

# Exact, purely azimuthal stratified equatorial flows in cylindrical coordinates

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ABSTRACT. This paper is concerned with the derivation of an exact solution to the governing equations for geophysical fluid dynamics expressed in terms of cylindrical coordinates. It is demonstrated, by way of a functional analytic approach, that there is a well-defined relationship between the imposed pressure at the free-surface and the resulting distortion of the surface, and it is established that this relationship exhibits the expected monotonicity properties. This exact solution for stratified fluid flow is then subjected to a short-wavelength perturbation stability analysis.

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## 1. Introduction

This paper is concerned with the derivation of a new exact solution to the geophysical fluid dynamics (GFD) governing equations which are formulated in terms of cylindrical coordinates in a rotating frame. This solution corresponds to a steady, purely-azimuthal equatorial flow with an associated free-surface, with stratification incorporated in the fluid by modelling the density distribution as a linear function

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of depth. GFD is the study of fluid motion whereby the effect of the earth's rotation plays a significant role, necessitating the inclusion of Coriolis forces in the governing equations. The dynamics of the ocean near the equator presents some unique and complex characteristics from a modelling perspective, among these being pronounced stratification and the presence of non-uniform underlying currents [8, 15, 32, 33, 35, 39].

Even for the relatively simplified regime of an inviscid, incompressible and laminar fluid, the GFD equations of motion are highly nonlinear and intractable [13, 18, 40]. In this context it is remarkable that a number of recent developments have produced an assortment of exact solutions to the GFD governing equations in various forms, cf. [3, 4, 5, 7, 8, 9, 10, 11, 12, 19, 20, 25, 26, 37, 38]; surveys of these results can be found in [21, 34]. Exact solutions are extremely rare in fluid mechanics, in general, and they offer an invaluable insight into the mathematical structure of a given problem.

In this paper we extend this body of work by constructing a new exact solution which incorporates the effects of stratification. Employing a cylindrical coordinate framework has a number of advantages: it offers a more transparent insight into the properties of the fluid flow compared to spherical coordinates (cf. [24]) while still retaining an appreciable amount of the mathematical structure of the full problem. An artefact of the construction of the exact solution below is a Bernoulli-type relation at the free-surface: this provides a constraint on the existence of a solution. The mathematical formulation of this Bernoulli relation is greatly complicated by the presence of fluid stratification, as can be seen by comparison with the homogeneous fluid setting [8]. This intricate and involved expression provides an implicit prescription of the relationship between the imposed pressure, and the resulting distortion, at the free-surface.

We demonstrate, by way of applying a functional analytic approach to the Bernoulli relation, that there is a well-defined relationship between the imposed pressure at the free-surface and the resulting distortion of the surface's shape. Additionally, we establish that this relationship exhibits physically-expected monotonicity properties. The presentation is concluded by subjecting this new exact solution for stratified fluid flow to a short-wavelength stability analysis. The short-wavelength perturbation method is a particularly elegant analytical approach which has proven to be highly applicable to a variety of recently derived exact solutions of the GFD governing equations, as can be seen in [6, 17, 22, 23, 27, 28, 30, 31]. Although the mathematical implementation of this technique is significantly complicated by the presence of fluid stratification, we succeed in deriving a physically interesting result.

## 2. The governing equations

In the following, the geophysical fluid dynamics (GFD) governing equations are presented in terms of a cylindrical coordinate system which is oriented as follows. The great circle of the sphere which corresponds to the equator is "straightened out", and this line (which is parallel to the  $z$ -axis) generates the cylinder. The interior of the cylinder (which corresponds to the interior of the sphere) is represented by standard polar coordinates. Thus, in a right handed system, the coordinates are  $(r, \theta, z)$ , where  $r$  is the distance to the centre of the disc (representing the Earth),  $\theta \in (-\pi/2, \pi/2)$  is increasing from North to South and measures the deflection

from the Equator, and the positive  $z$ -direction denotes azimuthal flow from West to East. The line of the Equator is chosen to be associated with  $\theta = 0$ . The corresponding unit vectors in the  $(r, \theta, z)$  system are  $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$  and the velocity components with respect to  $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$  are  $(u, v, w)$ . The GFD governing equations in a coordinate system with its origin at the centre of the cylinder are the Eulers equation, which is expressed in terms of cylindrical coordinates as

$$\begin{aligned}
 (2.1) \quad & u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} = -\frac{1}{\rho}p_r + F_r \\
 & v_t + uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}p_\theta + F_\theta \\
 & w_t + uw_r + \frac{v}{r}w_\theta + ww_z = -\frac{1}{\rho}p_z + F_z,
 \end{aligned}$$

and the equation of mass conservation

$$(2.2) \quad \frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r}v_\theta + w_z = 0.$$

