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Stability of Hyperbolic-Parabolic Mixed Type Equations

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Abstract. In this paper, we are concerned with entropy solution of hyperbolic-parabolic mixed type equations. Although we can define its trace on the boundary $\partial\Omega$, the Dirichlet boundary value condition may be overdetermined. The main feature which distinguishes this paper from other related works lies in the fact that the stability of weak solution is established based on the partial boundary value condition for a more general case where the convection term $\vec{b}$ is dependent on u, x and t. We also show that in some special cases, the stability of weak solution can be proved without any boundary condition.

Contents

1. Introduction

Consider the hyperbolic-parabolic mixed type equation [**1, 2**]:

$$\frac{\partial u}{\partial t} = \mathrm{div}(a(u,x,t)\nabla u) + \mathrm{div}(\vec{b}(u,x,t)), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

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where $\vec{b} = \{b_i\}$, and $\Omega \subset \mathbb{R}^N$ is an open bounded domain with a C^2 smooth boundary $\partial\Omega$. Throughout the paper, we denote

$$A(u,x,t) = \int_0^u a(s,x,t)ds,$$

and assume that

$$a(\cdot,x,t) \geq 0. \tag{1.3}$$

Condition (1.3) implies that the equation may be degenerate in the interior of Ω. How to establish the well-posedness theory of equation (1.1) has been an interesting and long-standing problem. The history can be traced back to the Fichera-Oleinik theory on the second order equations with the nonnegative characteristic form [**3, 4, 5, 6**]. Let us briefly review the Fichera-Oleinik theory. For a linear degenerate elliptic equation [**3, 6**]:

$$\sum_{r,s=1}^{N+1} a^{rs}(x)\frac{\partial^2 u}{\partial x_r \partial x_s} + \sum_{r=1}^{N+1} b_r(x)\frac{\partial u}{\partial x_r} + c(x)u = f(x), \ \ x \in \widetilde{\Omega} \subset \mathbb{R}^{N+1}, \tag{1.4}$$

the symmetric matrix $(a^{rs}(x))$ has nonnegative characteristic values. To study the well-posedness problem, one may give a partial boundary condition [**2**]. Specifically, let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote

$$\Sigma_2 = \left\{x \in \partial\widetilde{\Omega} : a^{rs}n_r n_s = 0, \ (b_r - a^{rs}_{x_s})n_r < 0\right\},$$

$$\Sigma_3 = \left\{x \in \partial\widetilde{\Omega} : a^{rs}n_s n_r > 0\right\}.$$

Throughout this paper, when we mention the double indices r and s, it always means from 1 to $N+1$, and the double indices i and j means from 1 to N. To ensure the well-posedness of solutions to equation (1.4), according to the Fichera-Oleinik theory, we can find a suitable boundary value condition as

$$u|_{\sum_2 \bigcup \sum_3} = g(x). \tag{1.5}$$

In particular, if the matrix $(a^{rs}(x))$ is positive definite, condition (1.5) actually is the Dirichlet boundary condition. However, in view of equation (1.1), it is nonlinear and may be degenerate, including the extreme case of $a(u,x,t) \equiv 0$, so the Fichera-Oleinik theory becomes invalid, and the corresponding problem becomes much complicated. In the case of $a(u,x,t) \equiv 0$, equation (1.1) actually reduces to a first-order hyperbolic equation, and it is well-known that a smooth solution is constant along the maximal segment of the characteristic line in Q_T. When this segment intersects both $\{0\} \times \Omega$ and $\partial\Omega$, the Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.6}$$

is overdetermined if condition (1.6) is fulfilled in the sense of the trace. Thus, instead of the usual homogeneous boundary condition (1.6), we may conjecture that only a partial boundary condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{1.7}$$

should be imposed, where $\Sigma_p \subseteq \partial\Omega$ is a relatively open subset. In the previous works [**7, 8**], Zhan etc studied equation (1.1) where the convection term $\vec{b}$ is independent of x and t. Much earlier, Wu-Zhao [**9, 10**] had considered an interesting case on a special kind of domain in $\mathbb{R}^N$.

In this paper, we consider equation (1.1) with a more general convection term and will present an explicit formula of Σ_p by proposing a novel theoretical idea, and establish the stability of weak solutions according to classifications of boundary value conditions. In the past decades continuous attention has been devoted to the well-posedness and stability of equation (1.1) with boundary value condition (1.6) in a weaker sense than the traditional trace. One can refer to Chen-DiBenedetto [**11**], Clendenen et al [**12**], Carrillo [**13**], Bendahmane-Karlsen [**14**], Andreianov et al [**15**], Karlsen-Ulusoy [**16**], Li-Wang [**17**], Mascia et al [**18**], Michel-Vovelle [**19**], Hao-Zhu [**20**] and references therein.

The rest of the paper is organized as follows. In Section 2, we present the definition of entropy solution of equation (1.1) and summarize our main results. In Section 3, we consider the existence of weak solution of an associated regularized problem. In Section 4, we prove the stability of entropy solution of equation (1.1) without any boundary value condition. Section 5 is dedicated to the stability of entropy solution of equation (1.1) with a partial boundary value condition. At the end of the paper, an appendix on entropy solution is provided.

2. Entropy Solution and Main Results

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau \quad \text{and} \quad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{\mid s \mid}{\eta}\right)_+.$$

Clearly, $h_\eta(s) \in C(\mathbb{R})$, and we have

$$h_\eta(s) \geq 0, \qquad \mid s h_\eta(s) \mid \leq 1, \qquad \mid S_\eta(s) \mid \leq 1,$$

and

$$\lim_{\eta\to 0} S_\eta(s) = \text{sgn} s, \qquad \lim_{\eta\to 0} sS'_\eta(s) = 0.$$

Definition 2.1. A function $u \in BV(Q_T) \cap L^\infty(Q_T)$ is said to be an entropy solution of equation (1.1) with the initial condition (1.2), provided that

1. There exist $g^i \in L^2(Q_T)$ $(i = 1, 2, \cdots, N)$ such that for any $\varphi(x,t) \in C_0^1(Q_T)$ there holds

$$\iint_{Q_T} \varphi(x,t) g^i(x,t)dxdt = \int\int_{Q_T} \varphi(x,t)\widehat{\sqrt{a(u,x,t)}}\frac{\partial u}{\partial x_i}dxdt,$$

where

$$\widehat{\sqrt{a(u,x,t)}}(u,x,t) = \int_0^1 \sqrt{a(\tau u^+ + (1-\tau)u^-, x, t)}d\tau,$$

which is called the composite mean value of $\sqrt{a(u,x,t)}$. Here, u^+ and u^- are the approximate limits of the BV function u.

