Dynamics of PDE, Vol.16, No.3, 225-251, 2019

Asymptotic stability of viscous shock profiles for the 1D compressible Navier-Stokes-Korteweg system with boundary effect

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Communicated by Jiahong Wu, received December 1, 2018.

Abstract. This paper is concerned with the time-asymptotic behavior of strong solutions to an initial-boundary value problem of the compressible Navier-Stokes-Korteweg system on the half line $\mathbb{R}^+$. The asymptotic profile of the problem is shown to be a shifted viscous shock profile, which is suitably away from the boundary. Moreover, we prove that if the initial data around the shifted viscous shock profile and the strength of the shifted viscous shock profile are sufficiently small, then the problem has a unique global strong solution, which tends to the shifted viscous shock profile as time goes to infinity. The analysis is based on the elementary L^2-energy method and the key point is to deal with the boundary estimates.

Contents

1991 *Mathematics Subject Classification.* (2010MSC:) Primary 35Q35, 35B40, Secondary 35L65.

Key words and phrases. Compressible Navier-Stokes-Korteweg system; Viscous shock profiles; Asymptotic stability; Boundary effect.

The authors would like to thank the anonymous referees for their valuable comments and suggestions which lead to the improvement of the presentation of this paper. The first author would like to thank Professor Huijiang Zhao for some useful discussions on the topic of this paper. Chen and Sheng was supported by the National Natural Science Foundation of China (Grant No. 11501003). Li is supported in part by the National Science Foundation of China (Grant No. 11671134).

1. Introduction

In this study, we consider the following one-dimensional compressible Navier-Stokes-Korteweg system on the half line $\mathbb{R}^+ = [0, +\infty)$ in the Lagrangian coordinates (see [**5, 7**]):

$$\begin{cases} v_t - u_x = 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u_t + p(v)_x = \mu \left(\dfrac{u_x}{v}\right)_x + \kappa \left(\dfrac{-v_{xx}}{v^5} + \dfrac{5v_x^2}{2v^6}\right)_x, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \end{cases} \tag{1.1}$$

with the initial and boundary conditions:

$$\begin{cases} (v,u)(x,0) = (v_0,u_0)(x), & x \geq 0, \\ u(0,t) = 0, \quad v_x(0,t) = 0, & t \geq 0, \\ (v,u)(+\infty,t) = (v_+,u_+), & t \geq 0. \end{cases} \tag{1.2}$$

Here the unknown functions are the specific volume $v(t,x) > 0$, the velocity $u(t,x)$ and the pressure $p = p(v(t,x))$ of the fluids. The constants $\mu > 0$ and $\kappa > 0$ are the viscosity coefficient and the capillary coefficient, respectively, and $v_+ > 0$, u_+ are given constants. The boundary conditions $u(0,t) = 0$ and $v_x(0,t) = 0$ means that the wall $x = 0$ is impermeable and there is no change of the specific volume at the boundary $x = 0$, respectively.

Throughout this paper, we assume that the pressure $p(v)$ is a positive smooth function for $v > 0$ satisfying

$$p'(v) < 0, \quad p''(v) > 0, \quad \forall\, v > 0, \tag{1.3}$$

and the initial data $u_0(x)$ and $v_0(x)$ satisfy

$$u_0(0) = 0, \quad v_{0x}(0) = 0 \tag{1.4}$$

as compatibility conditions.

The compressible Navier-Stokes-Korteweg system can be used to describe the motion of the compressible isothermal viscous fluids with internal capillarity, see the pioneering works by Van der Waals [**45**], Korteweg [**25**] and Dunn and Serrin [**11**] for the derivations on the compressible Navier-Stokes-Korteweg system. Notice that when $\kappa = 0$, the system (1.1) is reduced to the compressible Navier-Stokes system.

There have been extensive studies on the mathematical aspects of the compressible Navier-Stokes-Korteweg system. For the initial value problem, Hattori and Li [**16, 17**] proved the local existence and global existence of smooth solutions around constant states in the Sobolev space $H^s(\mathbb{R}^n)$ with $s \geq 4 + \frac{n}{2}$ and $n = 2, 3$. Since then, many authors studied the global existence and large-time behavior of strong solutions in the Sobolev space framework, cf. [**3, 4, 8, 9, 18, 30, 42, 43, 48, 49**] and the references therein. Danchin and Desjardins [**10**] and Haspot [**14, 15**] obtained the global existence and uniqueness of strong solutions in Besov space. Bian, Yao and Zhu [**2**] studied the capillarity limit of the compressible fluid models of Korteweg type to the Navier-Stokes equations. Li and Yong [**33**] discussed zero Mach number limit of the compressible Navier-Stokes-Korteweg equations. Concerning the global existence of weak solutions to the initial value problem of the Korteweg system, the readers are referred to [**10, 13**] for small initial data in the whole space $\mathbb{R}^2$, [**1**] for large initial data in a periodic domain $\mathbb{T}^d$ with $d = 2$ or 3, and [**12, 13**] for large initial data in the whole space $\mathbb{R}$.

For the initial-boundary value problem, fewer results have been obtained so far. Kotschote [**26, 27**] proved the local well-posedness of strong solutions to the compressible Korteweg system in a bounded domain of $\mathbb{R}^n$ with $n \geq 1$. Later, the global existence and exponential decay of strong solutions to the Korteweg system in a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) were obtained in [**28**]. Tsyganov [**44**] discussed the global existence and time-asymptotic behavior of weak solutions for an isothermal system on the interval $[0, 1]$. The time-asymptotic profiles in both [**28**] and [**44**] are non-constant stationary solutions.

In the one-dimensional case, the time-asymptotic nonlinear stability of some elementary waves (such as rarefaction wave, viscous shock wave and viscous contact discontinuity, etc.) to the initial value problem of the compressible Navier-Stokes-Korteweg system has been studied in [**5, 6, 7, 34**]. However, to the best of our knowledge, there is no result on the larger-time behavior of solutions for the initial-boundary value problem of the compressible Korteweg system toward these wave-like profiles. The main purpose of this paper is devoted to this problem and as a first step to this goal, we will show the large-time behavior of strong solutions of the impermeable wall problem (1.1)-(1.2) toward the viscous shock profiles. More precisely, we shall prove that the asymptotic profile of (1.1)-(1.2) is a viscous shock profile, which is suitably away from the boundary, and if the initial data is a small perturbation of the shifted viscous shock profile and the strength of the shifted viscous shock profile is sufficiently small, then the problem (1.1)-(1.2) has a unique global (in time) strong solution, which tends toward the shifted viscous shock profile as time goes to infinity. The precise statement of our main result can be found in Theorem 2.1 below.

The asymptotic nonlinear stability of viscous shock profiles for various equations from fluid mechanics in the half space $\mathbb{R}^+$ with boundary effect has been investigated by many authors. Liu and Yu [**31**] first studied the stability of the viscous shock profile for the Burgers equation with a Dirichlet boundary condition. Then such a result was extended by Liu and Nishihara [**32**] to the generalized Burgers equation. For the case of system, Matsumura and Mei [**38**] considered the stability of viscous shock wave to the p-system with viscosity and a Dirichlet boundary condition. Matsumura [**35**] gave, in 2001, a classification of the large-time behavior of the solutions in terms of the far-field state and boundary data. Kawashima, Nishibata and Zhu [**22**] investigated the asymptotic stability of the stationary solution to an outflow problem of the compressible Navier-Stokes equations in the half space. Huang, Matsumura and Shi [**20**] obtained the nonlinear stability of viscous shock wave for an inflow problem of the isentropic compressible Navier-Stokes equations. The stability of a supposition of two viscous shock waves for an initial-boundary value problem of the full compressible Navier-Stokes system was shown by Huang and Matsumura [**21**]. For the other results on the asymptotic stability of solutions with boundary effect, we refer to [**19, 23, 24, 36, 37, 40, 41**] and the references therein.

The present paper is motivated by the works [**38**] and [**7**]. We first use the ideas of [**38**] to identify the asymptotic profile for the original problem (1.1)-(1.2), which is a shifted viscous shock profile. Then by employing an energy method similar to [**7**], we show the time-asymptotic stability of this profile provided that the strength of the shifted viscous shock profile and the initial perturbation are sufficiently small. The main result of this paper can be viewed as an extension of [**38**] to the case of compressible fluid with capillarity effect. In the present paper, the smallness conditions of the strength of the viscous shock profile and the initial perturbation are mainly used to control the possible growth of solutions caused by the highly nonlinearity of the system and the boundary terms. Notice that the nonlinear stability of viscous shock wave for the Cauchy problem of 1D compressible Navier-Stokes-Korteweg system was also obtained under these smallness conditions [**7**]. However, for the 1D compressible Navier-Stokes system, there are some asymptotic stability results of viscous shock profile with large strength of shock profile [**39, 47, 50**], or certain class of large initial data [**46**]. Thus it is interesting to use the techniques in [**39, 46, 47, 50**] to remove the smallness condition of the strength of the viscous shock profile or the initial perturbation of the present paper, which is left for the future study.

Compared with the case of [**38**], there are two additional difficulties in our analysis. The first one is to control the boundary terms caused by the Korteweg tensor and the estimates of the unknown functions ψ_{xx} and ϕ_{xxx}, such as $|\int_0^t (\frac{\phi_{xx}\psi_{xx}}{v^5})|_{x=0} d\tau|$ and $|\int_0^t (\frac{\phi_{xxx}\phi_{txx}}{v^5})|_{x=0} d\tau|$. We control these boundary terms by making use of the estimates of $A(t)$ (see (3.22)), the first equation of system (3.3) and the a priori assumption (3.13) repeatedly (see the proof of (3.17)-(3.21) for details). The second difficulty is to estimate the highly nonlinear term F in (3.4). Since F contains some nonlinear terms arising from the Korteweg tensor, such as $\kappa V'' \left(\frac{1}{V^5} - \frac{1}{v^5}\right)$, $\frac{5\kappa}{2}\frac{\phi_{xx}^2 + 2\phi_{xx}V'}{v^6}$ and $\frac{5\kappa}{2}(V')^2\left(\frac{1}{V^6} - \frac{1}{v^6}\right)$ (see (3.4) for details), it is more complicated than the counterpart of the compressible Navier-Stokes equations [**38**]. We deal with the nonlinear term F by some delicate energy-type estimates.

The layout of this paper is as follows. In Sections 2, we first go over the existence and properties of viscous shock profiles to system (1.1) in the whole space $\mathbb{R}$. Then we identify the appropriate asymptotic profile for the original problem (1.1)-(1.2) and state our main result. In Section 3, we reformulate the original problem, and then perform the a priori estimates to the reformulated system. The proof of our main result is given at the end of this section.

Notations: Throughout this paper, C denotes some generic constant which may vary in different estimates. If the dependence needs to be explicitly pointed out, the notation $C(\cdot, \cdots, \cdot)$ or $C_i(\cdot, \cdots, \cdot)(i \in \mathbb{N})$ is used. For function spaces, $L^p(\mathbb{R}^+)$ $(1 \le p \le +\infty)$ denotes the standard Lebesgue space with the norm

$$\|f\|_{L^p(\mathbb{R}^+)} = \left(\int_0^\infty |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

$H^l(\mathbb{R}^+)$ is the usual l-th order Sobolev space with its norm

$$\|f\|_l = \left(\sum_{i=0}^{l} \|\partial_x^i f\|^2\right)^{\frac{1}{2}} \quad \text{with} \quad \|\cdot\| \triangleq \|\cdot\|_{L^2(\mathbb{R}^+)}.$$

2. Preliminaries and main result

In this section, we shall present our main result on the time-asymptotic behavior of solutions to the initial-boundary value problem (1.1)-(1.2). More precisely, we first go over the existence and properties of viscous shock profiles to system (1.1) in the whole space $\mathbb{R}$, and then identify the appropriate asymptotic profile for the initial-boundary value problem (1.1)-(1.2). Finally, we give the main result of this paper at the end of this section.