Here  $p(r, \theta, \varphi)$  is denotes the pressure in the fluid,  $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\varphi\mathbf{e}_z$  is the body-force vector, and  $\rho$  denotes the (variable) fluid density. The effect of the Earth's rotation is incorporated into the equations of motion through associating the orthogonal unit vector system  $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$  with a point fixed on the surface of the earth, which is rotating about its polar axis. This results in additional Coriolis force ( $2\boldsymbol{\Omega} \times \mathbf{u}$ ) and centripetal acceleration ( $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ ) terms. With  $\mathbf{r} = r\mathbf{e}_r$ ,  $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$ , and  $\boldsymbol{\Omega} = -\Omega(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)$ , where  $\Omega \approx 7.29 \times 10^{-5}$  rad  $s^{-1}$  is the constant rate of rotation of the Earth, the additional rotational terms which must be added to the left-hand side of (2.1) are given by

$$2\Omega(-w \cos\theta\mathbf{e}_r + w \sin\theta\mathbf{e}_\theta + (u \cos\theta - v \sin\theta)\mathbf{e}_z) + r\Omega^2(-(\cos\theta)^2\mathbf{e}_r + \sin\theta \cos\theta\mathbf{e}_\theta).$$

Assuming that the body-force is due only to gravity, the Euler equation is therefore

$$\begin{aligned}
 (2.3) \quad & u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} - 2w\Omega \cos\theta - r\Omega^2 \cos^2\theta = -\frac{1}{\rho}p_r - g \\
 & v_t + uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} + 2w\Omega \sin\theta + r\Omega^2 \sin\theta \cos\theta = -\frac{1}{\rho}\frac{1}{r}p_\theta \\
 & w_t + ww_r + \frac{v}{r}w_\theta + ww_z + 2\Omega(u \cos\theta - v \sin\theta) = -\frac{1}{\rho}p_z.
 \end{aligned}$$

Complementing the GFD governing equations (2.2) and (2.3) are boundary conditions associated with the free surface and the sea-bed: the free-surface is denoted  $r = R + h(\theta, \varphi)$ , where  $R \approx 6378$  km is the radius of the earth and  $h(\theta, \varphi)$  represents the deviation of the free-surface from a perfect sphere; the bottom of the ocean is an impermeable solid boundary denoted by the equation  $r = d(\theta, \varphi)$ . At the free-surface we have the kinematic boundary condition

$$(2.4) \quad u = wh_z + \frac{1}{r}vh_\theta$$

together with the dynamic boundary condition

$$(2.5) \quad p = P(\theta).$$

The kinematic boundary condition on the sea-bed is given by

$$(2.6) \quad u = wd_z + \frac{1}{r}vd_\theta,$$

where we observe that (2.6) holds trivially for flows which are motionless at great depths.

### 3. Existence of exact, stratified flows

In this section we derive an exact solution of the GFD governing equations (2.2)–(2.3) for a stratified fluid, where the fluid density exhibits a vertical variation which is a linear function of depth. The fluid density is therefore prescribed as  $\rho = b - ar$ , for  $a, b > 0$  any constants such that  $\rho > 0$ . The exact solution we present has a simplified prescription in the sense that it is a steady flow moving in the azimuthal direction, with no variations in this direction. Hence, the velocity field  $(u, v, w)$  satisfies

$$(3.1) \quad u = v = 0 \quad \text{and} \quad w = w(r, \theta),$$

with the free-surface described by  $r = R + h(\theta)$  and the sea-bed represented as  $r = d(\theta)$ . For equatorial flows the polar angle  $\theta$  is confined to an interval  $[-\varepsilon, \varepsilon]$ , for some  $\varepsilon > 0$  whose choice is motivated by geophysical considerations: setting  $\varepsilon = 0.016$  corresponds to a strip of 100km width about the Equator, cf. [8]. With this form of velocity field, the system (2.3) becomes