2. If $\varphi \in C_0^2(Q_T)$ and $\varphi \geq 0$, for $k \in \mathbb{R}$ and for any small $\eta > 0$ there holds

$$\begin{aligned}&\iint_{Q_T} \Big[I_\eta(u-k)\varphi_t - B_\eta^i(u,x,t,k)\varphi_{x_i} \\ &+ A_\eta(u,x,t,k)\Delta\varphi - \sum_{i=1}^{N} S'_\eta(u-k) \mid g^i \mid^2 \varphi \Big] dxdt \\ &+ \iint_{Q_T} \int_k^u a_{x_i}(s,x,t) S_\eta(s-k) ds \varphi_{x_i} dxdt \\ &+ \iint_{Q_T} \int_k^u b_{ix_i} S'_\eta(s-k) ds \varphi dxdt \geq 0, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} B_\eta^i(u,x,t,k) &= \int_k^u \frac{\partial b_i(s,x,t)}{\partial s} S_\eta(s-k) ds, \\ A_\eta(u,x,t,k) &= \int_k^u a(s,x,t) S_\eta(s-k) ds, \end{aligned}$$

for $i \in \{1, 2, \cdots, N\}$, and

$$I_\eta(u-k) = \int_0^{u-k} S_\eta(s) ds.$$

3. The initial value is satisfied in the sense of

$$\lim_{t \to 0} \int_\Omega \mid u(x,t) - u_0(x) \mid dx = 0. \tag{2.2}$$

There are a number of applications of entropy solutions in linear and nonlinear degenerate hyperbolic systems [**21, 22**]. A justification on Definition 2.1 is given in the Appendix.

Let us summarize our main results as follows.

THEOREM 2.2. *Suppose that $A(r,x,t) \in C^3$, $b_i(r,x,t) \in C^2$, $u_0(x) \in L^\infty(\Omega)$, and there is a constant $\delta_1 > 0$ such that*

$$a(r,x,t) - \delta_1 \sum_{s=1}^{N+1} (a_{x_s}(r,x,t))^2 \geq 0. \tag{2.3}$$

Then equation (1.1) with the initial condition (1.2) has an entropy solution in the sense of Definition 2.1, provided that

$$a(0,x,t) = 0, \ \ (x,t) \in Q_T \tag{2.4}$$

for $i = 1, 2, \cdots, N$.

THEOREM 2.3. *Suppose that $A(s,x,t)$ is a $C^2(\mathbb{R} \times Q_T)$ function and $b_i(s,x,t)$ is a $C^1(\mathbb{R} \times Q_T)$ function. Suppose that there exists a constant $\delta_2 > 0$ such that*

$$\left| \sqrt{a(u,x,t)} - \sqrt{a(u,y,\tau)} \right| \leq C \mid x - y \mid^{2+\delta_2}, \tag{2.5}$$

where $C = C\left(\|u\|_{L^\infty}, \|v\|_{L^\infty}, T\right)$. If $u(x,t)$ and $v(x,t)$ are two solutions of equation (1.1) with the different initial values $u_0(x)$ and $v_0(x)$ respectively, and with the same homogeneous boundary condition

$$u(x,t) = v(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,T), \tag{2.6}$$

then we have

$$\int_\Omega \mid u(x,t)-v(x,t)\mid dx \le c\int_\Omega \mid u_0(x)-v_0(x)\mid dx, \tag{2.7}$$

where c is a positive constant.

Apparently, there are many types of the function $a(u,x,t)$ satisfying condition (2.5). Here we give a simple example of $a(u,x,t)$ satisfying condition (2.5):

$$a(u,x,t)=\left[f(u,t)x^{2+\delta_2}\right]^2,$$

where $f(s,t)$ is a smooth nonnegative function. Moreover, if $f(0,t)=0$, then $a(u,x,t)$ satisfies both conditions (2.4) and (2.5).

However, since the equation may be strongly degenerate, the usual Dirichlet homogeneous value condition (2.6) may be overdetermined. As we had conjectured, in such a case only a partial boundary value condition (1.7) might be imposed. In the following theorem, our aim is to depict the geometric characteristic of Σ_p, to establish the stability of weak solutions based on the partial boundary value condition (1.7).

THEOREM 2.4. *Suppose that $A(s,x,t)$ is a $C^2(\mathbb{R}\times Q_T)$ function and $b_i(s,x,t)$ is a $C^1(\mathbb{R}\times Q_T)$ function. Suppose that condition (2.5) holds and there exist smooth functions $c_A(x)$ and $c_{b_i}(x)>0$ $(i\in\{1,2,\cdots,N\})$ such that*

$$|A(u,x,t)-A(v,x,t)|\le c_A(x)|u-v|, \tag{2.8}$$

$$|b_i(u,x,t)-b_i(v,x,t)|\le c_{b_i}(x)|u-v|. \tag{2.9}$$

If $u(x,t)$ and $v(x,t)$ are two solutions of equation (1.1) with the different initial values $u_0(x)$ and $v_0(x)$ respectively, and with the same homogeneous boundary condition

$$u(x,t)=v(x,t)=0,\ \ (x,t)\in\Sigma_p\times(0,T), \tag{2.10}$$

then the stability (2.7) is true, where the partial boundary is given by

$$\Sigma_p=\left\{x\in\partial\Omega : c_A(x)\Delta d+\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))>0\right\}, \tag{2.11}$$

where $d=dist(x,\partial\Omega)$ is the distance function from the boundary.

The following theorem is regarding the stability of entropy solution of equation (1.1) without any boundary value condition.

THEOREM 2.5. *Suppose that $A(s,x,t)$ is a $C^2(\mathbb{R}\times Q_T)$ function and $b_i(s,x,t)$ is a $C^1(\mathbb{R}\times Q_T)$ function. Suppose that condition (2.5) holds and there exists another constant $\delta_3>0$ such that*

$$|a_{x_i}(\cdot,x,t)|\le cd^{\delta_3}(x),\quad |A(u,x,t)-A(v,x,t)|\le cd(x), \tag{2.12}$$

and

$$|b_i(\cdot,x,t)\le cd(x). \tag{2.13}$$

If $u(x,t)$ and $v(x,t)$ are two solutions of equation (1.1) with the different initial values $u_0(x)$ and $v_0(x)\in L^\infty(\Omega)$ respectively, but without any boundary value condition, then the stability (2.7) is true.

REMARK 2.6. In [**8**], we once studied the stability of entropy solution under the partial boundary condition for the hyperbolic-parabolic mixed type equation with the convection term $\vec{b}$ independent of x and t, but the obtained stability (formula (22), [**8**]) looks relatively stronger and complicated. In this paper we extend the method used in [**8**] and propose a novel analytical idea to prove the stability of entropy solution of equation (1.1) without any boundary value condition. As we have seen, the present formula (2.7) appears neat and succinct.

3. Existence of Weak Solution

For the sake of convenience, we denote by Γ_u the set of all jump points of $u \in BV(Q_T)$ [**1, 23**]. Let v represent the normal of Γ_u at $X = (x,t)$, and $u^+(X)$ and $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(v, Y-X) > 0$ and $(v, Y-X) < 0$, respectively. For the continuous functions $p(u,x,t)$ and $u \in BV(Q_T)$, we define

$$\widehat{p}(u,x,t) = \int_0^1 p(\tau u^+ + (1-\tau)u^-, x, t)d\tau,$$

which is called the composite mean value of p. If $f(s) \in C^1(\mathbb{R})$ and $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \quad i = 1, 2, \cdots, N, N+1,$$

where $x_{N+1} = t$.