2.1. Viscous shock profiles. Viscous shock profiles are the traveling wave solutions of system (1.1) in the whole space $\mathbb{R}$ of the form:

$$(v,u)(x,t)=(V,U)(\xi),\quad \xi=x-st, \tag{2.1}$$

which satisfies

$$\begin{cases} -sV_\xi-U_\xi=0,\\ -sU_\xi+p(V)_\xi=\mu\left(\dfrac{U_\xi}{V}\right)_\xi+\kappa\left(\dfrac{-V_{\xi\xi}}{V^5}+\dfrac{5V_\xi^2}{2V^6}\right)_\xi \end{cases} \tag{2.2}$$

with the boundary conditions

$$(V,U)(\xi)\longrightarrow(v_\pm,u_\pm),\quad as\ \xi\to\pm\infty, \tag{2.3}$$

where s is the wave speed, ξ is the traveling wave variable and $v_\pm>0, u_\pm$ are given constants.

Integrating (2.2) with respect to ξ once, we obtain

$$\begin{cases} sV+U=a_1,\\ \mu\dfrac{U_\xi}{V}-\kappa\dfrac{V_{\xi\xi}}{V^5}+\dfrac{5\kappa V_\xi^2}{2V^6}=-sU+p(V)+a_2, \end{cases} \tag{2.4}$$

where a_1,a_2 are constants satisfying

$$\begin{cases} a_1=sv_++u_+=sv_-+u_-,\\ a_2=su_+-p(v_+)=su_--p(v_-), \end{cases} \tag{2.5}$$

which shows that s and $v_\pm,u_\pm$ satisfy the Rankine-Hugoniot condition for system (1.1):

$$\begin{cases} s(v_+-v_-)=(u_--u_+),\\ s(u_+-u_-)=p(v_+)-p(v_-). \end{cases} \tag{2.6}$$

Moreover, (2.6) can be reduced to

$$s^2=\frac{p(v_-)-p(v_+)}{v_+-v_-}. \tag{2.7}$$

In this paper, we only consider the case $s>0$, i.e., the 2-shock wave, and the results for the case $s<0$ follow similarly. The Lax entropy condition for the 2-shock (cf. [**29**]) is

$$\sqrt{-p'(v_+)}=\lambda_2(v_+)<s<\lambda_2(v_-)=\sqrt{-p'(v_-)}, \tag{2.8}$$

then it follows from (1.3), $(2.5)_1$ and (2.7) that

$$0<v_-<v_+,\quad u_+<u_-. \tag{2.9}$$

We assume that $u_-=0$ throughout this paper. Then for any given (v_+,u_+) with $v_+>0$ and $u_+<0$, the constants $v_-(0<v_-<v_+)$ and $s>0$ can be uniquely determined by the Rankine-Hugoniot condition (2.6).

The existence and properties of the traveling wave solutions to system (1.1) are summarized in the following theorem.

PROPOSITION 2.1. ([**7**]) Let (1.3) and (2.9) hold. If

$$\frac{\mu^2 s^2 v_-^8}{\kappa} - \left(\frac{10v_+}{v_-} - 6\right) v_+^5 \left(p'(v_+) + s^2\right) > 0,$$

then there exists a monotone viscous shock profile $(V, U)(x - st)$ to system (1.1), which is unique up to a shift and satisfies $V_\xi > 0$, $U_\xi < 0$, and

$$\begin{cases} |V(\xi) - v_-| \le C\delta e^{-c_1|\xi|}, \quad |U(\xi) - u_-| \le C\delta e^{-c_1|\xi|}, \quad \forall \xi \le 0, \\ |V(\xi) - v_+| \le C\delta e^{-c_1|\xi|}, \quad |U(\xi) - u_+| \le C\delta e^{-c_1|\xi|}, \quad \forall \xi \ge 0, \\ \left|\frac{d^k}{d\xi^k} V(\xi)\right| + \left|\frac{d^k}{d\xi^k} U(\xi)\right| \le C\delta^2 e^{-c_1|\xi|}, \quad \forall \xi \in \mathbb{R}, \forall k \ge 1, \end{cases} \tag{2.10}$$

where $\delta := v_+ - v_-$, and c_1, C are two positive constants depending only on v_+, v_-, s, μ and κ.

2.2. Asymptotic profile for the original problem. For the initial-boundary value problem (1.1)-(1.2), there exists an initial boundary layer $(u(x,t) - U(x - st))|_{(x,t)=(0,0)} = u_- - U(0) = -U(0)$ since $U(0)$ is always less than $u_- = 0$. Consequently, the solutions of the original problem (1.1)-(1.2) may not converge to a shifted viscous shock profile $(V, U)(x - st + \alpha)$ with the shift α, which is determined by the initial data and the viscous shock profile. Borrowing the arguments of [**38**], we consider the solution of the original problem (1.1)-(1.2) in a neighborhood of $(V, U)(x - st + \alpha - \beta)$, where α is a shift to be determined later, and $\beta \gg 1$ is a constant such that the initial boundary layer around the shifted wave $|(u(x,t) - U(x - st - \beta)|_{(x,t)=(0,0)} = |(u_- - U(-\beta)| = |U(-\beta)| \ll 1$. In the following, let us determine the shift α for some given constant $\beta \gg 1$.

First, it follows from $(1.1)_1$ and $(2.2)_1$ that

$$(v - V)_t = (u - U)_x, \quad (V, U) = (V, U)(x - st + \alpha - \beta). \tag{2.11}$$

Integrating (2.11) with respect to t and x over $[0, t] \times \mathbb{R}^+$ and using the boundary condition $(1.2)_2$, we have

$$\int_0^\infty [v(x,t) - V(x - st + \alpha - \beta)]dx = \int_0^\infty [v_0(x) - V(x + \alpha - \beta)]dx + \int_0^t U(-s\tau + \alpha - \beta)d\tau. \tag{2.12}$$

Moreover, due to conservation of mass principle, we can suppose that

$$\int_0^\infty [v(x,t) - V(x - st + \alpha - \beta)]dx \to 0, \quad \text{as } t \to \infty. \tag{2.13}$$

Then we set

$$I(\alpha) = \int_0^\infty [v_0(x) - V(x + \alpha - \beta)]dx + \int_0^\infty U(-s\tau + \alpha - \beta)d\tau. \tag{2.14}$$

We see from (2.12) and (2.13) that the shift α must satisfies $I(\alpha) = 0$. Further, differentiating (2.14) with respect to α, we have

$$\begin{aligned} I'(\alpha) &= -\int_0^\infty V'(x + \alpha - \beta)]dx + \int_0^\infty U'(-s\tau + \alpha - \beta)d\tau \\ &= -v_+ + V(\alpha - \beta) + \frac{1}{s}U(\alpha - \beta) = -v_+ + \frac{1}{s}(sv_- + u_-) = v_- - v_+, \end{aligned} \tag{2.15}$$

where we have used $(2.4)_1$ and the assumption that $u_- = 0$. Integrating (2.15) with respect to α over $[0, \alpha]$ yields

$$
\begin{aligned}
I(\alpha) &= I(0) + (v_- - v_+)\alpha \\
&= \int_0^\infty [v_0(x) - V(x-\beta)]dx + \int_0^\infty U(-st-\beta)dt + (v_- - v_+)\alpha. \qquad (2.16)
\end{aligned}
$$

Since $I(\alpha) = 0$, the shift $\alpha = \alpha(\beta)$ is determined explicitly by

$$
\alpha = \frac{1}{v_+ - v_-}\left[\int_0^\infty [v_0(x) - V(x-\beta)]dx + \int_0^\infty U(-st-\beta)dt\right]. \qquad (2.17)
$$

Then we deduce from (2.13), (2.14) and $I(\alpha) = 0$ that

$$
\begin{aligned}
\int_0^\infty [v(x,t) - V(x-st+\alpha-\beta)]dx &= I(\alpha) - \int_t^\infty U(-s\tau+\alpha-\beta)d\tau \\
&= -\int_t^\infty U(-s\tau+\alpha-\beta)d\tau \to 0, \quad t \to \infty, \qquad (2.18)
\end{aligned}
$$

which, together with (2.12) implies that

$$
\int_0^\infty [v_0(x) - V(x+\alpha-\beta)]dx = -\int_0^\infty U(-s\tau+\alpha-\beta)d\tau. \qquad (2.19)
$$

By integrating the equation:

$$
(u-U)_t + (p(v) - p(V))_x = \mu\left(\frac{u_x}{v} - \frac{U_x}{V}\right)_x + \kappa\left(-\frac{v_{xx}}{v^5} + \frac{5v_x^2}{2v^6} + \frac{V_{xx}}{V^5} - \frac{5V_x^2}{2V^6}\right)_x
$$

with respect to t and x over $\mathbb{R}^+ \times \mathbb{R}^+$, and using the assumption:

$$
\int_0^\infty [u(x,t) - U(x-st+\alpha-\beta)]dx \to 0, \quad \text{as } t \to \infty,
$$

we have

$$
\begin{aligned}
&\int_0^\infty [u_0(x) - U(x+\alpha-\beta)]\,dx + \int_0^\infty [p(v(0,t)) - p(V(-st+\alpha-\beta))]\,dt \\
&-\mu\int_0^\infty \left[\frac{u_x(0,t)}{v(0,t)} - \frac{U'(-st+\alpha-\beta)}{V(-st+\alpha-\beta)}\right]dt \\
&+\kappa\int_0^\infty \left[\frac{v_{xx}(0,t)}{v(0,t)^5} - \frac{V''(-st+\alpha-\beta)}{V^5(-st+\alpha-\beta)} + \frac{5}{2}\frac{(V'(-st+\alpha-\beta))^2}{V^6(-st+\alpha-\beta)}\right]dt = 0. \qquad (2.20)
\end{aligned}
$$

Here we expect that $v(0,t)$, $u_x(0,t) = v_t(0,t)$ and $v_{xx}(0,t)$ can be controlled by the effects of boundary, viscosity and capillarity such that (2.20) holds with the same shift α defined by (2.17). This is possible because $v(0,t)$ is not specified.

Thus, by the above heuristical analysis, we expect the asymptotic profile for the original problem (1.1)-(1.2) is the shifted traveling wave $(V,U)(x-st+\alpha-\beta)$ with the shift $\alpha = \alpha(\beta) \ll 1$ and the constant $\beta \gg 1$.

2.3. Main result. To state our main result, we suppose that for some constant $\beta > 0$,

$$
v_0(x) - V(x-\beta) \in H^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad u_0(y) - U(y-\beta) \in H^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+). \qquad (2.21)
$$

Set

$$
(\Phi_0, \Psi_0)(x) = -\int_x^\infty \Big(v_0(y) - V(y-\beta), u_0(y) - U(y-\beta)\Big)dy, \qquad (2.22)
$$

and assume that

$$(\Phi_0, \Psi_0)(x) \in L^2(\mathbb{R}^+). \tag{2.23}$$

Then our main result is stated as follows.

