V_{div}^{'})), G(t,0) \in L^4(\Omega; L^2(0,T;L_2(K,G_{div})))$$

and

$$(u_0,\varphi_0) \in L^4(\Omega,\mathcal{F}_0,\mathbb{P}; G_{div} \times D(\mathcal{B}))$$

luca.lorenzi@unipr.itluca.lorenzi@unipr.it such that $F(\varphi_0) \in L^2(\Omega; L^1(\mathcal{O}))$. *Then there exists a unique variational solution* (u, φ) *of problem (2.1). Moreover*

$$(u, \varphi) \in L^4(\Omega; \mathcal{C}([0,T]; \mathbb{Y})) \cap L^4(\Omega; L^2(0,T; \mathbb{V})), B(u) \in M^2_{\mathcal{F}_t}(0,T; V'_{div})$$

and

$$\mu \nabla \varphi \in M^2_{\mathcal{F}_t}(0,T; V'_{div}).$$

In fact, there exists a constant $c > 0$ *depending on* $T, \mathcal{O}, l_h$ *and* l_g, c_7, c_8, κ, J, ν *such that*

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} |(u(t),\varphi(t))|^4_{\mathbb{Y}}\right] + \mathbb{E}\left[\int_0^T |(u(s),\varphi(s))|^2_{\mathbb{V}} ds\right]^2 \\ &\leq c\left(\mathbb{E}(|u_0|^4 + |\varphi_0|^4) + \mathbb{E}|F(\varphi_0)|^2_{L^1(\mathcal{O})}\right) \\ &+ c\left(\mathbb{E}\left[\int_0^T \|h(s,0)\|^2_{V'_{div}} ds\right]^2 + \mathbb{E}\left[\int_0^T \|G(s,0)\|^2_{L_2(K,G_{div})} ds\right]^2\right). \end{aligned} \tag{2.19}$$

Remark 2.6. We assume that the hypotheses $(H_2) - (H_5)$ hold. If $(u_0, \varphi_0) \in G_{div} \times H$, and h and G are deterministic Lipschitz and sublinear mapping, then we can prove the existence of a unique strong solution (in the probabilistic sense) for the problem (2.1). Indeed for $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\mathcal{O})$, we define $\varphi_{0m} \in D(\mathcal{B})$ as

$$\varphi_{0m} = \left(I + \frac{1}{m}\mathcal{B}\right)^{-1} \varphi_0.$$

Since $\mathcal{B}$ is maximal and monotone, we then deduce from the maximal operator theory that $\varphi_{0m} \to \varphi$ in H as $m \to \infty$. Let (u_m, φ_m) be the unique variational solution of problem (2.1) corresponding to the initial data u_0 and φ_{0m}. As in [**13**], we can prove the tightness of the law of (u_m, φ_m) in $L^{2-\gamma}(0,T; G_{div}) \times L^2(0,T; H)$ with $0 < \gamma < \frac{2}{3}$. The Skorokhod embedding theorem enables us to prove the existence of a global existence of a weak martingale solution to problem (2.1). From the Yamada-Watanabe famous result, we can conclude that problem (2.1) has a unique strong with initial data $(u_0, \varphi_0) \in G_{div} \times H$.

3. Proof of the main result

In this section, we prove the existence and uniqueness of a variational solution to problem (2.1).

3.1. Proof of the existence.

Proof. The proof follows similar steps as in [**5**]. But let us note that the coupling between the stochastic Navier-Stokes equations and the nonlocal Cahn-Hilliard system makes the analysis of the problem more involved compared to the stochastic standard Cahn-Hilliard Navier-Stokes system [**35**].

Step 1: Galerkin approximation scheme. Assume that $\varphi_0 \in L^4(\Omega; D(\mathcal{B}))$. Let

$$\{(w_i, \psi_i),\ i = 1, 2, 3..., \} \subset \mathbb{V}$$

be an orthonormal basis of $\mathbb{Y}$, where $\{w_i,\ i = 1, 2..., \}$, $\{\psi_i, i = 1, 2..., \}$ are eigenvectors of A and $\mathcal{B}$, respectively.

We set $\mathbb{V}_m = \mathbb{Y}_m = \text{span}\{(w_1, \psi_1), ...(w_m, \psi_m)\}$.

We look for $(u_m, \varphi_m) \in \mathbb{Y}_m$ which solves the following approximating problem

$$
(3.1)\quad \begin{cases} \dfrac{du_m}{dt} + \mathcal{P}_m^1[\nu A u_m + B(u_m) - \mu_m \nabla \varphi_m] = \mathcal{P}_m^1(h(t,u_m) + G(t,u_m)\dot{W}_t), \\ \dfrac{d\varphi_m}{dt} + \mathcal{P}_m^2(u_m.\nabla)\varphi_m + \mathcal{P}_m^2\mathcal{A}(\rho(.,\varphi_m) - J * \varphi_m) = 0, \\ \rho(.,\varphi_m) := a(.)\varphi_m + F'(\varphi_m), \\ \mu_m = \mathcal{P}_m^2(\rho(.,\varphi_m) - J * \varphi_m), \\ (u_{0m}, \varphi_{0m}) = (u_m(0), \varphi_m(0)) = \mathcal{P}_m(u_0, \varphi_0), \end{cases}
$$

where $\mathcal{P}_m = (\mathcal{P}_m^1, \mathcal{P}_m^2) : G_{div} \times H \to \mathbb{V}_m$ is the orthogonal projection.
System (3.1) is a system of stochastic differential equations in a finite dimensional Banach spaces with locally Lipschitz coefficients ($F' \in C^1(\mathbb{R})$), $\mathcal{P}_m^1 B(u_m)$ and $\mathcal{P}_m^1 \mu_m \nabla \varphi_m$. The terms $\mathcal{P}_m^1 h(t,u_m)$ and $\mathcal{P}_m^1 G(t,u_m)$ are globally Lipschitz. Hence by a well-known result about existence and uniqueness of solution to stochastic differential equations [**27, 25**], there exists on a short interval $[0, T_m)$, $T_m \leq T$ and a sequence of continuous processes (u_m, φ_m) solving (3.1). It will follow from a priori estimates below that (u_m, φ_m) exists on $[0, T]$.

Step 2: Some estimates for the approximating sequence. Applying the Itô formula, we obtain

$$
(3.2)\quad \begin{aligned} & |u_m(t)|^2 + 2\nu \int_0^t \|u_m(s)\|^2 ds + 2\int_0^t \int_{\mathcal{O}} (u_m(s).\nabla \mu_m(s))\varphi_m(s) dx ds \\ & = 2\int_0^t \langle h(s,u_m(s)), u_m(s)\rangle \, ds + \sum_{j,k=1}^m \int_0^t [\langle G(s,u_m(s))e_j, w_k\rangle]^2 \, ds \\ & + 2\sum_{j=1}^m \int_0^t \langle G(s,u_m(s))e_j, u_m(s)\rangle \, d\beta_s^j. \end{aligned}
$$

By using μ_m as a test function in $(3.1)_2$ and using also the fact that $\partial_\eta \mu_m = 0$ on $\partial\mathcal{O}$ we obtain

$$
(3.3)\quad \begin{aligned} (\varphi_m', \mu_m) - \int_0^t \int_{\mathcal{O}} (u_m(s).\nabla\mu_m(s))\varphi_m(s) dx ds + (\nabla\rho(.,\varphi_m), \nabla\mu_m) \\ = \int_{\mathcal{O}} (\nabla J * \varphi_m).\, \nabla\mu_m \, dx. \end{aligned}
$$

Note that

$$
\begin{aligned}
&\tfrac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)(\varphi_m(x)-\varphi_m(y))^2dxdy+\int_{\mathcal{O}} F(\varphi_m(x))dx\\
&=\tfrac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)(\varphi_m(x))^2dxdy-\frac{1}{2}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)\varphi_m(x)\varphi_m(y)dxdy\\
&+\tfrac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)(\varphi_m(y))^2dxdy+\int_{\mathcal{O}} F(\varphi_m(x))dx\\
&=\tfrac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)(\varphi_m(x))^2dxdy-\frac{1}{2}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)\varphi_m(x)\varphi_m(y)dxdy\\
&+\tfrac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(y-x)(\varphi_m(x))^2dxdy+\int_{\mathcal{O}} F(\varphi_m(x))dx\\
&=\tfrac{1}{2}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)(\varphi_m(x))^2dxdy-\frac{1}{2}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)\varphi_m(x)\varphi_m(y)dxdy\\
&+\int_{\mathcal{O}} F(\varphi_m(x))dx \text{ since } J(y-x)=J(-(x-y))=J(x-y)\\
&=\tfrac{1}{2}\int_{\mathcal{O}} a(x)(\varphi_m(x))^2dx-\frac{1}{2}\int_{\mathcal{O}}(J*\varphi_m,\varphi_m)dx+\int_{\mathcal{O}} F(\varphi_m(x))dx\\
&=\tfrac{1}{2}|\sqrt{a}\varphi_m|^2+\int_{\mathcal{O}}\left[F(\varphi_m)-\frac{1}{2}(J*\varphi_m,\varphi_m)\right]dx.
\end{aligned}
\tag{3.4}
$$

We mention that in (3.4) we have used the fact that $J(x) = J(-x)$ in conjunction with $a(x) = \int_{\mathcal{O}} J(x-y)dy$. It then follows that

$$
\begin{aligned}
&(\varphi_m', \mu_m) = (\varphi_m', a\varphi_m + F'(\varphi_m) - J*\varphi_m)\\
&= \frac{d}{dt}\left[\frac{1}{2}|\sqrt{a}\varphi_m|^2 + \int_{\mathcal{O}}[F(\varphi_m) - \frac{1}{2}(\varphi_m, J*\varphi_m)]dx\right]\\
&= \frac{d}{dt}\left[\frac{1}{4}\int_{\mathcal{O}\times\mathcal{O}} J(x-y)(\varphi_m(x)-\varphi_m(y))^2dxdy + \int_{\mathcal{O}} F(\varphi_m(x))dx\right]\\
&= \frac{d}{dt}\mathcal{E}(\varphi_m(.)).
\end{aligned}
\tag{3.5}
$$

Furthermore, observe that

$$
\begin{aligned}
(\nabla\rho(.,\varphi_m), \nabla\mu_m) &= (\nabla\mu_m + \nabla\mathcal{P}_m^2(J*\varphi_n), \nabla\mu_m)\\
&= |\nabla\mu_m|^2 + \big(\nabla(\mathcal{P}_m^2(J*\varphi_m)), \nabla\mu_m\big)\\
&= (\nabla\rho_m, \nabla\mu_m),
\end{aligned}
\tag{3.6}
$$

where $\rho_m := \mathcal{P}_m^2\rho(.,\varphi_m) = \mu_m + \mathcal{P}_m^2(J*\varphi_m)$.
For all $m \in \mathbb{N}$ and all $k > 0$ let us define

$$
\tau_m^k := \inf\{t \geq 0 : |u_m(t)| \geq k\} \wedge T. \tag{3.7}
$$

Since the process $\{u_m(t)\}_{t\in[0,T]}$ is $\mathbb{F}$-adapted and continuous, τ_m^k is a stopping time. Moreover, since the process (u_m) is continuous on $[0,T]$, the trajectories $t \mapsto u_m(t)$ are bounded on $[0,T]$, $\mathbb{P}$-a.s. Hence $\tau_m^k \uparrow T$, $\mathbb{P}$-a.s., as $k \uparrow \infty$.

Now it follows from (3.2), (3.3), (3.5) and (3.6) that for all $t \in [0,T]$

$$
\begin{aligned}
&\mathcal{E}_{tot}(u_m(t\wedge\tau_m^k),\varphi_m(t\wedge\tau_m^k)) + 2\int_0^{t\wedge\tau_m^k}[\nu\|u_m(s)\|^2 + |\nabla\mu_m(s)|^2] \\
&= \mathcal{E}_{tot}(u_{0m},\varphi_{0m}) - 2\int_0^{t\wedge\tau_m^k}\big((\nabla\mathcal{P}_m^2(J*\varphi_m(s))),\nabla\mu_m(s)\big)\,ds \\
&+2\int_0^{t\wedge\tau_m^k}\langle h(s,u_m(s)),u_m(s)\rangle\,ds \\
&+2\int_0^{t\wedge\tau_m^k}\int_{\mathcal{O}}(\nabla J*\varphi_m(x,s)).\nabla\mu_m(x,s)dxds \\
&+2\sum_{j=1}^m\int_0^{t\wedge\tau_m^k}\langle G(s,u_m(s))e_j,u_m(s)\rangle\,d\beta_s^j \\
&+\sum_{j,k=1}^m\int_0^{t\wedge\tau_m^k}(G(s,u_m(s))e_j,w_k)^2\,ds.
\end{aligned}
\tag{3.8}
$$

We proceed to treat each term of the left and right hand-side of (3.8) as follows: Using (2.16) and the Young inequality, we obtain for all $t \in [0,T]$

$$
\begin{aligned}
2\mathcal{E}(\varphi_m(t)) &= 2|\sqrt{a}\varphi_m(t)|^2 + 2\int_{\mathcal{O}}F(\varphi_m(x,t))dx - (\varphi_m(t),J*\varphi_m(t)) \\
&\geq \int_{\mathcal{O}}(a(x)-|J|_{L^1})|\varphi_m(x,t)|^2dx + 2c_7|\varphi_m(t)|_{L^{2\kappa+2}(\mathcal{O})}^{2+2\kappa} - 2c_8|\mathcal{O}| \\
&\geq c_7|\varphi_m(t)|_{L^{2\kappa+2}(\mathcal{O})}^{2+2\kappa} - \mathcal{K}_d,
\end{aligned}
\tag{3.9}
$$

with

$$
\mathcal{K}_d = \left[2c_8 + \frac{\kappa}{c_7^{1/\kappa}}\left(\frac{|J|_{L^1}}{\kappa+1}\right)^{(\kappa+1)/\kappa}\right]|\mathcal{O}|.
$$

Hence, for all $t \in [0,T]$, we have

$$
\begin{aligned}
\mathcal{E}_{tot}(u_m(t),\varphi_m(t)) + \mathcal{K}_d &= |u_m(t)|^2 + 2\mathcal{E}(\varphi_m(t)) + \mathcal{K}_d \\
&\geq |u_m(t)|^2 + c_7|\varphi_m(t)|_{L^{2\kappa+2}(\mathcal{O})}^{2+2\kappa}.
\end{aligned}
\tag{3.10}
$$

Using the Cauchy Schwarz inequality, Young's inequality and (H_1) we obtain

$$
\begin{aligned}
\left|\left\langle\mathcal{P}_m^1h(s,u_m(s)),u_m(s)\right\rangle\right| &\leq \|h(s,u_m(s))\|_{V'_{div}}\|u_m(s)\| \\
&\leq \frac{\nu}{2}\|u_m(s)\|^2 + \frac{1}{\nu}|h(s,0)|_{V'_{div}}^2 + \frac{1}{\nu}l_h^2|u_m(s)|^2.
\end{aligned}
\tag{3.11}
$$

From the hypothesis (H_1) in Assumption 2.4, we deduce that

$$
\begin{aligned}
\sum_{j,k=1}^m(G(s,u_m(s))e_j,w_k)^2 &\leq \|G(s,u_m(s))\|_{L_2(K,G_{div})}^2 \\
&\leq 2l_g^2|u_m(s)|^2 + 2\|G(s,0)\|_{L_2(K,G_{div})}^2.
\end{aligned}
\tag{3.12}
$$

Using the Young inequality, we infer that

$$
\begin{aligned}
&\left|((\nabla \mathcal{P}_m^2(J*\varphi_m(s))), \nabla \mu_m(s))\right| \\
&\le |\nabla \mathcal{P}_m^2(J*\varphi_m(s))||\nabla \mu_m(s)| \\
&\le \left|\mathcal{B}^{1/2}\mathcal{P}_m^2(J*\varphi_m(s))\right| |\nabla \mu_m(s)| \\
&\le (|\nabla J*\varphi_m(s)| + |J*\varphi_m(s)|)|\nabla \mu_m(s)| \\
&\le |J|_{W^{1,1}(\mathbb{R}^2,\mathbb{R})}|\varphi_m(s)||\nabla \mu_m(s)| \\
&\le \frac{1}{4}|\nabla \mu_m(s)|^2 + c|\varphi_m(s)|^2,
\end{aligned} \tag{3.13}
$$

Proceeding similarly as in (3.11), we obtain

$$
\int_{\mathcal{O}} (\nabla J*\varphi_m(x,s)).\ \nabla \mu_m(x,s)dx \le \frac{1}{4}|\nabla \mu_m(s)|^2 + c|\varphi_m(s)|^2. \tag{3.14}
$$

By invoking Burkhölder-Davis-Gundy's inequality we get that

$$
\begin{aligned}
&2\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left|\int_0^s \sum_{j=1}^m \langle G(r,u_m(r))e_j, u_m(r)\rangle\, d\beta_r^j\right| \\
&\le 6\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}\sum_{j=1}^m (G(r,u_m(r))e_j, u_m(r))^2 dr\right)^{\frac{1}{2}} \\
&\le 6\mathbb{E}\left(\int_0^{t\wedge\tau_m^k} \|G(r,u_m(r))\|^2_{L_2(K,G_{div})}|u_m(r)|^2 dr\right)^{\frac{1}{2}} \\
&\le 6\mathbb{E}\left[\sup_{r\in[0,t\wedge\tau_m^k]}|u_m(r)|^2\int_0^{t\wedge\tau_m^k} \|G(r,u_m(r))\|^2_{L_2(K,G_{div})}dr\right]^{\frac{1}{2}} \\
&\le \tfrac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|u_m(s)|^2 + 36\mathbb{E}\int_0^T \|G(s,0)\|^2_{L_2(K,G_{div})}ds \\
&+36 l_g^2\mathbb{E}\int_0^{t\wedge\tau_m^k}|u_m(s)|^2 ds.
\end{aligned} \tag{3.15}
$$

Using the Fubini theorem, the Hölder and the Young inequality, we infer that for all $t \in [0,T]$

$$
\begin{aligned}
c\mathbb{E}\int_0^t |\varphi_m(s)|^2 ds &\le c\mathbb{E}\int_0^t\int_{\mathcal{O}} |\varphi_m(s,x)|^2 dxds \\
&\le c|\mathcal{O}|^{\frac{\kappa}{\kappa+1}}\mathbb{E}\int_0^t |\varphi_m(s)|^{\frac{2\kappa+2}{\kappa+1}}_{L^{2\kappa+2}(\mathcal{O})}ds \\
&\le c|\mathcal{O}|^{\frac{\kappa}{\kappa+1}}\int_0^t \left(\mathbb{E}|\varphi_m(s)|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{O})}\right)^{\frac{1}{\kappa+1}} ds \\
&\le \tfrac{c_7}{2}\mathbb{E}\int_0^t |\varphi_m(s)|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{O})}ds + \tilde{c}_1 c_7^{-1/\kappa}|\mathcal{O}|T,
\end{aligned} \tag{3.16}
$$

where $\tilde{c}_1$ is a positive constant depending on κ.
Note that

$$
\begin{aligned}
\mathcal{E}_{tot}(u_{0m},\varphi_{0m}) &\le |u_{0m}|^2 + 2|J|_{L^1(\mathbb{R}^2)}|\varphi_{0m}|^2 + 2\int_{\mathcal{O}} F(\varphi_{0m}(x))dx \\
&\le |u_0|^2 + 2|J|_{L^1(\mathbb{R}^2)}|\varphi_0|^2 + 2\int_{\mathcal{O}} F(\varphi_0(x))dx.
\end{aligned} \tag{3.17}
$$

In (3.17) we have used the fact that, since $\varphi_0 \in D(\mathcal{B})$ .a.s., then we have $\varphi_{0m} \to \varphi_0$ in $H^2(\mathcal{O})$ a.s. and hence also in $L^\infty(\mathcal{O})$ a.s..