LEMMA 3.1. *Let u be a solution of equation (1.1). Then we have*

$$a(r,x,t) = 0, \quad r \in I(u^+(x,t), u^-(x,t)) \ \text{a.e. on } \Gamma_u, \tag{3.1}$$

where $I(\alpha,\beta)$ denotes the closed interval with endpoints α and β, and (3.1) is true in the sense of the Hausdorff measure $H_N(\Gamma_u)$.

This lemma can be proved in an analogous manner as shown in [**8**], so we omit details here. The following lemma plays a crucial role in the proof of Theorem 2.2.

LEMMA 3.2. [**24**] *Assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set and f_k, $f \in L^q(\Omega)$, as $k \to \infty$, $f_k \rightharpoonup f$ weakly in $L^q(\Omega)$ $(1 \le q < \infty)$. Then we have*

$$\lim_{k\to\infty}\inf \ \| f_k \|^q_{L^q(\Omega)} \ge \| f \|^q_{L^q(\Omega)} .$$

We start with the existence of weak solution of a related regularized problem:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a(u,x,t)\frac{\partial u}{\partial x_i}\right) + \varepsilon\Delta u + \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{3.2}$$

with the initial-boundary conditions

$$u(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{3.3}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.4}$$

where $u_{0\varepsilon}(x)$ is a mollified function of u_0. We know that there exists a classical solution $u_\varepsilon \in C^2(\overline{Q_T}) \bigcap C^3(Q_T)$ satisfying

$$| u_\varepsilon | \le \| u_0 \|_{L^\infty} \le c, \tag{3.5}$$

provided that both $a(u,x,t)$ and $b_i(u,x,t)$ satisfy the assumptions described in Theorem 2.2. For details, one can refer to [**1**] or Chapter 8 of [**2**].

Lemma 3.3. [**9**] *Let u_ε be the solution of equation (3.2) with the initial-boundary conditions (3.3)-(3.4). If the assumptions given in Theorem 2.2 hold, then we have*

$$\varepsilon\int_\Sigma\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma\le c_1+c_2\left(|\nabla u_\varepsilon|_{L^1(\Omega)}+\left|\frac{\partial u_\varepsilon}{\partial t}\right|_{L^1(\Omega)}\right),\tag{3.6}$$

where the constants c_i $(i=1,2)$ are independent of ε, and n is the outer normal vector of Ω.

Lemma 3.4. *Let u_ε be the solution of the problem (3.2)-(3.4). If the assumptions given in Theorem 2.2 hold, then we have*

$$|gradu_\varepsilon|_{L^1(\Omega)}\le c,\tag{3.7}$$

where c is independent of ε, and

$$|gradu_\varepsilon|^2=\sum_{i=1}^N\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^2+\left|\frac{\partial u}{\partial t}\right|^2.$$

Lemma 3.4 can be proved in a similar manner as shown in [**8**].
Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. In view of Lemma 3.2, by multiplying both sides of equation (3.2) with u_ε, it is easy to see that

$$\iint_{Q_T}a(u_\varepsilon,x,t)|\nabla u_\varepsilon|^2dxdt\le c.\tag{3.8}$$

Then, $\sqrt{a(u_\varepsilon,x,t)}\frac{\partial u_\varepsilon}{\partial x_i}$ is weakly compact in $L^2(Q_T)$. By choosing a subsequence, for convenience of notations, still denoted by $\sqrt{a(u_\varepsilon,x,t)}\frac{\partial u_\varepsilon}{\partial x_i}$, we are able to show that

$$\sqrt{a(u_\varepsilon,x,t)}\frac{\partial u_\varepsilon}{\partial x_i}\rightharpoonup\widehat{\sqrt{a(u,x,t)}}\frac{\partial u}{\partial x_i},\ \text{in } L^2(Q_T),$$

where u satisfies Part (1) of Definition 2.1.

Let $\varphi\in C_0^2(Q_T)$, $\varphi\ge0$, and $\{n_i\}$ be the inner normal vector of Ω. Multiplying both sides of equation (3.2) by $\varphi S_\eta(u_\varepsilon-k)$ and integrating it over Q_T yields

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi S_\eta(u_\varepsilon-k)dxdt\\ =&\iint_{Q_T}\frac{\partial}{\partial x_i}\left(a(u_\varepsilon,x,t)\frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi S_\eta(u_\varepsilon-k)dxdt\\ &+\varepsilon\iint_{Q_T}\Delta u_\varepsilon\varphi S_\eta(u_\varepsilon-k)dxdt+\iint_{Q_T}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\varphi S_\eta(u_\varepsilon-k)dxdt.\end{aligned}\tag{3.9}$$

From (3.9), using integration by parts gives

$$
\begin{aligned}
&\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_t dxdt + \iint_{Q_T} A_\eta(u_\varepsilon, x, t, k)\triangle\varphi dxdt \\
&- \iint_{Q_T} B^i_\eta(u_\varepsilon, x, t, k)\varphi_{x_i} dxdt - \varepsilon \iint_{Q_T} \nabla u_\varepsilon \cdot \nabla\varphi S_\eta(u_\varepsilon - k) dxdt \\
&-\varepsilon \iint_{Q_T} \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi dxdt \\
&+ \iint_{Q_T} \int_k^{u_\varepsilon} a_{x_i}(s, x, t) S_\eta(s - k) ds \varphi_{x_i} dxdt \\
&- \iint_{Q_T} a(u_\varepsilon, x, t) \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi dxdt \\
&+ \iint_{Q_T} \int_k^{u_\varepsilon} b_{ix_i}(u_\varepsilon, x, t) S'_\eta(s - k) ds \varphi dxdt = 0.
\end{aligned}
$$

By virtue of Lemma 3.2, we obtain

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} \iint_{Q_T} S'_\eta(u_\varepsilon - k) a(u_\varepsilon, x, t) \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i}\varphi dxdt \\
&\geq \sum_{i=1}^N \iint_{Q_T} S'_\eta(u - k) \mid g^i \mid^2 \varphi dxdt.
\end{aligned}
\tag{3.10}
$$

Letting $\varepsilon \to 0$ in (3.10), we arrive at (2.1). By virtue of Lemmas 3.3 and 3.4, the proof of inequality (2.2) can be carried out in a closely similar manner as the one given in [**8, 1**], so we omit it. □

4. Proof of Theorem 2.5

We once explored the stability of entropy solution under a partial boundary condition for the hyperbolic-parabolic mixed type equation in the case where the convection term $\vec{b}$ is independent of x and t, but the established stability (formula (22), [**8**]) looks quite stronger and complicated. In this section, we develop the method described in [**8**] and introduce a novel analytical idea to prove the stability of entropy solution of equation (1.1) without any boundary value condition.

Proof of Theorem 2.5. Let $u(x,t)$ and $v(x,t)$ be two entropy solutions of equation (1.1) with the initial condition:

$$
u(x, 0) = u_0(x) \text{ and } v(x, 0) = v_0(x),
$$

respectively, but without any boundary condition.

