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Localization estimate and global attractor for the damped and forced Zakharov-Kuznetsov equation in $\mathbb{R}^2$

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Abstract. In the present paper, we show that solutions are spatially localized for the damped and forced Zakharov-Kuznetsov equation in $\mathbb{R}^2$. This result leads to the global attractor in the strong topology of the Sobolev space without weight.

Contents

1. Introduction and main results

In the present paper, we consider the existence of the global attractor associated with the Cauchy problem of the following damped and forced Zakharov-Kuznetsov equation in $\mathbb{R}^2$.

$$\partial_t u + (\partial_{x_1}^3 + \partial_{x_2}^3)u + (\partial_{x_1} + \partial_{x_2})(u^2) + \gamma u = f, \qquad t > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.1}$$

$$u(0, x) = u_0(x), \tag{1.2}$$

where γ is a positive constant and f is a time-independent external forcing term. We can rewrite the Zakharov-Kuznetsov equation as the symmetrized form of (1.1),

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though it is originally asymmetric in x_1 and x_2 variables (see [**6**]). In two dimensions, the local and global well-posedness of (1.1)-(1.2) below energy is known (for details, see Linares and Pastor [**7**], Grünrock and Herr [**6**], Molinet and Pilod [**9**] and Shan [**10, 11**]). In [**11**], the second author has proved the existence of the global attractor in the weak topology of $H^s \equiv H^s(\mathbb{R}^2)$ for $1 \geq s > 10/11$. In his proof, he uses the weak compactness of Sobolev spaces due to the lack of compactness in the $\mathbb{R}^2$ case. By a weak global attractor, we mean a global attractor in the weak topology. The global attractor in the strong topology is sometimes called a strong global attractor. On the one hand, the weak global attractor given in [**11**] is important and interesting in such a sense that it attracts solutions below energy space. On the other hand, the weak global attractor might be too weak. Indeed, it can not capture solutions running away toward infinity. So, it is natural to ask whether or not the weak global attractor proved in [**11**] is a global attractor in the strong topology. In the present paper, we show that it is a strong global attractor in H^s for $1 > s > 10/11$.

THEOREM 1.1. *We assume that $1 > s > 10/11$ and $f \in H^1$. Then, there exists the unique global attractor in the strong topology of H^s for the two dimensional Zakharov-Kuznetsov equation with damping and forcing terms.*

To prove Theorem 1.1, we need the localization estimate of the solution in H^s, $1 > s > 10/11$ for (1.1)-(1.2). Let ϕ be a function in $C^\infty([0,\infty))$ such that $\phi(r) = 1$ $(r \geq 2)$ and $\phi(r) = 0$ $(0 \leq r \leq 1)$. We put $\varphi_R(x) = \phi(|x|/R)$ for $R > 0$ and put $\psi(x) = 1 - \phi(|x|)$. For a positive number N, we define the operators $P_{\geq N}$ and $P_{<N}$ as follows.

$$P_{\geq N} f = \mathcal{F}^{-1}\big[\phi(|\xi|/N)\hat{f}\,\big], \quad P_{<N} f = \mathcal{F}^{-1}\big[\psi(\xi/N)\hat{f}\,\big].$$

For the localization estimate of the solution u to (1.1)-(1.2), we have only to prove the localization estimate of the low frequency part of the solution. This is because the high frequency part $P_{\geq N}u$ of the solution gets smaller and smaller as N increases (see [**11**]).

PROPOSITION 1.2. *We have two positive constants κ_0 and K with the following properties: For any $L > 0$ and $N > 0$, there exists a $T > 0$ such that if $\|u_0\|_{H^s} \leq L$, then the solution u of (1.1)-(1.2) satisfies the following estimate.*

$$\begin{aligned}\|\varphi_R P_{<N} u(t)\|_2^2 \lesssim \|u(T)\|_{L^2}^2 e^{-\gamma(t-T)} + R^{-1}K^2 + (R^{-1} + N^{-\kappa_0})K^3 \\ + \|\varphi_R f\|_2^2 + (NR)^{-2}\|f\|_2^2 \qquad (R \geq 1,\ t \geq T),\end{aligned}$$

where K depends only on γ and $\|f\|_{H^1}$.