$$(3.2) \quad \begin{cases} -2w\Omega \cos \theta - r\Omega^2 \cos^2 \theta & = -\frac{1}{\rho}p_r - g \\ 2w\Omega \sin \theta + r\Omega^2 \sin \theta \cos \theta & = -\frac{1}{\rho}\frac{1}{r}p_\theta \\ 0 & = p_z, \end{cases}$$

and we see immediately that the equation of mass conservation (2.2) as well as the kinematic boundary conditions (2.4) and (2.6) are automatically satisfied. Differentiating the first equation in (3.2) with respect to  $\theta$  and the second with respect to  $r$  we obtain

$$(3.3) \quad -2\Omega w_\theta \cos \theta + 2\Omega w \sin \theta + 2r\Omega^2 \cos \theta \sin \theta = -\frac{1}{\rho}p_{r\theta}$$

and

$$(3.4) \quad 2\Omega r w_r \sin \theta + r\Omega^2 \sin \theta \cos \theta = -\frac{1}{\rho}p_{\theta r} + \frac{1}{\rho} \cdot \frac{1}{r}p_\theta + \frac{\rho_r}{\rho^2}p_\theta,$$

which, after inserting the expression for  $\frac{1}{r}p_\theta$  from the second equation in (3.2), becomes

$$(3.5) \quad 2\Omega r w_r \sin \theta + 2\Omega w \sin \theta + 2r\Omega^2 \sin \theta \cos \theta = -\frac{1}{\rho}p_{\theta r} + \frac{\rho_r}{\rho^2}p_\theta.$$

Subtracting (3.3) from (3.5) we obtain

$$2\Omega r w_r \sin \theta + 2\Omega w_\theta \cos \theta = \frac{\rho_r}{\rho^2}p_\theta,$$

which can be rewritten as

$$2r w_r \sin \theta + 2w_\theta \cos \theta = -\frac{r\rho_r \sin \theta}{\rho}(2w + \Omega r \cos \theta).$$

Using the specific choice of the density distribution  $\rho(r) = b - ar$ , the latter equation becomes

$$2rw_r \sin \theta + 2w_\theta \cos \theta = \frac{ar \sin \theta}{b - ar} (2w + \Omega r \cos \theta).$$

This equation may be solved by way of the method of characteristics: defining

$$r = r(s), \quad \theta = \theta(s), \quad z(s) = w(r(s), \theta(s))$$

with

$$\dot{r}(s) = 2r(s) \sin \theta(s), \quad \dot{\theta}(s) = 2 \cos(\theta(s))$$

we obtain the equation

$$(3.6) \quad \dot{z}(s) = \frac{ar \sin \theta(s)}{b - ar(s)} (2z(s) + \Omega r(s) \cos \theta(s)).$$

Noticing that

$$\frac{d}{ds} (r(s) \cos \theta(s)) = 0 \quad \text{for all } s,$$

we can recast equation (3.6) as

$$\frac{1}{2} \frac{d}{ds} (\ln(2z(s) + \Omega r(s) \cos \theta(s))) = -\frac{1}{2} \frac{d}{ds} (\ln(b - ar(s))) \quad \text{for all } s,$$

whose general solution is

$$(3.7) \quad w(r, \theta) = -\frac{\Omega r \cos \theta}{2} + \frac{F(r \cos \theta)}{2(b - ar)},$$

for some function  $F$ .

REMARK 3.1. The impact of stratification on the form of the azimuthal velocity (3.7) is immediately apparent: the second term is essentially divided by the non-constant density  $\rho$ . This differs quite significantly from the homogeneous fluid regime [8].

REMARK 3.2. It is clear from (3.7) that the azimuthal flow velocity is determined by prescribing it at the equator: if the equatorial flow is given by  $w(r, 0) = W(r)$ , then we simply choose  $F(r) = (b - ar)(2W(r) + \Omega r)$ , and the azimuthal flow velocity  $w(r, \theta)$  in the neighbouring equatorial region is then prescribed by (3.7). More precisely, we have the following relation between the above mentioned flow velocities:

$$(3.8) \quad w(r, \theta) = w(r, 0) \cos \theta + \frac{1}{2(b - ar)} [F(r \cos \theta) - F(r) \cos \theta].$$

This is particularly relevant in the context of adapting (3.7) to provide an elementary model for equatorial flows; cf. [8] for the homogeneous fluid setting.

