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Asymptotic stability of harmonic maps between 2D hyperbolic spaces under the wave map equation. II. Small energy case

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ABSTRACT. In this paper, we prove that the small energy harmonic maps from $\mathbb{H}^2$ to $\mathbb{H}^2$ are asymptotically stable under the wave map equation in the subcritical perturbation class. This result may be seen as an example supporting the soliton resolution conjecture for geometric wave equations without equivariant assumptions on the initial data. In this paper, we construct Tao's caloric gauge in the case when nontrivial harmonic map occurs. With the "dynamic separation" the master equation of the heat tension field appears as a semilinear magnetic wave equation. By the endpoint and weighted Strichartz estimates for magnetic wave equations obtained by the first author [**38**], the asymptotic stability follows by a bootstrap argument.

Contents

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1. Introduction

Let (M, h) and (N, g) be two Riemannian manifolds without boundary. A wave map is a map from the Lorentz manifold $\mathbb{R} \times M$ into N,

$$u : \mathbb{R} \times M \to N,$$

which is locally a critical point for the functional

$$F(u) = \int_{\mathbb{R}\times M} \Big(-\langle \partial_t u, \partial_t u \rangle_{u^* g} + h^{ij} \big\langle \partial_{x_i} u, \partial_{x_j} u \big\rangle_{u^* g} \Big) \mathrm{dtdvol_h}. \tag{1.1}$$

Here $h_{ij} dx^i dx^j$ is the metric tension under a local coordinate $(x^1, ..., x^m)$ for M. In a coordinate free expression, the integrand in the functional $F(u)$ is the energy density of u under the Lorentz metric of $\mathbb{R} \times M$,

$$\eta = -dt \otimes dt + h_{ij} dx^i \otimes dx^j.$$

Given a local coordinate $(y^1, ..., y^n)$ for N, the Euler-Lagrange equation for (1.1) is given by

$$\Box u^k + \eta^{\alpha\beta} \overline{\Gamma}^k_{ij}(u) \partial_\alpha u^i \partial_\beta u^j = 0, \tag{1.2}$$

where $\Box = -\partial_t^2 + \Delta_M$ is the D'Alembertian on $\mathbb{R} \times M$, $\overline{\Gamma}^k_{ij}(u)$ are the Christoffel symbols at the point $u(t, x) \in N$. In this paper, we consider the case $M = \mathbb{H}^2$, $N = \mathbb{H}^2$.

The wave map equation on a flat spacetime, which is sometimes known as the nonlinear σ-model, arises as a model problem in general relativity and particle physics, see for instance [**39**]. The wave map equation on curved spacetime is related to the wave map-Einstein system and the Kerr Ernst potential, see [**1, 17, 13**]. We remark that the case where the background manifold is the hyperbolic space is of particular interest. Indeed, the anti-de Sitter space (AdSn), which is the exact solution of Einstein's field equation for an empty universe with a negative cosmological constant, is asymptotically hyperbolic.

There exist plenty of works on the Cauchy problem, the long dynamics and blow up for wave maps on $\mathbb{R}^{1+m}$. We first recall the non-exhaustive lists of results on equivariant maps. The critical well-posedness theory was initially considered by Christodoulou, Tahvildar-Zadeh [**7**] for radial wave maps and Shatah, Tahvildar-Zadeh [**47**] for equivariant wave maps. The global well-posedness result of [**7**] was recently improved to scattering by Chiodaroli, Krieger, Luhrmann [**6**]. The bubbling theorem of wave maps was proved by Struwe [**51**]. The explicit construction of blow up solutions behaving as a perturbation of the rescaling harmonic map was achieved by Krieger, Schlag, Tataru [**29**], Raphael, Rodnianski [**41**], and Rodnianski, Sterbenz [**43**] for the $\mathbb{S}^2$ target in the equivariant class. And the ill-posedness theory was studied in D'Ancona, Georgiev [**10**] and Tao [**52**].