In view of Definition 2.1, for any $\varphi \in C_0^2(Q_T)$, we have

$$
\begin{aligned}
&\iint_{Q_T} \left[I_\eta(u - k)\varphi_t + A_\eta(u, x, t, k)\Delta\varphi - \sum_{i=1}^N S'_\eta(u - k) \mid g^i \mid^2 \varphi \right] dxdt \\
&+ \iint_{Q_T} \int_k^u a_{x_i}(s, x, t) S_\eta(s - k) ds \varphi_{x_i} dxdt - \iint_{Q_T} B^i_\eta(u, x, t, k)\varphi_{x_i} dxdt \\
&+ \iint_{Q_T} \int_k^u b_{ix_i}(s, x, t) S'_\eta(s - k) ds \varphi dxdt \geq 0,
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&\iint_{Q_T}\left[I_\eta(v-l)\varphi_\tau+A_\eta(v,y,\tau,l)\Delta\varphi-\sum_{i=1}^{N}S'_\eta(v-l)\mid g^i\mid^2\varphi\right]dyd\tau\\
&+\iint_{Q_T}\int_l^v a_{y_i}(s,y,\tau)S_\eta(s-l)ds\varphi_{y_i}dyd\tau-\iint_{Q_T}B^i_\eta(v,y,\tau,l)\varphi_{y_i}dyd\tau\\
&+\iint_{Q_T}\int_l^v b_{iy_i}(s,y,\tau)S'_\eta(s-l)ds\varphi dyd\tau\ge 0.
\end{aligned}\tag{4.2}
$$

Let

$$\psi(x,t,y,\tau)=\phi(x,t)j_h(x-y,t-\tau),$$

with $\phi(x,t)\ge 0,\ \phi(x,t)\in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau)=\omega_h(t-\tau)\Pi_{i=1}^N\omega_h(x_i-y_i),$$

where

$$\omega_h(s)=\frac{1}{h}\omega\left(\frac{s}{h}\right),\quad \omega(s)\in C_0^\infty(R),\quad \omega(s)\ge 0$$

and $\omega(s)$ satisfies

$$\omega(s)=0 \text{ if } \mid s\mid>1 \text{ and } \int_{-\infty}^{\infty}\omega(s)ds=1.$$

Moreover, for any given positive constant δ there holds

$$\lim_{h\to 0}\omega'_h(s)s^{2+\delta}=0.\tag{4.3}$$

We choose

$$k=v(y,\tau),\ \ l=u(x,t)\ \text{ and }\ \varphi=\psi(x,t,y,\tau)$$

in (4.1) and (4.2). Integrating it over Q_T yields

$$
\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\Big\{I_\eta(u-v)(\psi_t+\psi_\tau)+A_\eta(u,x,t,v)\Delta_x\psi+A_\eta(v,y,\tau,u)\Delta_y\psi\\
&+\int_v^u a_{x_i}(s,x,t)S_\eta(s-v)ds\psi_{x_i}+\int_u^v a_{y_i}(s,y,\tau)S_\eta(s-u)ds\psi_{y_i}\\
&-\left(B^i_\eta(u,x,t,v)\psi_{x_i}+B^i_\eta(v,y,\tau,u)\psi_{y_i}\right)\\
&+\int_k^u b_{ix_i}(s,x,t)S'_\eta(s-k)ds+\int_l^v b_{iy_i}(s,y,\tau)S'_\eta(s-l)ds\\
&-\sum_{i=1}^{N}S'_\eta(u-v)\left(\mid g^i(u,x,t)\mid^2+\mid g^i(v,y,\tau)\mid^2\right)\psi\Big\}dxdtdyd\tau\ge 0.
\end{aligned}\tag{4.4}
$$

By a straightforward calculation, we deduce that

$$
\begin{aligned}
&\frac{\partial j_h}{\partial t}+\frac{\partial j_h}{\partial \tau}=0,\\
&\frac{\partial j_h}{\partial x_i}+\frac{\partial j_h}{\partial y_i}=0,\\
&\frac{\partial \psi}{\partial t}+\frac{\partial \psi}{\partial \tau}=\frac{\partial \phi}{\partial t}j_h,
\end{aligned}
$$

and

$$\frac{\partial\psi}{\partial x_i}+\frac{\partial\psi}{\partial y_i}=\frac{\partial\phi}{\partial x_i}j_h,$$

for $i=1,\cdots,N$, and

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\psi_{y_i}dxdtdyd\tau\\
=\;&-\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds(\phi j_h)_{x_i}dxdtdyd\tau\\
&+\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi_{x_i}j_hdxdtdyd\tau\\
=\;&-\iint_{Q_T}\int_\Sigma\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi j_hn_idxdtdyd\tau\\
&-\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_hdxdtdyd\tau\\
&+\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi_{x_i}j_hdxdtdyd\tau\\
=\;&-\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_hdxdtdyd\tau\\
&+\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(v,y,\tau)S_\eta(s-u)ds\phi_{x_i}j_hdxdtdyd\tau.
\end{aligned}\tag{4.5}$$

It follows from (4.4) that

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\Big\{I_\eta(u-v)(\psi_t+\psi_\tau)+A_\eta(u,x,t,v)\psi_{x_ix_i}+A_\eta(v,y,\tau,u)\psi_{y_iy_i}\\
&-\sum_{i=1}^N S'_\eta(u-v)\left(|\,g^i(u,x,t)\,|^2+|\,g^i(v,y,\tau)\,|^2\right)\psi\\
&+\int_v^u a_{x_i}S_\eta(s-v)ds\phi_{x_i}j_h+\int_u^v a_{y_i}S_\eta(s-u)ds\phi_{x_i}j_h\\
&+\int_k^u b_{ix_i}(s,x,t)S'_\eta(s-k)ds\psi+\int_l^v b_{iy_i}(s,y,\tau)S'_\eta(s-l)ds\psi\\
&+\int_v^u a_{x_i}S_\eta(s-v)ds\phi j_{hx_i}-\int_u^v a_{y_i}S'_\eta(s-u)dsu_{x_i}\phi j_h\\
&-\left(B^i_\eta(u,x,t,v)\psi_{x_i}+B^i_\eta(v,y,\tau,u)\psi_{y_i}\right)\Big\}dxdtdyd\tau\geq 0.
\end{aligned}\tag{4.6}$$

For simplicity, we denote

$$I_3 = \iint_{Q_T} \iint_{Q_T} [A_\eta(u,x,t,v)\psi_{x_ix_i} + A_\eta(v,y,\tau,u)\psi_{y_iy_i}]dxdtdyd\tau,$$

$$I_4 = -\sum_{i=1}^{N} \iint_{Q_T} \iint_{Q_T} S'_\eta(u-v)\left(\mid g^i(u,x,t)\mid^2 + \mid g^i(v,y,\tau)\mid^2\right)\psi dxdtdyd\tau,$$

$$I_5 = \iint_{Q_T} \iint_{Q_T} \int_v^u a_{x_i} S_\eta(s-v)ds\phi_{x_i} j_h dxdtdyd\tau,$$

$$I_6 = \iint_{Q_T} \iint_{Q_T} \int_u^v a_{y_i} S_\eta(s-u)ds\phi_{x_i} j_h dxdtdyd\tau,$$

$$I_7 = \iint_{Q_T} \iint_{Q_T} \int_v^u a_{x_i} S_\eta(s-v)ds\phi j_{hx_i} dxdtdyd\tau,$$

$$I_8 = -\iint_{Q_T} \iint_{Q_T} \int_u^v a_{y_i} S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau.$$