REMARK 1.3. There are many papers about global attractors for reaction diffusion equations in unbounded domains (see, e.g., Babin and Vishik [**1**], Merino [**8**] and Wang [**15**]). In [**15**] the global attractor is constructed within the framework of the standard L^2 space while the weighted L^2 space and the uniform Hölder space are used in [**1**] and [**8**], respectively. In [**1**] and [**8**], the precompactness of orbits is recovered by the use of the weighted L^2 space (see [**1**, Lemma 2.16 on page 237]) and the super- and sub-solutions (see [**8**, Theorem 2.4 on page 92]), respectively. In [**15**], such a localization estimate as Proposition 1.2 plays an important role for the proof of precompactness of orbits (see, e.g., [**15**, Lemma 5 on page 46]). But, for nonlinear parabolic equations, the proof of the localization estimate is rather straightforward because parabolic equations have an explicit smoothing effect.

The plan of this paper is as follows. In Section 2, we prove Proposition 1.2 concerning the localization estimate of solution for (1.1)-(1.2). In Section 3, by using Proposition 1.2, we show that the solution map generated by (1.1) is asymptotically compact in H^s for $1 > s > 10/11$, which leads to the global attractor in the strong topology of H^s for $1 > s > 10/11$.

2. Proof of localization estimate

In this section, we show Proposition 1.2. To make the notation simple, we suppress the suffix R indicating the dependence of φ on R. We denote $P_{<N}$ by J and set $v = \varphi J u$. The Cauchy problem (1.1) and (1.2) is rewritten as follows.

$$\partial_t v + (\partial_{x_1}^3 + \partial_{x_2}^3)v + \gamma v + \varphi J\big[(\partial_{x_1} + \partial_{x_2})(u^2)\big] - \sum_{j=1}^{2}\big(3\varphi_{x_j} J u_{x_j x_j} + 3\varphi_{x_j x_j} J u_{x_j} + \varphi_{x_j x_j x_j} J u\big) = \varphi J f, \quad t > 0,\ x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.1}$$

$$v(0,x) = v_0(x) := \varphi J u_0(x). \tag{2.2}$$

We begin with the following lemma concerning *a priori* estimates of solutions, which ensures the existence of the global attractor in the weak topology (see [**11**] and also Tsugawa [**14**, §4. Proof of Theorem 1.1 on pages 312–313]).

LEMMA 2.1. *We have two constants $K_1, K_2 > 0$ with the following properties: Let $L > 0$. We assume that $u_0 \in H^s$ and $\|u_0\|_{H^s} \le L$. Let u be the solution of* (1.1)-(1.2).

(i) There exists a $T_1 > 0$ depending only on L, γ and $\|f\|_{H^1}$ such that

$$\|u(t)\|_{H^s} \le K_1 \qquad (t \ge T_1).$$

(ii) For any $N > 0$, there exists a $T_2 > 0$ depending only on L, N, γ and $\|f\|_{H^1}$ such that

$$\|P_{<N} u(t)\|_{H^1} \le K_2 \qquad (t \ge T_2).$$

PROOF. In the proof of Theorem 1.2 of [**11**], it has been proved that for any $L > 0$, there exist two positive constants M, C depending only on γ, $\|f\|_{H^1}$ and a positive constant T depending also on L such that if $\|u_0\|_{H^s} \le L$, the solution map $S(t)$ satisfies

$$\|P_{<N(t)} S(t) u_0\|_{H^1} + N(t)^{1-s} \|P_{\ge N(t)} S(t) u_0\|_{H^s} \le M \qquad (t \ge T), \tag{2.3}$$

where $N(t)^{1-s} = Ce^{\gamma(t-T)}$ (for the KdV equation, see [**14**, line 21 and 22 on page 313] and Yang [**16**, lines 15 and 16 on page 284]). Lemma 2.1 follows immediately from (2.3). In fact, one can take $K_1 = M \max\{1, C^{-1}\}$ and $T_1 = T$ in (i), while in (ii) it is enough to set $K_2 = M$ and $T_2 = \min\{t \in [T, \infty) \mid N(t) \ge N\}$ for given $N > 0$. □