THEOREM 2.1. Assume that $v_+ > 0$, $u_+ < 0$ and $\beta > 0$ are constants. Let (1.4), (2.21)-(2.23) and the conditions listed in Proposition 2.1 hold, and $(V, U)(x-st)$ be the viscous shock profile obtained in Proposition 2.1. Then there exists two small positive constants δ_0 and ε_0 such that if $0 < \delta := v_+ - v_- \leq \delta_0$ and

$$\|\Phi_0\|_3 + \|\Psi_0\|_2 + \beta^{-1} \leq \varepsilon_0, \tag{2.24}$$

then the initial-boundary value problem (1.1)-(1.2) admits a unique global solution $(u, v)(t, x)$ satisfying

$$\begin{aligned} &v(x,t) - V(x - st + \alpha - \beta) \in C\left([0,\infty; H^2(\mathbb{R}^+)\right) \cap L^2\left([0,\infty; H^3(\mathbb{R}^+)\right), \\ &u(x,t) - U(x - st + \alpha - \beta) \in C\left([0,\infty; H^1(\mathbb{R}^+)\right) \cap L^2\left([0,\infty; H^2(\mathbb{R}^+)\right), \end{aligned} \tag{2.25}$$

where $\alpha = \alpha(\beta)$ is the shift determined by (2.17). Moreover, the following asymptotic behavior of solutions holds:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^+} |(v,u)(x,t) - (V,U)(x - st + \alpha - \beta)| = 0. \tag{2.26}$$

3. Proof of Theorem 2.1

This section is devoted to proving Theorem 2.1. We begin with a reformulation of the original problem (1.1)-(1.2).

3.1. Reformulation of the original problem. First, we define the perturbation functions $(\phi, \psi)(t, x)$ as follows:

$$(\phi,\psi)(x,t) = -\int_x^\infty (v(y,t) - V(y - st + \alpha - \beta), u(y,t) - U(y - st + \alpha - \beta))\,dy \tag{3.1}$$

for $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then, we have

$$(v,u)(x,t) = (\phi_x(x,t) + V(x - st + \alpha - \beta),\ \psi_x(x,t) + U(x - st + \alpha - \beta)). \tag{3.2}$$

Substituting (3.2) into (1.1), using (2.2) and integrating the system in x over $[x, +\infty)$, we obtain

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + p'(V)\phi_x - \dfrac{\mu}{V}\psi_{xx} + \dfrac{\kappa\phi_{xxx}}{v^5} + \mu\dfrac{U'\phi_x}{V^2} = F, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} F = &-\{p(v) - p(V) - p'(V)\phi_x\} - \frac{\mu\psi_{xx}\phi_x}{vV} + \frac{\mu U'\phi_x^2}{vV^2} \\ &+ \kappa V''\left(\frac{1}{V^5} - \frac{1}{v^5}\right) + \frac{5\kappa}{2}\frac{\phi_{xx}^2 + 2\phi_{xx}V'}{v^6} + \frac{5\kappa}{2}(V')^2\left(\frac{1}{V^6} - \frac{1}{v^6}\right). \end{aligned} \tag{3.4}$$

The initial data of system (3.3) satisfy

$$\begin{aligned}
\phi_0(x) \quad &:= \quad \phi|_{t=0} = -\int_x^\infty [v_0(y) - V(y-\beta)]dy + \int_x^\infty [V(y+\alpha-\beta) - V(y-\beta)]dy \\
&= \quad \Phi_0(x) + \int_x^\infty [V(y+\alpha-\beta) - V(y-\beta)]dy \\
&= \quad \Phi_0(x) + \int_x^\infty \int_0^\alpha V'(y+\theta-\beta)d\theta dy \\
&= \quad \Phi_0(x) + \int_0^\alpha \int_x^\infty V'(y+\theta-\beta)dyd\theta \\
&= \quad \Phi_0(x) + \int_0^\alpha [v_+ - V(x+\theta-\beta)]d\theta, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\psi_0(x) \quad &:= \quad \psi|_{t=0} = -\int_x^\infty [u_0(y) - U(y+\alpha-\beta)]\ dy \\
&= \quad -\int_x^\infty [u_0(y) - U(y-\beta)]\ dy + \int_x^\infty [U(y+\alpha-\beta) - U(y-\beta)]\ dy \\
&= \quad \Psi_0(x) + \int_0^\alpha [u_+ - U(x+\theta-\beta)]\ d\theta, \qquad (3.6)
\end{aligned}$$

where the functions $(\Phi_0, \Psi_0)(x)$ are defined in (2.22).

Next, using (3.1), (3.2) and (2.18), we have

(3.7)

$$\phi(0,t) = -\int_0^\infty [v(y,t) - V(y-st+\alpha-\beta)]\ dy = \int_t^\infty U(-s\tau+\alpha-\beta)\ d\tau := A(t),$$

and

(3.8)

$$\psi_x(0,t) = u(0,t) - U(-st+\alpha-\beta) = u_- - U(-st+\alpha-\beta) = -U(-st+\alpha-\beta) = A'(t).$$

Moreover, it follows from $v_x(0,t) = 0$ that

$$0 = v_x|_{x=0} = \phi_{xx}|_{x=0} + V'(x-st+\alpha-\beta)|_{x=0} = \phi_{xx}(0,t) + V'(-st+\alpha-\beta),$$

consequently,

$$\begin{aligned}
\phi_{xx}(0,t) \quad &= \quad -V'(-st+\alpha-\beta) = -\frac{1}{s}(sV')(-st+\alpha-\beta) \\
&= \quad \frac{1}{s}U'(-st+\alpha-\beta) = \frac{1}{s^2}A''(t), \qquad (3.9)
\end{aligned}$$

where we have used $(2.2)_1$ and (3.8). We assume that $\phi_0(0) = A(0), \psi_{0x} = A'(0)$ and $\phi_{0xx} = \frac{1}{s^2}A''(0)$ as compatible conditions.

In what follows, we seek the solutions of the initial-boundary value problem (3.4)-(3.9) in the following function space:

(3.10)

$$X_M(0,T) = \left\{ (\phi,\psi)(t,x) \;\middle|\; \begin{array}{c} \phi(t,x) \in C(0,T;H^3(\mathbb{R})), \phi_x(t,x) \in L^2(0,T;H^3(\mathbb{R})), \\ \psi(t,x) \in C(0,T;H^2(\mathbb{R})), \psi_x(t,x) \in L^2(0,T;H^2(\mathbb{R})), \\ \sup_{t\in[0,T]} \{\|\phi(t)\|_3 + \|\psi(t)\|_2\} \le M, \end{array} \right\}$$

where $0 \le T \le \infty$, and M is some positive constant.

For the problem (3.3)-(3.9), we have the following theorem, which leads to Theorem 2.1 immediately.

THEOREM 3.1. Suppose that the conditions of Theorem 2.1 hold, then there exists positive constants δ_0, ε_1 and C_0 which are independent of the time t and the initial data ϕ_0, ψ_0 such that if $0 < \delta := v_+ - v_- \leq \delta_0$ and $\|\phi_0\|_3 + \|\psi_0\|_2 + \beta^{-1} \leq \varepsilon_1$, then the initial-boundary value problem (3.4)-(3.9) has a unique global solution $(\phi, \psi)(t, x) \in X_{\hat{M}}[0, \infty)$ with $\hat{M} = 2\sqrt{C_0(\|\phi_0\|_3^2 + \|\psi_0\|_2^2 + e^{-c_1\beta})}$. Moreover, it holds that

$$\|\phi(t)\|_3^2 + \|\psi(t)\|_2^2 + \int_0^t [\|\phi_x(\tau)\|_3^2 + \|\psi_x(\tau)\|_2^2] \, d\tau \leq C_0(\|\phi_0\|_3^2 + \|\psi_0\|_2^2 + e^{-c_1\beta}) \tag{3.11}$$

for any $t \in [0, +\infty)$, and

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}^+} |(\phi_x, \psi_x)(x, t)| = 0. \tag{3.12}$$

Notice that the initial conditions in Theorems 3.1 and 2.1 are different. The following lemma state the relation between them.

LEMMA 3.1. Suppose that the condition (2.21) holds. Then we have

(i) The shift $\alpha \to 0$ if $\|\Phi_0\|_2 \to 0$ and $\beta \to \infty$;
(ii) $\|\phi_0\|_3 + \|\psi_0\|_2 \to 0$ if $\|\Phi_0\|_3 + \|\Psi_0\|_2 \to 0$ and $\beta \to \infty$.

Since the proof of Lemma 3.1 (i) and (ii) are almost the same as those of Lemma 2.1 and Lemma 3.2 in [**38**] respectively, we omit the details here for brevity.

Theorem 3.1 will be obtained by combining the following local existence and a priori estimates.

PROPOSITION 3.1 (Local existence). Under the assumptions of Theorem 3.1, suppose that $\|\phi_0\|_3 + \|\psi_0\|_2 \leq M$, then there exists a positive constant t_0 depending only on M such that the initial-boundary value problem (3.4)-(3.9) admits a unique solution $(\phi, \psi)(x, t) \in X_{2M}(0, t_0)$.

PROPOSITION 3.2 (A priori estimates). Suppose that $(\phi, \psi)(x, t) \in X_M(0, T)$ is a solution of the initial-boundary value problem (3.4)-(3.9) obtained in Proposition 3.1 for some positive constants T and M. Then there exists three positive constants $\delta_0 \ll 1$, $\varepsilon_1 \ll 1$ and C_0 which are independent of T such that if $0 < \delta \leq \delta_0$ and

$$N(T) := \sup_{t \in [0, T]} \{\|\phi(t)\|_3 + \|\psi(t)\|_2\} \leq \varepsilon \tag{3.13}$$

with the constant ε satisfying $0 < \varepsilon \leq \varepsilon_1$, the solution (ϕ, ψ) of the problem (3.4)-(3.9) satisfies the estimates (3.11) for all $t \in [0, T]$.

Proposition 3.1 can be proved by using the dual argument and iteration technique, which is similar to that of Theorem 1.1 in [**16**], the details are omitted here. The proof of Proposition 3.2 is more subtle and will be given in the next subsection.

3.2. A priori estimates. Due to Lemma 3.1 (i) and the assumptions in Theorem 3.1 and Proposition 3.2, without loss of generality, we assume that $|\alpha| < 1, \beta > 1$ and $N(t) < 1$ hereafter.

Let $(\phi, \psi)(x, t) \in X_M(0, T)$ be a solution of the initial-boundary value problem (3.4)-(3.9) for some positive constants T and M. Then we have from Proposition

2.1, the Sobolev inequality $\|f\|_{L^\infty} \le \|f\|_1$ for $f(x) \in H^1$, and the smallness of $\varepsilon_1 > 0$ that

$$v(x,t) = V(x,t) + \phi_x(x,t) \le v_+ + \|\phi_x(t)\|_1 \le \frac{3}{2}v_+, \ \forall\,(x,t) \in [0,T] \times \mathbb{R}^+, \tag{3.14}$$

and

$$v(x,t) = V(x,t) + \phi_x(x,t) \ge v_- - \|\phi_x(t)\|_1 \ge \frac{v_-}{2}, \ \forall\,(x,t) \in [0,T] \times \mathbb{R}^+. \tag{3.15}$$

We first give the following boundary estimates.