Without equivariant assumptions on the initial data the sharp subcritical well-posedness theory was developed by Klainerman, Machedon [**22, 23**] and Klainerman, Selberg [**25**]. The small data critical case was started by Tataru [**57**] in the critical Besov space, and then completed by Tao [**53, 54**] for wave maps from $\mathbb{R}^{1+d}$ to $\mathbb{S}^m$ in the critical Sobolev space. The small data theory in critical Sobolev space for general targets was considered by Krieger [**27, 26**], Klainerman, Rodnianski [**24**], Shatah, Struwe [**44**], Nahmod, Stefanov, Uhlenbeck [**40**], and Tataru [**59**].

The dynamic behavior for wave maps on $\mathbb{R}^{1+2}$ with general data was obtained by Krieger, Schlag [**28**] for the $\mathbb{H}^2$ targets, Sterbenz, Tataru [**49, 48**] for compact Riemann manifolds and initial data below the threshold, and Tao [**55**] for the $\mathbb{H}^n$ targets. In fact, Sterbenz, Tataru [**49, 48**] proved that for any initial data with energy less than that of the minimal energy nontrivial harmonic map evolves to a global and scattering solution.

The works on the wave map equations on curved spacetime were relatively less. The existence and orbital stability of equivariant time periodic wave maps from $\mathbb{R} \times \mathbb{S}^2$ to $\mathbb{S}^2$ were proved by Shatah, Tahvildar-Zadeh [**46**], see Shahshahani [**51**] for an generalization of $\mathbb{S}^2$. The critical small data Cauchy problem for wave maps on small asymptotically flat perturbations of $\mathbb{R}^4$ to compact Riemann manifolds was studied by Lawrie [**30**]. The soliton resolution and asymptotic stability of harmonic maps under wave maps on $\mathbb{H}^2$ to $\mathbb{S}^2$ or $\mathbb{H}^2$ in the 1-equivariant case were established by Lawrie, Oh, Shahshahani [**31, 32, 34, 35**], see also [**33**] for critical global well-posedness for wave maps from $\mathbb{R} \times \mathbb{H}^d$ to compact Riemann manifolds with $d \geq 4$.

In this paper, we study the asymptotic stability of harmonic maps to (1.1). The motivation is the so called soliton resolution conjecture in dispersive PDEs which claims that every global bounded solution splits into the superposition of divergent solitons with a radiation part plus an asymptotically vanishing remainder term as $t \to \infty$. The version for wave maps and hyperbolic Yang-Mills has been verified by Cote [**9**] and Jia, Kenig [**19**] for equivariant maps along a time sequence, see also [**20, 21**] for exotic-ball wave maps and [**42**] for wormholes. Recently Duyckaerts, Jia, Kenig, Merle [**11**] obtained the universal blow up profile for type II blow up solutions to wave maps $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{S}^2$ with initial data of energy slightly above the ground state. For wave maps from $\mathbb{R} \times \mathbb{H}^2$ to $\mathbb{H}^2$, Lawrie, Oh, Shahshahani [**33, 34**] raised the following soliton resolution conjecture,

Conjecture 1.1 Consider the Cauchy problem for wave map $u : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$ with finite energy initial data (u_0, u_1). Suppose that outside some compact subset $\mathcal{K}$ of $\mathbb{H}^2$ for some harmonic map $Q : \mathbb{H}^2 \to \mathbb{H}^2$ we have

$$u_0(x) = Q(x), \text{ for } x \in \mathbb{H}^2 \backslash \mathcal{K}.$$

Then the unique solution $(u(t), \partial_t u(t))$ to the wave map scatters to $(Q(x), 0)$ as $t \to \infty$.

In this paper, we consider the easiest case of Conjecture 1.1, i.e., when the initial data is a small perturbation of harmonic maps with small energy. In order to state our main result, we introduce the notion of admissible harmonic maps.

DEFINITION 1.1. Let $D = \{z : |z| < 1\}$ with the hyperbolic metric be the Poincare disk. We say the harmonic map $Q : D \to D$ is admissible if $Q(D)$ is a compact subset of D covered by a geodesic ball centered at 0 of radius R_0, $\|\nabla^k dQ\|_{L^2} < \infty$ for $k = 0, 1, 2$, and there exists some $\varrho > 0$ such that $e^{\varrho r}|dQ| \in L^\infty$, where r is the distance between $x \in D$ and the origin point in D.