Taking the supremum, then the mathematical expectation and using (3.10)-(3.17) we infer from (3.8) that

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left(|u_m(s)|^2 + c_7|\varphi_m(s)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})}\right)\\
&+\frac{3}{2}\mathbb{E}\int_0^{t\wedge\tau_m^k}(\nu\|u_m(s)\|^2 + |\nabla\mu_m(s)|^2)ds\\
&\leq \mathbb{E}\mathcal{K}(u_0,\varphi_0) + \mathcal{K}_d + \tilde{c}_1 c_7^{-1/\kappa}|\mathcal{O}|T \\
&+\mathbb{E}\int_0^T\left[\frac{1}{\nu}\|h(s,0)\|^2_{V'_{div}} + 38\|G(s,0)\|^2_{L_2(K,G_{div})}\right]ds\\
&+\left(\frac{1}{\nu}l_h^2 + 38l_g^2 + 1\right)\mathbb{E}\int_0^{t\wedge\tau_m^k}(|u_m(s)|^2 + c_7|\varphi_m(s)|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{O})})ds,
\end{aligned}
\tag{3.18}
$$

where

$$
\mathcal{K}(u_0,\varphi_0) = |u_0|^2 + 2|J|_{L^1(\mathbb{R}^2)}|\varphi_0|^2 + 2\int_{\mathcal{O}}F(\varphi_0(x))dx. \tag{3.19}
$$

Dropping off the second term in the left hand side of the estimate (3.18) and invoking the deterministic Gronwall's lemma yield

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left(|u_m(s)|^2 + c_7|\varphi_m(s)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})}\right)\\
&\leq c\mathbb{E}\mathcal{K}(u_0,\varphi_0) + c\mathbb{E}\int_0^T\|h(s,0)\|^2_{V'_{div}}ds + c\mathbb{E}\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds.
\end{aligned}
$$

Letting now $k\uparrow\infty$ in this last estimate, using the fact that $\tau_m^k\uparrow T$, $\mathbb{P}$-a.s., as $k\uparrow\infty$, we infer that

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\left(|u_m(s)|^2 + c_7|\varphi_m(s)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})}\right)\\
&\leq c\mathbb{E}\mathcal{K}(u_0,\varphi_0) + c\mathbb{E}\int_0^T[\|h(s,0)\|^2_{V'_{div}} + \|G(s,0)\|^2_{L_2(K,G_{div})}]ds.
\end{aligned}
\tag{3.20}
$$

We deduce from (3.18) and (3.20) that

$$
\begin{aligned}
\mathbb{E}\int_0^T(\nu\|u_m(s)\|^2 + |\nabla\mu_m(s)|^2)ds &\leq c\mathbb{E}\mathcal{K}(u_0,\varphi_0)\\
&+c\mathbb{E}\int_0^T\|h(s,0)\|^2_{V'_{div}}ds + c\mathbb{E}\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds.
\end{aligned}
\tag{3.21}
$$

We now give a higher estimate for the sequences u_m and φ_m.
We first note that from (3.8), (3.10)-(3.14) and (3.17), it is easy to see that

$$\begin{aligned}|u_m(t\wedge\tau_m^k)|^2 \quad & +c_7|\varphi_m(t\wedge\tau_m^k)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})} \\ & +\tfrac{3}{2}\int_0^{t\wedge\tau_m^k}(\nu\|u_m(s)\|^2+|\nabla\mu_m(s)|^2)ds \\ & \leq |u_0|^2+2|J|_{L^1(\mathbb{R}^2)}|\varphi_0|^2+2\int_{\mathcal{O}}F(\varphi_0(x))dx \\ & +\tfrac{1}{\nu}\int_0^T\|h(s,0)\|^2_{V'_{div}}ds+\mathcal{K}_d \\ & +(2l_g^2+\tfrac{1}{\nu}l_h^2)\int_0^{t\wedge\tau_m^k}|u_m(s)|^2ds+2\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds \\ & +c\int_0^{t\wedge\tau_m^k}|\varphi_m(s)|^2ds+2\int_0^{t\wedge\tau_m^k}\sum_{j=1}^m(G(s,u_m(s))e_j,u_m(s))d\beta_s^j.\end{aligned}$$

Now raising both side to the power $\frac{p}{2}>1$, taking supremum over $s\in[0,t\wedge\tau_m^k]$ and taking mathematical expectation, we infer that

$$\begin{aligned} & \mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left(|u_m(s)|^2+c_7|\varphi_m(s)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})}\right)^{\frac{p}{2}} \\ & +\left(\tfrac{3}{2}\right)^{\frac{p}{2}}\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}(\nu\|u_m(s)\|^2+|\nabla\mu_m(s)|^2)ds\right)^{\frac{p}{2}} \\ & \leq c\mathbb{E}\left[\mathcal{K}(u_0,\varphi_0)\right]^{\frac{p}{2}}+c\mathcal{K}_d^{\frac{p}{2}}+c\mathbb{E}\left(\int_0^T\|h(s,0)\|^2_{V'_{div}}ds\right)^{\frac{p}{2}} \\ & +c\mathbb{E}\left(\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds\right)^{\frac{p}{2}} \\ & +c\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}|u_m(s)|^2ds\right)^{\frac{p}{2}}+c\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}|\varphi_m(s)|^2ds\right)^{\frac{p}{2}} \\ & +c\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left|\int_0^s\sum_{j=1}^m\langle G(r,u_m(r))e_j,u_m(r)\rangle\,d\beta_r^j\right|^{\frac{p}{2}}.\end{aligned}\tag{3.22}$$

We see from Burkhölder-Davis-Gundy's inequality that

$$\begin{aligned} & c\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left|\int_0^s\sum_{j=1}^m\langle G(r,u_m(r))e_j,u_m(r)\rangle\,d\beta_r^j\right|^{p/2} \\ & \leq c\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}\|G(r,u_m(r))\|^2_{L_2(K,G_{div})}|u_m(r)|^2dr\right)^{p/4}.\end{aligned}\tag{3.23}$$

This gives

$$
\begin{aligned}
&c\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left|\int_0^s\sum_{j=1}^m\langle G(r,u_m(r))e_j,u_m(r)\rangle\,d\beta_r^j\right|^{p/2}\\
&\le\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|u_m(s)|^p\\
&\quad+c\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}\left(2l_g^2|u_m(s)|^2+2\|G(s,0)\|^2_{L_2(K,G_{div})}ds\right)\right)^{p/2}\\
&\le\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|u_m(s)|^p\\
&\quad+c\mathbb{E}\left(\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds\right)^{\frac{p}{2}}+c\mathbb{E}\int_0^{t\wedge\tau_m^k}|u_m(s)|^pds.
\end{aligned}
\tag{3.24}
$$

Using the Hölder inequality, we get

$$
c\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}|u_m(s)|^2ds\right)^{p/2}\le c\mathbb{E}\int_0^{t\wedge\tau_m^k}|u_m(s)|^pds.
\tag{3.25}
$$

Arguing similarly as in (3.16), we infer that

$$
\begin{aligned}
&c\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}|\varphi_m(s)|^2ds\right)^{\frac{p}{2}}\le ct^{\frac{p-2}{2}}\mathbb{E}\int_0^{t\wedge\tau_m^k}|\varphi_m(s)|^pds\\
&\le ct^{\frac{p-2}{2}}|\mathcal{O}|^{\frac{p\kappa}{2(\kappa+1)}}\mathbb{E}\int_0^{t\wedge\tau_m^k}|\varphi_m(s)|^p_{L^{2\kappa+2}}ds\\
&\le c_7\mathbb{E}\int_0^{t\wedge\tau_m^k}|\varphi_m(s)|^{p(\kappa+1)}_{L^{2\kappa+2}(\mathcal{O})}ds+\frac{c\kappa c_7^{-1/\kappa}}{(\kappa+1)^{\frac{\kappa+1}{\kappa}}}|\mathcal{O}|^{\frac{p}{2}}T^{\frac{p(\kappa+1)-2}{2\kappa}}.
\end{aligned}
\tag{3.26}
$$

Inserting now the estimates (3.24)-(3.26) in (3.22), we obtain

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left(|u_m(s)|^2+c_7|\varphi_m(s)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})}\right)^{\frac{p}{2}}\\
&+\left(\frac{3}{2}\right)^{\frac{p}{2}}\mathbb{E}\left(\int_0^{t\wedge\tau_m^k}(\nu\|u_m(s)\|^2+|\nabla\mu_m(s)|^2)ds\right)^{\frac{p}{2}}\\
&\le c\mathbb{E}\left[\mathcal{K}(u_0,\varphi_0)\right]^{\frac{p}{2}}+c\mathcal{K}_d^{\frac{p}{2}}+c\mathbb{E}\left(\int_0^T\|h(s,0)\|^2_{V'_{div}}ds\right)^{\frac{p}{2}}\\
&+c\mathbb{E}\left(\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds\right)^{\frac{p}{2}}+\frac{c\kappa c_7^{-1/\kappa}}{(\kappa+1)^{\frac{\kappa+1}{\kappa}}}|\mathcal{O}|^{\frac{p}{2}}T^{\frac{p(\kappa+1)-2}{2\kappa}}\\
&+\left(c+\frac{1}{c_7^{\frac{p-2}{2}}}\right)\mathbb{E}\int_0^{t\wedge\tau_m^k}\left(|u_m(s)|^2+|\varphi_m(s)|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{O})}\right)^{p/2}ds.
\end{aligned}
\tag{3.27}
$$

Now arguing similarly as in (3.20), we derive that

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\left(|u_m(s)|^2+c_7|\varphi_m(s)|^{2+2\kappa}_{L^{2\kappa+2}(\mathcal{O})}\right)^{\frac{p}{2}}\\
&\leq c\mathbb{E}\left[\mathcal{K}(u_0,\varphi_0)\right]^{\frac{p}{2}}+c\mathcal{K}_d^{\frac{p}{2}}+c\mathbb{E}\left(\int_0^T\|h(s,0)\|^2_{V'_{div}}ds\right)^{\frac{p}{2}}\\
&+c\mathbb{E}\left(\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds\right)^{\frac{p}{2}}+\frac{c\kappa c_7^{-1/\kappa}}{(\kappa+1)^{\frac{\kappa+1}{\kappa}}}|\mathcal{O}|^{\frac{p}{2}}T^{\frac{p(\kappa+1)-2}{2\kappa}}.
\end{aligned}
\tag{3.28}
$$

With the use of (3.27), (3.28) and the fact that $\tau_m^k\uparrow T$, $\mathbb{P}$-a.s., as $k\uparrow\infty$, we derive that

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T(\nu\|u_m(s)\|^2+|\nabla\mu_m(s)|^2)ds\right)^{\frac{p}{2}}\\
&\leq c\mathbb{E}\left[\mathcal{K}(u_0,\varphi_0)\right]^{\frac{p}{2}}+c\mathcal{K}_d^{\frac{p}{2}}+c\mathbb{E}\left(\int_0^T\|h(s,0)\|^2_{V'_{div}}ds\right)^{\frac{p}{2}}\\
&+c\mathbb{E}\left(\int_0^T\|G(s,0)\|^2_{L_2(K,G_{div})}ds\right)^{\frac{p}{2}}+\frac{c\kappa c_7^{-1/\kappa}}{(\kappa+1)^{\frac{\kappa+1}{\kappa}}}|\mathcal{O}|^{\frac{p}{2}}T^{\frac{p(\kappa+1)-2}{2\kappa}}.
\end{aligned}
\tag{3.29}
$$

for all $\frac{p}{2}>1$.
By (H_3), using the Young inequality we have

$$
\begin{aligned}
(\mu_m,-\Delta\varphi_m)=(\nabla\mu_m,\nabla\varphi_m)&=(\nabla\varphi_m,(F''(\varphi_m)+a)\nabla\varphi_m+\varphi_m\nabla a-\nabla(J*\varphi_m))\\
&\geq c_0|\nabla\varphi_m|^2+(\nabla\varphi_m,\varphi_m\nabla a-\nabla(J*\varphi_m))\\
&\geq c_0|\nabla\varphi_m|^2-2|\nabla J|_{L^1}|\varphi_n||\nabla\varphi_m|\\
&\geq\frac{c_0}{2}|\nabla\varphi_m|^2-\frac{2}{c_0}|\nabla J|^2_{L^1}|\varphi_m|^2.
\end{aligned}
$$

Hence

$$
\frac{c_0}{2}|\nabla\varphi_m|^2-\frac{2}{c_0}|\nabla J|^2_{L^1}|\varphi_m|^2\leq(\nabla\mu_m,\nabla\varphi_m)\leq\frac{c_0}{4}|\nabla\varphi_m|^2+\frac{1}{c_0}|\nabla\mu_m|^2.
$$

From this last inequality, we obtain

$$
|\nabla\varphi_m|^2\leq\frac{8}{c_0^2}|\varphi_m|^2+\frac{4}{c_0^2}|\nabla\mu_m|^2.
\tag{3.30}
$$

Now from (3.20), (3.21) and (3.30), using also the fact that the domain $\mathcal{O}\subset\mathbb{R}^2$ is bounded, we infer that

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}|(u_m(s),\varphi_m(s))|^2_{\mathbb{Y}}=\mathbb{E}\sup_{s\in[0,T]}\left(|u_m(s)|^2+|\varphi_m(s)|^2\right)\leq C,\\
&\mathbb{E}\int_0^T\|(u_m(s),\varphi_m(s))\|^2_{\mathbb{V}}ds\leq C.
\end{aligned}
\tag{3.31}
$$

Also from (3.28), (3.29) and (3.30), using also the fact that the domain $\mathcal{O} \subset \mathbb{R}^2$ is bounded, we derive that

$$\begin{aligned} &\mathbb{E} \sup_{s\in[0,T]} |(u_m(s), \varphi_m(s))|^4_{\mathbb{Y}} + \mathbb{E}\left(\int_0^T \|(u_m(s), \varphi_m(s))\|^2_{\mathbb{V}} ds\right)^2 \\ &\leq c\mathbb{E}\left[\mathcal{K}(u_0, \varphi_0)\right]^2 + c\mathcal{K}_d^2 + c\mathbb{E}\left(\int_0^T \|h(s,0)\|^2_{V'_{div}} ds\right)^2 \\ &+c\mathbb{E}\left(\int_0^T \|G(s,0)\|^2_{L_2(K,G_{div})} ds\right)^2 + \frac{c\kappa c_7^{-1/\kappa}}{(\kappa+1)^{\frac{\kappa+1}{\kappa}}} |\mathcal{O}|^{\frac{p}{2}} T^{\frac{4(\kappa+1)-2}{2\kappa}}. \end{aligned} \tag{3.32}$$

We also note that

$$\begin{aligned} \|\mathcal{P}^1_m B(u_m)\|_{V'_{div}} &\leq c|u_m|\|u_m\|, \\ \|\mathcal{P}^2_m (u_m.\nabla)\varphi_m\|_{V'_0} &\leq c|\varphi_m|_{L^{2\kappa+2}(\mathcal{O})}\|u_m\|, \\ \|\mathcal{P}^1_m (\mu_m \nabla\varphi_m)\|_{V'_{div}} &\leq c|\varphi_m|_{L^{2\kappa+2}(\mathcal{O})}|\nabla\mu_m|. \end{aligned} \tag{3.33}$$

From (H_1), we have

$$\begin{aligned} \|G(s, u_m(s))\|^2_{L_2(K,G_{div})} &\leq c l_g^2 |u_m(s)|^2 + \|G(s,0)\|^2_{L_2(K,G_{div})}, \\ \|h(s, u_m(s))\|^2_{V'_{div}} &\leq c l_h^2 |u_m(s)|^2 + \|h(s,0)\|^2_{V'_{div}}. \end{aligned} \tag{3.34}$$

We will now prove that the following estimate holds

$$\mathbb{E} \sup_{s\in[0,T]} |F(\varphi_m(s))|^{p/2}_{L^1(\mathcal{O})} \leq C_p. \tag{3.35}$$

To prove (3.35), we first assume that $F(\varphi_m) \geq 0$. Then from (3.8), we have for fix $s \leq t \wedge \tau_m^k$

$$\begin{aligned} 2|F(\varphi_m(s))|_{L^1(\mathcal{O})} &\leq \mathcal{E}_{tot}(u_{0m}, \varphi_{0m}) + |(\varphi_m(s), J * \varphi_m(s))| \\ &+2\int_0^s \left|\left((\nabla\mathcal{P}^2_m(J * \varphi_m(r))), \nabla\mu_m(r)\right)\right| dr \\ &+2\int_0^s |\langle h(r, u_m(r)), u_m(r)\rangle| \, dr \\ &+2\int_0^s \int_{\mathcal{O}} |(\nabla J * \varphi_m(x,r)).\nabla\mu_m(x,r)| \, dxdr \\ &+\sum_{j,k=1}^m \int_0^s \left(G(r, u_m(r))e_j, w_k\right)^2 dr \\ &+2\sum_{j=1}^m \int_0^s \langle G(s, u_m(r))e_j, u_m(r)\rangle \, d\beta_r^j. \end{aligned} \tag{3.36}$$