Having prescribed the velocity field in the stratified fluid by way of (3.1), (3.7), attention must now be focused on the associated fluid pressure distribution. Substituting the expressions (3.1), (3.7) for the velocity field into (3.2), the gradient of the pressure is expressed as  $p_z = 0$ ,

$$p_r = \Omega(\cos \theta)F(r \cos \theta) - (b - ar)g,$$

and

$$p_\theta = -\Omega r(\sin \theta)F(r \cos \theta).$$

This can be solved directly leading to the following formulation for the pressure function  $p$ :

$$p(r, \theta) = A - gbr + \frac{agr^2}{2} + \Omega \left[ \int_{c \cos \theta}^{r \cos \theta} F(y) dy - \int_0^\theta c \sin \tilde{\theta} F(c \cos \tilde{\theta}) d\tilde{\theta} \right],$$

where  $A$  and  $c$  are real constants. This expression, in combination with the dynamic boundary condition on the free-surface (2.5), prescribe the relationship between the imposed pressure at the surface of the ocean and the resulting deformation of that surface. The associated Bernoulli-type relation at the surface is given by

$$(3.9) \quad P(\theta) = A - gb[R + h(\theta)] + \frac{ga[R + h(\theta)]^2}{2} + \Omega \left[ \int_{c \cos \theta}^{[R+h(\theta)] \cos \theta} F(y) dy - \int_0^\theta c \sin \tilde{\theta} F(c \cos \tilde{\theta}) d\tilde{\theta} \right].$$

#### 4. Functional analysis of the Bernoulli relation

Although the velocity field of the exact solution assumes a relatively simplified form (whereby two of the components are zero), the corresponding Bernoulli relation (3.9) is highly convoluted and must be subjected to a careful analysis: this is the aim of this section. In particular, relation (3.9) prescribes the surface pressure in terms of the azimuthal velocity  $w$  and the free-surface  $h$ . We will recast (3.9) in a form which is amenable to an application of the implicit function theorem [14], thereby establishing that the distortion of the free surface, implicitly determined by way of (3.9), is uniquely prescribed by a given pressure distribution  $P(\theta)$ . This result is achieved through performing a nondimensionalisation procedure that allows a meaningful comparison of the involved physical quantities, and which will lead to an expression which characterises the relation between variations of the pressure at the free surface and variations of the shape of the free surface. This relation will then be subjected to a theoretical analysis, without resorting to approximation, by means of the implicit function theorem.

REMARK 4.1. A special solution can be obtained which determines the pressure required to maintain the free surface undisturbed, that is, a surface shadowing the curvature of the Earth away from the Equator. For this purpose we set  $h \equiv 0$  in (3.9) and obtain the pressure necessary to maintain the unperturbed free surface as

$$(4.1) \quad P_0(\theta) = A - gbR + \frac{gaR^2}{2} + \Omega \left[ \int_{c \cos \theta}^{R \cos \theta} F(y) dy - \int_0^\theta c \sin \tilde{\theta} F(c \cos \tilde{\theta}) d\tilde{\theta} \right].$$

Consequently, if the pressure at the Equator (described by  $\theta = 0$ ) equals the atmospheric pressure  $P_{atm}$ , we obtain the expression

$$(4.2) \quad P_{atm} = A - gbR + \frac{gaR^2}{2} + \Omega \int_c^R F(y) dy.$$

The identity (4.2) provides a characterisation of the atmospheric pressure, and it is this quantity that we divide (3.9) by in order to nondimensionalise it, leading

to:

$$(4.3) \quad \alpha - \beta[1 + \mathfrak{h}(\theta)] + \gamma[1 + \mathfrak{h}(\theta)]^2 + \frac{\Omega}{P_{atm}} \left[ \int_{c \cos \theta}^{[1+\mathfrak{h}(\theta)]R \cos \theta} F(y) dy - \int_0^\theta c \sin \tilde{\theta} F(c \cos \tilde{\theta}) d\tilde{\theta} \right] - \mathfrak{P}(\theta) = 0.$$

Here  $\mathfrak{h}$  and  $\mathfrak{P}$  are nondimensional functions, defined as

$$\mathfrak{h}(\theta) := \frac{h(\theta)}{R}, \quad \mathfrak{P}(\theta) := \frac{P(\theta)}{R},$$

and the nondimensional constants  $\alpha, \beta$  and  $\gamma$  are

$$\alpha := \frac{A}{P_{atm}}, \quad \beta := \frac{gbR}{P_{atm}}, \quad \gamma := \frac{gaR^2}{2P_{atm}}.$$