We now prove the following lemma concerning the localization estimate of v.

LEMMA 2.2. *Let $L > 0$ and $N > 0$. We assume that $u_0 \in H^s$ and $\|u_0\|_{H^s} \le L$. There exists a positive constant T such that the solution v of* (2.1)-(2.2) *satisfies the following estimate:*

$$\|v(t)\|_2^2 \lesssim \|v(T)\|_2^2 e^{-\gamma(t-T)} + \gamma^{-1}(R^{-1} + N^{-\kappa_0})K_1^3 + \gamma^{-1}R^{-1}K_2^2 + \gamma^{-2}\|\varphi J f\|_2^2 \qquad (R \ge 1,\ t \ge T),$$

where K_1, K_2 are positive constants defined as in Lemma 2.1, and κ_0 is an absolute positive constant.

PROOF. We multiply (2.1) by v and integrate the resulting equation in the spatial variables to have

$$\begin{aligned}(2.4)\qquad &\frac{d}{dt}\|v\|_2^2 + 2\gamma\|v\|_2^2 + (A(\partial_{x_1}+\partial_{x_2})J(u^2), Ju) + (BJu, Ju)\\ &+ 6(\varphi\varphi_{x_1}Ju_{x_1}, Ju_{x_1}) + 6(\varphi\varphi_{x_2}Ju_{x_2}, Ju_{x_2}) = 2(\varphi Jf, v),\end{aligned}$$

where

$$A(x) = 2\varphi^2, \qquad B(x) = -(\partial_{x_1}^3 + \partial_{x_2}^3)(\varphi^2).$$

By the definition of function φ, we have

$$|A(x)| \lesssim 1, \quad |B(x)| \lesssim R^{-3} \qquad (x \in \mathbb{R}^2).$$

For the third term on the left side of (2.4), we have

$$\begin{aligned}&(A(\partial_{x_1}+\partial_{x_2})J(u^2), Ju)\\ =&\big(A(\partial_{x_1}+\partial_{x_2})J\{(P_{\geq N}u)^2 + 2(P_{\geq N}u)(Ju)\},\ Ju\big)\\ &+\big(A(\partial_{x_1}+\partial_{x_2})(Ju)^2,\ Ju\big) - \big(A(\partial_{x_1}+\partial_{x_2})P_{\geq N}(Ju)^2,\ Ju\big)\\ =:&\ I_1 + I_2 + I_3.\end{aligned}$$

We now estimate the terms I_1, I_2 and I_3. By integration by parts, we have

$$|I_2| \lesssim R^{-1}\|Ju\|_{H^s}^3.$$

For the term I_3, by the fractional Leibniz rule and the Sobolev embedding we can easily see that

$$\begin{aligned}(2.5)\qquad |I_3| =&\big|\big(A(\partial_x+\partial_y)P_{\geq N}(Ju)^2,\ Ju\big)\big|\\ =&\big|\big((-\Delta)^{-s/2}(\partial_{x_1}+\partial_{x_2})P_{\geq N}(Ju)^2,\ (-\Delta)^{s/2}(AJu)\big)\big|\\ \lesssim&\big\|(-\Delta)^{(1-s)/2}P_{\geq N}(Ju)^2\big\|_2\\ &\times\big(\|A\|_\infty\|(-\Delta)^{s/2}Ju\|_2 + \|A\|_{\dot{B}^s_{\infty,2}}\|Ju\|_2\big)\\ \lesssim&N^{-\kappa_0}\|(-\Delta)^{(1-s)/2+\kappa_0/2}P_{\geq N}(Ju)^2\|_2\|(1-\Delta)^{s/2}Ju\|_2\\ \lesssim&N^{-\kappa_0}\|Ju\|_{H^s}^3,\end{aligned}$$