For any given admissible harmonic map Q, we define the space $\mathbf{H^k} \times \mathbf{H^{k-1}}$ by (2.8). Our main theorem is as follows.

THEOREM 1.2. *Fix any $R_0 > 0$. Assume the given admissible harmonic map Q in Definition 1.1 satisfies*

$$\|dQ\|_{L^2_x} < \mu_1, \ \|e^{\varrho r}|dQ|\|_{L^\infty_x} < \mu_1, \ \|\nabla^2 dQ\|_{L^\infty_x} + \|\nabla dQ\|_{L^\infty_x} < \mu_1. \tag{1.3}$$

And assume that the initial data $(u_0, u_1) \in \mathbf{H^3} \times \mathbf{H^2}$ *to (1.2) with* $u_0 : \mathbb{H}^2 \to \mathbb{H}^2$, $u_1(x) \in T_{u_0(x)}N$ *for each* $x \in \mathbb{H}^2$ *satisfy*

$$\|(u_0, u_1) - (Q, 0)\|_{\mathbf{H^2}\times\mathbf{H^1}} < \mu_2. \tag{1.4}$$

Then if $\mu_1 > 0$ *and* $\mu_2 > 0$ *are sufficiently small depending only on* R_0*, (1.2) has a global solution* $(u(t), \partial_t u(t))$ *which converges to the harmonic map* $Q : \mathbb{H}^2 \to \mathbb{H}^2$ *as* $t \to \infty$*, i.e.,*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{H}^2} d_{\mathbb{H}^2}(u(t,x), Q(x)) = 0.$$

The initial data considered in this paper are perturbations of harmonic maps in the $\mathbf{H^2}$ norm. If one considers perturbations in the energy critical norm H^1, the S_k v.s. N_k norm constructed by Tataru [**57**] and Tao [**54**] should be built for the hyperbolic background.

Remark 1.1 Notice that the limit harmonic map coincides with the unperturbed harmonic map in Theorem 1.1. The reason for this coincidence is that the $\mathbf{H^2}\times\mathbf{H^1}$ norm assume the initial data coincide with Q at the infinity, then the uniqueness of harmonic maps with prescribed boundary map shows the limit harmonic map is exactly the unperturbed one.

Remark 1.2(Examples for the admissible harmonic maps)
Denote $D = \{z : |z| < 1\}$ to be the Poincare disk. Then any holomorphic map $f : D \to D$ is a harmonic map. If we assume that $f(z)$ can be analytically extended into a larger disk than the unit disk, then $\mu_1 f : D \to D$ satisfies all the conditions in Definition 1.1 and Theorem 1.1 if $0 < \mu_1 \ll 1$. Hence the harmonic maps involved in Theorem 1.1 are relatively rich. See [Appendix,[**37**]] for the proof of these facts. It is important to see in theses examples that the dependence of μ_1 on R_0 is neglectable.

Remark 1.3(Examples for the perturbations of admissible harmonic maps) Since we have global coordinates for $\mathbb{H}^2$ given by (2.1), the perturbation in the sense of (1.4) is nothing but perturbations of $\mathbb{R}^2$-valued functions.

Since we are dealing with non-equivariant data where the linearization method seems to be hard to apply, we use the caloric gauge technique introduced by Tao [**56**] to prove Theorem 1.1. The caloric gauge of Tao was applied to solve the global regularity of wave maps from $\mathbb{R}^{2+1}$ to $\mathbb{H}^n$ in the heat-wave project. We briefly recall the main idea of the caloric gauge. Given a solution to the wave map $u(t,x) : \mathbb{R}^{1+2} \to \mathbb{H}^n$, suppose that $\widetilde{u}(s,t,x)$ solves the heat flow equation with initial data $u(t,x)$

$$\begin{cases} \partial_s \widetilde{u}(s,t,x) = \sum_{i=1}^2 \nabla_i \partial_i \widetilde{u} \\ \widetilde{u}(s,t,x) \restriction_{s=0} = u(t,x). \end{cases}$$