Inserting now the estimates (3.11)-(3.14) in (3.36) and taking the supremum over $s \in [0, t \wedge \tau_m^k]$, we infer that

$$\begin{aligned} &\sup_{s\in[0,t\wedge\tau_m^k]} |F(\varphi_m(s))|_{L^1(\mathcal{O})} \leq C\left(\mathcal{K}(u_0, \varphi_0) + \sup_{s\in[0,t\wedge\tau_m^k]} |\varphi_m(s)|^2\right) \\ &+C\int_0^{t\wedge\tau_m^k} \left(\|u_m(s)\|^2 + |\varphi_m(s)|^2\right) ds \\ &+C\int_0^{t\wedge\tau_m^k} \left(\|h(s,0)\|^2_{V'_{div}} + \|G(s,0)\|^2_{L_2(K,G_{div})} + |\nabla\mu_m(s)|^2\right) ds \\ &+C \sup_{s\in[0,t\wedge\tau_m^k]} \left|\sum_{j=1}^m \int_0^s \langle G(s, u_m(r))e_j, u_m(r)\rangle \, d\beta_r^j\right| \end{aligned} \tag{3.37}$$

where $\mathcal{K}(u_0,\varphi_0)$ is given by (3.19) and the constant $C := C_{\lambda_1,J,\nu,l_h,l_g}$.
Now raising both side to the power $p/2$ and taking the expectation, we get

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|F(\varphi_m(s))|_{L^1(\mathcal{O})}^{p/2}\leq C\mathbb{E}[\mathcal{K}(u_0,\varphi_0)]^{p/2}\\
&+C\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|\varphi_m(s)|^p+C\mathbb{E}\left[\int_0^{t\wedge\tau_m^k}\left(\|u_m(s)\|^2+|\nabla\mu_m(s)|^2\right)ds\right]^{p/2}\\
&+C\mathbb{E}\left[\int_0^{t\wedge\tau_m^k}\left(|\varphi_m(s)|^2+\|h(s,0)\|_{V'_{div}}^2\right)ds\right]^{p/2}\\
&+C\mathbb{E}\left[\int_0^{t\wedge\tau_m^k}\|G(s,0)\|_{L_2(K,G_{div})}^2ds\right]^{p/2}\\
&+C\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left|\sum_{j=1}^m\int_0^s\langle G(s,u_m(r))e_j,u_m(r)\rangle\,d\beta_r^j\right|^{p/2},
\end{aligned}\tag{3.38}$$

where the positive constant $C := C_{\lambda_1,J,\nu,l_h,l_g,p}$.
By (3.16), (3.24), (3.27), (3.28), (3.38) and some well known inequalities, we infer that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|F(\varphi_m(s))|_{L^1(\mathcal{O})}^{p/2}\leq C+C\mathbb{E}[\mathcal{K}(u_0,\varphi_0)]^{p/2}\\
&+C\mathbb{E}\left[\int_0^{t\wedge\tau_m^k}\left(\|u_m(s)\|^2+|\nabla\mu_m(s)|^2\right)ds\right]^{p/2}\\
&+C\mathbb{E}\left[\int_0^T\|h(s,0)\|_{V'_{div}}^2\,ds\right]^{p/2}+C\mathbb{E}\left[\int_0^T\|G(s,0)\|_{L_2(K,G_{div})}^2ds\right]^{p/2}\\
&+C\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}|u_m(s)|^p+C\mathbb{E}\sup_{s\in[0,t\wedge\tau_m^k]}\left(|\varphi_m|_{L^{2\kappa+2}}^{2\kappa+2}\right)^{p/2}\\
&\leq C_{\lambda_1,J,\nu,l_h,l_g,p,\kappa,\mathcal{O},T}.
\end{aligned}\tag{3.39}$$

Note that in (3.39) the constant $C_{\lambda_1,J,\nu,l_h,l_g,p,\kappa,\mathcal{O},T}$ is independent of $m\in\mathbb{N}$, $k>0$ and $t\in[0,T]$. Hence passing to the limit as $k\to\infty$, we get

$$\mathbb{E}\sup_{s\in[0,T]}|F(\varphi_m(s))|_{L^1(\mathcal{O})}^{p/2}\leq C_p.\tag{3.40}$$

For the case $F(\varphi_m)\leq 0$, we infer from (3.9) that

$$\begin{aligned}
2|F(\varphi_m)(s)|_{L^1(\mathcal{O})}&\leq 2|\sqrt{a}\varphi_m(s)|^2-(\varphi_m(s),J*\varphi_m(s))\\
&\quad-c_7|\varphi_m(s)|_{L^{2\kappa+2}}^{2\kappa+2}+\mathcal{K}_d\\
&\leq 3|J|_{L^1}|\varphi_m(s)|^2+c_7|\varphi_m(s)|_{L^{2\kappa+2}}^{2\kappa+2}+\mathcal{K}_d\\
&\leq C_{J,\kappa,\mathcal{O}}|\varphi_m(s)|_{L^{2\kappa+2}}^2+C_{c_7}|\varphi_m(s)|_{L^{2\kappa+2}}^{2\kappa+2}+\mathcal{K}_d\\
&\leq C_{\kappa,J,\mathcal{O},c_7}(1+\mathcal{K}_d+|\varphi_m(s)|_{L^{2\kappa+2}}^{2\kappa+2}).
\end{aligned}\tag{3.41}$$

Now raising both sides to the power $p/2$, taking supremum over $t \in [0, t \wedge \tau_m^k]$, the expectation and using also (3.28), we obtain

$$\begin{aligned} &\mathbb{E} \sup_{s\in[0,t\wedge\tau_m^k]} |F(\varphi_m(s))|_{L^1(\mathcal{O})}^{p/2} \leq C_{\kappa,J,\mathcal{O},c_7,p,c_8,T}\mathbb{E}\left[1+[\mathcal{K}(u_0,\varphi_0)]^{p/2}\right] \\ &+C_{\kappa,J,\mathcal{O},c_7,p,c_8,T}\mathbb{E}\left(\int_0^T \|h(s,0)\|_{V'_{div}}^2 ds\right)^{p/2} \\ &+C_{\kappa,J,\mathcal{O},c_7,p,c_8,T}\mathbb{E}\left(\int_0^T \|G(s,0)\|_{L_2(K,G_{div})}^2 ds\right)^{p/2}, \end{aligned} \tag{3.42}$$

for some constant $C_{\kappa,J,\mathcal{O},c_7,p,c_8,T}$ independent of k, $t \in [0,T]$ and $m \in \mathbb{N}$. Now using the fact that $t \wedge \tau_m^k \to T$ as $k \to \infty$, we infer from (3.42) and the dominated convergent theorem that

$$\mathbb{E} \sup_{s\in[0,T]} |F(\varphi_m(s))|_{L^1(\mathcal{O})}^{p/2} \leq C_p. \tag{3.43}$$

This completes the proof of (3.35). That is

$$\mathbb{E} \sup_{s\in[0,T]} |F(\varphi_m(s))|_{L^1(\mathcal{O})}^{p/2} \leq C_p, \quad \text{for both cases.} \tag{3.44}$$

By means of (H_5), (3.28) and (3.44), we have

$$\mathbb{E} \sup_{s\in[0,T]} |\rho(.,\varphi_m(s))|_{L^r(\mathcal{O})}^{p} \leq C_p. \tag{3.45}$$

We first note that from (H_5) and the Young inequality, it is easy to see that for all $s \in [0,T]$

$$|F'(\varphi_m(s))| \leq \frac{1}{r}|F'(\varphi_m(s))|^r + \frac{r-1}{r} \leq \frac{c_3}{r}|F(\varphi_m(s))| + \frac{c_4}{r} + \frac{r-1}{r}. \tag{3.46}$$

By (3.46), using the Poincaré-Wirtinger inequality and the Young inequality, we infer that

$$\begin{aligned} &|\mu_m(s)| \leq c\left(|\nabla\mu_m(s)| + \tfrac{1}{|\mathcal{O}|}\left|\int_{\mathcal{O}} \mu_m(x,s)dx\right|\right) \\ &= c(|\nabla\mu_m(s)| + \tfrac{1}{|\mathcal{O}|}|(\mu_m(s),1)|) \\ &\leq c|\nabla\mu_m(s)| \\ &+\tfrac{c}{|\mathcal{O}|}\left|\int_{\mathcal{O}} |a(x)\varphi_m(x,s) + F'(\varphi_m(x,s)) - J*\varphi_m(x,s)|dx\right| \\ &\leq c|\nabla\mu_m(s)| \\ &+\tfrac{c}{|\mathcal{O}|^{\frac12}}(|a|_{L^\infty} + |J|_{L^1(\mathbb{R}^2)})|\varphi_n(s)| + \tfrac{c_3}{r|\mathcal{O}|}|F(\varphi_m(s))|_{L^1} + \tfrac{c_4+r-1}{r}. \end{aligned} \tag{3.47}$$

From (3.47), we obtain

$$\begin{aligned} |\mu_m(s)|^2 &\leq c\left[|\nabla\mu_m(s)|^2 + \tfrac{1}{|\mathcal{O}|}(|a|_{L^\infty} + |J|_{L^1(\mathbb{R}^d)})^2|\varphi_m(s)|^2\right] \\ &+c\left[\frac{c_3^2}{r^2|\mathcal{O}|^2}|F(\varphi_m(s))|_{L^1(\mathcal{O})}^2 + \left(\frac{c_4+r-1}{r}\right)^2\right]. \end{aligned} \tag{3.48}$$

By (3.48), (3.44) and (3.29) we infer that:

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \|\mu_m(s)\|_U^2 ds\right]^{\frac{p}{2}} = \mathbb{E}\left[\int_0^T |\mu_m(s)|^2 ds + \int_0^T |\nabla\mu_m(s)|^2 ds\right]^{\frac{p}{2}} \\ &\leq C_p\mathbb{E}\left(\int_0^T |\mu_m(s)|^2 ds\right)^{\frac{p}{2}} + C_p\mathbb{E}\left(\int_0^T |\nabla\mu_m(s)|^2 ds\right)^{\frac{p}{2}} \\ &\leq C_p. \end{aligned} \tag{3.49}$$

By (3.20), (3.49), $(3.33)_3$, using the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \|\mathcal{P}_m^1(\mu_m(s)\nabla\varphi_m(s))\|_{V'_{div}}^2 ds \\
&\le c\mathbb{E}\int_0^T |\varphi_m(s)|_{L^{2+2\kappa}(\mathcal{O})}^2 \|\mu_m(s)\|_U^2 dt \\
&\le c\mathbb{E}\left[\sup_{s\in[0,T]} |\varphi_m(s)|_{L^{2+2\kappa}(\mathcal{O})}^2 \int_0^T \|\mu_m(s)\|_U^2 dt\right] \qquad (3.50)\\
&\le c\left[\mathbb{E}\sup_{s\in[0,T]} |\varphi_m(s)|_{L^{2+2\kappa}(\mathcal{O})}^{2+2\kappa}\right]^{\frac{1}{1+\kappa}} \left[\mathbb{E}\left(\int_0^T \|\mu_m(s)\|_U^2 dt\right)^{\frac{1+\kappa}{\kappa}}\right]^{\frac{\kappa}{1+\kappa}} \\
&\le C.
\end{aligned}$$

From (3.20), (3.29), $(3.33)_2$ and the Hölder inequality we get

$$\begin{aligned}
&\mathbb{E}\int_0^T \|\mathcal{P}_m^2(u_m.\nabla)\varphi_m\|_{V'_0}^2 ds \\
&\le c\mathbb{E}\int_0^T |\varphi_m(s)|_{L^{2+2\kappa}(\mathcal{O})}^2 \|u_m(s)\|^2 ds \\
&\le c\mathbb{E}\left[\sup_{s\in[0,T]} |\varphi_m(s)|_{L^{2+2\kappa}(\mathcal{O})}^2 \int_0^T \|u_m(s)\|^2 ds\right] \qquad (3.51)\\
&\le c\left[\mathbb{E}\sup_{s\in[0,T]} |\varphi_m(s)|_{L^{2+2\kappa}(\mathcal{O})}^{2+2\kappa}\right]^{\frac{1}{1+\kappa}} \left[\mathbb{E}\left(\int_0^T \|u_m(s)\|^2 ds\right)^{\frac{1+\kappa}{\kappa}}\right]^{\frac{\kappa}{1+\kappa}} \\
&\le C.
\end{aligned}$$

From (3.48), (3.44), and (3.21), we infer that

$$\mathbb{E}\int_0^T \|\mathcal{P}_m^2\Delta\mu_m(s)\|_{V'_0}^2 ds \le c\mathbb{E}\int_0^T \|\mu_m(s)\|_U^2 ds \le C. \qquad (3.52)$$

By (3.34), (3.21), $(3.31)_1$ and (3.32) we infer that

$$\begin{aligned}
&\mathbb{E}\int_0^T \|\mathcal{P}_m^1 B(u_m(s))\|_{V'_{div}}^2 ds \\
&\le c\mathbb{E}\sup_{s\in[0,T]} |u_m(s)|^2 \int_0^T \|u_m(s)\|^2 ds \qquad (3.53)\\
&\le c\left[\mathbb{E}\sup_{s\in[0,T]} |u_m(s)|^4\right]^{1/2} \left[\mathbb{E}\left(\int_0^T \|u_m(s)\|^2 ds\right)^2\right]^{1/2} \le C,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\int_0^T \|\mathcal{P}_m^1 h(s,u_m(s))\|_{V'_{div}}^2 ds \\
&\le cl_h^2\mathbb{E}\int_0^T |u_m(s)|^2 ds + c\mathbb{E}\int_0^T \|h(s,0)\|_{V'_{div}}^2 ds \qquad (3.54)\\
&\le cl_h^2\mathbb{E}\int_0^T \|u_m(s)\|^2 ds + c\mathbb{E}\int_0^T \|h(s,0)\|_{V'_{div}}^2 ds \le C,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\int_0^T \|\mathcal{P}_m^1 G(s,u_m(s))\|_{L_2(K,G_{div})}^2 ds \qquad (3.55)\\
&\le cl_g^2\mathbb{E}\int_0^T \|u_m(s)\|^2 ds + c\mathbb{E}\int_0^T \|G(s,0)\|_{L_2(K,G_{div})}^2 ds \le C,
\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\int_0^T \|J*\varphi_m(s)\|_U^2 ds = \mathbb{E}\int_0^T (|J*\varphi_m(s)|^2 + |\nabla J*\varphi_m(s)|^2)ds \\ &\le (|J|_{L^1}^2 + |\nabla J|_{L^1}^2)T\mathbb{E}\sup_{s\in[0,T]}|\varphi_m(s)|^2 \le C.\end{aligned} \tag{3.56}$$

Step 3: Taking limits in the finite dimensional equations. From $(3.31)_2$, $(3.31)_1$, (3.28), (3.44), (3.45), (3.50), (3.51), (3.53) and the Banach-Alaoglu theorem, we can find a subsequence (still denoted) $\{(u_m,\varphi_m)\}$ such that

$$\begin{aligned}(u_m,\varphi_m) &\rightharpoonup (\bar u,\bar\varphi) \text{ in } M^2_{\mathcal{F}_t}(0,T;\mathbb{V}),\\ (u_m,\varphi_m) &\text{ weakly star } (\bar u,\bar\varphi) \text{ in } L^2(\Omega;L^\infty(0,T;G_{div}\times H)),\\ \varphi_m &\rightharpoonup \bar\varphi \text{ in } L^{2\kappa+2}(\Omega,\mathbb{P};L^\infty(0,T;L^{2\kappa+2}(\mathcal{O}))),\\ (u_m(0),\varphi_m(0)) &\rightharpoonup (\zeta_1,\zeta_2) \text{ in } L^4(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{Y}),\\ \mathcal{P}^1_m B(u_m) &\rightharpoonup \bar K_0 \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_{div}),\\ \mathcal{P}^1_m(\mu_m\nabla\varphi_m) &\rightharpoonup \bar K_1 \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_{div}),\\ \mathcal{P}^2_m((u_m.\nabla)\varphi_m) &\rightharpoonup \bar K_2 \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_0),\\ \mathcal{P}^1_m h(t,u_m) &\rightharpoonup \bar h \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_{div}),\\ \mathcal{P}^1_m G(t,u_m) &\rightharpoonup \bar G \text{ in } M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div})),\\ \rho(.,\varphi_m) &\overset{\star}{\rightharpoonup} \bar\rho \text{ in } L^r(\Omega,\mathbb{P};L^\infty(0,T;L^r(\mathcal{O}))),\\ J*\varphi_m &\rightharpoonup \bar J \text{ in } M^2_{\mathcal{F}_t}(0,T;U),\\ \mu_m &\rightharpoonup \bar\mu \text{ in } M^2_{\mathcal{F}_t}(0,T;U).\end{aligned} \tag{3.57}$$

We note that in fact $\bar J = J*\bar\varphi$, since the map $J*.: L^2 \to U = H^1$ is linear and bounded, and $\varphi_m \rightharpoonup \bar\varphi$ in $M^2_{\mathcal{F}_t}(0,T;V_0) \subset M^2_{\mathcal{F}_t}(0,T;L^2)$.