where $\dot{B}^s_{p,q}$ denotes the Besov space (for the definition of the Besov space, see Bergh and Löfström [**3**] and Triebel [**13**]). In a similar manner, we see that

$$|I_1| \lesssim N^{-\kappa_0}\|u\|_{H^s}^3.$$

These three inequalities and Lemma 2.1 (i) yield, for some $T > 0$,

$$(2.6)\qquad \big|(A(\partial_{x_1}+\partial_{x_2})J(u^2), Ju)\big| \lesssim (R^{-1} + N^{-\kappa_0})K_1^3 \qquad (t \geq T).$$

On the other hand, by Lemma 2.1 (ii) we have, for some $T > 0$ depending also on N,

$$\begin{aligned}(2.7)\qquad &|(BJu, Ju)| + 6|(\varphi\varphi_{x_1}Ju_{x_1}, Ju_{x_1})| + 6|(\varphi\varphi_{x_2}Ju_{x_2}, Ju_{x_2})|\\ &\lesssim R^{-1}\|P_{<N}u\|_{H^1}^2 \leq R^{-1}K_2^2 \qquad (t \geq T).\end{aligned}$$

Equality (2.4) and estimates (2.6) and (2.7) complete the proof of Lemma 2.2. □

Remark 2.3. At the last inequality of (2.5), we have only used the Sobolev embedding which is based on the H^s bound of u. If we would use the bound of Ju in H^1, then the estimate (2.5) would be improved with respect to the inverse power κ_0 of N.

Finally, we prove the following lemma, which enables us to bound the L^2 norm of φJf by that of f.

Lemma 2.4.

$$\|\varphi Ju\|_2 \lesssim \|J(\varphi u)\|_2 + (NR)^{-1}\|u\|_2.$$

Proof. We have only to show the following commutator estimate.

$$\big\|[\varphi, J]u\big\|_2 \lesssim (NR)^{-1}\|u\|_2. \tag{2.8}$$

By the mean value theorem, we have

$$\begin{aligned}
&\big|\varphi(x)(Ju)(x) - \{J(\varphi u)\}(x)\big| \\
&\le \int_{\mathbb{R}^2} \big|(\varphi(x) - \varphi(y))N^2\check{\psi}(N(x-y))u(y)\big|\, dy \\
&\lesssim (NR)^{-1}\|\nabla\phi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^2} H(x-y)|u(y)|\, dy,
\end{aligned}$$

where $H(x) = N^2|Nx||\check{\psi}(Nx)|$. Since $\psi \in C_0^\infty(\mathbb{R}^2)$, we can conclude that $H \in L^1(\mathbb{R}^2)$ with a uniform bound in N, which implies (2.8). □

Proposition 1.2 follows from Lemma 2.2 and Lemma 2.4.

3. Proof of strong global attractor

In this section, we prove by using Proposition 1.2 that the weak global attractor given by the paper [**11**] is a global attractor in the strong topology of H^s. We have only to show that the solution map generated by (1.1) is asymptotically compact in H^s (see, e.g., [**12**, Remark 1.4 on page 26]). We begin with the definition of the asymptotic compactness.

Definition 3.1. Let X be a Banach space and let $S(t)$ be a continuous map from $[0,\infty) \times X \to X$ with semi-group properties: $S(t+s) = S(t)S(s)$ $(t, s \ge 0)$ and $S(0) = I$, where I is the identity operator in X. The semi-group $\{S(t)\}_{t\ge 0}$ is said to be asymptotically compact in X if for every bounded sequence $\{x_k\}$ in X and every sequence $t_k \to \infty$, $\{S(t_k)x_k\}$ is relatively compact in X.