Since there exists no nontrivial finite energy harmonic map from $\mathbb{R}^2$ to $\mathbb{H}^n$, one can expect that the corresponding heat flow $\widetilde{u}(s,t,x)$ converges to a fixed point Q as $s \to \infty$. For any given orthonormal frame at the point Q, one can pullback the orthonormal frame parallel with respect to s along the heat flow to obtain the frame at $\widetilde{u}(s,t,x)$, particularly $u(t,x)$ when $s = 0$. Then rewriting (1.2) under the constructed frame will give us a scalar system for the differential fields and connection coefficients. Despite the fact that the caloric gauge can be viewed as a nonlinear Littlewood-Paley decomposition, the essential advantage of the caloric gauge is that it removes some troublesome frequency interactions, which is of fundamental importance for critical problems in low dimensions.

Generally the caloric gauge was used in the case where no harmonic map occurs, for instance energy critical geometric wave equations with energy below the threshold. In our case nontrivial harmonic exists no mater how small the data one considers. However, as observed in our work [**37**], the caloric gauge is still extraordinarily powerful. In fact, denoting the solution of the heat flow with initial data $u(0,x)$ by $U(s,x)$, it is known that $U(s,x)$ converges to some harmonic map $Q(x)$ as $s\to\infty$. And one can expect that the solution $u(t,x)$ of (1.2) also converges to the same harmonic map $Q(x)$ as $t\to\infty$. This heuristic idea combined with the caloric gauge reduces the convergence of solutions to (1.2) to proving the decay of the heat tension filed.

There are three main ingredients in our proof. The first is to guarantee that all the heat flows initiated from $u(t,x)$ for different t converge to the same harmonic map. This enables us to construct the caloric gauge. The second is to derive the master equation for the heat tension field, which finally reduces to a linear wave equation with a small magnetic potential. The third is to design a suitable closed bootstrap program. All these ingredients are used to overcome the difficulty that no integrability with respect to t is available for the energy density because the harmonic maps prevent the energy from decaying to zero as $t\to\infty$.

The key for the first ingredient is using the decay of $\partial_t u$ along the heat flow. In order to construct the caloric gauge, one has to prove the heat flow initiated from $u(t,x)$ converges to the same harmonic map independent of t. If one only considers t as a smooth parameter, i.e., in the homotopy class, the corresponding limit harmonic map yielded by the heat flow initiated from $u(t,x)$ can be different when t varies. Indeed, there exist a family of harmonic maps $\{Q_\lambda\}$ which depend smoothly with respect to $\lambda\in(0,1)$. Therefore the heat flow with initial data Q_λ remains to be Q_λ, which changes according to the variation of λ. This tells us the structure of (1.2) should be considered. The essential observation is $\partial_t u$ decays fast along the heat flow as $s\to\infty$. By a monotonous property observed initially by Hartman [**14**] and the decay estimates of the heat semigroup, we can prove the distance between the heat flows initiated from $u(t_1)$ and $u(t_2)$ goes to zero as $s\to\infty$. Therefore the limit harmonic map for the heat flow generated from $u(x,t)$ are all the same for different t. Similar idea works for the Landau-Lifshitz flow, see our paper [**37**]. And we remark that this part can be adapted to energy critical wave maps form $\mathbb{R}\times\mathbb{H}^2$ to $\mathbb{H}^2$ since essentially we only use the L^2_x norm of $\partial_t u$ in the arguments which is bounded by the energy.

Different from the usual papers on the asymptotic stability, we will not use the linearization arguments involving spectrum analysis of the linearized operator and modulation equations. But the master equation appears naturally as a semilinear wave equation with a small magnetic potential. Indeed, the main equation we need to consider is the nonlinear wave equation for the heat tension filed. The point is that although the nonlinear part of this equation is not controllable, one can separate part of them to be a magnetic potential with a remainder likely to be controllable. This is why we need the Strichartz estimates for magnetic wave equations.