Let us now recall that Problem (3.1) can be rewritten as

$$\begin{aligned}&(u_m(t),w_k) + \int_0^t \langle \nu A u_m(s) + B(u_m(s)) - \mu_m(s)\nabla\varphi_m(s), w_k\rangle\, ds \\ &\quad = (u_m(0),w_k) + \int_0^t \langle h(s,u_m(s)),w_k\rangle\, ds \\ &\quad + \sum_{j=1}^m \int_0^t (G(s,u_m(s))e_j,w_k)d\beta_s^j,\\ &(\varphi_m(t),\psi_k) + \int_0^t (\nabla\rho(.,\varphi_m(s)),\nabla\psi_k)\, ds + \int_0^t \langle (u_m(s).\nabla)\varphi_m(s),\psi_k\rangle\, ds \\ &= (\varphi_m(0),\psi_k) + \int_0^t\int_{\mathcal{O}} (\nabla J*\varphi_m(x,s)).\nabla\psi_k dxds,\\ &\rho(.,\varphi_m) := a(.)\varphi_m + F'(\varphi_m),\\ &\mu_m = \mathcal{P}^2_m(\rho(.,\varphi_m) - J*\varphi_m),\\ &(u_{0m},\varphi_{0m}) = (u_m(0),\varphi_m(0)) = \mathcal{P}_m(u_0,\varphi_0).\end{aligned} \tag{3.58}$$

Using a well known result in [**31, 32, 33**], we can check that $(\bar u,\bar\varphi) \in L^2(\Omega;\mathcal{C}(0,T;G_{div}\times H))$ and satisfies for all $t\in[0,T]$

$$\begin{aligned}&\bar u(t) + \int_0^t [\nu A\bar u(s) + \bar K_0(s) - \bar K_1(s)]ds = u_0 + \int_0^t \bar h(s)ds + \int_0^t \bar G(s)dW(s),\\ &\bar\varphi(t) + \int_0^t \bar K_2(s)ds + \int_0^t \mathcal{A}\bar\mu(s)ds = \varphi_0, \quad \bar\mu = \bar\rho - J*\bar\varphi,\end{aligned} \tag{3.59}$$

The $\mathcal{F}_t$-progressively measurability of the processes $\bar{u}$ and $\bar{\varphi}$ follows from the Hilbert space $M^2_{\mathcal{F}_t}(0,T;\mathbb{V})$ (see [**8**]), the convergence $(3.57)_1$ and the fact that the processes u_m and φ_m are $\mathcal{F}_t$- progressively measurable. We recall that you have applied the compactness theorem 3.18 in [**6**].
The fact that $(u,\varphi) \in L^2(\Omega; L^\infty(0,T; G_{div} \times H)$ follows from the estimate $(3.31)_2$ and the weak star convergence $(3.57)_2$.

Step 4: Identification of some parameters. Let us prove that $\bar{K}_0 = B(\bar{u},\bar{u})$, $\bar{K}_1 = (a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi}$, $\bar{h} = h(t,\bar{u})$, $\bar{G} = G(t,\bar{u})$, $\bar{K}_2 = (\bar{u}.\nabla)\bar{\varphi}$ and $\bar{\rho} = a(.)\bar{\varphi} + F'(\bar{\varphi})$.
For each $m \geq 1$, we define $(\tilde{u}_m, \tilde{\varphi}_m) = \mathcal{P}_m(\bar{u},\bar{\varphi})$, where $\mathcal{P}_m = (\mathcal{P}^1_m, \mathcal{P}^2_m) \in \mathcal{L}(\mathbb{Y}, \mathbb{Y}_m)$ is the orthogonal projection on $\mathbb{Y}$ onto $\mathbb{Y}_m$. It follows that

$$\begin{aligned}
&|(\tilde{u}_m, \tilde{\varphi}_m)|_{\mathbb{Y}} \leq |(\bar{u},\bar{\varphi})|_{\mathbb{Y}},\\
&\|(\tilde{u}_m, \tilde{\varphi}_m)\|_{\mathbb{V}} \leq c\|(\bar{u},\bar{\varphi})\|_{\mathbb{V}},\\
&(\tilde{u}_m, \tilde{\varphi}_m) \to (\bar{u},\bar{\varphi}) \ \text{ in } \ M^2_{\mathcal{F}_t}(0,T;\mathbb{V}).
\end{aligned} \tag{3.60}$$

From (3.34) and (3.44), it follows that for all $t \in [0,T]$, $1 \leq k \leq m$ and all $m \geq 1$

$$\begin{aligned}
&(\tilde{u}_m(t) - u_m(t), w_k) + \nu \int_0^t \langle A(\tilde{u}_m(s) - u_m(s)), w_k\rangle \, ds\\
&\quad + \int_0^t \left\langle \bar{K}_0(s) - B(u_m(s)), w_k \right\rangle ds\\
&\quad = \int_0^t \left\langle \bar{K}_1(s) - \mu_m(s)\nabla\varphi_m(s), w_k \right\rangle ds\\
&\quad + \int_0^t \left\langle \bar{h}(s) - h(s, u_m(s)), w_k \right\rangle ds\\
&\quad + \sum_{j=1}^m \int_0^t (\bar{G}(s)e_j - G(s, u_m(s))e_j, w_k) d\beta^j_s\\
&\quad + \sum_{j=m+1}^\infty \int_0^t (\bar{G}(s)e_j, w_k) d\beta^j_s,\\
&(\tilde{\varphi}_m(t) - \varphi_m(t), \psi_k) ds + \int_0^t \langle \mathcal{A}(\bar{\mu} - \mu_m), \psi_k\rangle\\
&\quad + \int_0^t \left\langle \bar{K}_2(s) - (u_m.\nabla)\varphi_m, \psi_k \right\rangle ds = 0,\\
&(\tilde{\mu}_m - \mu_m, \psi_k) = (\tilde{\rho}_m - \rho(.,\varphi_m), \psi_k) - (J * \bar{\varphi} - J * \varphi_m, \psi_k).
\end{aligned} \tag{3.61}$$

Let

$$\chi_m = a\tilde{\varphi}_m - J * \tilde{\varphi}_m + F'(\tilde{\varphi}_m). \tag{3.62}$$

We note that

$$\begin{aligned}
&\bar{K}_0 - B(u_m) = \bar{K}_0 - B(\tilde{u}_m) + B(\tilde{u}_m - u_m, \tilde{u}_m) + B(u_m, \tilde{u}_m - u_m),\\
&\bar{K}_2 - (u_m.\nabla)\varphi_m = \bar{K}_2 - (\tilde{u}_m.\nabla)\tilde{\varphi}_m\\
&\qquad + ((\tilde{u}_m - u_m).\nabla)\tilde{\varphi}_m + (u_m.\nabla)(\tilde{\varphi}_m - \varphi_m),\\
&\bar{K}_1 - \mu_m\nabla\varphi_m = \bar{K}_1 - \chi_m\nabla\tilde{\varphi}_m + \chi_m\nabla\tilde{\varphi}_m - \mu_m\nabla\varphi_m,
\end{aligned} \tag{3.63}$$

$$\begin{aligned}\left\langle \bar{K}_1 - K_1\varphi_m, \tilde{u}_m - u_m\right\rangle &= \left\langle \bar{K}_1 - K_1\tilde{\varphi}_m, \tilde{u}_m - u_m\right\rangle \\ &+ \int_{\mathcal{O}} a\tilde{\varphi}_m(\tilde{u}_m - u_m).\, \nabla(\tilde{\varphi}_m - \varphi_m)dx \\ &+ \int_{\mathcal{O}} a(\tilde{\varphi}_m - \varphi_m)(\tilde{u}_m - u_m).\, \nabla\varphi_m dx \\ &+ \tfrac{1}{2}\sum_{j=1}^{2}\int_{\mathcal{O}} a(\tilde{\varphi}_m - \varphi_m)(\tilde{\varphi}_m + \varphi_m)\partial x_j(\tilde{u}_m - u_m)dx \\ &+ \int_{\mathcal{O}} (\nabla J * \tilde{\varphi}_m)(\tilde{\varphi}_m - \varphi_m).\, (\tilde{u}_m - u_m)dx \\ &+ \int_{\mathcal{O}} (\nabla J * (\tilde{\varphi}_m - \varphi_m))\varphi_m.\, (\tilde{u}_m - u_m)dx,\end{aligned} \tag{3.64}$$

$$\begin{aligned}&\bar{\rho} - \rho(., \varphi_m) = \bar{\rho} - a\tilde{\varphi}_m - F'(\tilde{\varphi}_m) + a(\tilde{\varphi}_m - \varphi_m) \\ &\qquad + F'(\tilde{\varphi}_m) - F'(\varphi_m), \\ &J * \bar{\varphi} - J * \varphi_m = J * \bar{\varphi} - J * \tilde{\varphi}_m + J * (\tilde{\varphi}_m - \varphi_m).\end{aligned} \tag{3.65}$$

We set $\theta_m = \tilde{u}_m - u_m$, $\phi_m = \tilde{\varphi}_m - \varphi_m$, $\xi_m = \tilde{\mu}_m - \mu_m$.
Now, by Itô's formula

$$\begin{aligned}d(\theta_m(t), w_k)^2 &= 2(\theta_m(t), w_k)d(\theta_m(t), w_k) \\ &+ \sum_{j=1}^{m}(\bar{G}(t)e_j - G(t, u_m(t))e_j, w_k)^2 dt + \sum_{j=m+1}^{\infty}(\bar{G}(t)e_j, w_k)^2 dt,\end{aligned}$$

in $[0, T]$, for all $1 \le k \le m$ and all $m \ge 1$, and summing in k, we deduce that

$$\begin{aligned}&|\theta_m(t)|^2 + 2\nu\int_0^t \|\theta_m(s)\|^2 ds + 2\int_0^t \left\langle \bar{K}_0(s) - B(u_m(s)), \theta_m(s)\right\rangle ds \\ &= 2\int_0^t \left\langle \bar{K}_1(s) - \mu_m\nabla\varphi_m, \theta_m(s)\right\rangle ds \\ &+ 2\int_0^t \left\langle \bar{h}(s) - h(s, u_m(s)), \theta_m(s)\right\rangle ds \\ &+ 2\sum_{j=1}^{m}\int_0^t (\bar{G}(s)e_j - G(s, u_m(s))e_j, \theta_m(s))d\beta_s^j \\ &+ 2\sum_{j=m+1}^{\infty}\int_0^t (\bar{G}(s)e_j, \theta_m(s))d\beta_s^j \\ &+ \sum_{j=1}^{m}\int_0^t \left|\mathcal{P}_m^1(\bar{G}(s)e_j - G(s, u_m(s))e_j)\right|^2 ds \\ &+ \sum_{j=m+1}^{\infty}\int_0^t \left|\mathcal{P}_m^1(\bar{G}(s)e_j)\right|^2 ds.\end{aligned} \tag{3.66}$$

Replacing ψ_k in (3.61) by $\mathcal{A}_N^{-1}\phi_m$, we obtain

$$\begin{aligned}|\mathcal{A}_N^{-1/2}\phi_m(t)|^2 &+ 2\int_0^t \left\langle \bar{K}_2(s) - (u_m.\nabla)\varphi_m, \mathcal{A}_N^{-1}\phi_m\right\rangle ds \\ &+ 2\int_0^t (\bar{\mu} - \mu_m, \phi_m(s))ds = 0.\end{aligned} \tag{3.67}$$

We observe that

$$\begin{aligned}(\bar{\mu} - \mu_m, \phi_m) &= (\bar{\mu} - \chi_m + \chi_m - \mu_m, \phi_m) \\ &= (\bar{\mu} - \chi_m + a\phi_m + F'(\tilde{\varphi}_m) - F'(\varphi_m) - J * \phi_m, \phi_m)\end{aligned}$$

Inserting now this last equality in (3.67), we get

$$
\begin{aligned}
&|\mathcal{A}_N^{-1/2}\phi_m(t)|^2 + 2\int_0^t \left\langle \bar{K}_2(s) - (u_m(s).\nabla)\varphi_m(s), \mathcal{A}_N^{-1}\phi_m(s)\right\rangle ds \\
&+2\int_0^t (a\phi_m(s) + F'(\tilde{\varphi}_m(s)) - F'(\varphi_m(s)), \phi_m(s))ds \\
&= 2\int_0^t (J * \phi_m(s), \phi_m)ds - 2\int_0^t (\bar{\mu}(s) - \chi_m(s), \phi_m(s))ds.
\end{aligned} \tag{3.68}
$$

Applying the Lagrangian theorem to F', using (H_3), we infer that

$$
\begin{aligned}
&2\int_0^t (a\phi_m(s) + F'(\tilde{\varphi}_m(s)) - F'(\varphi_m(s)), \phi_m(s))ds \\
&= 2\int_0^t (a\phi_m(s) + F''(\tilde{\varphi}_m(s) + \theta\phi_m(s))\phi_m(s), \phi_m(s))ds \\
&\geq 2c_0\int_0^t |\phi_m(s)|^2 ds,
\end{aligned} \tag{3.69}
$$

with $0 < \theta < 1$.
Now, we state and prove the following lemma.

Lemma 3.1. *Let $m \in \mathbb{N}$. The following estimates hold*

$$
\begin{aligned}
&2\left\langle \bar{K}_0 - B(u_m), \theta_m\right\rangle \leq 2\left\langle \bar{K}_0 - B(\tilde{u}_m), \theta_m\right\rangle + \frac{\nu}{6}\|\theta_m\|^2 + c\|\tilde{u}_m\|^2|\theta_m|^2, \\
&2\left\langle h(s,\tilde{u}_m(s)) - h(s,u_m(s)), \theta_m(s)\right\rangle \leq \frac{\nu}{6}\|\theta_m\|^2 + cl_h^2|\theta_m|^2, \\
&2\left|(J * \phi_m, \phi_m)\right| \leq \frac{c_0}{5}|\phi_m|^2 + c|\mathcal{A}_N^{-1/2}\phi_m|^2,
\end{aligned} \tag{3.70}
$$

$$
\begin{aligned}
&\sum_{j=1}^m \left|\mathcal{P}_m^1(\bar{G}(s)e_j - G(s,u_m(s))e_j)\right|^2 \leq 2l_g^2|\bar{u}(s) - \tilde{u}_m(s)|^2 \\
&\quad + 2l_g^2|\tilde{u}_m(s) - u_m(s)|^2 - \|G(s,\bar{u}(s)) - \bar{G}(s)\|^2_{L_2(K,G_{div})} \\
&\quad + 2\left(\bar{G}(s) - G(s,u_m), \bar{G}(s) - G(s,\bar{u})\right)_{L_2(K,G_{div})},
\end{aligned} \tag{3.71}
$$

$$
\begin{aligned}
&2\left\langle \bar{K}_1 - \mu_m\nabla\varphi_m, \theta_m\right\rangle \leq 2\left\langle \bar{K}_1 - \chi_m\nabla\tilde{\varphi}_m, \theta_m\right\rangle + \frac{3c_0}{5}|\phi_m|^2 + \frac{\nu}{2}\|\theta_m\|^2 \\
&+ c(|\tilde{\varphi}_m + \varphi_m|^4_{L^4(\mathcal{O})} + |\tilde{\varphi}_m|^4_{L^4(\mathcal{O})} + |\varphi_m|^4_{L^4(\mathcal{O})})|\theta_m|^2,
\end{aligned} \tag{3.72}
$$

$$
\begin{aligned}
&2\left\langle \bar{K}_2 - (u_m.\nabla)\varphi_m, \mathcal{A}_N^{-1}\phi_m\right\rangle \leq 2\left\langle \bar{K}_2 - (\tilde{u}_m.\nabla)\tilde{\varphi}_m, \mathcal{A}_N^{-1}\phi_m\right\rangle + \frac{c_0}{5}|\phi_m|^2 \\
&+ \frac{\nu}{6}\|\theta_m\|^2 + c(|\tilde{\varphi}_m|^2_{L^4(\mathcal{O})} + |u_m|^4_{L^4(\mathcal{O})})|\mathcal{A}_N^{-1/2}\phi_m|^2,
\end{aligned} \tag{3.73}
$$

with $\theta_m = \bar{u}_m - u_m$, $\phi_m = \bar{\varphi}_m - \varphi_m$ and χ_m is given by (3.62).