Lemma 3.2 (Boundary estimates). *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that the following boundary estimates:*

$$\begin{aligned}&\left|\int_0^t (\phi\psi)|_{x=0}d\tau\right| + \left|\int_0^t (\psi\psi_x)|_{x=0}d\tau\right| + \left|\int_0^t (\psi_x\phi_x)|_{x=0}d\tau\right| + \left|\int_0^t (\psi_t\psi_x)|_{x=0}d\tau\right| \\ &+ \left|\int_0^t (\psi_x\psi_{xx})|_{x=0}d\tau\right| + \left|\int_0^t (\psi_{xt}\psi_{xx})|_{x=0}d\tau\right| \le Ce^{-c_1\beta},\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\left|\int_0^t \left[\frac{\phi_{xx}\psi}{p'(V)v^5}\right]\Big|_{x=0}d\tau\right| + \left|\int_0^t \left[\frac{\phi_x\phi_{xx}}{v^5}\right]\Big|_{x=0}d\tau\right| + \left|\int_0^t \left[\frac{\phi_{xx}\psi_{xx}}{v^5}\right]\Big|_{x=0}d\tau\right| \\ &+ \left|\int_0^t (\psi_t\phi_{xx})|_{x=0}d\tau\right| \le Ce^{-c_1\beta},\end{aligned} \tag{3.17}$$

$$\left|\int_0^t \left[\frac{\phi_{xxx}\phi_{txx}}{v^5}\right]\Big|_{x=0}d\tau\right| \le C\varepsilon^{\frac12}\int_0^t \|\phi_{xxxx}(\tau)\|^2\,d\tau + C\varepsilon^{\frac12}e^{-\frac43 c_1\beta}, \tag{3.18}$$

$$\left|\int_0^t (\psi_{tx}\phi_{xxx})\,|_{x=0}d\tau\right| \le \eta\int_0^t \|\phi_{xxxx}(\tau)\|^2\,d\tau + C_\eta\int_0^t \|\phi_{xxx}(\tau)\|^2\,d\tau + C_\eta e^{-c_1\beta}, \tag{3.19}$$

$$\left|\int_0^t [p'(V)\phi_{xx}\phi_{xxx}]\,|_{x=0}\,d\tau\right| \le \eta\int_0^t \|\phi_{xxxx}(\tau)\|^2\,d\tau + C_\eta\int_0^t \|\phi_{xx}(\tau)\|_1^2 d\tau, \tag{3.20}$$

and

$$\left|\int_0^t (\psi_{xx}\psi_{xxx})\,|_{x=0}d\tau\right| \le C\int_0^t \|\psi_{xx}(\tau)\|_1^2\,d\tau + Ce^{-c_1\beta} \tag{3.21}$$

hold for all $0 \le t \le T$, where $c_1 > 0$ is constant defined in Proposition 2.1, η is a small positive constant and C_η is a positive constant depending on η.

Proof. First, we show that the boundary data (3.7): $\phi(0,t) = A(t) := \int_t^\infty U(-s\tau + \alpha - \beta)\,d\tau$ satisfies that

$$\left|\frac{d^k}{dt^k}A(t)\right| \le Ce^{-c_1\beta}e^{-c_1st}(k = 0,1,2,3), \ \|A\|_{W^{3,1}} \le Ce^{-c_1\beta}. \tag{3.22}$$

In fact, since $|-s\tau + \alpha - \beta| = s\tau + \beta - \alpha$ because $s > 0$ and $\beta > \alpha$, it follows from Proposition 2.1 that

$$|U(-s\tau + \alpha - \beta)| \le Ce^{-c_1|-s\tau+\alpha-\beta|} \le Ce^{-c_1(\beta-\alpha)}e^{-c_1s\tau} \le Ce^{-c_1\beta}e^{-c_1s\tau}.$$

Consequently, we have

$$|A(t)| \le Ce^{-c_1\beta}e^{-c_1 st}. \tag{3.23}$$

Similarly, differentiating $A(t)$ with respect to t and using Proposition 2.1, we can also obtain that $A'(t), A''(t)$ and $A'''(t)$ are all bounded by $Ce^{-c_1\beta}e^{-c_1 st}$, thus the inequalities in (3.22) hold.

Moreover, from the Sobolev inequality, we have

$$\begin{aligned} &|\psi(0,t)| \le \sup_{x\in\mathbb{R}^+} |\psi(x,t)| \le CN(T), \\ &|\phi_x(0,t)| + |\psi_x(0,t)| \le \sup_{x\in\mathbb{R}^+} (|\phi_x(x,t)| + |\psi_x(x,t)|) \le CN(T). \end{aligned} \tag{3.24}$$

Now we should give the proofs of (3.16)-(3.21). For (3.16), since its proof is almost the same as that in Lemma 4.1 of [**38**], we omit the details here. Next, we focus on deducing (3.17)-(3.21), which are new boundary terms compared with the the case of compressible Navier-Stokes system (see Lemma 4.1 in [**38**]). Firstly, utilizing the boundary conditions (3.9), the estimates (3.22) and (3.24), we have

$$\begin{aligned} \left|\int_0^t \left(\frac{\phi_{xx}\psi}{p'(V)v^5}\right)\Big|_{x=0} d\tau\right| &\le C\int_0^t |\phi_{xx}\psi|_{x=0}\, d\tau \le C\int_0^t |A''(\tau)||\psi(0,\tau)|d\tau \\ &\le CN(T)\int_0^t |A''(\tau)|\, d\tau \le Ce^{-c_1\beta}, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \left|\int_0^t \left(\frac{\phi_x\phi_{xx}}{v^5}\right)\Big|_{x=0} d\tau\right| &\le C\int_0^t |\phi_x(0,\tau)\phi_{xx}(0,\tau)|\, d\tau \le C\int_0^t |\phi_x(0,\tau)||A''(\tau)|\, d\tau \\ &\le CN(T)\int_0^t |A''(\tau)|\, d\tau \le Ce^{-c_1\beta}. \end{aligned} \tag{3.26}$$

Moreover, making use of the relation $\phi_{tx} = \psi_{xx}$ (see $(3.3)_1$), integration by parts, Proposition 2.1 and (3.22)-(3.24), we obtain

$$\begin{aligned} &\left|\int_0^t \left(\frac{\phi_{xx}\psi_{xx}}{v^5}\right)\Big|_{x=0} d\tau\right| \\ = &\left|\int_0^t \left(\frac{\phi_{xx}\phi_{tx}}{v^5}\right)\Big|_{x=0} d\tau\right| = \left|\int_0^t \left[\left(\phi_{xx}\frac{\phi_x}{v^5}\right)_t - \left(\frac{\phi_{xx}}{v^5}\right)_t\phi_x\right]\Big|_{x=0} d\tau\right| \\ = &\left(\phi_{xx}\frac{\phi_x}{v^5}\right)(0,t) - \left(\frac{\phi_{xx}}{v^5}\phi_x\right)(0,0) - \int_0^t \left(\phi_{xxt}\frac{\phi_x}{v^5} + 5\frac{\phi_{xx}v_t\phi_x}{v^6}\right)\Big|_{x=0} d\tau \\ \le &C[|\phi_{xx}(0,t)\phi_x(0,t)| + |\phi_{xx}(0,0)||\phi_x(0,0)|] \\ &+C\int_0^t |\phi_{xxt}(0,\tau)\phi_x(0,\tau)|\, d\tau + C\left|\int_0^t \left(\frac{\phi_{xx}v_t}{v^6}\phi_x\right)(0,\tau)d\tau\right| \\ \le &C(|A''(t)||\phi_x(0,t)| + |A''(0)||\phi_x(0,0)|) + C\int_0^t |A'''(\tau)||\phi_x(0,\tau)|d\tau \\ &+C\left|\int_0^t \left(\frac{\phi_{xx}v_t}{v^6}\phi_x\right)(0,\tau)d\tau\right| \\ \le &CN(T)e^{-c_1\beta} + C\left|\int_0^t \left(\frac{\phi_{xx}v_t}{v^6}\phi_x\right)(0,\tau)d\tau\right|, \end{aligned} \tag{3.27}$$

and

$$
\begin{aligned}
(3.28)\quad & \left|\int_0^t \left(\frac{\phi_{xx}v_t}{v^6}\phi_x\right)(0,\tau)\ d\tau\right| \\
\leq\ & \left|\int_0^t \frac{\phi_{xx}(0,\tau)}{v^6(0,\tau)}\phi_x(0,\tau)(U'+\psi_{xx}(0,\tau))d\tau\right| \\
\leq\ & C\delta^2\int_0^t |\phi_{xx}(0,\tau)\phi_x(0,\tau)|\ d\tau + \left|\int_0^t \frac{\phi_{xx}(0,\tau)\psi_{xx}(0,\tau)}{v^5(0,\tau)}d\tau\right|\cdot\left\|\frac{\phi_x}{v}(0,t)\right\|_{L_t^\infty} \\
\leq\ & C\delta^2 N(T)\int_0^t |A''(\tau)|d\tau + CN(T)\left|\int_0^t \left(\frac{\phi_{xx}\psi_{xx}}{v^5}\right)\bigg|_{x=0} d\tau\right| \\
\leq\ & C\delta^2 e^{-c_1\beta} + C\varepsilon\left|\int_0^t \left(\frac{\phi_{xx}\psi_{xx}}{v^5}\right)\bigg|_{x=0} d\tau\right|.
\end{aligned}
$$

Therefore, combining (3.27)-(3.28), and using the smallness of ε, we show

$$
\left|\int_0^t \left(\frac{\phi_{xx}\psi_{xx}}{v^5}\right)\bigg|_{x=0} d\tau\right| \leq Ce^{-c_1\beta}. \tag{3.29}
$$

Similarly, we have

$$
\begin{aligned}
\left|\int_0^t (\psi_t\phi_{xx})\,|_{x=0}d\tau\right| &= \left|\int_0^t \psi_t(0,\tau)\frac{1}{s^2}A''(\tau)\ d\tau\right| \\
&= \frac{1}{s^2}\left|\int_0^t [(\psi(0,\tau)A''(\tau))_\tau - \psi(0,\tau)A'''(\tau)]d\tau\right| \\
&= \frac{1}{s^2}\left|\psi(0,t)A''(t)-\psi(0,0)A''(0)-\int_0^t \psi(0,\tau)A'''(\tau)d\tau\right| \\
&\leq CN(T)e^{-c_1\beta}+CN(T)\int_0^t |A'''(\tau)|d\tau \leq Ce^{-c_1\beta}. \qquad (3.30)
\end{aligned}
$$

Thus (3.17) follows from (3.25), (3.26), (3.29) and (3.30) immediately.

Next, we can derive from the equalities $\phi_{txx}(0,t)=A'''(t)$, $\psi_{tx}(0,t)=A''(t)$, the Sobolev inequality, the Cauchy inequality and (3.22) that

$$
\begin{aligned}
\left|\int_0^t \left(\frac{\phi_{xxx}\phi_{txx}}{v^5}\right)\bigg|_{x=0} d\tau\right| &\leq C\int_0^t |\phi_{xxx}(0,\tau)||A'''(\tau)|\ d\tau \\
&\leq C\int_0^t \|\phi_{xxx}(\tau)\|^{\frac12}\|\phi_{xxxx}(\tau)\|^{\frac12}|A'''(\tau)|\ d\tau \\
&\leq C\varepsilon^{\frac12}\int_0^t \|\phi_{xxxx}(\tau)\|^2\ d\tau + C\varepsilon^{\frac12}e^{-\frac43 c_1\beta}, \qquad (3.31)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_0^t (\psi_{tx}\phi_{xxx})|_{x=0} d\tau\right| &\le \int_0^t |\phi_{xxx}(0,\tau)||A''(\tau)|\ d\tau \\
&\le \int_0^t \|\phi_{xxx}(\tau)\|^{\frac{1}{2}}\|\phi_{xxxx}(\tau)\|^{\frac{1}{2}}|A''(\tau)|\ d\tau \\
&\le \eta\int_0^t \|\phi_{xxxx}(\tau)\|^2\ d\tau + C_\eta\int_0^t (\|\phi_{xxx}(\tau)\|^2 + |A''(\tau)|^2)\ d\tau \\
&\le \eta\int_0^t \|\phi_{xxxx}(\tau)\|^2\ d\tau + C_\eta\int_0^t \|\phi_{xxx}(\tau)\|^2\ d\tau + C_\eta e^{-c_1\beta}, \qquad (3.32)
\end{aligned}
$$

which imply (3.18) and (3.19), respectively. Similarly, using

$$
\begin{aligned}
\left|\int_0^t (p'(V)\phi_{xx}\phi_{xxx})|_{x=0}\ d\tau\right| &\le C\int_0^t |\phi_{xx}(0,\tau)||\phi_{xxx}(0,\tau)|d\tau \\
&\le C\int_0^t \|\phi_{xx}(\tau)\|^{\frac{1}{2}}\|\phi_{xxx}(\tau)\|\|\phi_{xxxx}(\tau)\|^{\frac{1}{2}}d\tau \\
&\le \eta\int_0^t \|\phi_{xxxx}(\tau)\|^2\ d\tau + C_\eta\int_0^t \|\phi_{xx}(\tau)\|_1^2 d\tau, \qquad (3.33)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_0^t (\psi_{xx}\psi_{xxx})|_{x=0} d\tau\right| &= \left|\int_0^t (\psi_{xx}\phi_{txx})|_{x=0} d\tau\right| \le C\int_0^t |\psi_{xx}(0,\tau)||A'''(\tau)|\,d\tau \\
&\le C\int_0^t \|\psi_{xx}(\tau)\|^{\frac{1}{2}}\|\psi_{xxx}(\tau)\|^{\frac{1}{2}}|A'''(\tau)|\ d\tau \\
&\le C\int_0^t \|\psi_{xx}(\tau)\|_1^2\ d\tau + C\int_0^t |A'''(\tau)|\ d\tau \\
&\le C\int_0^t \|\psi_{xx}(\tau)\|_1^2\ d\tau + Ce^{-c_1\beta}, \qquad (3.34)
\end{aligned}
$$

we can show (3.20) and (3.21). This completes the proof of Lemma 3.2.