Defining the functional

$$\mathcal{F}(\mathfrak{h}, \mathfrak{P}) := \alpha - \beta[1 + \mathfrak{h}(\theta)] + \gamma[1 + \mathfrak{h}(\theta)]^2 + \frac{\Omega}{P_{atm}} \left[ \int_{c \cos \theta}^{[1+\mathfrak{h}(\theta)]R \cos \theta} F(y) dy - \int_0^\theta c \sin \tilde{\theta} F(c \cos \tilde{\theta}) d\tilde{\theta} \right] - \mathfrak{P}(\theta),$$

it follows that  $\mathcal{F}$  defines a continuously differentiable map

$$\mathcal{F}(\mathfrak{h}, \mathfrak{P}) : C([0, \varepsilon]) \rightarrow C([0, \varepsilon]),$$

where  $C([0, \varepsilon])$  denotes the space of continuous functions  $f : [0, \varepsilon] \rightarrow \mathbb{R}$  endowed with the (usual) supremum norm, and  $B$  denotes the open ball  $\{f \in C([0, \varepsilon]) : \|f\| < 10^{-2}\}$ . Moreover, the identity (4.3) can be formulated as the functional equation

$$(4.4) \quad \mathcal{F}(\mathfrak{h}, \mathfrak{P}) = 0,$$

from which it follows immediately that  $\mathcal{F}(0, \mathfrak{P}_0) = 0$  where

$$\mathfrak{P}_0(\theta) := \alpha - \beta + \gamma + \frac{\Omega}{P_{atm}} \left[ \int_{c \cos \theta}^{R \cos \theta} F(y) dy - \int_0^\theta c \sin \tilde{\theta} F(c \cos \tilde{\theta}) d\tilde{\theta} \right]$$

is the non-dimensionalised version of relation (4.1), which represents the imposed surface–pressure distribution required to maintain an undisturbed free–surface. In order to employ the implicit function theorem we compute the derivative

$$D_{\mathfrak{h}} \mathcal{F}(0, \mathfrak{P}_0)(\mathfrak{h}) := \lim_{s \rightarrow 0} \frac{\mathcal{F}(s\mathfrak{h}, \mathfrak{P}_0) - \mathcal{F}(0, \mathfrak{P}_0)}{s}.$$

A straightforward calculation shows that

$$\begin{aligned} D_{\mathfrak{h}} \mathcal{F}(0, \mathfrak{P}_0)(\mathfrak{h}) &= \left( -\beta + 2\gamma + \frac{\Omega R \cos \theta}{P_{atm}} F(R \cos \theta) \right) \mathfrak{h} \\ &= \frac{b - aR}{P_{atm}} [-gR + \Omega R \cos \theta (2w(R, \theta) + \Omega R \cos \theta)] \mathfrak{h}, \end{aligned}$$

where the last equality is obtained though using formula (3.7) and the definition of the constants  $\alpha, \beta, \gamma$ . From a comparison of the physical orders of magnitude, it is clear that there exists a constant  $e < 0$  such that

$$-gR + \Omega R(\cos \theta)(2w + \Omega R \cos \theta) \leq e < 0.$$

We infer from the latter inequality that

$$D_{\mathfrak{h}}\mathcal{F}(0, \mathfrak{P}_0) : C([0, \varepsilon]) \rightarrow C([0, \varepsilon])$$

is a surjective linear homeomorphism, and hence the implicit function theorem may be applied. We formulate the result as follows.

**THEOREM 4.2.** *For any sufficiently small deviation  $\mathfrak{P}$  from  $\mathfrak{P}_0$ , there exists a unique  $\mathfrak{h} \in C([0, \varepsilon])$  such that*

$$\begin{aligned} &\alpha - \beta[1 + \mathfrak{h}(\theta)] + \gamma[1 + \mathfrak{h}(\theta)]^2 \\ &+ \frac{\Omega}{P_{atm}} \left[ \int_{c \cos \theta}^{[1 + \mathfrak{h}(\theta)]R \cos \theta} F(y) dy - \int_0^\theta c(\sin \tilde{\theta}) F(c \cos \tilde{\theta}) d\tilde{\theta} \right] - \mathfrak{P}(\theta) = 0. \end{aligned}$$