Proof. We have

$$\begin{aligned}
&2\left\langle \bar{K}_0 - B(u_m), \theta_m \right\rangle \\
&= 2\left\langle \bar{K}_0 - B(\tilde{u}_m), \theta_m \right\rangle + 2\left\langle B(\theta_m, \tilde{u}_m), \theta_m \right\rangle, \\
&\leq 2\left\langle \bar{K}_0 - B(\tilde{u}_m), \theta_m \right\rangle + c\|\theta_m\|\|\tilde{u}_m\||\theta_m| \\
&\leq 2\left\langle \bar{K}_0 - B(\tilde{u}_m), \theta_m \right\rangle + \frac{\nu}{6}\|\theta_m\|^2 + c\|\tilde{u}_m\|^2|\theta_m|^2, \\
&2\left\langle h(s, \tilde{u}_m(s)) - h(s, u_m(s)), \theta_m(s) \right\rangle \\
&\leq 2l_h|\theta_m(s)|\|\theta_m(s)\| \\
&\leq \frac{\nu}{6}\|\theta_m\|^2 + cl_h^2|\theta_m|^2,
\end{aligned} \tag{3.74}$$

$$\begin{aligned}
&\sum_{j=1}^{m} \left|\mathcal{P}_m^1(\bar{G}(s)e_j - G(s, u_m(s))e_j)\right|^2 \leq \|\bar{G}(s) - G(s, u_m(s))\|^2_{L_2(K, G_{div})} \\
&\leq \|G(s, \bar{u}(s)) - G(s, u_m(s))\|^2_{L_2(K, G_{div})} - \|G(s, \bar{u}(s)) - \bar{G}(s)\|^2_{L_2(K, G_{div})} \\
&+ 2\left(\bar{G}(s) - G(s, u_m), \bar{G}(s) - G(s, \bar{u})\right)_{L_2(K, G_{div})} \\
&\leq 2\|G(s, \bar{u}(s)) - G(s, \tilde{u}_m(s))\|^2_{L_2(K, G_{div})} \\
&+ 2\|G(s, \tilde{u}_m(s)) - G(s, u_m(s))\|^2_{L_2(K, G_{div})} \\
&+ 2\left(\bar{G}(s) - G(s, u_m), \bar{G}(s) - G(s, \bar{u})\right)_{L_2(K, G_{div})} \\
&- \|G(s, \bar{u}(s)) - \bar{G}(s)\|^2_{L_2(K, G_{div})} \\
&\leq 2l_g^2|\bar{u}(s) - \tilde{u}_m(s)|^2 + 2l_g^2|\tilde{u}_m(s) - u_m(s)|^2 \\
&- \|G(s, \bar{u}(s)) - \bar{G}(s)\|^2_{L_2(K, G_{div})} \\
&+ 2\left(\bar{G}(s) - G(s, u_m), \bar{G}(s) - G(s, \bar{u})\right)_{L_2(K, G_{div})}.
\end{aligned} \tag{3.75}$$

Hence, from (3.74)-(3.75), we obtain the estimates $(3.70)_{1,2}$ and (3.71).
We now move to the proof of (3.72).
We begin by making the following observation: from Remark 2.3 and the fact that $\nabla.u = 0$, for all $u \in V_{div}$, we infer that

$$\begin{aligned}
&\left\langle \bar{K}_1 - \mu_m \nabla\varphi_m, \theta_m \right\rangle = \left\langle \bar{K}_1 - \chi_m \nabla\tilde{\varphi}_m, \theta_m \right\rangle + \left\langle \chi_m \nabla\tilde{\varphi}_m - \mu_m \nabla\varphi_m, \theta_m \right\rangle \\
&= \left\langle \bar{K}_1 - \chi_m \nabla\tilde{\varphi}_m, \theta_m \right\rangle + \left\langle \nabla(F(\tilde{\varphi}_m) - F(\varphi_m) + \frac{a}{2}\tilde{\varphi}_m^2 - \frac{a}{2}\varphi_m^2), \theta_m \right\rangle \\
&- \left\langle \phi_m(\tilde{\varphi}_m + \varphi_m)\frac{\nabla a}{2}, \theta_m \right\rangle - \left\langle (J * \phi_m)\nabla\tilde{\varphi}_m, \theta_m \right\rangle + \left\langle (J * \varphi_m)\nabla\phi_m, \theta_m \right\rangle \\
&= \left\langle \bar{K}_1 - \chi_m \nabla\tilde{\varphi}_m, \theta_m \right\rangle - \left\langle \phi_m(\tilde{\varphi}_m + \varphi_m)\frac{\nabla a}{2}, \theta_m \right\rangle - \left\langle (J * \phi_m)\nabla\tilde{\varphi}_m, \theta_m \right\rangle \\
&+ \left\langle (J * \varphi_m)\nabla\phi_m, \theta_m \right\rangle.
\end{aligned} \tag{3.76}$$

Now, we will estimate the last three terms in the right-hand side of (3.76) as follows:

$$\begin{aligned}
&|\langle\phi_m(\tilde{\varphi}_m+\varphi_m)\nabla a,\theta_m\rangle| \\
&\leq |\nabla a|_{L^\infty}|\phi_m||\tilde{\varphi}_m+\varphi_m|_{L^4(\mathcal{O})}|\theta_m|_{L^4(\mathcal{O})} \\
&\leq c|\nabla a|_{L^\infty}|\phi_m||\tilde{\varphi}_m+\varphi_m|_{L^4(\mathcal{O})}|\theta_m|^{1/2}\|\theta_m\|^{1/2} \\
&\leq \frac{c_0}{5}|\phi_m|^2+c|\tilde{\varphi}_m+\varphi_m|^2_{L^4(\mathcal{O})}|\theta_m|\|\theta_m\| \\
&\leq \frac{c_0}{5}|\phi_m|^2+\frac{\nu}{6}\|\theta_m\|^2+c|\tilde{\varphi}_m+\varphi_m|^4_{L^4(\mathcal{O})}|\theta_m|^2.
\end{aligned} \tag{3.77}$$

By integration by parts and the fact that $u_m=0$ on $\partial\mathcal{O}$, $\text{div}u_m=\nabla.u_m=0$, we obtain

$$-2\left\langle (J*\phi_m)\nabla\tilde{\varphi}_m,\theta_m\right\rangle = 2\left\langle (\nabla J*\phi_m)\tilde{\varphi}_m,\theta_m\right\rangle.$$

Hence from this last equality, using the Hölder inequality, the Ladyzhenskaya inequality and the Young inequality, we infer that

$$\begin{aligned}
&2\left|\langle (J*\phi_m)\nabla\tilde{\varphi}_m,\theta_m\rangle\right| \\
&\leq 2|\tilde{\varphi}_m|_{L^4}|\theta_m|_{L^4}|\nabla J*\phi_m| \\
&\leq c|\tilde{\varphi}_m|_{L^4}|\theta_m|^{1/2}\|\theta_m\|^{1/2}|\nabla J*\phi_m| \\
&\leq \frac{c_0}{5}|\phi_m|^2+c|\tilde{\varphi}_m|^2_{L^4}|\theta_m||\nabla J|^2_{L^1}\|\theta_m\| \\
&\leq \frac{c_0}{5}|\phi_m|^2+\frac{\nu}{6}\|\theta_m\|^2+c|\tilde{\varphi}_m|^4_{L^4(\mathcal{O})}|\theta_m|^2.
\end{aligned} \tag{3.78}$$

Arguing similarly as in (3.78), we derive that

$$2\left|\langle (J*\varphi_m)\nabla\phi_m,\theta_m\rangle\right| \leq \frac{c_0}{5}|\phi_m|^2+\frac{\nu}{6}\|\theta_m\|^2+c|\varphi_m|^4_{L^4(\mathcal{O})}|\theta_m|^2. \tag{3.79}$$

Inserting now these estimates (3.77)-(3.79) in (3.76), we obtain (3.72).
For the proof of estimate (3.73), we also make the following observation: we have

$$\begin{aligned}
\left\langle \bar{K}_2-(u_m.\nabla)\varphi_m,\mathcal{A}_N^{-1}\phi_m\right\rangle &= \left\langle \bar{K}_2-(\tilde{u}_m.\nabla)\tilde{\varphi}_m,\mathcal{A}_N^{-1}\phi_m\right\rangle \\
&\quad+\left\langle (\theta_m.\nabla)\tilde{\varphi}_m,\mathcal{A}_N^{-1}\phi_m\right\rangle+\left\langle (u_m.\nabla)\phi_m,\mathcal{A}_N^{-1}\phi_m\right\rangle.
\end{aligned} \tag{3.80}$$

The goal now is to estimate the last two terms in the right-hand side of (3.80).
By integration by parts, using the fact that $\nabla.u_m=0$ and $u_m|_{\partial\mathcal{O}}=0$, we obtain

$$\begin{aligned}
-2\left\langle (\theta_m.\nabla)\tilde{\varphi}_m,\mathcal{A}_N^{-1}\phi_m\right\rangle &= -2\int_{\mathcal{O}}\mathcal{A}_N^{-1}(\phi_m)\tilde{\varphi}_m\nabla.\theta_m dx \\
&= 2\left\langle \theta_m.\nabla\mathcal{A}_N^{-1}\phi_m,\tilde{\varphi}_m\right\rangle.
\end{aligned}$$

Owing to (2.7), Ladyzhenskaya's inequality and Young's inequality, we have

$$\begin{aligned}
2\left|\left\langle (\theta_m.\nabla)\tilde{\varphi}_m,\mathcal{A}_N^{-1}\phi_m\right\rangle\right| &= 2\left|\left\langle \theta_m.\nabla\mathcal{A}_N^{-1}\phi_m,\tilde{\varphi}_m\right\rangle\right| \\
&\leq 2|\tilde{\varphi}_m|_{L^4}|\theta_m|_{L^4}|\nabla\mathcal{A}_N^{-1}\phi_m| \\
&\leq c|\tilde{\varphi}_m|_{L^4}|\theta_m|^{\frac{1}{2}}\|\theta_m\|^{\frac{1}{2}}|\nabla\mathcal{A}_N^{-1}\phi_m| \\
&\leq c|\tilde{\varphi}_m|_{L^4(\mathcal{O})}\|\theta_m\||\nabla\mathcal{A}_N^{-1}\phi_m| \\
&\leq \frac{\nu}{6}\|\theta_m\|^2+c|\tilde{\varphi}_m|^2_{L^4}|\mathcal{A}_N^{-1/2}\phi_m|^2.
\end{aligned} \tag{3.81}$$

Proceeding similarly as in (3.81), we infer that

$$
\begin{aligned}
2\left|\left\langle (u_m.\nabla)\phi_m, \mathcal{A}_N^{-1}\phi_m\right\rangle\right| &= 2\left|\left(u_m.\nabla\mathcal{A}_N^{-1}\phi_m, \phi_m\right)\right| \\
&\leq 2|\phi_m||u_m|_{L^4}|\nabla\mathcal{A}_N^{-1}\phi_m|_{L^4} \\
&\leq \frac{c_0}{10}|\phi_m|^2 + c|u_m|^2_{L^4}|\nabla\mathcal{A}_N^{-1}\phi_m|^2_{L^4(\mathcal{O})} \\
&\leq \frac{c_0}{10}|\phi_m|^2 + c|u_m|^2_{L^4}|\nabla\mathcal{A}_N^{-1}\phi_m||\nabla\mathcal{A}_N^{-1}\phi_m|_{H^1(\mathcal{O})},
\end{aligned} \tag{3.82}
$$

where we have used the Galiardo-Nirenberg inequality (see [**11**, Theorem 2.1]). One can see that the H^2-norm of ϕ in $D(\mathcal{A}_N)$ is equivalent to the L^2-norm of $(\mathcal{A}_N+I)\phi$, i.e., $|\phi|_{H^2} \cong |(\mathcal{A}_N + I)\phi|_{L^2}$.
Now since $\mathcal{A}_N^{-1}\phi_m \in D(\mathcal{A}_N)$, we have

$$
\begin{aligned}
|\nabla\mathcal{A}_N^{-1}\phi_m|_{H^1(\mathcal{O})} \leq |\mathcal{A}_N^{-1}\phi_m|_{H^2(\mathcal{O})} &\leq c|(\mathcal{A}_N + I)\mathcal{A}_N^{-1}\phi_m| \\
&\leq c|\phi_m| + c|\mathcal{A}_N^{-1}\phi_m| \\
&\leq c|\phi_m| + c\|\mathcal{A}_N^{-1}\phi_m\|_{V_0} \\
&= c|\phi_m| + c|\mathcal{A}_N^{1/2}\mathcal{A}_N^{-1}\phi_m|.
\end{aligned}
$$

But, since $L^2_{(0)}(\mathcal{O}) \equiv (L^2_{(0)}(\mathcal{O}))' \hookrightarrow V_0'$, we have

$$
\begin{aligned}
|\mathcal{A}_N^{1/2}\mathcal{A}_N^{-1}\phi_m| &= \left[\left(\mathcal{A}_N\mathcal{A}_N^{-1}\phi_m, \mathcal{A}_N^{-1}\phi_m\right)\right]^{1/2} \\
&= \left[(\phi_m, \mathcal{A}_N^{-1}\phi_m)\right]^{1/2} \\
&= \left[(\phi_m, \phi_m)_{V_0'}\right]^{1/2} \\
&= \|\phi_m\|_{V_0'} \leq c|\phi_m|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\nabla\mathcal{A}_N^{-1}\phi_m|_{H^1(\mathcal{O})} &\leq c|\phi_m| + c|\mathcal{A}_N^{1/2}\mathcal{A}_N^{-1}\phi_m| \\
&\leq c|\phi_m|.
\end{aligned}
$$

Substituting this previous estimate in (3.82), using (2.7) and the Young inequality we obtain

$$
\begin{aligned}
2\left|\left\langle (u_m.\nabla)\phi_m, \mathcal{A}_N^{-1}\phi_m\right\rangle\right| &\leq \frac{c_0}{10}|\phi_m|^2 + c|u_m|^2_{L^4}|\nabla\mathcal{A}_N^{-1}\phi_m||\phi_m| \\
&\leq \frac{c_0}{10}|\phi_m|^2 + c|u_m|^2_{L^4}|\mathcal{A}_N^{-1/2}\phi_m||\phi_m| \\
&\leq \frac{c_0}{5}|\phi_m|^2 + c|u_m|^4_{L^4}|\mathcal{A}_N^{-1/2}\phi_m|^2.
\end{aligned} \tag{3.83}
$$

From (3.80) and the estimates (3.81) and (3.83), we get (3.73).
Finally, using the Young inequality and the Young inequality for convolution, we

obtain

$$
\begin{aligned}
2\left|(J*\phi_m,\phi_m)\right| &\le 2\left|\left(\mathcal{A}_N^{1/2}(J*\phi_m),\mathcal{A}_N^{-1/2}\phi_m\right)\right|\\
&= 2\left|\left(\mathcal{A}_N^{1/2}(J*\phi_m-\langle J*\phi_m\rangle),\mathcal{A}_N^{-1/2}\phi_m\right)\right|\\
&\le 2|\nabla(J*\phi_m-\langle J*\phi_m\rangle)||\mathcal{A}_N^{-1/2}\phi_m|\\
&\le 2|\nabla J|_{L^1}|\phi_m||\mathcal{A}_N^{-1/2}\phi_m|\\
&\le \frac{c_0}{5}|\phi_m|^2+c|\mathcal{A}_N^{-1/2}\phi_m|^2,
\end{aligned}
\tag{3.84}
$$

which proves $(3.70)_3$ and the proof of Lemma 3.1 is now complete. ■

We denote $z(t)=e^{-z_1t-\int_0^t\mathcal{Z}(s)ds},\ \ 0\le t\le T,$ with $z_1=2l_g^2+cl_h^2+c$ and

$$
\mathcal{Z}(t)=c(\|\tilde{u}_m\|^2+|\tilde{\varphi}_m+\varphi_m|_{L^4}^4+|\tilde{\varphi}_m|_{L^4}^4+|\varphi_m|_{L^4}^4+|\tilde{\varphi}_m|_{L^4}^2+|u_m|_{L^4}^4).
\tag{3.85}
$$

We also set

$$Z(t)=|\theta_m(t)|^2+\|\phi_m(t)\|_{V_0'}^2=|\tilde{u}_m(t)-u_m(t)|^2+|\mathcal{A}_N^{-1/2}(\tilde{\varphi}_m(t)-\varphi_m(t))|^2$$

and

$$\tilde{Z}(t)=\nu\|\theta_m(t)\|^2+c_0|\phi_m(t)|^2=\nu\|\tilde{u}_m(t)-u_m(t)\|^2+c_0|\tilde{\varphi}_m(t)-\varphi_m(t)|^2.$$

Applying now the Itô formula to the real process $z(t)Z(t)$ in conjunction with Lemma 3.1, we infer that

$$
\begin{aligned}
&z(t)Z(t)+\textstyle\int_0^t z(s)\tilde{Z}(s)ds+\displaystyle\int_0^t z(s)\|G(s,\bar{u}(s))-\bar{G}(s)\|_{L_2(K,G_{div})}^2ds\\
&\le 2\int_0^t z(s)\left\langle B(\tilde{u}_m)-\bar{K}_0,\theta_m\right\rangle ds-2\int_0^t z(s)\left\langle \bar{K}_1-\chi_m\nabla\tilde{\varphi}_m,\theta_m\right\rangle ds\\
&+2\int_0^t z(s)\left\langle \bar{h}(s)-h(s,\tilde{u}_m(s)),\theta_m(s)\right\rangle ds\\
&+2\sum_{j=1}^m\int_0^t z(s)\left(\bar{G}(s)e_j-G(s,u_m(s))e_j,\theta_m(s)\right)d\beta_s^j\\
&+2\sum_{j=m+1}^\infty\int_0^t z(s)(\bar{G}(s)e_j,\theta_m(s))d\beta_s^j-2\int_0^t(\bar{\mu}(s)-\chi_m(s),\phi_m(s))ds\\
&+2\int_0^t z(s)\left(\bar{G}(s)-G(s,u_m(s)),\bar{G}(s)-G(s,\bar{u}(s))\right)_{L_2(K,G_{div})}ds\\
&+\sum_{j=m+1}^\infty\int_0^t z(s)|\bar{G}(s)e_j|^2ds+2l_g^2\int_0^t z(s)|\bar{u}(s)-\tilde{u}_m(s)|^2ds\\
&-2\int_0^t z(s)\left\langle \bar{K}_2(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s),\mathcal{A}_N^{-1}\phi_m(s)\right\rangle ds,
\end{aligned}
\tag{3.86}
$$

for all $t\in[0,T]$ and all $m\ge1$.
Let $m\ge1$ be fix. For each $n\ge1$, we consider the $\mathcal{F}_t$-stopping time τ_n defined by:

$$
\begin{aligned}
\tau_n=\min(T,\inf\{t\in[0,T];\,&|(\bar{u}(t),\bar{\varphi}(t))|_{\mathbb{Y}}^2+|(u_m(t),\varphi_m(t))|_{\mathbb{Y}}^2\\
&+\int_0^t[\|(\bar{u},\bar{\varphi})\|_{\mathbb{V}}^2+\|(u_m,\varphi_m)\|_{\mathbb{V}}^2]ds\ge n^2\}).
\end{aligned}
\tag{3.87}
$$

It is straightforward to obtain from (3.86) and for all n, $m \geq 1$ that

$$
\begin{aligned}
&\mathbb{E}z(\tau_n)Z(\tau_n) + \mathbb{E}\int_0^{\tau_n} z(s)\tilde{Z}(s)ds \\
&\qquad +\mathbb{E}\int_0^{\tau_n} z(s)\|G(s,\bar{u}(s)) - \bar{G}(s)\|^2_{L_2(K,G_{div})}ds \\
&\leq 2\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\tilde{u}_m(s)) - \bar{K}_0(s), \theta_m(s)\right\rangle ds \\
&+2\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{h}(s) - h(s,\tilde{u}_m(s)), \theta_m(s)\right\rangle ds \\
&-2\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_1(s) - \chi_m(s)\nabla\tilde{\varphi}_m(s), \theta_m(s)\right\rangle ds \\
&-2\mathbb{E}\int_0^{\tau_n} (\bar{\mu}(s) - \chi_m(s), \phi_m(s))ds \\
&+2\mathbb{E}\int_0^{\tau_n} z(s)\left(\bar{G}(s) - G(s,u_m(s)), \bar{G}(s) - G(s,\bar{u}(s))\right)_{L_2(K,G_{div})} ds \\
&+\sum_{j=m+1}^{\infty}\mathbb{E}\int_0^T z(s)|\bar{G}(s)e_j|^2 ds + 2l_g^2\mathbb{E}\int_0^T z(s)|\bar{u}(s) - \tilde{u}_m(s)|^2 ds \\
&-2\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_2(s) - (\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s), \mathcal{A}_N^{-1}\phi_m(s)\right\rangle ds.
\end{aligned}
\tag{3.88}
$$

Claim 1. The right side of (3.88) goes to 0 as m goes to $+\infty$.