Once we have the estimates of boundary terms, we can continue a-priori estimates. First, let us give the L^2-estimates on $(\phi,\psi)(x,t)$.

Lemma 3.3. Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that for $0 \le t \le T$,

$$
\begin{aligned}
&\|(\phi,\psi,\phi_x)(t)\|^2 + \int_0^t \left\|(\sqrt{V}_x\psi,\psi_x)(\tau)\right\|^2\ d\tau \\
\le\ & C\left[\|(\phi_0,\psi_0,\phi_{0x})\|^2 + e^{-c_1\beta} + (\delta^2+\varepsilon)\int_0^t \|(\phi_x,\phi_{xx},\psi_{xx})(\tau)\|^2\ d\tau\right] \qquad (3.35)
\end{aligned}
$$

holds provided that ϵ and δ are suitably small.

Proof. Multiplying $(3.3)_1$ and $(3.3)_2$ by ϕ and $-\frac{1}{p'(V)}\psi$, respectively, then adding these equalities and integrating the resulting equation over $[0,t]\times\mathbb{R}^+$, we have

$$\begin{aligned}
&\int_0^\infty \left(\frac{\phi^2}{2}+\frac{\psi^2}{2(-p'(V))}\right)dx+\int_0^t\int_0^\infty \frac{sp''(V)V_x\psi^2}{2(p'(V))^2}dxd\tau\\
&+\int_0^t\int_0^\infty \frac{\mu\psi_x^2}{-p'(V)V}dxd\tau\\
=&\int_0^\infty \left(\frac{\phi_0^2}{2}+\frac{\psi_0^2}{2(-p'(V_0))}\right)dx+\mu\int_0^t\int_0^\infty \left(\frac{1}{p'(V)V}\right)_x\psi_x\psi dxd\tau\\
&+\kappa\int_0^t\int_0^\infty \frac{\phi_{xxx}\psi}{p'(V)v^5}dxd\tau+\mu\int_0^t\int_0^\infty \frac{U_x\phi_x\psi}{p'(V)V^2}dxd\tau+\int_0^t\int_0^\infty \frac{-F\psi}{p'(V)}dxd\tau\\
&+\mu\int_0^t \frac{\psi_x\psi}{p'(V)V}|_{x=0}\,d\tau+\int_0^t \psi\phi|_{x=0}d\tau \qquad (3.36)\\
=&\int_0^\infty \left(\frac{\phi_0^2}{2}+\frac{\psi_0^2}{2(-p'(V_0))}\right)dx+\sum_{i=1}^6 I_i,
\end{aligned}$$

where $V_0=V(x+\alpha-\beta)$. In the following, let us estimate $I_i(i=1,\cdots,6)$ one by one. The terms I_1,I_3,I_5 and I_6 are the same as the those of the compressible Navier-Stokes system [**38**]. For completeness, we give the estimates of these term as follows. From Proposition 2.1, (3.14), (3.15), (3.16) and the Cauchy inequality, it is easy to obtain

$$\begin{aligned}
|I_1|+|I_3| &\le C\int_0^t\int_0^\infty (|V_x\psi\psi_x|+|V_x\phi_x\psi|)dxd\tau\\
&\le \frac{1}{8}\int_0^t\int_0^\infty \frac{sp''(V)V_x\psi^2}{(p'(V))^2}dxd\tau+C\int_0^t\int_0^\infty \|V_x(\tau)\|_{L^\infty}(\psi_x^2+\phi_x^2)\,dxd\tau\\
&\le \frac{1}{8}\int_0^t\int_0^\infty \frac{sp''(V)V_x\psi^2}{(p'(V))^2}dxd\tau+C\delta^2\int_0^t\int_0^\infty \left(\psi_x^2+\phi_x^2\right)dxd\tau \qquad (3.37)
\end{aligned}$$

and

$$|I_5|+|I_6|\le Ce^{-c_1\beta}. \qquad (3.38)$$

The term I_2 and I_4 are two difficult terms caused by the Korteweg tensor. For I_2, note that

$$\begin{aligned}
I_2&=\kappa\int_0^t\int_0^\infty \left[\left(\frac{\kappa\phi_{xx}\psi}{p'(V)v^5}\right)_x-(\frac{\psi}{p'(V)v^5})_x\phi_{xx}\right]dxd\tau\\
&=-\kappa\int_0^t \frac{\phi_{xx}\psi}{p'(V)v^5}\bigg|_{x=0}d\tau-\kappa\int_0^t\int_0^\infty \left(\frac{\psi_x\phi_{xx}}{p'(V)v^5}+\psi\frac{-(p'(V)v^5)_x}{(p'(V)v^5)^2}\phi_{xx}\right)dxd\tau. \qquad (3.39)
\end{aligned}$$

Moreover, using $(3.3)_1$ and integration by parts, we have

$$
\begin{aligned}
&-\kappa\int_0^t\int_0^\infty \frac{\psi_x\phi_{xx}}{p'(V)v^5}\,dxd\tau = -\kappa\int_0^t\int_0^\infty \frac{\phi_t\phi_{xx}}{p'(V)v^5}\,dxd\tau\\
=\ &\kappa\int_0^t \frac{\phi_t\phi_x}{p'(V)v^5}\Big|_{x=0}\,d\tau+\kappa\int_0^t\int_0^\infty \frac{\phi_{tx}\phi_x}{p'(V)v^5}\,dxd\tau\\
&-\kappa\int_0^t\int_0^\infty \frac{\psi_x\phi_x(p'(V)v^5)_x}{(p'(V)v^5)^2}\,dxd\tau\\
=\ &\kappa\int_0^t \frac{\phi_t\phi_x}{p'(V)v^5}\Big|_{x=0}\,d\tau+\kappa\int_0^\infty \frac{\phi_x^2}{2p'(V)v^5}\,dx-\kappa\int_0^\infty \frac{\phi_{0x}^2}{2p'(V_0)v_0^5}dx\\
&+\frac{\kappa}{2}\int_0^t\int_0^\infty \phi_x^2\frac{-sV'p''(V)}{(p'(V))^2v^5}dxd\tau+\frac{5\kappa}{2}\int_0^t\int_0^\infty \phi_x^2\frac{U'+\psi_x}{p'(V)v^6}dxd\tau\\
&-\kappa\int_0^t\int_0^\infty \frac{\psi_x\phi_x(p'(V)v^5)_x}{(p'(V)v^5)^2}\,dxd\tau,
\end{aligned}
$$

which together with (3.39) yields

$$
\begin{aligned}
I_2 =\ &\kappa\int_0^\infty \frac{\phi_x^2}{2p'(V)v^5}\,dx-\kappa\int_0^\infty \frac{\phi_{0x}^2}{2p'(V_0)v_0^5}\,dx\\
&-\kappa\int_0^t \frac{\phi_{xx}\psi}{p'(V)v^5}\Big|_{x=0}d\tau+\kappa\int_0^t \frac{\psi_x\phi_x}{p'(V)v^5}|_{x=0}d\tau\\
&+\frac{\kappa}{2}\int_0^t\int_0^\infty \left(\phi_x^2\frac{-sV'p''(V)}{(p'(V))^2v^5}+5\phi_x^2\frac{U'+\psi_{xx}}{p'(V)v^6}\right)dxd\tau\\
&+\kappa\int_0^t\int_0^\infty \frac{p''(V)V'v^5+p'(V)5v^4(V'+\phi_{xx})}{(p'(V)v^5)^2}\phi_{xx}\psi\,dxd\tau\\
&-\kappa\int_0^t\int_0^\infty \psi_x\phi_x\frac{p''(V)V'v^5+p'(V)5v^4(V'+\phi_{xx})}{(p'(V)v^5)^2}\,dxd\tau\\
=\ &\kappa\int_0^\infty \frac{\phi_x^2}{2p'(V)v^5}\,dx-\kappa\int_0^\infty \frac{\phi_{0x}^2}{2p'(V_0)v_0^5}\,dx+\sum_{i=1}^5 I_{2i}. \qquad (3.40)
\end{aligned}
$$

Similar as (3.37) and (3.38), we have

$$
|I_{21}|+|I_{22}|\le Ce^{-c_1\beta}. \qquad (3.41)
$$

Using Proposition 2.1, the Sobolev inequality, the Cauchy inequality, (3.14), (3.15) and the a priori assumption (3.13), we get

$$
\begin{aligned}
I_{23} \le\ &C\int_0^t\int_0^\infty \left(\|(V',U')(\tau)\|_{L^\infty}\phi_x^2+\|\phi_x(\tau)\|_{L^\infty}|\phi_x\psi_{xx}|\right)dxd\tau\\
\le\ &C\delta^2\int_0^t\int_0^\infty \phi_x^2\,dxd\tau+CN(T)\int_0^t\int_0^\infty \left(\phi_x^2+\psi_{xx}^2\right)dxd\tau\\
\le\ &C(\delta^2+\varepsilon)\int_0^t\int_0^\infty \left(\phi_x^2+\psi_{xx}^2\right)dxd\tau, \qquad (3.42)
\end{aligned}
$$

and

$$\begin{aligned} I_{24} &\leq \eta \int_0^t \int_0^\infty V'\psi^2\ dxd\tau + C_\eta \int_0^t \int_0^\infty (V'\phi_{xx}^2 + \|\psi(\tau)\|_{L^\infty}\phi_{xx}^2) dxd\tau \\ &\leq \eta \int_0^t \int_0^\infty V'\psi^2\ dxd\tau + C_\eta(\delta^2+\varepsilon) \int_0^t \int_0^\infty \phi_{xx}^2 dxd\tau. \end{aligned} \tag{3.43}$$

Here and hereafter, η is a small positive constant and C_η is a positive constant depending on η. Similarly, we have

$$I_{25} \leq C(\delta^2+\varepsilon) \int_0^t \int_0^\infty \left(\psi_x^2 + \phi_x^2 + \phi_{xx}^2\right) dxd\tau. \tag{3.44}$$