Let $S(t)$ be the solution map associated with (1.1)-(1.2).

Proof of Theorem 1.1. Let $\{u_{0k}\}$ be a bounded sequence in H^s and let $\{t_k\}$ be a positive number sequence with $t_k \to \infty$. It suffices to show that $\{S(t_k)u_{0k}\}$ is totally bounded in H^s.

By (2.3) we have, for some $k_1 \in \mathbb{N}$,

$$\begin{aligned}
&\|P_{\ge N}S(t_k)u_{0k}\|_{H^s} \\
&\quad\le \|P_{\ge N}P_{\ge N(t_k)}S(t_k)u_{0k}\|_{H^s} + \|P_{\ge N}P_{<N(t_k)}S(t_k)u_{0k}\|_{H^s} \\
&\quad\lesssim \|P_{\ge N(t_k)}S(t_k)u_{0k}\|_{H^s} + N^{-(1-s)}\|P_{<N(t_k)}S(t_k)u_{0k}\|_{H^1} \\
&\quad\lesssim e^{-\gamma(t_k - T)}M + N^{-(1-s)}M \qquad (N > 0,\ k \ge k_1).
\end{aligned}$$

Furthermore, by Proposition 1.2 and Lemma 2.1 we have the following: For any $N > 0$ there is a $k_2 = k_2(N) \in \mathbb{N}$ such that

$$\|\varphi_R P_{<N} S(t_k) u_{0k}\|_{H^s} \lesssim K_2^s \|\varphi_R P_{<N} S(t_k) u_{0k}\|_2^{1-s},$$
$$\|\varphi_R P_{<N} S(t_k) u_{0k}\|_2^2 \lesssim K_1^2 e^{-\gamma(t_k - T)} + R^{-1}K^2 + (R^{-1} + N^{-\kappa_0})K^3 + \|\varphi_R f\|_2^2 + (NR)^{-2}\|f\|_2^2 \qquad (R \geq 1,\ k \geq k_2).$$

Hence, for any $\varepsilon > 0$ we can choose so large N, $R > 0$ and $k_0 \in \mathbb{N}$ that

$$\|P_{\geq N} S(t_k) u_{0k}\|_{H^s} + \|\varphi_R P_{<N} S(t_k) u_{0k}\|_{H^s} \leq \varepsilon/2 \qquad (k \geq k_0). \tag{3.1}$$

On the other hand, since the embedding $H^1(|x| < R) \hookrightarrow H^s(|x| < R)$ is compact, the set $\{(1 - \varphi_R) P_{<N} S(t_k) u_{0k}\}$, which is bounded in H^1 by Lemma 2.1 (ii), is relatively compact in H^s. Hence, $\{(1 - \varphi_R) P_{<N} S(t_k) u_{0k}\}$ has a finite collection of H^s-balls with radius $\varepsilon/2$ which cover it. This fact and (3.1) imply that $\{S(t_k) u_{0k}\}$ is covered by a finite collection of H^s-balls with radius ε. Therefore, we conclude that $\{S(t_k) u_{0k}\}$ is totally bounded in H^s, which completes the proof of Theorem 1.1. □

REMARK 3.2. It seems possible to prove Theorem 1.1 by using J. M. Ball's argument (see [**2**], [**4**] and [**5**]). For that purpose, it is necessary to show the global existence toward negative time of solution for the Cauchy problem (1.1)-(1.2) with $u_0 \in H^s$, $1 > s > 10/11$. The global solvability in negative time is not obvious for $1 > s > 10/11$ though it is easy to prove for $s = 1$. Our proof shows that solutions are spatially localized for the damped and forced Zakharov-Kuznetsov equation. In this respect, our proof is more direct than J. M. Ball's argument.

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