Specifically, by evaluating I_3, it gives

$$\begin{aligned}
I_3 \quad &= \quad \iint_{Q_T} \iint_{Q_T} [A_\eta(u,x,t,v)(\phi_{x_ix_i} j_h + 2\phi_{x_i} j_{hx_i} + \phi j_{hx_ix_i}) \\
&\quad\quad +A_\eta(v,y,\tau,u)\phi j_{hy_iy_i}]dxdtdyd\tau \\
&= \quad \iint_{Q_T} \iint_{Q_T} [A_\eta(u,x,t,v)(\phi_{x_ix_i} j_h + \phi_{x_i} j_{hx_i}) - A_\eta(v,y,\tau,u)\phi_{x_i} j_{hx_i} \\
&\quad\quad -\partial_{x_i} A_\eta(u,x,t,v)\phi j_{hx_i} - \partial_{x_i} A_\eta(v,y,\tau,u)\phi j_{hx_i}]dxdydtd\tau \\
&= \quad I_{31} + I_{32} - I_7, \qquad (4.7)
\end{aligned}$$

where

$$\begin{aligned}
I_{31} \quad &= \quad \iint_{Q_T} \iint_{Q_T} [A_\eta(u,x,t,v)(\phi_{x_ix_i} j_h + \phi_{x_i} j_{hx_i}) \\
&\quad\quad -A_\eta(v,y,\tau,u)\phi_{x_i} j_{hx_i}]dxdtdyd\tau, \\
I_{32} \quad &= \quad -\iint_{Q_T} \iint_{Q_T} \Big[\int_0^1 a(\sigma u^+ + (1-\sigma)u^-, x, t)S_\eta(\sigma u^+ + (1-\sigma)u^- - v)d\sigma \\
&\quad\quad +\int_0^1 \int_{\sigma u^+ + (1-\sigma)u^-}^{v} a(s,y,\tau)S_\eta(s - \sigma u^+ \\
&\quad\quad -(1-\sigma)u^-)d\sigma ds\Big] \frac{\partial u}{\partial x_i}\phi j_{hx_i} dxdtdyd\tau.
\end{aligned}$$

By evaluating $-I_4$, we get

$$\begin{aligned}
-I_4 \quad &= \quad \iint_{Q_T} \iint_{Q_T} S'_\eta(u-v)\left(\mid g^i(u,x,t)\mid^2 + \mid g^i(v,y,\tau)\mid^2\right)\psi dxdtdyd\tau \\
&= \quad \iint_{Q_T} \iint_{Q_T} S'_\eta(u-v)(\mid g^i(u,x,t)\mid - \mid g^i(v,y,\tau)\mid)^2 \psi dxdtdyd\tau \\
&\quad\quad +2\iint_{Q_T} \iint_{Q_T} S'_\eta(u-v)g^i(u,x,t)g^i(v,y,\tau)\psi dxdtdyd\tau \\
&= \quad I_{41} + I_{42}, \qquad (4.8)
\end{aligned}$$

where

$$\begin{aligned} I_{41} &= \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)(\mid g^i(u,x,t)\mid - \mid g^i(v,y,\tau)\mid)^2\psi dxdtdyd\tau, \\ I_{42} &= 2\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)g^i(u,x,t)g^i(v,y,\tau)\psi dxdtdyd\tau. \end{aligned}$$

By (2.5), (3.1) and (4.3), a direct calculation gives

$$\begin{aligned} &\lim_{h\to 0}\lim_{\eta\to 0}(I_{32}+I_{42}) \\ &= 2\lim_{h\to 0}\iint_{Q_T}\iint_{Q_T}\sqrt{a_{y_i}(u,y,\tau)}\mathrm{sgn}(v-u)\sqrt{a(u,x,t)}u_{x_i}\phi j_h dxdtdyd\tau \\ &= 2\iint_{Q_T}\left(\sqrt{a(u,x,t)}\right)_{x_i}\sqrt{a(u,x,t)}\mathrm{sgn}(v-u)u_{x_i}\phi dxdt, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} I_8 &= -\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(s,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau \\ &= -\iint_{Q_T}\iint_{Q_T}\int_u^v [a_{y_i}(s,y,\tau)-a_{y_i}(u,y,\tau)]S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau \\ &\quad -\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(u,y,\tau)S'_\eta(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau. \end{aligned} \tag{4.10}$$

Thus, we know

$$\begin{aligned} I_8 &\to -\iint_{Q_T}\iint_{Q_T}\int_u^v a_{y_i}(u,y,\tau)\mathrm{sgn}(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau, \text{ as } \eta\to 0 \\ &= -2\iint_{Q_T}\iint_{Q_T}\int_u^v \sqrt{a(u,y,\tau)}(\sqrt{a(u,y,\tau)})_{y_i}\mathrm{sgn}(s-u)dsu_{x_i}\phi j_h dxdtdyd\tau \\ &\to -2\iint_{Q_T}\int_u^v \sqrt{a(u,x,t)}(\sqrt{a(u,x,t)})_{x_i}\mathrm{sgn}(v-u)dsu_{x_i}\phi dxdt, \quad \text{as } h\to 0 \\ &= -\lim_{h\to 0}\lim_{\eta\to 0}(I_{32}+I_{42}). \end{aligned} \tag{4.11}$$

Notice that

$$\begin{aligned} \lim_{\eta\to 0}A_\eta(u,x,t,v) &= \lim_{\eta\to 0}A_\eta(v,y,\tau,u) \\ &= \mathrm{sgn}(u-v)(A(u,x,t)-A(v,y,\tau)). \end{aligned}$$

So it implies that

$$\lim_{\eta\to 0}(A_\eta(u,x,t,v)\phi_{x_i}j_{hx_i} - A_\eta(u,y,\tau,v)\phi_{x_i}j_{hx_i}) = 0. \tag{4.12}$$

Meanwhile, we have

$$\begin{aligned} &\lim_{\eta\to 0}B^i_\eta(u,x,t,v) = \mathrm{sgn}(u-v)(b_i(u,x,t)-b_i(v,x,t)), \\ &\lim_{\eta\to 0}B^i_\eta(v,y,\tau,u) = \mathrm{sgn}(v-u)(b_i(v,y,\tau)-b_i(u,y,\tau)), \end{aligned}$$

as $\eta \to 0$, and

$$\begin{aligned}
&\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T}[B^i_\eta(u,x,t,v)\psi_{x_i}+B^i_\eta(v,y,\tau,u)\psi_{y_i}]dxdtdyd\tau\\
=\ &\iint_{Q_T}\iint_{Q_T}\operatorname{sgn}(u-v)[b_i(u,x,t)-b_i(v,y,\tau)]\phi_{x_i}j_h dxdtdyd\tau\\
&+\iint_{Q_T}\iint_{Q_T}\operatorname{sgn}(u-v)[b_i(v,y,\tau)-b_i(v,x,t)]\phi_{x_i}j_h dxdtdyd\tau,
\end{aligned}$$

as $h \to 0$. Thus, we further get

$$\begin{aligned}
&\lim_{h\to 0}\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T}[B^i_\eta(u,x,t,v)\psi_{x_i}+B^i_\eta(v,y,\tau,u)\psi_{y_i}]dxdtdyd\tau\\
&=\iint_{Q_T}\operatorname{sgn}(u-v)[b_i(u,x,t)-b_i(v,x,t)]\phi_{x_i}dxdt. \qquad (4.13)
\end{aligned}$$