Hence it follows from (3.40)-(3.44) that

$$\begin{aligned} I_2 \ &\leq \kappa \int_0^\infty \frac{\phi_x^2}{2p'(V)v^5}\ dx - \kappa \int_0^\infty \frac{\phi_{0x}^2}{2p'(V_0)v_0^5}\ dx + \eta \int_0^t \int_0^\infty V'\psi^2\ dxd\tau \\ &\quad + C_\eta(\delta^2+\varepsilon) \int_0^t \int_0^\infty (\psi_x^2 + \phi_x^2 + \phi_{xx}^2 + \psi_{xx}^2)\ dxd\tau + Ce^{-c_1\beta}. \end{aligned} \tag{3.45}$$

Finally, for I_4, we have from the proof of Proposition 2.1 in [**7**] that $V'(x-st+\alpha-\beta) > 0$ and $\lim_{x\to\pm\infty} \frac{V''}{V'} = \lambda_\pm$, where $\lambda_\pm \in \mathbb{R}$ are two constants. Consequently, it follows from Proposition 2.1 that

$$F = O(1)\left(|\phi_x^2| + |\phi_{xx}^2| + |\psi_{xx}\phi_x| + |V'\phi_x| + |V''\phi_x| + |V'\phi_{xx}|\right). \tag{3.46}$$

Furthermore, similar to the estimate of I_2, we have

$$\begin{aligned} |I_4| \ &\leq \ C \int_0^t \int_0^\infty |F||\psi|\ dxd\tau \\ &\leq \ C \int_0^t \int_0^\infty (|\phi_x^2| + |\psi_{xx}\phi_x| + |\phi_{xx}^2| + |V'\phi_x| + |V''\phi_x| + |V'\phi_{xx}|)|\psi|\ dxd\tau \\ &\leq \ \frac{1}{8} \int_0^t \int_0^\infty \frac{sp''(V)V_x\psi^2}{(p'(V))^2}\ dxd\tau + C(\delta^2+\varepsilon) \int_0^t \int_0^\infty (\phi_x^2 + \phi_{xx}^2 + \psi_{xx}^2)\ dxd\tau. \end{aligned} \tag{3.47}$$

Inserting (3.37), (3.41), (3.45) and (3.47) into (3.36), and using Proposition 2.1, (3.14)-(3.15) and the smallness of η, ε and δ, we have (3.35) immediately. Thus the proof of Lemma 3.3 is completed.

The next lemma gives the estimate on $\|\phi_x(t)\|$.

Lemma 3.4. Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that for $0 \leq t \leq T$, it holds that

$$\begin{aligned} &\|\phi_x(t)\|^2 + \int_0^t \|\phi_x(\tau)\|_1^2\ d\tau \\ &\leq C\left[\|(\phi_{0x}, \psi_0, \phi_0)\|^2 + e^{-c_1\beta} + (\delta^2+\varepsilon)\int_0^t \|\psi_{xx}(\tau)\|^2 d\tau\right] \end{aligned} \tag{3.48}$$

provided that ε and δ are suitably small.

Proof. Multiplying $(3.3)_2$ by $-\phi_x$, and integrating the resulting equation with respect to x over $[0,+\infty)$ yield

$$\begin{aligned} & \int_0^\infty -p'(V)\phi_x^2\,dx - \kappa\int_0^\infty \frac{\phi_{xxx}\phi_x}{v^5}\,dx \\ = & \int_0^\infty \psi_t\phi_x\,dx - \int_0^\infty \frac{\mu}{V}\psi_{xx}\phi_x\,dx - \mu\int_0^\infty \frac{U'\phi_x^2}{V^2}\,dx - \int_0^\infty F\phi_x\,dx \\ = & :I_7+I_8+I_9+I_{10}. \end{aligned} \tag{3.49}$$

Here $-\kappa\int_0^\infty \frac{\phi_{xxx}\phi_x}{v^5}dx$ and $-\int_0^\infty F\phi_x dx$ are two new terms compared with the case of the compressible Navier-Stokes system [**38**], the others are the same as those of [**38**]. First, from Proposition 2.1, (3.14)-(3.15) and integration by parts, we have

$$\begin{aligned} & -\kappa\int_0^\infty \frac{\phi_{xxx}\phi_x}{v^5}\,dx \\ = & -\kappa\int_0^\infty \left(\phi_{xx}\frac{\phi_x}{v^5}\right)_x dx + \kappa\int_0^\infty \frac{\phi_{xx}^2}{v^5}\,dx - 5\kappa\int_0^\infty \frac{\phi_{xx}(\phi_x+V_x)\phi_x}{v^6}dx \\ \geq & \kappa\phi_{xx}\frac{\phi_x}{v^5}\Big|_{x=0} + \kappa\int_0^\infty \frac{\phi_{xx}^2}{v^5}\,dx - C(\delta^2+\varepsilon)\int_0^\infty (\phi_{xx}^2+\phi_x^2)dx. \end{aligned} \tag{3.50}$$

Moreover, using $(3.3)_1$ and integration by parts, it is easy to obtain

$$\begin{aligned} I_7 &= \int_0^\infty [(\psi\phi_x)_t - \psi\phi_{xt}]dx = \frac{d}{dt}\int_0^\infty \psi\phi_x\,dx - \int_0^\infty [(\psi\psi_x)_x - \psi_x^2]dx \\ &= \frac{d}{dt}\int_0^\infty \psi\phi_x\,dx + \psi\psi_x|_{x=0} + \int_0^\infty \psi_x^2 dx. \end{aligned} \tag{3.51}$$

Next, using Cauchy inequality and Proposition 2.1, one gets

$$\begin{aligned} I_8 &= -\mu\int_0^\infty \frac{\phi_{tx}\phi_x}{V}\,dx = -\frac{\mu}{2}\frac{d}{dt}\int_0^\infty \frac{\phi_x^2}{V}\,dx + \frac{\mu}{2}\int_0^\infty \frac{V'}{V^2}\phi_x^2 dx \\ &\leq -\frac{\mu}{2}\frac{d}{dt}\int_0^\infty \frac{\phi_x^2}{V}\,dx + C\delta^2\int_0^\infty \phi_x^2\,dx, \end{aligned} \tag{3.52}$$

and

$$I_9 \leq C\delta^2\int_0^\infty \phi_x^2\,dx. \tag{3.53}$$

Finally, noting (3.46), we have

$$\begin{aligned} I_{10} &= \int_0^\infty |F||\phi_x|\,dx \leq C\int_0^\infty (|\phi_x^2| + |\psi_{xx}\phi_x| + |\phi_{xx}^2| + |V'\phi_x| + |V'\phi_{xx}|)|\phi_x|dx \\ &\leq C(\delta^2+\varepsilon)\int_0^\infty (\phi_x^2+\psi_{xx}^2+\phi_{xx}^2)dx. \end{aligned} \tag{3.54}$$

Substituting (3.50)-(3.52) into (3.49), we get by the smallness of δ and ε that

$$\begin{aligned} & \frac{\mu}{2}\frac{d}{dt}\int_0^\infty \frac{\phi_x^2}{V}\,dx + \int_0^\infty -p'(V)\phi_x^2\,dx + \kappa\int_0^\infty \frac{\phi_{xx}^2}{v^5}\,dx \\ \leq & \frac{d}{dt}\int_0^\infty \psi\phi_x\,dx + \psi\psi_x|_{x=0} - \kappa\frac{\phi_x\phi_{xx}}{v^5}\Big|_{x=0} + \int_0^\infty \psi_x^2 dx. \end{aligned} \tag{3.55}$$

Integrating (3.55) in t over $[0,t]$ gives

$$\begin{aligned}
&\frac{\mu}{4}\int_0^\infty \frac{\phi_x^2}{V}\,dx+\int_0^t\int_0^\infty\left(-p'(V)\phi_x^2+\kappa\frac{\phi_{xx}^2}{v^5}\right)dxd\tau\\
\le\ & C\left(\|(\phi_{0x},\psi_0)\|^2+\|\psi(t)\|^2\right)+\int_0^t\|\psi_x(\tau)\|^2\,d\tau\\
&+\int_0^t\psi\psi_x|_{x=0}\,d\tau-\kappa\int_0^t\left.\frac{\phi_x\phi_{xx}}{v^5}\right|_{x=0}d\tau,
\end{aligned}\tag{3.56}$$

where we have used the fact that

$$\int_0^\infty\psi\phi_x dx\le\frac{\mu}{4}\int_0^\infty\frac{\phi_x^2}{V}\,dx+C\|\psi(t)\|^2.$$

Thus (3.48) follows from (3.56), (3.16), (3.17), Lemma 3.3 and the smallness of ε and δ immediately. This completes the proof of Lemma 3.4.

Furthermore, we estimate $\|\psi_x(t)\|$ in the following lemma.

Lemma 3.5. Under the assumptions of Proposition 3.2, there exists a positive constant $C>0$ which is independent of α,β and T such that for $0\le t\le T$, it holds that

$$\begin{aligned}
&\|(\psi_x,\phi_{xx})(t)\|^2+\int_0^t\|\psi_{xx}(\tau)\|^2\,d\tau\\
\le\ & C\left(\|\phi_0\|_2^2+\|\psi_0\|_1^2+e^{-c_1\beta}+(\delta^2+\varepsilon)\int_0^t\|\phi_{xxx}(\tau)\|^2\,d\tau\right)
\end{aligned}\tag{3.57}$$

provided that ε and δ are suitably small.

Proof. Multiplying $(3.3)_2$ by $-\psi_{xx}$, and integrating the resulting equation in x over $[0,+\infty)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\infty\psi_x^2\,dx+\int_0^\infty\frac{\mu}{V}\psi_{xx}^2dx\\
=\ &\frac{1}{2}\int_0^\infty\psi_{0x}^2\,dx-\psi_t\psi_x|_{x=0}+\int_0^\infty p'(V)\phi_x\psi_{xx}\,dx\\
&+\kappa\int_0^\infty\frac{\phi_{xxx}\psi_{xx}}{v^5}\,dx+\mu\int_0^\infty\frac{U'\phi_x\psi_{xx}}{V^2}\,dx-\int_0^\infty F\psi_{xx}\,dx\\
=:\ &\frac{1}{2}\int_0^\infty\psi_{0x}^2\,dx-\psi_t\psi_x|_{x=0}+J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.58}$$

Here J_2 and J_4 are two terms caused by the Korteweg tensor, the others are the same as those of the compressible Navier-Stokes system [**38**]. Now we control them one by one. The Cauchy inequality implies that

$$J_1\le\eta\int_0^\infty\psi_{xx}^2\,dx+C_\eta\int_0^\infty(p'(V))^2\phi_x^2dx\le\eta\int_0^\infty\psi_{xx}^2\,dx+C_\eta\int_0^\infty\phi_x^2\,dx,\tag{3.59}$$

and

$$J_3\le\eta\int_0^\infty\psi_{xx}^2\,dx+C_\eta\delta^2\int_0^\infty\phi_x^2dx.\tag{3.60}$$

Using $(3.3)_1$, we get

$$\begin{aligned} J_2 &= \kappa \int_0^\infty \frac{\phi_{xxx}\phi_{tx}}{v^5}\ dx = \kappa \int_0^\infty [(\phi_{xx}\frac{\phi_{tx}}{v^5})_x - \phi_{xx}(\frac{\phi_{tx}}{v^5})_x]\ dx \\ &= -\frac{\kappa}{2}\frac{d}{dt}\int_0^\infty \frac{\phi_{xx}^2}{v^5}\ dx - \kappa\phi_{xx}\frac{\psi_{xx}}{v^5}\Big|_{x=0} + J_{21} + J_{22} \end{aligned}$$

with

$$J_{21} = -\frac{5\kappa}{2}\int_0^\infty \phi_{xx}^2 \frac{\psi_{xx}+U'}{v^6}\ dx,\ \ J_{22} = 5\kappa \int_0^\infty \phi_{xx}\psi_{xx}\frac{\phi_{xx}+V'}{v^6}\ dx.$$