As a by-product of the identities derived above, we can establish that the relationship between  $\mathfrak{P}$  and  $\mathfrak{h}$  prescribed by (3.9) exhibits the expected monotonicity properties. We first observe that the smoothness properties established for  $\mathfrak{P}$  can be transferred to  $\mathfrak{h}$  via an iterative bootstrapping procedure, cf. [2]. Thus, we can differentiate with respect to  $\theta$  in relation (4.3) and obtain that

$$\begin{aligned} \mathfrak{P}' &= \left[ -\beta + 2\gamma(1 + h) + \frac{\Omega R \cos \theta}{P_{atm}} F((1 + \mathfrak{h})R \cos \theta) \right] \mathfrak{h}' \\ &\quad - \frac{\Omega R \sin \theta}{P_{atm}} (1 + \mathfrak{h}) F((1 + \mathfrak{h})R \cos \theta) \\ &= \frac{\rho(R + h)}{P_{atm}} \left\{ \left[ -gR + \Omega R \cos \theta (2w(R + h, \theta) + \Omega(R + h) \cos \theta) \right] \mathfrak{h}' \right. \\ &\quad \left. - \Omega(R + h) \sin \theta (2w(R + h, \theta) + \Omega(R + h) \cos \theta) \right\} \end{aligned}$$

The latter relation allows us to infer that

$$(4.5) \quad \mathfrak{P}'(\theta) < 0 \quad \text{if} \quad \mathfrak{h}'(\theta) \geq 0 \quad \text{for some} \quad \theta \in (0, \varepsilon),$$

and

$$(4.6) \quad \mathfrak{h}'(\theta) < 0 \quad \text{if} \quad \mathfrak{P}'(\theta) \geq 0 \quad \text{for some} \quad \theta \in (0, \varepsilon).$$

The mathematical relations (4.5) and (4.6) establish that  $\mathfrak{P}$  and  $\mathfrak{h}$  possess monotonicity properties which concur with physical expectations.



### 5. Local stability of perturbations along the azimuthal flow

We conclude the presentation of the exact solution (3.1) and (3.7) by subjecting it to a short-wavelength stability analysis. The short-wavelength stability approach, developed independently by Bayly [1], Friedlander and Vishik [16] and Lifschitz and Hameiri [36], applies for general three-dimensional flows, and it examines the time growth of the amplitude of perturbations to basic flows having a velocity field  $\mathbf{u}$  which obeys the Euler equations (2.1) and the equation of mass conservation (2.2). More precisely, the basic flow  $\mathbf{u}$  is called stable with respect to the short-wavelength perturbation (5.3)-(5.4) if, for any initial data, the amplitude  $\mathbf{A}$  is uniformly bounded in time. Recent studies concerning the short-wavelength stability/instability of geophysical water flows were undertaken in [6, 17, 22, 23, 27, 28, 30, 31]; cf. the survey paper [29].

Let  $\mathbf{U} = U\mathbf{e}_r + V\mathbf{e}_\theta + W\mathbf{e}_z$  be a perturbation of the azimuthal flow whose velocity field is  $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$  with  $u, v, w$  given by (3.1) and (3.7). This amounts to seeking  $U, V, W$  and a pressure function  $P$  such that  $\mathbf{U} + \mathbf{u}, P + p$  satisfy (2.1), (2.2). Ignoring quadratic terms, we see that  $\mathbf{U}$  and  $P$  satisfy

$$(5.1) \quad \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U} + 2(\boldsymbol{\Omega} \times \mathbf{U}) = -\frac{\nabla P}{\rho},$$

$$(5.2) \quad \nabla \cdot (\rho \mathbf{U}) = 0,$$

subjected to an initial disturbance  $\mathbf{U}|_{t=0} = \mathbf{U}_0$ . We employ the (WKB) Ansatz, that is, we seek  $\mathbf{U}$  and  $P$  solutions of (5.1) and (5.2) having the specific form

$$(5.3) \quad \mathbf{U}(t, r, \theta, z) = \mathbf{A}(t, r, \theta, z)e^{\frac{i}{\epsilon}f(t, r, \theta, z)} + \mathcal{O}(\epsilon)$$

$$(5.4) \quad P(t, r, \theta, z) = \epsilon B(t, r, \theta, z)e^{\frac{i}{\epsilon}f(t, r, \theta, z)} + \mathcal{O}(\epsilon^2),$$

where

$$\mathbf{A}(t, r, \theta, z) = A_1(t, r, \theta, z)\mathbf{e}_r + A_2(t, r, \theta, z)\mathbf{e}_\theta + A_3(t, r, \theta, z)\mathbf{e}_\varphi,$$

and  $f = f(t, r, \theta, z)$  is a scalar function and  $\epsilon$  plays the role of a small parameter.