(1). We first note that

$$
\lim_{m\to+\infty}\left(\sum_{j=m+1}^{\infty}\mathbb{E}\int_0^T z(s)|\bar{G}(s)e_j|^2 ds + 2l_g^2\mathbb{E}\int_0^T z(s)|\bar{u}(s) - \tilde{u}_m(s)|^2 ds\right) = 0.
\tag{3.89}
$$

Also, as $G(s,u_m(s)) \rightharpoonup \bar{G}(s)$ in $M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div}))$ and $1_{[0,\tau_n]}z(s)(\bar{G}(s) - G(s,\bar{u}(s))) \in M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div}))$, we have

$$
\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left(\bar{G}(s) - G(s,u_m(s)), \bar{G}(s) - G(s,\bar{u}(s))\right)_{L_2(K,G_{div})} ds = 0.
\tag{3.90}
$$

(2). Let us prove that

$$
\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{h}(s) - h(s,\tilde{u}_m(s)), \theta_m(s)\right\rangle ds = 0.
$$

We recall that

$$
\begin{aligned}
&(u_m,\varphi_m) \rightharpoonup (\bar{u},\bar{\varphi}), (\tilde{u}_m,\tilde{\varphi}_m) \to (\bar{u},\bar{\varphi}) \text{ in } M^2_{\mathcal{F}_t}(0,T;\mathbb{V}), \\
&(\tilde{u}_m,\tilde{\varphi}_m) - (u_m,\varphi_m) \rightharpoonup (0,0) \text{ in } M^2_{\mathcal{F}_t}(0,T;\mathbb{V}).
\end{aligned}
\tag{3.91}
$$

From (3.91), the fact that $1_{[0,\tau_n]}z(s)(\bar{h}(s) - h(s,\bar{u}(s))) \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$ and $1_{[0,\tau_n]}z(s)(h(s,\bar{u}(s)) - h(s,\tilde{u}_m(s))) \to 0$ in $M^2_{\mathcal{F}_t}(0,T;V'_{div})$ as $m\to+\infty$, we derive that

$$
\begin{aligned}
&\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{h}(s) - h(s,\bar{u}(s)), \theta_m(s)\right\rangle ds = 0, \\
&\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} \langle h(s,\bar{u}(s)) - h(s,\tilde{u}_m(s)), \theta_m(s)\rangle ds = 0.
\end{aligned}
\tag{3.92}
$$

Hence, we derive that

$$\begin{aligned}
&\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{h}(s)-h(s,\tilde{u}_m(s)),\theta_m(s)\right\rangle ds\\
&=\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{h}(s)-h(s,\bar{u}),\theta_m(s)\right\rangle ds \qquad (3.93)\\
&+\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle h(s,\bar{u})-h(s,\tilde{u}_m),\theta_m(s)\right\rangle ds=0.
\end{aligned}$$

This proves **(2)**.

(3). We also prove that

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\tilde{u}_m(s))-\bar{K}_0(s),\theta_m(s)\right\rangle ds=0.$$

Note that

$$\begin{aligned}
&\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\tilde{u}_m(s))-\bar{K}_0(s),\theta_m\right\rangle ds\\
&=\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\tilde{u}_m)-B(\bar{u}),\theta_m)\right\rangle ds \qquad (3.94)\\
&+\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\bar{u})-\bar{K}_0,\theta_m\right\rangle ds.
\end{aligned}$$

By (3.91) and the fact that $1_{[0,\tau_n]}z(s)(B(\bar{u}(s))-\bar{K}_0(s))\in M^2_{\mathcal{F}_t}(0,T;V'_{div})$, we deduce that

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\bar{u}(s))-\bar{K}_0(s),\theta_m(s)\right\rangle ds=0. \qquad (3.95)$$

By (2.9) and (3.60), we infer that

$$\begin{aligned}
&\|1_{[0,\tau_n]}z(s)(B(\tilde{u}_m(s))-B(\bar{u}(s)))\|_{V'_{div}}\\
&\le 1_{[0,\tau_n]}(s)c|\tilde{u}_m(s)-\bar{u}(s)|^{1/2}\|\tilde{u}_m(s)-\bar{u}(s)\|^{1/2}(|\tilde{u}_m(s)|^{1/2}\|\tilde{u}_m(s)\|^{1/2}\\
&+|\bar{u}(s)|^{1/2}\|\bar{u}(s)\|^{1/2})\\
&\le 1_{[0,\tau_n]}(s)c|\tilde{u}_m(s)-\bar{u}(s)|^{1/2}\|\tilde{u}_m(s)-\bar{u}(s)\|^{1/2}|\bar{u}(s)|^{1/2}\|\bar{u}(s)\|^{1/2},
\end{aligned}$$

and thus, $\|1_{[0,\tau_n]}z(s)(B(\tilde{u}_m(s))-B(\bar{u}(s)))\|_{V'_{div}}\to 0$, as $m\to+\infty$, $dt\times d\mathbb{P}$-a.e., and

$$\begin{aligned}
&\|1_{[0,\tau_n]}z(s)(B(\tilde{u}_m(s))-B(\bar{u}(s)))\|_{V'_{div}}\\
&\le 1_{[0,\tau_n]}(s)c|\tilde{u}_m(s)-\bar{u}(s)|^{1/2}\|\tilde{u}_m(s)-\bar{u}(s)\|^{1/2}\|\bar{u}(s)\|\\
&\le cn\|\bar{u}(s)\|\in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}),
\end{aligned}$$

Hence

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\tilde{u}_m(s))-B(\bar{u}(s)),\theta_m(s)\right\rangle ds=0. \qquad (3.96)$$

Finally, we conclude from (3.94)-(3.96) that

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle B(\tilde{u}_m(s))-\bar{K}_0(s),\theta_m(s)\right\rangle ds=0. \qquad (3.97)$$

(4). Next we will prove that

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_1(s)-\chi_m(s)\nabla\tilde{\varphi}_m(s),\theta_m(s)\right\rangle ds=0. \qquad (3.98)$$

By (3.91), using the fact that $1_{[0,\tau_n]}z(s)(\bar{K}_1(s) - (a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi}) \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$,
we get

$$\lim_{m\to+\infty} \mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_1(s) - (a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi}, \theta_m(s)\right\rangle ds = 0. \tag{3.99}$$

Let $v \in V_{div}$. We have

$$\begin{aligned}
&|\langle (a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \chi_m\nabla\tilde{\varphi}_m, v\rangle| \\
&= \left|\left\langle (\bar{\varphi} - \tilde{\varphi}_m)(\bar{\varphi} + \tilde{\varphi}_m)\frac{\nabla a}{2}, v\right\rangle\right| \\
&- |\langle J * (\bar{\varphi} - \tilde{\varphi}_m)\nabla\tilde{\varphi}_m, v\rangle| + |\langle (J * \bar{\varphi})\nabla(\bar{\varphi} - \tilde{\varphi}_m), v\rangle| \\
&\leq c(|\nabla a|_{L^\infty}|\bar{\varphi} + \tilde{\varphi}_m||\bar{\varphi} - \tilde{\varphi}_m|_{L^4} + |J|_{L^1}|\nabla\tilde{\varphi}_m||\bar{\varphi} - \tilde{\varphi}_m|_{L^4} \\
&+ |J|_{L^1}|\nabla(\bar{\varphi} - \tilde{\varphi}_m)||\bar{\varphi}|_{L^4})|v|_{L^4} \\
&\leq c(|\nabla a|_{L^\infty}(|\bar{\varphi}| + |\nabla\tilde{\varphi}_m|)|\nabla(\bar{\varphi} - \tilde{\varphi}_m)| \\
&+ |J|_{L^1}|\nabla\tilde{\varphi}_m||\nabla(\bar{\varphi} - \tilde{\varphi}_m)| + |J|_{L^1}|\nabla(\bar{\varphi} - \tilde{\varphi}_m)||\nabla\bar{\varphi}|)\|v\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|(a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \tilde{\mu}_m\nabla\tilde{\varphi}_m\|_{V'_{div}} \\
&\leq c(|\nabla a|_{L^\infty}(|\bar{\varphi}| + |\nabla\tilde{\varphi}_m|) + |J|_{L^1}|\nabla\tilde{\varphi}_m| + |J|_{L^1}|\nabla\bar{\varphi}|)|\nabla(\bar{\varphi} - \tilde{\varphi}_m)|.
\end{aligned} \tag{3.100}$$

From (3.60), (3.100) and (3.91), we infer that

$$\begin{aligned}
&\|1_{[0,\tau_n]}z(s)(a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \tilde{\mu}_m\nabla\tilde{\varphi}_m\|_{V'_{div}} \\
&\leq 1_{[0,\tau_n]}(t)c(|\nabla a|_{L^\infty}|\tilde{\varphi}_m(t) + \bar{\varphi}(t)| + |\nabla J|_{L^1}|\bar{\varphi}(t)| + |\nabla J|_{L^1}|\tilde{\varphi}_m(t)|)|\tilde{\varphi}_m(t) - \bar{\varphi}(t)|_{L^4} \\
&\leq 1_{[0,\tau_n]}(s)c(|\nabla a|_{L^\infty}(|\bar{\varphi}| + |\nabla\bar{\varphi}|) + |J|_{L^1}|\nabla\bar{\varphi}| + |J|_{L^1}|\nabla\bar{\varphi}|)|\nabla(\bar{\varphi} - \tilde{\varphi}_m)|.
\end{aligned}$$

From this last inequality and (3.60), it follows that

$$\begin{aligned}
&\|1_{[0,\tau_n]}z(s)(a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \tilde{\mu}_m\nabla\tilde{\varphi}_m\|_{V'_{div}} \to 0 \text{ as } m \to +\infty,\ dt \times d\mathbb{P}\text{-a.e.}, \\
&\|1_{[0,\tau_n]}z(s)(a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \tilde{\mu}_m\nabla\tilde{\varphi}_m\|_{V'_{div}} \\
&\leq c(|\nabla a|_{L^\infty}(|\bar{\varphi}| + |\nabla\bar{\varphi}|) + |J|_{L^1}|\nabla\bar{\varphi}| + |J|_{L^1}|\nabla\bar{\varphi}|)n \in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}).
\end{aligned}$$

Thus

$$\lim_{m\to+\infty} \mathbb{E}\int_0^{\tau_n} z(s)\langle (a\bar{\varphi} - J * \bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \tilde{\mu}_m\nabla\tilde{\varphi}_m, \theta_m(s)\rangle ds = 0. \tag{3.101}$$

By (3.99) and (3.101), we infer that

$$\lim_{m\to+\infty} \mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_1(s) - \tilde{\mu}_m(s)\nabla\tilde{\varphi}_m(s), \theta_m(s)\right\rangle ds = 0.$$

which proves (3.98).

(5). We now check that

$$\lim_{m\to+\infty} \mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_2(s) - (\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s), \mathcal{A}_N^{-1}\phi_m(s)\right\rangle ds = 0. \tag{3.102}$$

By (3.91), using also the fact that $z(s)(\bar{K}_2(s) - (\bar{u}(s).\nabla)\bar{\varphi}(s)) \in M^2_{\mathcal{F}_t}(0,T;V'_0)$,
we derive that

$$\lim_{m\to+\infty} \mathbb{E}\int_0^{\tau_n} z(s)(\bar{K}_2(s) - (\bar{u}(s).\nabla)\bar{\varphi}(s), \mathcal{A}_N^{-1}\phi_m(s))ds = 0. \tag{3.103}$$

Also by (3.60), we have

$$\begin{aligned}&\|1_{[0,\tau_n]}z(s)((\bar{u}(s).\nabla)\bar{\varphi}(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s))\|_{V_0'}\\&\leq 1_{[0,\tau_n]}(s)c(|\bar{\varphi}(s)|_{L^{2\kappa+2}(\mathcal{O})}\|\bar{u}(s)-\tilde{u}_m(s)\|+|\bar{\varphi}(s)-\tilde{\varphi}_m(s)|_{L^{2\kappa+2}(\mathcal{O})}\|\tilde{u}_m(s)\|)\\&\leq 1_{[0,\tau_n]}(s)c(|\bar{\varphi}(s)|_{L^{2\kappa+2}(\mathcal{O})}\|\bar{u}(s)-\tilde{u}_m(s)\|+|\nabla(\bar{\varphi}(s)-\tilde{\varphi}_m(s))|\|\bar{u}(s)\|)\end{aligned}$$

which implies that

$$\begin{aligned}&\|1_{[0,\tau_n]}z(s)((\bar{u}(s).\nabla)\bar{\varphi}(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s))\|_{V_0'}\to 0 \quad \text{as} \quad m\to+\infty, dt\times d\mathbb{P}\text{-a.e.},\\&\|1_{[0,\tau_n]}z(s)((\bar{u}(s).\nabla)\bar{\varphi}(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s))\|_{V_0'}\\&\leq 1_{[0,\tau_n]}(s)c(|\bar{\varphi}(s)|_{L^{2\kappa+2}(\mathcal{O})}+\|\bar{u}(s)\|)n\in M^2_{\mathcal{F}_t}(0,T;\mathbb{R}).\end{aligned}$$

Hence

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)((\bar{u}(s).\nabla)\bar{\varphi}(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s)),\mathcal{A}_N^{-1}\phi_m(s))ds=0. \tag{3.104}$$

Therefore from (3.103) and (3.104), we derive that

$$\begin{aligned}&\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)\left\langle \bar{K}_2(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s),\mathcal{A}_N^{-1}\phi_m(s)\right\rangle ds\\&=\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)(\bar{K}_2(s)-(\bar{u}(s).\nabla)\bar{\varphi}(s),\mathcal{A}_N^{-1}\phi_m(s))ds\\&+\lim_{m\to+\infty}\mathbb{E}\int_0^{\tau_n} z(s)((\bar{u}(s).\nabla)\bar{\varphi}(s)-(\tilde{u}_m(s).\nabla)\tilde{\varphi}_m(s)),\mathcal{A}_N^{-1}\phi_m(s))ds=0,\end{aligned}$$

which proves (3.102).
(6). Let us also prove that

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{t}(\bar{\mu}(s)-\chi_m(s),\phi_m(s))ds=0. \tag{3.105}$$

We first observe that

$$\begin{aligned}\bar{\mu}-\chi_m=\bar{\rho}-J*\bar{\varphi}-\chi_m \quad &=\bar{\rho}-(a\bar{\varphi}+F'(\bar{\varphi}))+a(\bar{\varphi}-\tilde{\varphi}_m)\\&+F'(\bar{\varphi})-F'(\tilde{\varphi}_m)-J*(\bar{\varphi}-\tilde{\varphi}_m).\end{aligned}$$

From (3.91) and the fact that $1_{[0,\tau_n]}z(s)(\bar{\rho}-(a\bar{\varphi}+F'(\bar{\varphi})))\in M^2_{\mathcal{F}_t}(0,T;V_0)\subset M^2_{\mathcal{F}_t}(0,T;V_0')$, $1_{[0,\tau_n]}z(s)(F'(\bar{\varphi}(s))-F'(\tilde{\varphi}_m(s)))\to 0$ in $M^2_{\mathcal{F}_t}(0,T;H)$ and $1_{[0,\tau_n]}z(s)(J*(\bar{\varphi}(s)-\tilde{\varphi}_m(s)))\to 0$ in $M^2_{\mathcal{F}_t}(0,T;V_0)\subset M^2_{\mathcal{F}_t}(0,T;V_0')$ as $m\to+\infty$, we derive that

$$\lim_{m\to+\infty}\mathbb{E}\int_0^{t}(\bar{\mu}(s)-\chi_m(s),\phi_m(s))ds=0.$$

This proves (3.105).
Hence we conclude from **(1)-(6)** that the right-hand side of (3.88) tends to zero as m tends to infinity.
Now since

$$\begin{aligned}&e^{-z_1T-cc_2^2n^2-8cc_1^4c_2^2n^4-8cc_1^4n^4-\frac{cc_1^2c_2n^2}{2}T-\frac{cc_1^2c_2}{2}n^2-cc_1^4c_2^2n^4-cc_1^4n^4-2cn^4}\\&\leq 1|_{[0,\tau_n]}z(s)\leq 1,\end{aligned} \tag{3.106}$$

we infer from (3.88) that

$$\begin{aligned}
&\lim_{m\to\infty} \mathbb{E}Z(\tau_n) = 0\\
&\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \tilde{Z}(s)ds = \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \nu\|\theta_m(s)\|^2 + c_0|\phi_m(s)|^2 ds\\
&= \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \nu\|\tilde{u}_m(s) - u_m(s)\|^2 + c_0|\tilde{\varphi}_m(s) - \varphi_m(s)|^2 ds = 0,\\
&\mathbb{E}\int_0^{\tau_n} \|G(s,\bar{u}(s)) - \bar{G}(s)\|^2_{L_2(K,G_{div})} ds = 0.
\end{aligned} \tag{3.107}$$

We note that in (3.106) the constants c_1 and c_2 come from:

$$\begin{aligned}
&\|(\tilde{u}_m, \tilde{\varphi}_m)\|_{\mathbb{V}} \le c_2\|(\bar{u},\bar{\varphi})\|_{\mathbb{V}},\\
&|\varphi_m^i|_{L^4} \le c_1|\varphi_m^i|^{1/2}|\nabla\varphi_m^i|^{1/2},\ i = 1,2,\ \varphi_m^1 = \varphi_m \text{ and } \varphi_m^2 = \tilde{\varphi}_m.
\end{aligned}$$

Also the constant c appearing in (3.106) comes from (3.85) and $z_1 = 2l_g^2 + cl_h^2 + c$. In (3.106) we have also used the property of the stopping time τ_n defined in (3.87), the fact that $|u_m|_{L^4} \le 2^{1/4}|u_m|^{1/2}\|u_m\|^{1/2}$ (by Ladyzhenskaya's inequality) and $|\tilde{\varphi}_m + \varphi_m|_{L^4} \le 8(|\tilde{\varphi}_m|^4_{L^4} + |\varphi_m|^4_{L^4})$.