The second ingredient is to control the remained terms in the nonlinear part of the master equation after we separate the magnetic potential away. In fact, the terms involving one order derivatives of the heat tension filed can not be controlled only by Strichartz estimates, even if we are working in the subcritical regularity. In this paper, the one order derivative terms are controlled by the weighted Strichartz estimates and the exotic Strichartz estimates owned only by hyperbolic backgrounds compared with the flat case. These estimates were obtained in the first author's work [**38**].

The third ingredient is to close the bootstrap, by which the global spacetime norm bounds of the heat tension field follows. The caloric gauge yields the gauged equation for the corresponding differential fields $\phi_{x,t}$, connection coefficients $A_{x,t}$ and the heat tension filed. It has been discovered in Tao [**55**] that the key field one needs to study is the heat tension field which satisfies a semilinear wave equation. And for the small data Cauchy problem of wave maps on $\mathbb{R}\times\mathbb{H}^4$, Lawrie, Oh, Shahshahani [**33**] shows in order to close the bootstrap arguments it suffices to firstly proving a global spacetime bound for the heat tension filed ϕ_s. In our case, since the energy will not decay, one has to get rid of the inhomogeneous terms which involve only the differential fields ϕ_x in the master equation. Furthermore, these troublesome terms involving only ϕ_x are much more serious in the study of the equation of wave map tension filed. This difficulty is overcome by using identities from intrinsic geometry to gain some cancelation and adding a space-time bound for $|\partial_t u|$ on the basis of the bootstrap arguments of [**33, 55**].

This paper is organized as follows. In Section 2, we recall some notations and notions and prove an equivalence between the intrinsic and extrinsic Sobolev norms in some sense. In Section 3, we construct the caloric gauge and obtain the estimates of the connection coefficients. In Section 4, we we derive the master equation. In Section 5, we first recall the non-endpoint and endpoint Strichartz estimates, Morawetz inequality, and weighted Strichartz estimates for the linear magnetic wave equation. Then we close the bootstrap and deduce the global spacetime bounds for the heat tension field. In Section 6, we finish the proof of Theorem 1.1. In Section 7, we prove some remaining claims in the previous sections.

We denote the constants by $C(M)$ and they can change from line to line. Small constants are usually denoted by δ and it may vary in different lemmas. $A\lesssim B$ means there exists some constant C such that $A\le CB$.

2. Preliminaries

Some standard preliminaries on the geometric notions of the hyperbolic spaces, Sobolev embedding inequalities and an equivalence relationship for the intrinsic and extrinsic formulations of the Sobolev spaces are recalled first. As a corollary we prove the local well-posedness for initial data (u_0,u_1) in the $\mathbf{H^3}\times\mathbf{H^2}$ regularity and a conditional global well-posedness proposition. In addition, the smoothing effect of heat semigroup is recalled.

2.1. The global coordinates and definitions of the function spaces.

The covariant derivative in TN is denoted by $\widetilde{\nabla}$, the covariant derivative induced by u in $u^*(TN)$ is denoted by ∇. We denote the Riemann curvature tension of N by $\mathbf{R}$. The components of Riemann metric are denoted by h_{ij} for M and g_{ij} for N respectively. The Christoffel symbols on M and N are denoted by Γ^k_{ij} and $\overline{\Gamma}^k_{ij}$ respectively.

We recall some facts on hyperbolic spaces. Let $\mathbb{R}^{1+2}$ be the Minkowski space with Minkowski metric $-(dx^0)^2+(dx^1)^2+(dx^2)^2$. Define a bilinear form on $\mathbb{R}^{1+2}\times$

$\mathbb{R}^{1+2}$,

$$[x, y] = x^0 y^0 - x^1 y^1 - x^2 y^2.$$

The hyperbolic space $\mathbb{H}^2$ is defined by

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2+1} : [x, x] = 1 \text{ and } x^0 > 0\},$$

with a Riemannian metric being the pullback of the Minkowski metric by the inclusion map $\iota : \mathbb{H}^2 \to \mathbb{R}^{1+2}$. By Iwasawa decomposition we have a global system of coordinates. Indeed, the diffeomorphism $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{H}^2$ is given by

$$\Psi(x_1, x_2) = (\cosh x_2 + e^{-x_2}|x_1|^2/2, \sinh x_2 + e^{-x_2}|x_1|^2/2, e^{-x_2}x_1). \tag{2.1}$$