It follows from Proposition 2.1, the Sobolev inequality, the Cauchy inequality, (3.14), (3.15) and the a priori assumption (3.13) that

$$\begin{aligned} J_{21} &\le C\int_0^\infty \|\phi_{xx}(t)\|_{L^\infty}|\phi_{xx}\psi_{xx}| + \|U'(t)\|_{L^\infty}\phi_{xx}^2\ dx \\ &\le C\left(\delta^2\|\phi_{xx}(t)\|^2 + \|\phi_{xx}(t)\|^{\frac{3}{2}}\|\phi_{xxx}(t)\|^{\frac{1}{2}}\|\psi_{xx}(t)\|\right), \\ &\le C(\delta^2+\varepsilon)\|(\phi_{xx},\phi_{xxx},\psi_{xx})(t)\|^2, \end{aligned}$$

and

$$J_{22} \le C(\delta^2+\varepsilon)\|(\phi_{xx},\phi_{xxx},\psi_{xx})(t)\|^2.$$

Thus, we have

$$J_2 \le -\frac{\kappa}{2}\frac{d}{dt}\int_0^\infty \frac{\phi_{xx}^2}{v^5}\ dx - \kappa\phi_{xx}\frac{\psi_{xx}}{v^5}|_{x=0} + C(\delta^2+\varepsilon)\|(\phi_{xx},\phi_{xxx},\psi_{xx})(t)\|^2. \tag{3.61}$$

Using (3.46), J_4 can be controlled as follows:

$$\begin{aligned} J_4 &\le \int_0^\infty |F\psi_{xx}|dx \\ &\le \eta\int_0^\infty \psi_{xx}^2\ dx + C_\eta\int_0^\infty |F|^2dx \\ &\le \eta\int_0^\infty \psi_{xx}^2\ dx + C_\eta\int_0^\infty \big[\|\phi_x(t)\|_{L^\infty}^2(\phi_x^2+\psi_{xx}^2) \\ &\qquad + \|V'(t)\|_{L^\infty}^2(\phi_x^2+\phi_{xx}^2) + \|\phi_{xx}(t)\|_{L^\infty}^2\phi_{xx}^2\big]\ dx \\ &\le \eta\|\psi_{xx}(t)\|^2 + C_\eta(\delta^2+\varepsilon)\|(\phi_x,\psi_{xx},\phi_{xx})(t)\|^2. \end{aligned} \tag{3.62}$$

Inserting (3.59)-(3.62) into (3.58), and integrating the resulting inequality in t over $[0,t]$, we have by the smallness of η,δ and ε that

$$\begin{aligned} &\|\psi_x(t)\|^2 + \|\phi_{xx}(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2\ d\tau \\ \le\ & C\left[\|(\psi_{0x},\phi_{0xx})\|^2 + \int_0^t \|\phi_x(\tau)\|_1^2\ d\tau\right] + C(\delta^2+\varepsilon)\int_0^t \|\phi_{xxx}(\tau)\|^2\ d\tau \\ &+ \int_0^t \left(-\psi_t\psi_x - \kappa\phi_{xx}\frac{\psi_{xx}}{v^5}\right)\Big|_{x=0}\ d\tau. \end{aligned} \tag{3.63}$$

Then (3.57) can be obtained by (3.63), (3.16),(3.17), Lemma 3.4 and the smallness of ε and δ. This completes the proof of Lemma 3.5.

Now, let us focus on the estimate of $\int_0^t \|\phi_{xx}(\tau)\|_1^2\ d\tau$.

LEMMA 3.6. Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that for $0 \le t \le T$,

$$\|\phi_{xx}(t)\|^2 + \int_0^t \|\phi_{xx}(\tau)\|_1^2 d\tau \le C(\|\phi_0\|_2^2 + \|\psi_0\|_1^2 + e^{-c_1\beta}) \tag{3.64}$$

holds provided that ε and δ are suitably small.

Proof. Multiplying $(3.3)_2$ by ϕ_{xxx} and utilizing $(3.3)_1$, we have

$$\begin{aligned}
& -p'(V)\phi_{xx}^2 + \kappa\frac{\phi_{xxx}^2}{v^5} + \left(\frac{\mu\phi_{xx}^2}{2V}\right)_t - (\psi_x\phi_{xx})_t \\
= \ & \left(\frac{\mu}{V}\psi_{xx}\phi_{xx}\right)_x - \left(\frac{\mu}{V}\right)_x \phi_{xx}\psi_{xx} + \left(\frac{\mu}{V}\right)_t \frac{\phi_{xx}^2}{2} - (p'(V)\phi_x\phi_{xx})_x \\
& + p''(V)V_x\phi_x\phi_{xx} - (\psi_t\phi_{xx})_x - (\psi_x\psi_{xx})_x + \psi_{xx}^2 - \mu\frac{U'\phi_x}{V^2}\phi_{xxx} + F\phi_{xxx}
\end{aligned} \tag{3.65}$$

Integrating (3.65) over $[0,t] \times \mathbb{R}^+$ and using the Cauchy inequality, we obtain

$$\begin{aligned}
& \frac{1}{4}\int_0^\infty \frac{\mu\phi_{xx}^2}{V}\, dx + \int_0^t \int_0^\infty \left(-p'(V)\phi_{xx}^2 + \frac{\kappa\phi_{xxx}^2}{v^5}\right) dx d\tau \\
\le \ & C\|(\phi_{0xx}, \psi_{0x})\|^2 + C\|\psi_x(t)\|^2 + \left|\int_0^t K_1\ d\tau\right| + \int_0^t \|\psi_{xx}(\tau)\|^2\ d\tau \\
& + \left|\int_0^t \psi_t\phi_{xx}|_{x=0}\ d\tau\right| + \int_0^t \int_0^\infty (|K_2| + |K_3|)\, dx d\tau,
\end{aligned} \tag{3.66}$$

where

$$K_1 = \left(-\frac{\mu}{V}\psi_{xx}\phi_{xx} + \psi_x\psi_{xx} + p'(V)\phi_x\phi_{xx}\right)\Big|_{x=0},$$

$$K_2 = \frac{\mu V_x}{V^2}\phi_{xx}\psi_{xx} - \frac{\mu}{2V^2}V_t\phi_{xx}^2 + p''(V)V_x\phi_x\phi_{xx},\ K_3 = F\phi_{xxx} - \mu\frac{U'\phi_x\phi_{xxx}}{V^2}.$$

Here K_1 and K_3 are new terms compared with the case of compressible Navier-Stokes system [**38**], and K_2 can be bounded by the method of [**38**]. Notice that K_1 has been dealt with in Lemma 3.2 before. Moreover, similar to the estimates of J_{21}, we have

$$\begin{aligned}
\int_0^t \int_0^\infty |K_2|\ dx d\tau \ & \le \ C\int_0^t \int_0^\infty \left[|V_x\phi_{xx}\psi_{xx}| + |V'\phi_{xx}^2| + |V_x\phi_x\phi_{xx}|\right]\ dx d\tau \\
& \le \ C\delta^2 \int_0^t \|(\phi_{xx}, \psi_{xx}, \phi_x)(\tau)\|^2\ d\tau,
\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}
\int_0^t \int_0^\infty |K_3|\ dx d\tau \ & \le \ C\int_0^t \int_0^\infty \left(|\phi_x^2| + |\psi_{xx}\phi_x| + |\phi_{xx}^2| + |V'(\phi_x, \phi_{xx})|\right)|\phi_{xxx}|\ dx d\tau \\
& \le \ C(\delta^2 + \varepsilon)\int_0^t \|(\phi_x, \psi_{xx}, \phi_{xx}, \phi_{xxx})(\tau)\|^2\ d\tau.
\end{aligned} \tag{3.68}$$

Combining (3.66)-(3.68), using (3.16)-(3.17) and the smallness of δ, ε yield (3.64) at once. This finishes the proof of Lemma 3.6.

As a consequence of Lemmas 3.3-3.6, we have

COROLLARY 3.1. Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that for $\forall t \in [0,T]$, it holds that

$$\begin{aligned}&\|\phi(t)\|_2^2 + \|\psi(t)\|_1^2 + \int_0^t \left(\|\phi_x(\tau)\|_2^2 + \|\psi_x(\tau)\|_1^2\right) d\tau \\ &\leq C[\|\phi_0\|_2^2 + \|\psi_0\|_1^2 + e^{-c_1\beta}].\end{aligned} \tag{3.69}$$

About the estimate of $\|\psi_{xx}(\tau)\|$, we have

LEMMA 3.7. Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that for $0 \leq t \leq T$,

$$\begin{aligned}&\|(\phi_{xxx}, \psi_{xx})(t)\|^2 + \int_0^t \|\psi_{xxx}(\tau)\|^2 \, d\tau \\ &\leq C\left(\|\phi_0\|_3^2 + \|\psi_0\|_2^2 + e^{-c_1\beta} + \varepsilon^{\frac{1}{2}} \int_0^t \|\phi_{xxx}(\tau)\|^2 \, d\tau\right)\end{aligned} \tag{3.70}$$

holds provided that ε and δ are suitably small.

Proof. Differentiating $(3.3)_2$ with respect to x once, then multiplying the resulting equation by $-\psi_{xxx}$ and integrating over $[0,t] \times \mathbb{R}^+$, we get

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \psi_{xx}^2 \, dx + \int_0^t \int_0^\infty \frac{\mu}{V}\psi_{xxx}^2 \, dxd\tau \\ =\; &\frac{1}{2}\int_0^\infty \psi_{0xx}^2 \, dx - \int_0^t \psi_{xt}\psi_{xx}|_{x=0} \, d\tau + \kappa \int_0^t \int_0^\infty \left(\frac{\phi_{xxx}}{v^5}\right)_x \psi_{xxx} dxd\tau \\ &- \int_0^t \int_0^\infty F_x \psi_{xxx} \, dxd\tau + \int_0^t \int_0^\infty Q_1 dxd\tau,\end{aligned} \tag{3.71}$$

where

$$Q_1 = p''(V)V_x\phi_x\psi_{xxx} - p'(V)\phi_{xx}\psi_{xxx} - \left(\frac{\mu}{V}\right)_x \psi_{xx}\psi_{xxx} - \mu\left(\frac{U'\phi_x}{v^2}\right)_x \psi_{xxx}.$$

Here $\kappa \int_0^t \int_0^\infty \left(\frac{\phi_{xxx}}{v^5}\right)_x \psi_{xxx} dxd\tau$ and $\int_0^t \int_0^\infty F_x\psi_{xxx} \, dxd\tau$ are two new terms caused by the Korteweg tensor, the others are similar as those of [**38**]. First, using $(3.3)_1$ and integration by parts, we have

$$\begin{aligned}&\kappa \int_0^t \int_0^\infty \left(\frac{\phi_{xxx}}{v^5}\right)_x \psi_{xxx} \, dxd\tau = \kappa \int_0^t \int_0^\infty \left(\frac{\phi_{xxx}}{v^5}\right)_x \phi_{txx} \, dxd\tau \\ =\; &\kappa \int_0^t \int_0^\infty \left[\left(\frac{\phi_{xxx}}{v^5}\phi_{txx}\right)_x - \frac{\phi_{xxx}}{v^5}\phi_{txxx}\right] dxd\tau \\ =\; &-\kappa \int_0^t \frac{\phi_{xxx}}{v^5}\phi_{txx}\Big|_{x=0} d\tau - \kappa \int_0^t \int_0^\infty \left[\left(\frac{\phi_{xxx}^2}{2v^5}\right)_t + \frac{5\phi_{xxx}^2 v_t}{2v^6}\right] dxd\tau \\ =\; &-\kappa \int_0^t \frac{\phi_{xxx}}{v^5}\phi_{txx}\Big|_{x=0} d\tau - \kappa \int_0^\infty \frac{\phi_{xxx}^2}{2v^5} dx + \kappa \int_0^\infty \frac{\phi_{0xxx}^2}{2v_0^5} dx \\ &- \frac{5\kappa}{2} \int_0^t \int_0^\infty \frac{\phi_{xxx}^2 u_x}{v^6} dxd\tau.\end{aligned} \tag{3.72}$$