In view of (4.7)-(4.13), letting $\eta \to 0$ and $h \to 0$ in (4.12) leads to

$$\begin{aligned}
&\iint_{Q_T}\Big\{\mid u(x,t)-v(x,t)\mid\phi_t+\operatorname{sgn}(u-v)(A(u,x,t)-A(v,x,t))\Delta\phi\\
&+\int_v^u a_{x_i}(s,x,t)\operatorname{sgn}(s-v)ds\phi_{x_i}+\int_u^v a_{x_i}(s,x,t)\operatorname{sgn}(s-u)ds\phi_{x_i}\\
&-\operatorname{sgn}(u-v)(b_i(u,x,t)-b_i(v,x,t))\phi_{x_i}\\
&+b_{ix_i}(v,x,t)\operatorname{sgn}(u-v)\phi+b_{ix_i}(u,x,t)\operatorname{sgn}(v-u)\phi\Big\}dxdt\geq 0. \qquad (4.14)
\end{aligned}$$

We now choose ϕ in (4.14) as

$$\phi(x,t)=\omega_\lambda(x)\eta(t),$$

where $\eta(t)\in C_0^\infty(0,T)$ and $\omega_\lambda(x)\in C_0^1(\Omega)$ are defined as follows: for any given small enough $\lambda>0$, let $0\leq\omega_\lambda\leq 1$ and $\omega_\lambda|_{\partial\Omega}=0$. If $d(x)=\operatorname{dist}(x,\partial\Omega)\geq\lambda$, then

$$\omega_\lambda(x)=1.$$

If $0\leq d(x)\leq\lambda$, then

$$\omega_\lambda(d(x))=1-\frac{(d(x)-\lambda)^2}{\lambda^2}.$$

We denote

$$\Omega_\lambda=\{x\in\Omega, d(x,\partial\Omega)<\lambda\},$$

where λ is a small enough positive constant. By the fact that $|\nabla d|=1$, and

$$|\Delta d|\leq c, x\in\Omega_\lambda,$$

then we have

$$\begin{aligned}
&\int_\Omega\operatorname{sgn}(u-v)(A(u,x,t)-A(v,x,t))\Delta\phi dx\\
=\ &-2\eta(t)\int_{\Omega_\lambda}|A(u,x,t)-A(v,x,t)|\left(\frac{|\nabla d|^2}{\lambda^2}+\frac{d-\lambda}{\lambda^2}\Delta d\right)dx\\
\leq\ &\frac{2\eta(t)}{\lambda}\int_{\Omega_\lambda}|A(u,x,t)-A(v,x,t)||\Delta d|dx. \qquad (4.15)
\end{aligned}$$

Since

$$|A(u,x,t)-A(v,x,t)|\leq cd(x),$$

we get

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{2\eta(t)}{\lambda}\int_{\Omega\setminus\Omega_\lambda}|A(u,x,t)-A(v,x,t)||\Delta d|dx\\
\leq\;& c\lim_{\lambda\to 0}\int_{\Omega\setminus\Omega_\lambda}dx\\
=\;& 0.
\end{aligned}\tag{4.16}$$

In view of $a(\cdot,x,t)\in C^1$ and $|a_{x_i}|\leq cd^{\delta_3}(x)$, it follows that

$$\begin{aligned}
&\left|\int_\Omega\int_v^u a_{x_i}(s,x,t)\mathrm{sgn}(s-v)ds\phi_{x_i}dx\right|\\
=\;&\left|\int_{\Omega_\lambda}\int_v^u a_{x_i}(s,x,t)\mathrm{sgn}(s-v)ds\phi_{x_i}dx\right|\\
\leq\;&\frac{1}{\lambda}\sum_{i=1}^N\int_{\Omega_\lambda}\left|\int_v^u a_{x_i}(s,x,t)\mathrm{sgn}(s-v)ds\right|dx\\
\leq\;& c\lambda^{\delta_3}\frac{1}{\lambda}\int_{\Omega_\lambda}|u-v|dx\\
\to\;& 0,
\end{aligned}\tag{4.17}$$

as $\lambda\to 0$.

Similarly, we can derive that

$$\lim_{\lambda\to 0}\left|\int_\Omega\int_u^v a_{x_i}(s,x,t)\mathrm{sgn}(s-u)ds\phi_{x_i}dx\right|=0.\tag{4.18}$$

Due to the fact of $|b_i(\cdot,x,t)|\leq cd(x)$, we find

$$\left|\int_\Omega sgn(u-v)(b_i(u,x,t)-b_i(v,x,t))\phi_{x_i}dx\right|\leq c\eta(t)\int_{\Omega_\lambda}dx,\tag{4.19}$$

which goes to zero as $\lambda\to 0$.

It follows the inequality $|b_{ix_i}(u,x,t)-b_{ix_i}(v,x,t)|\leq c|u-v|$ that

$$\left|\int_\Omega b_{ix_i}(v,x,t)\mathrm{sgn}(u-v)\phi-b_{ix_i}(u,x,t)\mathrm{sgn}(u-v)\phi dx\right|\leq c\eta(t)\int_\Omega|u-v|dx.\tag{4.20}$$

Let $\lambda\to 0$ in (4.14). Making use of (4.15)-(4.20), we have

$$\iint_{Q_T}\mid u(x,t)-v(x,t)\mid\eta_t'dxdt+c\iint_{Q_T}\mid u(x,t)-v(x,t)\mid\eta(t)dxdt\geq 0.\tag{4.21}$$

Let $0<s<\tau<T$, and

$$\eta(t)=\int_{\tau-t}^{s-t}\alpha_\varepsilon(\sigma)d\sigma\quad\text{and}\quad\varepsilon<\min\{\tau,T-s\},$$

where $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t)=0$ for $t\notin(-\varepsilon,\varepsilon)$.

As $\varepsilon\to 0$ in (4.21), we obtain

$$\int_\Omega|u(x,s)-v(x,s)|dx\leq\int_\Omega|u(x,\tau)-v(x,\tau)|dx+c\int_\tau^s|u(x,t)-v(x,t)|dxdt.$$

Consequently, by Gronwall's inequality, the desired result follows by letting $\tau\to 0$. $\square$

5. Proofs of Theorems 2.4 and 2.3

In this section, we show the stability of entropy solution of equation (1.1) with a partial boundary value condition.