The Riemannian metric with respect to this coordinate system is given by

$$e^{-2x_2}(dx_1)^2 + (dx_2)^2.$$

The corresponding Christoffel symbols are

$$\Gamma^1_{2,2} = \Gamma^2_{2,1} = \Gamma^2_{2,2} = \Gamma^1_{1,1} = 0;\ \Gamma^1_{2,1} = -1,\ \Gamma^2_{1,1} = e^{-2x_2}. \tag{2.2}$$

For any (t, x) and $u : [0, T] \times \mathbb{H}^2 \to \mathbb{H}^2$, we define an orthonormal frame at $u(t, x)$ by

$$\Theta_1(u(t, x)) = e^{u^2(t,x)}\frac{\partial}{\partial y_1};\ \Theta_2(u(t, x)) = \frac{\partial}{\partial y_2}. \tag{2.3}$$

where (u^1, u^2) denotes the coordinate of u given by (2.1). **Throughout this paper we will use coordinates (2.1) for both the target manifold $N = \mathbb{H}^2$ and the starting manifold $M = \mathbb{H}^2$.** Recall also the identity for Riemannian curvature on $N = \mathbb{H}^2$

$$\mathbf{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X. \tag{2.4}$$

We have a useful identity for $X, Y, Z \in u^*(TN)$

$$\nabla_i (\mathbf{R}(X, Y) Z) = \mathbf{R}(X, \nabla_i Y) Z + \mathbf{R}(\nabla_i X, Y) Z + \mathbf{R}(X, Y) \nabla_i Z. \tag{2.5}$$

For simplicity, denote $(X \wedge Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X$.

Let $H^k(\mathbb{H}^2; \mathbb{R})$ be the usual Sobolev space for scalar functions defined on manifolds. We also recall the norm of H^k:

$$\|f\|^2_{H^k} = \sum_{l=1}^{k} \|\nabla^l f\|^2_{L^2_x},$$

where $\nabla^l f$ is the covariant derivative. For maps $u : \mathbb{H}^2 \to \mathbb{H}^2$, we define the intrinsic Sobolev semi-norm $\mathfrak{H}^k$ by

$$\|u\|^2_{\mathfrak{H}^k} = \sum_{i=1}^{k} \int_{\mathbb{H}^2} |\nabla^{i-1} du|^2 \mathrm{dvol_h}.$$

The map $u : \mathbb{H}^2 \to \mathbb{H}^2$ is associated with a vector-valued function $u : \mathbb{H}^2 \to \mathbb{R}^2$ by (2.1). Indeed, the vector $(u^1(x), u^2(x))$ is defined by $\Psi(u^1(x), u^2(x)) = u(x)$ for any $x \in \mathbb{H}^2$. Let $Q : \mathbb{H}^2 \to \mathbb{H}^2$ be an admissible harmonic map in Definition 1.1. Then the extrinsic Sobolev space is defined by

$$\mathbf{H^k_Q} = \{u : u^1 - Q^1(x), u^2 - Q^2(x) \in H^k(\mathbb{H}^2; \mathbb{R})\}, \tag{2.6}$$

where $(Q^1(x), Q^2(x)) \in \mathbb{R}^2$ is the corresponding components of $Q(x)$ under the coordinate (2.1). Denote the set of smooth maps which coincide with Q outside of some compact subset of $M = \mathbb{H}^2$ by $\mathcal{D}$. Let $\mathcal{H}^k_Q$ be the completion of $\mathcal{D}$ under the metric given by

$$\mathrm{dist}_{k,Q}(u,w) = \sum_{j=1}^{2} \|u^j - w^j\|_{H^k(\mathbb{H}^2;\mathbb{R})}, \tag{2.7}$$