It follows from the Sobolev inequality and Proposition 2.1 that

$$\begin{aligned}
&\left|\int_0^t\int_0^\infty \frac{\phi_{xxx}^2 u_x}{v^6}\ dxd\tau\right| \\
\le\ & C\int_0^t\int_0^\infty \left|\phi_{xxx}^2(U'+\psi_{xx})\right| dxd\tau \\
\le\ & C\delta^2\int_0^t \|\phi_{xxx}(\tau)\|^2\ d\tau + C\int_0^t \|\psi_{xx}(\tau)\|^{\frac12}\|\psi_{xxx}(\tau)\|^{\frac12}\|\phi_{xxx}(\tau)\|^2\ d\tau \\
\le\ & C\delta^2\int_0^t \|\phi_{xxx}(\tau)\|^2\ d\tau + \eta\int_0^t \|\psi_{xxx}(\tau)\|^2\ d\tau \\
& +C_\eta\int_0^t \left(\|\psi_{xx}(\tau)\|^2 + \|\phi_{xxx}(\tau)\|^4\right) d\tau. \qquad (3.73)
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
F_x =& -[p(v)-p(V)-p'(V)\phi_x]_x - \left(\frac{\mu\psi_{xx}\phi_x}{vV}\right)_x \\
&+\left[\kappa V''\left(\frac{1}{V^5}-\frac{1}{v^5}\right)+\frac{5\kappa}{2}\frac{\phi_{xx}^2+2\phi_{xx}V'}{v^6}+\frac{5\kappa}{2}(V')^2\left(\frac{1}{v^6}-\frac{1}{V^6}\right)+\frac{\mu U'\phi_x^2}{vV^2}\right]_x \\
=&O(1)\Big(|(V_x,U'')|\phi_x^2+|\phi_{xx}\phi_x^2|+|\phi_x\phi_{xx}|+|\psi_{xxx}\phi_x|+|\psi_{xx}\phi_{xx}|+|V_x\phi_x\psi_{xx}| \\
&+|\phi_{xx}\phi_x\psi_{xx}|+|(V''',V''V',(V')^3)||\phi_x|+|(V'',(V')^2)||\phi_{xx}|+|V'\phi_{xxx}| \\
&+|\phi_{xx}\phi_{xxx}|+|\phi_{xx}^3|+|\phi_{xx}^2V'|\Big),
\end{aligned}$$

we obtain from the Cauchy inequality, the Sobolev inequality, Proposition 2.1 and the a priori assumption (3.13) that

$$\begin{aligned}
&\left|-\int_0^t\int_0^\infty F_x\psi_{xxx}\ dxd\tau\right| \qquad (3.74)\\
\le\ & \eta\int_0^t\int_0^\infty \psi_{xxx}^2\ dxd\tau + C_\eta\int_0^t\int_0^\infty |F_x|^2\ dxd\tau \\
\le\ & \eta\int_0^t\int_0^\infty \psi_{xxx}^2\ dxd\tau + C_\eta\int_0^t\int_0^\infty \Big[|(V',U'')|^2\phi_x^4+|\phi_{xx}|^2\phi_x^4+|\phi_x^2\phi_{xx}^2| \\
&+|\psi_{xx}^2\phi_x^2|+|\psi_{xx}^2\phi_{xx}^2|+|V_x^2\phi_x^2\psi_{xx}^2|+|\phi_{xx}\phi_x\psi_{xx}|^2+|(V''',V''V',(V')^3)|^2|\phi_x^2| \\
&+|(V'',(V')^2)|^2|\phi_{xx}|^2+|V'\phi_{xxx}|^2+|\phi_{xx}\phi_{xxx}|^2+|\phi_{xx}^6|+|\phi_{xx}^4V'^2|\Big]dxd\tau \\
\le\ & \eta\int_0^t \|\psi_{xxx}(\tau)\|^2 d\tau + C_\eta(\delta^2+\varepsilon^2)\int_0^t \|(\phi_x,\phi_{xx},\phi_{xxx},\psi_{xxx})(\tau)\|^2 d\tau.
\end{aligned}$$

Similarly, it holds

$$\int_0^t\int_0^\infty |Q_1| dxd\tau \le \eta\int_0^t \|\psi_{xxx}(\tau)\|^2 d\tau + C_\eta\int_0^t \|(\phi_x,\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau. \qquad (3.75)$$

Putting (3.72)-(3.75) into (3.71), then (3.70) follows from Corollary 3.1, (3.16), (3.18) and the smallness of δ,ε and η. This completes the proof of Lemma 3.7.

Finally, we give the estimate of $\int_0^t \|\phi_{xxx}(\tau)\|_1^2\ d\tau$ in the following lemma, which can be achieved by making use of the effect of the Korteweg tensor.

LEMMA 3.8. Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ which is independent of α, β and T such that for $0 \le t \le T$, it holds

$$\|\phi_{xxx}(t)\|^2 + \int_0^t \|\phi_{xxx}(\tau)\|_1^2 \, d\tau \le C\left(\|\phi_0\|_3^2 + \|\psi_0\|_2^2 + e^{-c_1\beta}\right) \tag{3.76}$$

provided that ε and δ are suitably small.

Proof. Differentiating $(3.3)_2$ with respect to x once, then multiplying the resulting equation by ϕ_{xxx} and integrating over $[0,t] \times \mathbb{R}^+$ gives

$$\begin{aligned} &\frac{1}{2}\int_0^\infty \frac{\mu}{V}\phi_{xxx}^2 dx + \int_0^t\int_0^\infty \left(-p'(V)\phi_{xxx}^2 + \frac{\kappa\phi_{xxxx}^2}{v^5}\right) dx d\tau - \int_0^\infty \psi_{xx}\phi_{xxx} dx \\ = \; &\frac{1}{2}\int_0^\infty \frac{\mu}{V_0}\phi_{0xxx}^2 dx - \int_0^\infty \psi_{0xx}\phi_{0xxx} dx + \int_0^t\int_0^\infty \sum_{i=2}^4 Q_i dx d\tau, \end{aligned} \tag{3.77}$$

where

$$Q_2 = \left(p'(V)\phi_{xx}\phi_{xxx} + \psi_{tx}\phi_{xxx} + \psi_{xx}\psi_{xxx} - \frac{\mu}{V}\phi_{txx}\phi_{xxx}\right)\Big|_{x=0},$$

$$Q_3 = p''(V)V_x\phi_x\phi_{xxx} + p'(V)\phi_{xxx}^2 - \left(\frac{\mu}{V}\right)_x \psi_{xxx}\phi_{xxx} + \left(\frac{\mu}{V}\right)_t \frac{\phi_{xxx}^2}{2},$$

and

$$\begin{aligned} Q_4 = &- p'(V)\phi_{xx}\phi_{xxx} - p''(V)V'\phi_x\phi_{xxxx} + \left(\frac{\mu}{V}\right)_x \psi_{xx}\phi_{xxxx} + 5\kappa\phi_{xxx}\phi_{xxxx}\frac{v_x}{v^6} \\ &- \mu\left(\frac{U'\phi_x}{V^2}\right)_x \phi_{xxxx} + F_x\phi_{xxxx}. \end{aligned}$$

The estimates (3.18)-(3.21) imply that

$$\begin{aligned} \left|\int_0^t\int_0^\infty Q_2 dx d\tau\right| \le &2\eta\int_0^t \|\phi_{xxxx}(\tau)\|^2 \, d\tau + C_\eta\left(\int_0^t \|\phi_{xx}(\tau)\|_1^2 \, d\tau + e^{-c_1\beta}\right) \\ &+ C\left(\int_0^t \|\psi_{xx}(\tau)\|_1^2 \, d\tau + \varepsilon^{\frac{1}{2}}\int_0^t \|\phi_{xxxx}(\tau)\|^2 \, d\tau\right). \end{aligned} \tag{3.78}$$

Similar to (3.73)-(3.74), we have

$$\left|\int_0^t\int_0^\infty Q_3 dx d\tau\right| \le C\int_0^t \|(\phi_x, \phi_{xxx}, \psi_{xxx})(\tau)\|^2 \, d\tau, \tag{3.79}$$

$$\left|\int_0^t\int_0^\infty Q_4 dx d\tau\right| \le \eta\int_0^t\int_0^\infty \phi_{xxxx}^2 \, dx d\tau + C_\eta\int_0^t \left(\|\phi_x(\tau)\|_2^2 + \|\psi_{xx}(\tau)\|^2\right) d\tau, \tag{3.80}$$

and

$$\left|-\int_0^\infty \psi_{xx}\phi_{xxx} dx\right| \le \frac{1}{4}\int_0^\infty \frac{\mu}{V}\phi_{xxx}^2 \, dx + C\|\psi_{xx}(t)\|^2. \tag{3.81}$$

Combining (3.77)-(3.81), and using Corollary 2.1, Lemma 3.7 and the smallness of ε and η, we obtain (3.76). This completes the proof of Lemma 3.8.

Now let us give the proof of Proposition 3.2 in the following.

Proof of Proposition 3.2. Proposition 3.2 follows from Corollary 3.1 and Lemmas 3.7-3.8 immediately.

3.3. Proof of Theorem 3.1. To complete the proof Theorem 3.1, we first observe that the a priori assumption (3.13) can be closed by choosing $\|\phi_0\|, \|\psi_0\|_1$ and β^{-1} sufficiently small such that

$$\|\phi_0\|_2^2 + \|\psi_0\|_1^2 + e^{-c_1\beta} < \frac{\varepsilon_1^2}{4C_0}.$$

Then based on Propositions 3.1 and 3.2, the standard continuation argument asserts that there exists a unique global (in time) solution $(\phi, \psi)(t,x) \in X_{\tilde{M}}(0, +\infty)$ to the initial-boundary value problem (3.4)-(3.9), where $\hat{M} = 2\sqrt{C_0(\|\phi_0\|_3^2 + \|\psi_0\|_2^2 + e^{-c_1\beta})}$. Moreover, we can derive from (3.11) and the system (3.3) that

$$\int_0^{+\infty} \left[\|\phi_x(t)\|_2^2 + \|\psi_x(t)\|_1^2 + \left| \frac{d}{dt}\left(\|\phi_x(t)\|_2^2 + \|\psi_x(t)\|_1^2\right) \right| \right] dt < \infty, \tag{3.82}$$

which implies that $\|\phi_x(t)\|_2 + \|\psi_x(t)\|_1 \to 0$ as $t \to +\infty$. Then by the Sobolev inequality, we have

$$\|\phi_x(t)\|_{L^\infty} \leq \|\phi_x(t)\|^{\frac{1}{2}} \|\phi_{xx}(t)\|^{\frac{1}{2}} \to 0 \quad \text{as } t \to +\infty, \tag{3.83}$$

and

$$\|\psi_x(t)\|_{L^\infty} \leq \|\psi_x(t)\|^{\frac{1}{2}} \|\psi_{xx}(t)\|^{\frac{1}{2}} \to 0 \quad \text{as } t \to +\infty. \tag{3.84}$$

Thus (3.12) is proved and the proof of Theorem 3.1 is completed.

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