REMARK 5.1. In [24] it is proven that the remainder terms in (5.3) and (5.4) are well behaved, meaning that their  $L^2$  norm is uniformly bounded (with respect to  $\epsilon$ ) on any time interval  $[0, T]$ , (with  $T > 0$ ), for given initial data. Hence the stability of the basic flow is determined by the boundedness of the amplitude  $\mathbf{A}$  in (5.3), cf. [36]. The time growth of the amplitude  $\mathbf{A}$  will be analysed in the remaining part of the paper.

We note that, given the particular formulation of the flow field  $u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$  prescribed by (3.1), we can compute

$$(5.5) \quad \begin{aligned} (\mathbf{u} \cdot \nabla)\mathbf{U} &= \left( uU_r + \frac{v}{r}U_\theta + wU_z - \frac{vV}{r} \right) \mathbf{e}_r + \left( uV_r + \frac{v}{r}V_\theta + wV_z + \frac{vU}{r} \right) \mathbf{e}_\theta \\ &+ \left( uW_r + \frac{v}{r}W_\theta + wW_z \right) \mathbf{e}_z = w(U_z\mathbf{e}_r + V_z\mathbf{e}_\theta + W_z\mathbf{e}_z) \end{aligned}$$

and

$$(5.6) \quad \begin{aligned} (\mathbf{U} \cdot \nabla) \mathbf{u} &= \left( Uu_r + \frac{V}{r}u_\theta + Wu_z - \frac{Vv}{r} \right) \mathbf{e}_r + \left( Uv_r + \frac{V}{r}v_\theta + Wv_z + \frac{Vu}{r} \right) \mathbf{e}_\theta \\ &+ \left( Uw_r + \frac{V}{r}w_\theta + Ww_z \right) \mathbf{e}_z = \left( Uw_r + \frac{V}{r}w_\theta \right) \mathbf{e}_z. \end{aligned}$$

Inserting the WKB Ansatz (5.3)-(5.4) in (5.1), and taking into account (5.5) and (5.6), we obtain, after suitable identifications of the coefficients,

$$(5.7) \quad \begin{aligned} A_{1,t} + wA_{1,z} - 2\Omega A_3 \cos \theta &= -i \frac{B}{\rho} f_r \\ A_{2,t} + wA_{2,z} + 2\Omega A_3 \sin \theta &= -i \frac{B}{\rho} \frac{f_\theta}{r} \\ A_{3,t} + wA_{3,z} + A_1 w_r + \frac{A_2}{r} w_\theta + 2\Omega(A_1 \cos \theta - A_2 \sin \theta) &= -i \frac{B}{\rho} f_z \end{aligned}$$

and

$$(5.8) \quad A_1(f_t + wf_z) = A_2(f_t + wf_z) = A_3(f_t + wf_z) = 0.$$

Since the vector  $\mathbf{A}$  is not zero, we have from (5.8) that

$$(5.9) \quad f_t + wf_z = 0.$$

To solve for  $f$  in (5.9) we notice that the position vector of a particle is given by

$$r(t)\mathbf{e}_r + z(t)\mathbf{e}_z.$$

Therefore,

$$\begin{aligned} \mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z &= \frac{d}{dt}(r(t)\mathbf{e}_r + z(t)\mathbf{e}_z) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r + \dot{z}\mathbf{e}_z \\ &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z. \end{aligned}$$

Thus, the equations of a streamline for the azimuthal flow (3.1) and passing through  $(r_0, \theta_0, z_0)$  are

$$\frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = 0, \quad \frac{dz}{dt} = w,$$

$$r(0) = r_0, \quad \theta(0) = \theta_0, \quad z(0) = z_0,$$

whose general solution is

$$(5.10) \quad r(t) \equiv r_0, \quad \theta(t) \equiv \theta_0, \quad z(t) = \int_0^t w(r(s), \theta(s)) ds + z_0.$$

Moreover, the equation (5.9) satisfied by the phase  $f$  has the general solution

$$(5.11) \quad f = \mathfrak{F} \left( z - \int_0^t w(r(s), \theta(s)) ds \right).$$