Proof of Theorem 2.4. Let $u(x,t)$ and $v(x,t)$ be two entropy solutions of equation (1.1) with the initial condition:

$$u(x,0)=u_0(x) \quad \text{and} \quad v(x,0)=v_0(x),$$

respectively. We choose ϕ in (4.14) as

$$\phi(x,t)=\omega_\lambda(x)\eta(t),$$

where $\omega_\lambda(x)\in C_0^1(\Omega)$ and $\eta(t)\in C_0^\infty(0,T)$ are the same as defined in the preceding section. Then we have

$$\begin{aligned}&\iint_{Q_T}\Big\{\mid u(x,t)-v(x,t)\mid \omega_\lambda(x)\eta'(t)+\eta(t)|A(u,x,t)-A(v,x,t)|\Delta\omega_\lambda(x)\\&+\eta(t)\int_v^u a_{x_i}(s,x,t)\mathrm{sgn}(s-v)ds\omega_{\lambda x_i}+\eta(t)\int_u^v a_{x_i}(s,x,t)\mathrm{sgn}(s-u)ds\omega_{\lambda x_i}\\&-\eta(t)\mathrm{sgn}(u-v)(b_i(u,x,t)-b_i(v,x,t))\omega_{\lambda x_i}\\&+b_{ix_i}(v,x,t)\mathrm{sgn}(u-v)\phi+b_{ix_i}(u,x,t)\mathrm{sgn}(v-u)\eta(t)\omega_\lambda(x)\Big\}dxdt\geq 0.\end{aligned}\tag{5.1}$$

In $\Omega_\lambda=\{x\in\Omega: d(x)<\lambda\}$, we have

$$\omega_{\lambda x_i}=-2\frac{d(x)-\lambda}{\lambda^2}d_{x_i},$$

and

$$\Delta\omega_\lambda(x)=-\frac{2}{\lambda^2}|\nabla d|^2-2\frac{d(x)-\lambda}{\lambda^2}\Delta d.$$

By a straightforward calculation we deduce that

$$\begin{aligned}&\eta(t)|A(u,x,t)-A(v,x,t)|\Delta\omega_\lambda(x)\eta(t)\\\leq\;&-2\eta(t)\frac{d(x)-\lambda}{\lambda^2}\Delta d|A(u,x,t)-A(v,x,t)|\\\leq\;&\eta(t)\frac{c_A(x)}{\lambda}|u-v|\Delta d,\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}&\eta(t)\int_v^u a_{x_i}(s,x,t)\mathrm{sgn}(s-v)ds\omega_{\lambda x_i}\\&+\eta(t)\int_u^v a_{x_i}(s,x,t)\mathrm{sgn}(s-u)ds\omega_{\lambda x_i}\\&-\eta(t)\mathrm{sgn}(u-v)(b_i(u,x,t)-b_i(v,x,t))\omega_{\lambda x_i}\\\leq\;&\eta(t)\frac{\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))}{\lambda}|u-v|.\end{aligned}\tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1) leads to

$$\begin{aligned}
&\iint_{Q_T} \mid u(x,t)-v(x,t) \mid \omega_\lambda(x)\eta'(t)dxdt \\
&+\int_0^T \int_{\Omega_\lambda} \left[c_A(x)\Delta d+\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))\right]\frac{\eta(t)}{\lambda}|u-v|dxdt \\
&+\iint_{Q_T}[b_{ix_i}(v,x,t)\mathrm{sgn}(u-v)\eta(t)\omega_\lambda(x) \\
&-b_{ix_i}(u,x,t)\mathrm{sgn}(u-v)\eta(t)\omega_\lambda(x)]dxdt \\
\geq\ & 0.
\end{aligned}$$

Denote

$$\Omega_{1\lambda}=\left\{x\in\Omega_\lambda : c_A(x)\Delta d+\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))>0\right\}.$$

Then, by (2.10) and (2.11), we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_0^T \int_{\Omega_\lambda} \left[c_A(x)\Delta d+\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))\right]\frac{\eta(t)}{\lambda}|u-v|dxdt \\
\leq\ &\lim_{\lambda\to 0}\int_0^T \frac{\eta(t)}{\lambda}\int_{\Omega_{1\lambda}} \left[c_A(x)\Delta d+\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))\right]|u-v|dxdt \\
=\ &\int_0^T \int_{\Sigma_p} \left[c_A(x)\Delta d+\sum_{i=1}^N(|a_{x_i}|+c_{b_i}(x))\right]|u-v|d\Sigma dt \\
=\ & 0. \qquad (5.4)
\end{aligned}$$

Let $\lambda\to 0$ in (5.1). It follows from (5.4) that

$$\iint_{Q_T} \mid u(x,t)-v(x,t) \mid \eta_t' dxdt + c\iint_{Q_T} \mid u(x,t)-v(x,t) \mid \eta(t)dxdt \geq 0.$$

The rest discussion is closely analogous to what we do in the proof of Theorem 2.5, and the inequality (2.7) can be obtained accordingly. □

Proof of Theorem 2.3. Similar to the proof of Theorem 2.4, inequality (4.14) still holds. We then choose ϕ in (4.14) as

$$\phi(x,t)=\omega_\lambda(x)\eta(t),$$

where $\omega_\lambda(x)\in C_0^1(\Omega)$ and $\eta(t)\in C_0^\infty(0,T)$ are the same as defined in the preceding section.

By the fact $|\nabla d|=1$, we denote

$$\Omega_\lambda=\{x\in\Omega, d(x,\partial\Omega)<\lambda\} \text{ and } \ |\Delta d|\leq c.$$

Using the homogeneous boundary value condition (2.6), we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\int_\Omega \operatorname{sgn}(u-v)(A(u,x,t)-A(v,x,t))\Delta\phi dx\right| \\
= &\lim_{\lambda\to 0}\left|-2\eta(t)\int_{\Omega_\lambda}|A(u,x,t)-A(v,x,t)|\left(\frac{|\nabla d|^2}{\lambda^2}+\frac{d-\lambda}{\lambda^2}\Delta d\right)dx\right| \\
\le &\lim_{\lambda\to 0}\frac{2\eta(t)}{\lambda}\int_{\Omega_\lambda}|A(u,x,t)-A(v,x,t)||\Delta d|dx \\
\le &\lim_{\lambda\to 0}c\frac{2\eta(t)}{\lambda}\int_{\Omega_\lambda}|u-v|dx \\
= &\ 2\eta(t)\int_{\partial\Omega}|u-v|d\Sigma \\
= &\ 0. \qquad (5.5)
\end{aligned}$$

Since $a(\cdot,x,t)\in C^1$ and $|a_{x_i}|\le c$, by using the homogeneous boundary condition we can derive that

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\int_\Omega\int_v^u a_{x_i}(s,x,t)\operatorname{sgn}(s-v)ds\phi_{x_i}dx\right| \\
= &\lim_{\lambda\to 0}\left|\int_{\Omega_\lambda}\int_v^u a_{x_i}(s,x,t)\operatorname{sgn}(s-v)ds\phi_{x_i}dx\right| \\
\le &\lim_{\lambda\to 0}\frac{1}{\lambda}\sum_{i=1}^N\int_{\Omega_\lambda}\left|\int_v^u a_{x_i}(s,x,t)\operatorname{sgn}(s-v)ds\right|dx \\
\le &\ c\int_{\partial\Omega}|u-v|d\Sigma \\
= &\ 0. \qquad (5.6)
\end{aligned}$$