Claim 2. The following hold true

$$\begin{aligned}
&G(t,\bar{u}) = \bar{G}(t) \text{ in } M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div})),\\
&h(t,\bar{u}) = \bar{h}(t) \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_{div}),\\
&B(\bar{u},\bar{u}) \equiv B(\bar{u}) = \bar{K}_0 \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_{div}),\\
&(a\bar{\varphi} - J*\bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} = \bar{K}_1 \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_{div}),\\
&(\bar{u}.\nabla)\bar{\varphi} = \bar{K}_2 \text{ in } M^2_{\mathcal{F}_t}(0,T;V'_0),\\
&a(.)\bar{\varphi} + F'(\bar{\varphi}) = \bar{\rho} \text{ in } L^r(\Omega,\mathbb{P};L^\infty(0,T;L^r(\mathcal{O}))),\\
&\bar{\mu} = \bar{\rho} - J*\bar{\varphi} = a\bar{\varphi} - J*\bar{\varphi} + F'(\bar{\varphi} \text{ in } M^2_{\mathcal{F}_t}(0,T;U).
\end{aligned} \tag{3.108}$$

By $(3.107)_3$ and the fact that the sequence $\{\tau_n : n \ge 1\}$ is increasing to T, we infer that

$$\bar{G}(t) = G(t,\bar{u}) \text{ in } M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div})),$$

which proves $(3.108)_1$.
We observe that owing to $(3.60)_3$ and $(3.107)_2$, we have

$$\begin{aligned}
&u_m|_{[0,\tau_n]} \to \bar{u}|_{[0,\tau_n]} \text{ in } M^2_{\mathcal{F}_t}(0,T;V_{div}),\\
&\varphi_m|_{[0,\tau_n]} \to \bar{\varphi}|_{[0,\tau_n]} \text{ in } M^2_{\mathcal{F}_t}(0,T;L^2_{(0)}(\mathcal{O})).
\end{aligned} \tag{3.109}$$

Let us move to the proof of $(3.108)_2$. For any $w \in M^\infty_{\mathcal{F}_t}(0,T;V_{div})$, we have using also the Lipschitz condition on h

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n} \langle h(s,\bar{u}(s)) - h(s,u_m(s)), w\rangle\, ds\\
&\le cl_h\|w\|_{M^\infty_{\mathcal{F}_t}(0,T;V_{div})}\mathbb{E}\int_0^{\tau_n} |\bar{u}(s) - u_m(s)|ds\\
&\le cl_h\|w\|_{M^\infty_{\mathcal{F}_t}(0,T;V_{div})}\mathbb{E}\int_0^{\tau_n} \|\bar{u}(s) - u_m(s)\|ds,
\end{aligned}$$

which implies that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \langle h(s,\bar{u}(s)) - h(s,u_m(s)), w\rangle\, ds = 0. \tag{3.110}$$

By $(3.57)_6$ and (3.110) we have for any $w \in M^\infty_{\mathcal{F}_t}(0,T;V_{div})$

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \left\langle \bar{h}(s) - h(s,\bar{u}(s)), w\right\rangle ds = 0.$$

Hence from this last inequality, using the fact that $\tau_n \uparrow T$ and $M^\infty_{\mathcal{F}_t}(0,T;V_{div})$ is dense in $M^2_{\mathcal{F}_t}(0,T;V_{div})$, we infer that

$$\bar{h}(t) = h(t,\bar{u}) \ \text{ in } \ M^2_{\mathcal{F}_t}(0,T;V'_{div}).$$

This proves $(3.108)_2$.
Let us move to the proof of $(3.108)_3$. We note that for any $w \in M^\infty_{\mathcal{F}_t}(0,T;V_{div})$

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n} \langle B(\bar{u}(s),\bar{u}(s)) - B(u_m(s),u_m(s)), w\rangle\, ds \\
&\le c\|w\|_{M^\infty_{\mathcal{F}_t}(0,T;V_{div})}\mathbb{E}\int_0^{\tau_n} |\bar{u}(s)-u_m(s)|^{\frac12}\|\bar{u}(s)-u_m(s)\|^{\frac12}(\|\bar{u}(s)\|+\|u_m(s)\|)ds.
\end{aligned}$$

Thus

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \langle B(\bar{u}(s),\bar{u}(s)) - B(u_m(s),u_m(s)), w\rangle\, ds = 0. \tag{3.111}$$

From (3.111) and $(3.57)_3$, we derive that for any $w \in M^\infty_{\mathcal{F}_t}(0,T;V_{div})$

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \left\langle B(\bar{u}(s),\bar{u}(s)) - \bar{K}_0(s), w\right\rangle ds = 0.$$

From this last inequality, using the fact that $\tau_n \uparrow T$ and $M^\infty_{\mathcal{F}_t}(0,T;V_{div})$ is dense in $M^2_{\mathcal{F}_t}(0,T;V_{div})$, we infer that

$$B(\bar{u},\bar{u}) = \bar{K}_0 \ \text{ in } \ M^2_{\mathcal{F}_t}(0,T;V'_{div}).$$

This proves $(3.108)_3$.
To proves $(3.108)_4$, we note that

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n} \langle (a\bar{\varphi} - J*\bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \mu_m\nabla\varphi_m, w\rangle\, ds \\
&\le c\|w\|_{M^\infty_{\mathcal{F}_t}(0,T;V_{div})}\mathbb{E}\int_0^{\tau_n} \left(|\bar{\varphi}+\varphi_m| + |\nabla\varphi_m| + |\bar{\varphi}|\right)|\bar{\varphi}-\varphi_m|_{L^4} ds \\
&\le c\|w\|_{M^\infty_{\mathcal{F}_t}(0,T;V_{div})}\mathbb{E}\int_0^{\tau_n} \left(|\bar{\varphi}| + |\varphi_m| + |\nabla\varphi_m|\right)|\bar{\varphi}-\varphi_m|^{\frac12}|\nabla(\bar{\varphi}-\varphi_m)|^{\frac12} ds
\end{aligned}$$

which gives

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \langle (a\bar{\varphi} - J*\bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \mu_m\nabla\varphi_m, w\rangle\, ds = 0. \tag{3.112}$$

From $(3.57)_4$ and (3.112), we infer that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \left\langle (a\bar{\varphi} - J*\bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} - \bar{K}_1, w\right\rangle ds = 0 \ \text{ for all } \ w \in M^\infty_{\mathcal{F}_t}(0,T;V_{div}).$$

Hence

$$(a\bar{\varphi} - J*\bar{\varphi} + F'(\bar{\varphi}))\nabla\bar{\varphi} = \bar{K}_1 \ \text{ in } \ M^2_{\mathcal{F}_t}(0,T;V'_{div}).$$

This proves $(3.108)_4$.
Let us move to the proof of $(3.108)_5$.
Using the Gagliardo-Nirenberg inequality, we have for any $\psi \in M^{\infty}_{\mathcal{F}_t}(0,T;V_0)$

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n} \langle(\bar{u}.\nabla)\bar{\varphi} - (u_m.\nabla)\varphi_m, \psi\rangle\, ds \\
&\leq c\|\psi\|_{M^{\infty}_{\mathcal{F}_t}(0,T;V_0)}\mathbb{E}\int_0^{\tau_n} (|\bar{\varphi}|_{L^{2\kappa+2}}\|\bar{u} - u_m\| + \|u_m\||\bar{\varphi}(s) - \varphi_m(s)|_{L^{2\kappa+2}})ds \\
&\leq c\|\psi\|_{M^{\infty}_{\mathcal{F}_t}(0,T;V_0)}\mathbb{E}\int_0^{\tau_n} (|\bar{\varphi}|_{L^{2\kappa+2}}\|\bar{u} - u_m\| + \|u_m\||\bar{\varphi} - \varphi_m|^{\frac{1}{\kappa+1}}|\nabla(\bar{\varphi} - \varphi_m)|^{\frac{\kappa}{\kappa+1}})ds
\end{aligned}$$

which implies that for any $\psi \in M^{\infty}_{\mathcal{F}_t}(0,T;V_0)$

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \langle(\bar{u}(s).\nabla)\bar{\varphi}(s) - (u_m(s).\nabla)\varphi_m(s), \psi\rangle\, ds = 0. \tag{3.113}$$

From $(3.57)_5$ and (3.113), we derive that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \left\langle(\bar{u}(s).\nabla)\bar{\varphi}(s) - \bar{K}_2(s), \psi\right\rangle ds = 0.$$

Finally from this last inequality, using the fact that $\tau_n \uparrow T$ and $M^{\infty}_{\mathcal{F}_t}(0,T;V_0)$ is dense in $M^2_{\mathcal{F}_t}(0,T;V_0)$, we infer that

$$(\bar{u}.\nabla)\bar{\varphi} = \bar{K}_2 \quad \text{in} \quad M^2_{\mathcal{F}_t}(0,T;V_0').$$

This proves $(3.108)_5$.
The proof of $(3.108)_6$ follows easily from $(3.57)_8$ and the fact that $\varphi_m|_{[0,\tau_n]} \to \bar{\varphi}|_{[0,\tau_n]}$ in $M^2_{\mathcal{F}_t}(0,T;L^2_{(0)}(\mathcal{O}))$.
Finally the proof of $(3.108)_7$ is just a consequence of the proof of $(3.108)_6$.
Hence, we conclude from **Claim 1** and **Claim 2** that $(\bar{u}, \bar{\varphi})$ is a solution of (2.17).
We note that the estimate (2.19) follows from (3.32). The fact that $B(\bar{u}) \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$, $\bar{\mu}\nabla\bar{\varphi} \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$ follows respectively from the estimates (3.53), (3.50).
Since $B(\bar{u})$, $\bar{\mu}\nabla\bar{\varphi} \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$, $A\bar{u} + h(t,\bar{u}) \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$,
$G(t,\bar{u}) \in M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div}))$ and $\bar{\mu}\nabla\bar{\varphi} \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$ then from Theorem 4.2.5 in [**32**], $\bar{u}$ is $\mathbb{P}$-a.s. continuous with values in G_{div}.
Similarly owing to the fact that $\bar{\varphi} \in M^2_{\mathcal{F}_t}(0,T;V_0)$ and $\bar{\varphi}_t \in M^2_{\mathcal{F}_t}(0,T;V_0')$, we deduce from Lemma 1.2 in [**37**] that, $\bar{\varphi}$ is $\mathbb{P}$-a.s. continuous with values in H. This completes the proof of the existence. □

3.2. Proof of the uniqueness.

PROPOSITION 3.2. *We assume that the hypotheses of Theorem 2.5 hold. Let (u,φ) be a variational solution to (2.17), then (u,φ) satisfies*

$$\begin{aligned}
&\mathcal{E}_{tot}(u(t),\varphi(t)) + 2\int_0^t \left(\nu\|u(s)\|^2 + |\nabla\mu(s)|^2\right) ds = \mathcal{E}_{tot}(u_0,\varphi_0) \\
&+2\int_0^t \langle h(s,u(s)), u(s)\rangle\, ds + \int_0^t \|G(s,u(s))\|^2_{L_2(K,G_{div})} ds \\
&+2\int_0^t \langle G(s,u(s))dW(s), u(s)\rangle\, ds, \\
&\mathbb{E}\int_0^t \langle G(s,u(s))dW(s), u(s)\rangle = 0,
\end{aligned} \tag{3.114}$$

with $t \in [0,T]$ and $\mathcal{E}_{tot}$ is given by (2.14)$_2$.

Proof. From (2.17)$_1$ and the fact that $B(u) \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$, $\mu\nabla\varphi \in M^2_{\mathcal{F}_t}(0,T;V'_{div})$, we can apply the Itô formula to the process $|u|^2$ (see Theorem 4.2.5 in [**32**]) and derive that

$$\begin{aligned} &|u(t)|^2 + 2\int_0^t \nu\|u(s)\|^2 ds = |u_0|^2 + 2\int_0^t \langle\mu(s)\nabla\varphi(s), u(s)\rangle\, ds \\ &+2\int_0^t \langle h(s,u(s)), u(s)\rangle\, ds + \int_0^t \|G(s,u(s))\|^2_{L_2(K,G_{div})} ds \\ &+2\int_0^t \langle G(s,u(s))dW(s), u(s)\rangle\,,\ \forall t\in[0,T]. \end{aligned} \tag{3.115}$$

We have $\mu \in H^1$ a.e. (can be derived from the estimates (3.49)). We also note that since the following Gelfand chain holds

$$U \subset H = H' \subset H',$$

where each space is densely and compactly embedded into the next one, the duality pairing between U and U' can be also viewed as a scalar product in H. Hence we can multiply $\partial_t\varphi$ by μ in H and derive that

$$\frac{d}{dt}\mathcal{E}(\varphi) + \langle(u.\nabla)\varphi, \mu\rangle + |\mathcal{A}^{1/2}\mu|^2 = 0. \tag{3.116}$$

Integrating (3.116) from 0 to t, multiplying the result by 2 and adding the resulting equation to (3.115), we derive (3.114)$_1$.
Note that (3.114)$_2$ follows from the fact that

$$G(t,u) \in M^2_{\mathcal{F}_t}(0,T;L_2(K,G_{div})) \ \text{ and } \ u \in L^4(\Omega;\mathcal{C}([0,T];G_{div})).$$

The proof of Proposition 3.2 is now complete. □

Proposition 3.3. *Assume that the hypotheses of Theorem 2.5 are satisfied. There exists a unique variational solution to problem (2.17).*

Proof. Let (u_1,φ_1), (u_2,φ_2) be two variational solutions to (2.17). Let $(u,\varphi,\mu) = (u_1,\varphi_1,\mu_1) - (u_2,\varphi_2,\mu_2)$. Then (u,φ) satisfies

$$\begin{cases} du + [\nu Au + B(u_2,u) + B(u,u_1)]dt = [-\varphi(\varphi_1+\varphi_2)\dfrac{\nabla a}{2}]dt \\ \qquad -[(J*\varphi)\nabla\varphi_2 + (J*\varphi_1)\nabla\varphi dt + [h(t,u_1) - h(t,u_2)]dt \\ \qquad +[G(t,u_1) - G(t,u_2)]dW, \\ \dfrac{d\varphi}{dt} + (u.\nabla)\varphi_1 + (u_2.\nabla)\varphi + \mathcal{A}\mu = 0, \\ \mu = a\varphi - J*\varphi + F'(\varphi_1) - F'(\varphi_2), \\ (u,\varphi)(0) = (0,0). \end{cases} \tag{3.117}$$

Applying the Itô formula to the process $|u(t)|^2$ and integrating the result between 0 and t, we obtain

$$
\begin{aligned}
&|u(t)|^2 + 2\nu \int_0^t \|u\|^2 ds + 2\int_0^t \langle B(u,u_1),u\rangle\, ds \\
&= -\int_0^t (\varphi(\varphi_1+\varphi_2)\nabla a, u)\, ds + 2\int_0^t \langle h(s,u_1) - h(s,u_2), u\rangle\, ds \\
&-2\int_0^t ((J*\varphi)\nabla\varphi_2 + (J*\varphi_1)\nabla\varphi, u)\, ds \\
&+\int_0^t \|G(s,u_1(s)) - G(s,u_2(s))\|^2_{L_2(K,G_{div})} ds \\
&+2\int_0^t \langle G(s,u_1(s)) - G(s,u_2(s))dW(s), u\rangle\, .
\end{aligned}
\tag{3.118}
$$

We apply the Lagrangian theorem to F' (F is regular enough to do so), and get

$$F'(\varphi_1) - F'(\varphi_2) = F''(\varphi_2 + \theta\varphi)\varphi, \quad \text{with} \ \ 0 < \theta < 1.$$

We take the duality of (3.117) with $\mathcal{A}_N^{-1}\varphi$ and obtain after integrating over 0 and t

$$
\begin{aligned}
&|\mathcal{A}_N^{-1/2}\varphi(t)|^2 + 2\int_0^t (a\varphi(s) + F''(\varphi_2(s)+\theta\varphi(s))\varphi(s),\varphi(s))\, ds \\
&= -2\int_0^t \left\langle (u.\nabla)\varphi_1, \mathcal{A}_N^{-1}\varphi\right\rangle ds - 2\int_0^t \left\langle (u_2.\nabla)\varphi, \mathcal{A}_N^{-1}\varphi\right\rangle ds \\
&+2\int_0^t (J*\varphi(s),\varphi(s))\, ds.
\end{aligned}
\tag{3.119}
$$

Summing (3.118) and (3.119), we derive that

$$
\begin{aligned}
&|u(t)|^2 + |\mathcal{A}_N^{-1/2}\varphi(t)|^2 + 2\nu \textstyle\int_0^t \|u\|^2 ds + 2\displaystyle\int_0^t \langle B(u,u_1),u\rangle\, ds \\
&+2\int_0^t (a\varphi(s) + F''(\varphi_2(s)+\theta\varphi(s))\varphi(s),\varphi(s))\, ds \\
&= -\int_0^t (\varphi(\varphi_1+\varphi_2)\nabla a, u)\, ds + 2\int_0^t \langle h(s,u_1) - h(s,u_2), u\rangle\, ds \\
&-2\int_0^t ((J*\varphi)\nabla\varphi_2 + (J*\varphi_1)\nabla\varphi, u)\, ds - 2\int_0^t \left\langle (u.\nabla)\varphi_1, \mathcal{A}_N^{-1}\varphi\right\rangle ds \\
&+\int_0^t \|G(s,u_1(s)) - G(s,u_2(s))\|^2_{L_2(K,G_{div})} ds \\
&-2\int_0^t \left\langle (u_2.\nabla)\varphi, \mathcal{A}_N^{-1}\varphi\right\rangle ds \\
&+2\int_0^t \langle G(s,u_1(s)) - G(s,u_2(s))dW(s), u\rangle + 2\int_0^t (J*\varphi(s),\varphi(s))\, ds.
\end{aligned}
\tag{3.120}
$$

Note that

$$2\int_0^t (a\varphi(s) + F''(\varphi_2(s)+\theta\varphi(s))\varphi(s),\varphi(s))\, ds \geq 2c_0\int_0^t |\varphi(s)|^2 ds, \tag{3.121}$$

$$2\,|\langle B(u(s),u_1(s)),u(s)\rangle| \leq \frac{\nu}{6}\|u(s)\|^2 + c\|u_1(s)\|^2|u(s)|^2, \tag{3.122}$$

$$\begin{aligned} 2\,|(J*\varphi,\varphi)| &\le 2\left|\left(\mathcal{A}_N^{1/2}(J*\varphi),\mathcal{A}_N^{-1/2}\varphi\right)\right| \\ &= 2\left|\left(\mathcal{A}_N^{1/2}(J*\varphi-\langle J*\varphi\rangle),\mathcal{A}_N^{-1/2}\varphi\right)\right| \\ &\le 2|\nabla(J*\varphi-\langle J*\varphi\rangle)||\mathcal{A}_N^{-1/2}\varphi| \\ &\le 2|\nabla J|_{L^1}|\varphi||\mathcal{A}_N^{-1/2}\varphi| \\ &\le \frac{c_0}{5}|\varphi|^2+c|\nabla J|_{L^1}^2|\mathcal{A}_N^{-1/2}\varphi|^2, \end{aligned} \tag{3.123}$$

where we have also used Hölder's, Ladyzhenskaya inequalities, and Young's inequality for convolution and the fact that $|\mathcal{A}_N^{1/2}\varphi|^2 = (\mathcal{A}_N\varphi,\varphi) = |\nabla\varphi|^2$, for all $\varphi \in D(\mathcal{A}_N)$ and hence $|\mathcal{A}_N^{1/2}\varphi| = |\nabla\varphi|$, which also holds, by density, for all $\varphi \in D(\mathcal{A}_N^{1/2}) = V_0$.