We see now immediately from (5.10)-(5.11) that  $f$  is constant along the streamlines of the azimuthal flow (3.1). The latter renders the system (5.7) satisfied by

the amplitude  $\mathbf{A}$  of the perturbation  $\mathbf{U}$  to

$$A_{1,t} + wA_{1,z} - 2\Omega A_3 \cos \theta = 0,$$

$$A_{2,t} + wA_{2,z} + 2\Omega A_3 \sin \theta = 0,$$

$$A_{3,t} + wA_{3,z} + A_1 w_r + \frac{A_2}{r} w_\theta + 2\Omega(A_1 \cos \theta - A_2 \sin \theta) = 0.$$

Hence, along the streamlines (5.10) of the azimuthal flow (3.1), the amplitude  $\mathbf{A}$  satisfies

$$\left\{ \begin{array}{l} \frac{d}{dt} A_1(t, r(t), \theta(t), z(t)) = 2\Omega A_3 \cos \theta_0, \\ \frac{d}{dt} A_2(t, r(t), \theta(t), z(t)) = -2\Omega A_3 \sin \theta_0, \\ \frac{d}{dt} A_3(t, r(t), \theta(t), z(t)) = -A_1 w_r(r_0, \theta_0) - \frac{A_2}{r_0} w_\theta(r_0, \theta_0) - 2\Omega(A_1 \cos \theta_0 - A_2 \sin \theta_0), \end{array} \right.$$

and noticing that

$$\frac{d}{dt} A_1(t, r(t), \theta(t), z(t)) = -(\cot \theta_0) \frac{d}{dt} A_2(t, r(t), \theta(t), z(t))$$

we can rewrite the above system as

$$\begin{pmatrix} \frac{d}{dt} A_2(t, r(t), \theta(t), z(t)) \\ \frac{d}{dt} A_3(t, r(t), \theta(t), z(t)) \end{pmatrix} = \mathcal{M} \begin{pmatrix} A_2(t, r(t), \theta(t), z(t)) \\ A_3(t, r(t), \theta(t), z(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ d \end{pmatrix},$$

where  $d$  is a constant depending on the initial data, and

$$\mathcal{M} = \left( \begin{array}{c|c} 0 & -2\Omega \sin \theta_0 \\ \hline -\frac{w_\theta(r_0, \theta_0)}{r_0} + w_r(r_0, \theta_0) \cot \theta_0 + \frac{2\Omega}{\sin \theta_0} & 0 \end{array} \right).$$

While the preceding considerations pertain to a flow with general azimuthal velocity given by (3.7), at this point, in order to analyse the eigenvalues of  $\mathcal{M}$ , we particularise to the flow

$$w(r, \theta) = -\frac{\Omega r \cos \theta}{2} + \frac{cr \cos \theta}{2(b - ar)},$$

for some constant  $c \in \mathbb{R}$ , for which we can compute

$$-\frac{w_\theta(r_0, \theta_0)}{r_0} = \frac{\sin \theta_0}{2} \left( -\Omega + \frac{c}{b - ar_0} \right),$$

and

$$w_r(r_0, \theta_0) \cot \theta_0 = \frac{\cos^2 \theta_0}{2 \sin \theta_0} \left( -\Omega + \frac{bc}{(b - ar_0)^2} \right).$$

It follows that the eigenvalues  $\lambda$  of  $\mathcal{M}$  satisfy the equation

$$(5.12) \quad \lambda^2 + 3\Omega^2 + \frac{c\Omega(b - ar_0 \sin^2 \theta_0)}{(b - ar_0)^2} = 0.$$

Since  $b - ar_0 \sin^2 \theta_0 > b - ar_0$ , the roots of (5.12) are purely imaginary in the case  $c > -3\frac{(b - ar_0)^2}{b - ar_0 \sin^2 \theta_0}$ . This leads us to the following result:

THEOREM 5.2. *The azimuthal flow with*

$$u = v = 0, \quad w(r, \theta) = \frac{-\Omega r \cos \theta}{2} + \frac{cr \cos \theta}{2(b - ar)}$$

*is linearly stable under short-wavelength perturbations for all values  $c$  such that*

$$c > -3\Omega \frac{(b - ar_0)^2}{b - ar_0 \sin^2 \theta_0}.$$

*It is (linearly) unstable if*

$$c < -3\Omega \frac{(b - ar_0)^2}{b - ar_0 \sin^2 \theta_0},$$

*where  $r_0$  and  $\theta_0$  represent the initial values of the streamlines of the azimuthal flow.*

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