Similarly, we have

$$\lim_{\lambda\to 0}\left|\int_\Omega\int_u^v a_{x_i}(s,x,t)\operatorname{sgn}(s-u)ds\phi_{x_i}dx\right|=0. \tag{5.7}$$

Using the inequality $|b_i(u,x,t)-b_i(v,x,t)|\le c|u-v|$, we deduce that

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\int_\Omega sgn(u-v)(b_i(u,x,t)-b_i(v,x,t))\phi_{x_i}dx\right| \\
\le &\ c\eta(t)\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\Omega_\lambda}|u-v|dx \\
= &\ c\eta(t)\int_{\partial\Omega}|u-v|d\Sigma \\
= &\ 0. \qquad (5.8)
\end{aligned}$$

It follows the inequality $|b_{ix_i}(u,x,t)-b_{ix_i}(v,x,t)|\le c|u-v|$ that

$$\left|\int_\Omega b_{ix_i}(v,x,t)\operatorname{sgn}(u-v)\phi-b_{ix_i}(u,x,t)\operatorname{sgn}(u-v)\phi dx\right|\le c\eta(t)\int_\Omega|u-v|dx. \tag{5.9}$$

Let $\lambda\to 0$ in (4.14). By (5.5)-(5.9), we obtain

$$\iint_{Q_T}\mid u(x,t)-v(x,t)\mid\eta_t'dxdt+c\iint_{Q_T}\mid u(x,t)-v(x,t)\mid\eta(t)dxdt\ge 0.$$

The rest of the proof can be processed in a way analogous to the proof of Theorem 2.5, and the desired inequality (2.7) can be obtained accordingly. □

Appendix: entropy solution

At the end of this paper, we give a brief explanation of Definition 2.1.

Consider the regularized equation:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a(u,x,t)\frac{\partial u}{\partial x_i}\right)+\varepsilon\Delta u+\frac{\partial b_i(u,x,t)}{\partial x_i},\quad (x,t)\in Q_T. \tag{5.10}$$

Multiplying both sides of equation (5.10) by $\varphi S_\varepsilon(u_\varepsilon-k)$ and integrating it over Q_T yields

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi S_\varepsilon(u_\varepsilon-k)dxdt\\ =&\iint_{Q_T}\frac{\partial}{\partial x_i}\left(a(u_\varepsilon,x,t)\frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi S_\varepsilon(u_\varepsilon-k)dxdt\\ &+\varepsilon\iint_{Q_T}\Delta u_\varepsilon\varphi S_\varepsilon(u_\varepsilon-k)dxdt+\iint_{Q_T}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\varphi S_\varepsilon(u_\varepsilon-k)dxdt.\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}&\iint_{Q_T}I_\varepsilon(u_\varepsilon-k)\varphi_t dxdt+\iint_{Q_T}A_\varepsilon(u_\varepsilon,x,t,k)\triangle\varphi dxdt\\ &-\iint_{Q_T}B^i_\varepsilon(u_\varepsilon,x,t,k)\varphi_{x_i}dxdt-\varepsilon\iint_{Q_T}\nabla u_\varepsilon\cdot\nabla\varphi S_\varepsilon(u_\varepsilon-k)dxdt\\ &-\varepsilon\iint_{Q_T}\mid\nabla u_\varepsilon\mid^2 S'_\varepsilon(u_\varepsilon-k)\varphi dxdt\\ &+\iint_{Q_T}\int_k^{u_\varepsilon}a_{x_i}(s,x,t)S_\varepsilon(s-k)ds\varphi_{x_i}dxdt\\ &-\iint_{Q_T}a(u_\varepsilon,x,t)\mid\nabla u_\varepsilon\mid^2 S'_\varepsilon(u_\varepsilon-k)\varphi dxdt\\ &+\iint_{Q_T}\int_k^{u_\varepsilon}b_{ix_i}S'_\varepsilon(s-k)ds\varphi dxdt=0.\end{aligned}\tag{5.11}$$

Since

$$\lim_{\varepsilon\to 0}\iint_{Q_T}a(u_\varepsilon,x,t)\mid\nabla u_\varepsilon\mid^2 S'_\varepsilon(u_\varepsilon-k)\varphi dxdt$$

is difficult to be evaluated, by discarding the two terms from the left hand side of (5.11):

$$-\iint_{Q_T}a(u_\varepsilon,x,t)\mid\nabla u_\varepsilon\mid^2 S'_\varepsilon(u_\varepsilon-k)\varphi dxdt,\tag{5.12}$$

and

$$-\varepsilon\iint_{Q_T}\mid\nabla u_\varepsilon\mid^2 S'_\varepsilon(u_\varepsilon-k)\varphi dxdt,$$

we have

$$\begin{aligned}
&\iint_{Q_T} I_\varepsilon(u_\varepsilon - k)\varphi_t dxdt + \iint_{Q_T} A_\varepsilon(u_\varepsilon, x, t, k)\triangle\varphi dxdt \\
&- \iint_{Q_T} B^i_\varepsilon(u_\varepsilon, x, t, k)\varphi_{x_i} dxdt - \varepsilon \iint_{Q_T} \nabla u_\varepsilon \cdot \nabla\varphi S_\varepsilon(u_\varepsilon - k)dxdt \\
&+ \iint_{Q_T} \int_k^{u_\varepsilon} a_{x_i}(s, x, t)S_\varepsilon(s - k)ds\varphi_{x_i} dxdt \\
&+ \iint_{Q_T} \int_k^{u_\varepsilon} b_{ix_i} S'_\varepsilon(s - k)ds\varphi dxdt \\
(5.13) \quad \geq \quad & 0.
\end{aligned}$$

Let $\varepsilon \to 0$ in (5.13). Then we obtain

$$\begin{aligned}
&\iint_{Q_T} \Big[|u - k|\varphi_t - |b_i(u, x, t) - b_i(k, x, t)|\varphi_{x_i} \\
&+|A(u, x, t) - A(k, x, t)|\Delta\varphi\Big] dxdt \\
&+ \iint_{Q_T} \int_k^u a_{x_i}(s, x, t)\text{sign}(s - k)ds\varphi_{x_i} dxdt \\
&+ \iint_{Q_T} b_{ix_i}(u, x, t)\text{sign}(u - k)\varphi dxdt \geq 0. \qquad (5.14)
\end{aligned}$$

Inequality (5.14) is actually the classical entropy solution defined and used in [**1, 9, 10, 23**] etc. However, since the term (5.12) is dropped, it becomes extremely difficult to prove the uniqueness of weak solution by using the entropy inequality (5.14).

We now make a change through multiplying both sides of (5.10) by $\varphi S_\eta(u_\varepsilon - k)$ rather than by $\varphi S_\varepsilon(u_\varepsilon - k)$, where η is a small positive constant independent of ε. Note that

$$\iint_{Q_T} a(u_\varepsilon, x, t) \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi dxdt \leq c \iint_{Q_T} a(u_\varepsilon, x, t) \mid \nabla u_\varepsilon \mid^2 dxdt \leq c.$$

Hence, we can employ Lemma 3.2 to obtain (3.10), and then Definition 2.1 follows naturally. By virtue of (3.10), the uniqueness of entropy solution can be proved by the Kružkov's method [**7, 8**].

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