Thanks to the Hölder, Young's inequality in conjunction with (2.7), the following two estimates hold:

$$\begin{aligned} 2|\left\langle (u(s).\nabla)\varphi_1(s),\mathcal{A}_N^{-1}\varphi(s)\right\rangle| &= 2\left|\left(u(s).\nabla\mathcal{A}_N^{-1}\varphi(s),\varphi_1(s)\right)\right| \\ &\le 2|\varphi_1(s)|_{L^4}|u(s)|_{L^4}|\nabla\mathcal{A}_N^{-1}\varphi(s)| \\ &\le \frac{\nu}{6}\|u(s)\|^2+c|\varphi_1(s)|_{L^4}^2|\mathcal{A}_N^{-1/2}\varphi(s)|^2, \end{aligned} \tag{3.124}$$

$$\begin{aligned} 2|\left\langle (u_2(s).\nabla)\varphi(s),\mathcal{A}_N^{-1}\varphi(s)\right\rangle| &= 2\left|-\left(u_2(s).\nabla\mathcal{A}_N^{-1}\varphi(s),\varphi(s)\right)\right| \\ &\le 2|\varphi(s)||u_2(s)|_{L^4}|\nabla\mathcal{A}_N^{-1}\varphi(s)|_{L^4} \\ &\le \frac{c_0}{5}|\varphi(s)|^2+c|u_2(s)|_{L^4}^4|\mathcal{A}_N^{-1/2}\varphi(s)|^2. \end{aligned} \tag{3.125}$$

By Hölder's, Ladyzhenskaya's and Young's inequalities, we have

$$\begin{aligned} |(\varphi(\varphi_1+\varphi_2)\nabla a,u)| &\le |\nabla a|_{L^\infty}|\varphi||\varphi_1+\varphi_2|_{L^4}|u|_{L^4} \\ &\le c|\nabla a|_{L^\infty}|\varphi||\varphi_1+\varphi_2|_{L^4}\|u\|^{1/2}|u|^{1/2} \\ &\le \frac{c_0}{5}|\varphi|^2+\frac{\nu}{6}\|u\|^2+c|\nabla a|_{L^\infty}^4|\varphi_1+\varphi_2|_{L^4}^4|u|^2 \\ &\le \frac{c_0}{5}|\varphi|^2+\frac{\nu}{6}\|u\|^2+c|\nabla J|_{L^1}^4|\varphi_1+\varphi_2|_{L^4}^4|u|^2, \end{aligned} \tag{3.126}$$

where we have also used the fact that $|\nabla a|_{L^\infty} \le |\nabla J|_{L^1}$.

In order to estimate $((J*\varphi)\nabla\varphi_2,u)$ and $((J*\varphi_1)\nabla\varphi,u)$, we rewrite these terms using an integration by parts and the divergence free condition as

$$\begin{aligned} ((J*\varphi)\nabla\varphi_2,u) &= -((\nabla J*\varphi)\varphi_2,u), \\ ((J*\varphi_1)\nabla\varphi,u) &= -((\nabla J*\varphi_1)\varphi,u). \end{aligned}$$

Now by Hölder's, the Ladyzhenskaya inequalities, and Young's inequality for convolution, the following estimate holds:

$$\begin{aligned} 2\,|((\nabla J*\varphi)\varphi_2,u)| &\le 2|\varphi_2|_{L^4}|u|_{L^4}|\nabla J*\varphi| \\ &\le c|\nabla J|_{L^1}|\varphi|\|u\|^{1/2}|u|^{1/2}|\varphi_2|_{L^4} \\ &\le \frac{c_0}{5}|\varphi|^2+\frac{\nu}{6}\|u\|^2+c|\nabla J|_{L^1}^4|\varphi_2|_{L^4}^4|u|^2. \end{aligned} \tag{3.127}$$

Making similar reasoning as in the previous estimate, we infer that

$$2|((\nabla J*\varphi_1)\varphi,u)| \le \frac{c_0}{5}|\varphi|^2+\frac{\nu}{6}\|u\|^2+c|\nabla J|_{L^1}^4|\varphi_1|_{L^4}^4|u|^2. \tag{3.128}$$

Owing to the assumptions on h and Young's inequality, we have

$$2\| \langle h(s,u_1(s)) - h(s,u_2(s)), u(s)\rangle \|_{V'_{div}} \begin{aligned}[t] &\leq 2l_h|u(s)|\|u(s)\| \\ &\leq \tfrac{\nu}{6}\|u(s)\|^2 + cl_h^2|u(s)|^2, \end{aligned} \tag{3.129}$$

$$\|G(s,u_1(s)) - G(s,u_2(s))\|^2_{L_2(K,G_{div})} \leq l_g^2|u(s)|^2. \tag{3.130}$$

Let

$$\mathcal{Z}_2(t) = |u(t)|^2 + |\mathcal{A}_N^{-1/2}\varphi(t)|^2 = |u(t)|^2 + \|\varphi(t)\|^2_{V'_0},$$

and

$$\begin{aligned} \mathcal{K}_1(t) \quad &= c(\|u_1(t)\|^2 + |\varphi_1(t)|^2_{L^4} + |u_2(t)|^4_{L^4} + |\nabla J|^4_{L^1}|\varphi_1(t) + \varphi_2(t)|^4_{L^4}) \\ &+c|\nabla J|^4_{L^1}(|\varphi_1(t)|^4_{L^4} + |\varphi_2(t)|^4_{L^4}), \end{aligned} \tag{3.131}$$

$$\mathcal{K}_2(t) = e^{-\int_0^t \mathcal{K}_1(s)ds}.$$

Applying Itô's formula to the process $\mathcal{K}_2(t)\mathcal{Z}_2(t)$ and using (3.120)-(3.130), we derive that

$$\begin{aligned} \mathcal{K}_2(t)\mathcal{Z}_2(t) \quad &+ \int_0^t \mathcal{K}_2(s)[\nu\|u(s)\|^2 + c_0|\varphi(s)|^2]ds \\ &\leq (c|\nabla J|^2_{L^1} + l_g^2 + cl_h^2)\int_0^t \mathcal{K}_2(s)\mathcal{Z}_2(s)ds \\ &+2\int_0^t \mathcal{K}_2(s)\langle G(s,u_1(s)) - G(s,u_2(s))dW(s), u(s)\rangle . \end{aligned} \tag{3.132}$$

As $0 < \mathcal{K}_2(t) \leq 1$, the expectation of the stochastic integral in (3.132) vanishes and it follows that

$$\mathbb{E}\mathcal{K}_2(t)\mathcal{Z}_2(t) \leq (c|\nabla J|^2_{L^1} + l_g^2 + cl_h^2)\mathbb{E}\int_0^t \mathcal{K}_2(s)\mathcal{Z}_2(s)ds, \ 0 \leq t \leq T. \tag{3.133}$$

Now applying the Gronwall lemma to (3.133), we derive that $\mathcal{Z}_2(t) = 0$ $\mathbb{P}$-a.s., for all $t \in [0,T]$, which gives $(u_1,\varphi_1) = (u_2,\varphi_2)$, $\mathbb{P}$-a.s., for all $t \in [0,T]$.
This completes the proof of Proposition 3.3. □

4. Convergence Results

The following proposition shows that the sequence of approximate solutions (u_m, ϕ_m) defines by the Galerkin approximation scheme converges to the unique variational solution of problem (2.1).

PROPOSITION 4.1. *Under the hypotheses of Theorem 2.5, the following convergence hold:*

$$\begin{aligned} &\lim_{m\to\infty} \mathbb{E}\left(|u_m(t) - \bar{u}(t)|^2 + \|\varphi_m(t) - \bar{\varphi}(t)\|^2_{V'_0}\right) = 0, \\ &\lim_{m\to\infty} \mathbb{E}\int_0^t \left(\|u_m(s) - \bar{u}(s)\|^2 + |\varphi_m(s) - \bar{\varphi}(s)|^2\right)ds = 0, \end{aligned} \tag{4.1}$$

for $m \to \infty$ and for all $t \in [0,T]$.

The following lemma will be useful for the proof of Proposition 4.1. The proof of this lemma can be found in [**5, 8, 35**].

LEMMA 4.2. *Let $\{Z_m : m \geq 1\} \subset M^2_{\mathcal{F}_t}(0,T;\mathbb{R})$ be a sequence of continuous real processes and let $\{\sigma_n : n \geq 1\}$ be a sequence of $\mathcal{F}_t$-stopping times such that $\sigma_n \uparrow T$; $\sup_{m\geq 1}\mathbb{E}|Z_m(T)|^2 < \infty$ and $\lim_{m\to\infty}|Z_m(\sigma_n)| = 0$, for $n \geq 1$. Then $\lim_{m\to\infty}\mathbb{E}|Z_m(T)| = 0$.*

We now give the proof of Proposition 4.1.

PROOF. Applying Lemma 4.2 to $Z_m(t) = \|(u_m(t), \varphi_m(t)) - (\bar{u}(t), \bar{\varphi}(t))\|^2_{G_{div}\times V_0'}$ and $\sigma_n = \tau_n$, using (2.19), (3.32), $(3.107)_1$ and the uniqueness of $(\bar{u}, \bar{\varphi})$, we derive that the whole sequence given by (3.58) satisfies

$$\lim_{m\to\infty} \mathbb{E}\|(u_m(t), \varphi_m(t)) - (\bar{u}(t), \bar{\varphi}(t))\|^2_{G_{div}\times V_0'} = 0 \quad \text{for all} \quad t \in [0, T].$$

Similarly, applying Lemma 4.2 to $Z_m(t) = \int_0^t \left(\|u_m(s) - \bar{u}(s)\|^2 + |\varphi_m(s) - \bar{\varphi}(s)|^2\right) ds$, using (2.19), $(3.107)_2$ and the fact that $V_0 \hookrightarrow L^2_{(0)}(\mathcal{O})$ continuously, we infer that the whole sequence (u_m, φ_m) converges to $(\bar{u}, \bar{\varphi})$ strongly in $M^2_{\mathcal{F}_t}(0, T; \mathbb{V}_1)$, i.e.,

$$\lim_{m\to\infty} \mathbb{E}\int_0^t \left(\|u_m(s) - \bar{u}(s)\|^2 + |\varphi_m(s) - \bar{\varphi}(s)|^2\right) ds = 0 \quad \text{for all} \quad t \in [0, T].$$

We note that $\mathbb{V}_1$ denotes a Hilbert space defined by

$$\mathbb{V}_1 = V_{div} \times L^2_{(0)}(\mathcal{O}), \tag{4.2}$$

endowed with the scalar product whose associated norm is

$$\|(v, \varphi)\|^2_{\mathbb{V}_1} = \|v\|^2 + |\varphi|^2. \tag{4.3}$$

The proof of Proposition 4.1 is now complete. □

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References

[1] H. Abels, On a diffusion interface model for the two-phase flows of viscous, incompressible fluids with matched densities, Arch. Ration. Mech. Anal. **194** (2009), 463-506.

[2] H. Abels, Longtime behavior of solutions of a Navier-Stokes/Cahn-Hilliard system, Proceedings of the Conference "Nonlocal and Abstract Parabolic Equations and their applications", Bedlevo, Banach Center Publ. **86** (2009), 9-19.

[3] P. W. Bates and J. Han, The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equations, J. Math. Anal. Appl. **331** (2005) (1), 289-312.

[4] F. Boyer, Mathematical study of multi-phase flow under shear through order parameter formulation, Asymptot. Anal. **20** (1999), 175-212.

[5] H. Breckner, Galerkin approximation and the strong solution of the stochastic Navier-Stokes equation, J. Appl. Math. Stoch. Anal. **13** (2000) (3), 239-259.

[6] H. Brezis, Functional Analysis, Sobolev spaces and Partial Differential Equations, Springer, 2010.

[7] P. Constantin and C. Foias, Navier-Stokes Equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1988).

[8] T. Caraballo, J. Real and T. Taniguchi, On the existence and uniqueness of solutions to stochastic three-dimensional Lagrangian averaged Navier-Stokes equations, Proceedings of the Royal Society A. **462** (2006) (2066), 459-479.

[9] P. Colli, S. Frigeri and M. Grasselli, Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes System, J. Appl. Math. Anal. **386** (2012), 428-444.

[10] F. Cornalba, A nonlocal stochastic Cahn-Hilliard equation. POLITECHNICO DI MILANO, PhD thesis, 2014-2015.

[11] E. DiBenedetto: Degenerate Parabolic Equations, Springer-Verlag, New-York, (1993).

[12] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992).

[13] G. Deugoué, A. Ndongmo Ngana and T. Tachim Medjo, Global existence of a weak martingale solution to a stochastic nonlocal Cahn-Hilliard Navier-Stokes model. Submitted.

[14] G. Deugoué and T. Tachim Medjo, The exponential behavior of a stochastic globally modified Cahn-Hilliard-Navier-Stokes model with multiplicative noise, J. Math. Anal. Appl. **460** (2018) (1), 140-163.
[15] G. Deugoué and T. Tachim Medjo, Convergence of the solution of the stochastic 3D globally modified Cahn-Hilliard-Navier-Stokes equations, J. Differential Equations. **265** (2018) (2), 545-592.
[16] Fang Li and Bo You, Random attractor for the stochastic Cahn-Hilliard-Navier-Stokes system with small additive noise, Stochastic Analysis and Applications. **36** (2018) (3).
[17] S. Frigeri, C. G. Gal and M. Grasselli, On nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions, Journal of Nonlinear Science. **26** (2016) (4), 847-893.
[18] S. Frigeri, M. Grasselli and P. Krejčí, Strong solutions for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems, Journal of Differential equat. **255** (2013) (9), 2597-2614.
[19] S. Frigeri, M. Grasselli, Nonlocal Cahn-Hilliard Navier-Stokes system with singular potentials. Dynamics of PDE. **9** (2012) (4), 273-304.
[20] C. G. Gal, M. Grasselli, Asymptotic behavior of a Cahn-Hilliard Navier-Stokes system in 2D, Ann. Inst. H. Poincaré Anal. Non linéaire. **27** (2010), 401-436.
[21] C. G. Gal, M. Grasselli, Trajectory attractors for binary fluid mixtures in 3D, Chinese Ann. Math. Ser. B **31** (2010), 655-678.
[22] M. E. Gurtin, D. Polignome, J. Viñals, Two phase binary fluids and immiscible fluids described by order parameter, Math. Models Meth. Appl. Sci. **6** (1996), 8-15.
[23] P. C. Hohenberd, B. I. Halperin, Theory of dynamical critical phenomena, Rev. Mod. Phys. **49** (1977), 435-479.
[24] M. Heida, J. Málek, K. R. Rajagopal, On the development and generalization of Cahn-Hilliard equations within a thermodynamic framework, Z. Angew, Math. Phys. **63** (2012), 145-169.
[25] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes. Second edition. **24** of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York, (1989).
[26] D. Jasnow, J. Vinals, Coarse-grained description of thermo-capillary flow, Phys. Fluids. **8** (1996), 660-669.
[27] H. Kunita, Stochastic flows and Stochastic Differential Equations. Cambridge University Press, Cambridge (199).
[28] E. M. Ouhabaz, ANALYSIS OF HEAT EQUATIONS ON DOMAINS. Vol. **31** of London Mathematical Society Monographs Series. Princeton University Press, Princeton, (2005).
[29] R. Mikulevicius, B. L. Rozovskii. Stochastic Navier-Stokes for turbulent flows. SIAM J. Math. Anal. **35** (2004), 1250-1310.
[30] A. Morro, Phase-field models of Cahn-Hilliard Fluids and extra fluxes, Adv. Theor. Appl. Mech. **3** (2010), 409-424.
[31] E. Pardoux: Equations aux Dérivées Partielles Stochastiques Non Linéaires Monotones. Thèse, Université Paris XI, 1975.
[32] C. Prévôt and M. Röckner: A Concise Course on Stochastic Partial Differential Equations, vol. **1905** of Lecture Notes in Mathematics, Springer-Verlag, Berlin, (2007).
[33] B. Rozowskii. Stochastic evolution systems, Mathematics and its Applications, no. 35, Kluwer Academic Publishers Group, Dordrecht, (1990).
[34] V. N. Starovoitov, The dynamics of a two-component fluid in the presence of capillary forces, Math. Notes **62** (1997), 244-254.
[35] T. Tachim Medjo, On the existence and uniqueness of solution to a stochastic 2D Cahn-Hilliard-Navier-Stokes model, J. Differential Equations. **262** (2017), 1028-1054.
[36] R. Temam, Infinite dimensional dynamical systems in mechanics and physics, vol. **68**. Appl. Math. Sci., Springer-Verlag, New York, second edition, (1997).
[37] R. Temam, Navier-Stokes, Theory and Numerical Analysis. **2** of studies in Mathematics and Its applications, North-Holland, Amsterdam, The Netherlands, 3rd edition, (1984).
[38] L. Zhao, H. Wa and H. Huang, Convergence to equilibrium for a phase-field model for the mixture of two viscous incompressible fluids, Commun. Math. Sci. **7** (2009), 939-962.

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