where $u, w \in \mathcal{H}^k_Q$. Since $C_c^\infty(\mathbb{H}^2;\mathbb{R})$ is dense in $H^k(\mathbb{H}^2;\mathbb{R})$ (see Hebey [**16**]), $\mathcal{H}^k_Q$ coincides with $\mathbf{H^k_Q}$. And for simplicity, we write $\mathbf{H^k}$ without confusions. If u is a map from $\mathbb{R}\times\mathbb{H}^2$ to $\mathbb{H}^2$, we define the space $\mathbf{H^k}\times\mathbf{H^{k-1}}$ by

$$\mathbf{H^k}\times\mathbf{H^{k-1}} = \left\{ u : \sum_{j=1}^{2} \|u^j - Q^j\|_{H^k(\mathbb{H}^2;\mathbb{R})} + \|\partial_t u^j\|_{H^{k-1}(\mathbb{H}^2;\mathbb{R})} < \infty \right\}. \tag{2.8}$$

The distance in $\mathbf{H^k}\times\mathbf{H^{k-1}}$ is given by

$$\mathrm{dist}_{\mathbf{H^k}\times\mathbf{H^{k-1}}}(u,w) = \sum_{j=1}^{2} \|u^j - w^j(x)\|_{H^k} + \|\partial_t u^j - \partial_t w^j\|_{H^{k-1}}. \tag{2.9}$$

2.2. Sobolev embedding and Equivalence lemma. The Fourier transform on hyperbolic spaces takes proper functions defined on $\mathbb{H}^2$ to functions defined on $\mathbb{R}\times\mathbb{S}^1$, see Helgason [**15**] for details. The operator $(-\Delta)^{\frac{s}{2}}$ is defined by the Fourier multiplier $\lambda \to (\frac{1}{4}+\lambda^2)^{\frac{s}{2}}$. We now recall the Sobolev inequalities of functions in H^k.

Lemma 2.1. *If $f \in C_c^\infty(\mathbb{H}^2;\mathbb{R})$, then for $1 < p < \infty$, $p \le q \le \infty$, $0 < \theta < 1$, $1 < r < 2$, $r \le l < \infty$, $\alpha > 1$, the following inequalities hold*

$$\|f\|_{L^2} \lesssim \|\nabla f\|_{L^2} \tag{2.10}$$

$$\|f\|_{L^q} \lesssim \|\nabla f\|_{L^2}^{\theta}\,\|f\|_{L^p}^{1-\theta} \quad \text{when } \frac{1}{p} - \frac{\theta}{2} = \frac{1}{q} \tag{2.11}$$

$$\|f\|_{L^l} \lesssim \|\nabla f\|_{L^r} \quad \text{when } \frac{1}{r} - \frac{1}{2} = \frac{1}{l} \tag{2.12}$$

$$\|f\|_{L^\infty} \lesssim \left\|(-\Delta)^{\frac{\alpha}{2}} f\right\|_{L^2} \quad \text{when } \alpha > 1 \tag{2.13}$$

$$\|\nabla f\|_{L^p} \sim \left\|(-\Delta)^{\frac{1}{2}} f\right\|_{L^p}. \tag{2.14}$$

For the proof, we refer to Bray [**5**] for (2.11), Ionescu, Pausader, Staffilani [**18**] for (2.12), Hebey [**16**] for (2.13), see also Lawrie, Oh, Shahshahani [**33**]. (2.14) is obtained in [**50**].

We also recall the diamagnetic inequality which sometimes refers to Kato's inequality (see [**33**]) and a more generalized Sobolev inequality (see [Proposition 2.2,[**2**]]).

Lemma 2.2. (*a*) *If T is a tension field defined on $\mathbb{H}^2$, then in the distribution sense, one has the diamagnetic inequality*

$$|\nabla|T|| \le |\nabla T|. \tag{2.15}$$

(*b*) *Let $1 < p, q < \infty$ and $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 - \sigma_2 \ge n/p - n/q \ge 0$. Then for all $f \in C_c^\infty(\mathbb{H}^n;\mathbb{R})$*

$$\|(-\Delta)^{\sigma_2} f\|_{L^q} \lesssim \|(-\Delta)^{\sigma_1} f\|_{L^p}.$$

REMARK 2.3. Lemma 2.1 and (2.15) have several useful corollaries, for instance for $f \in H^2$

$$\|f\|_{L_x^\infty} \lesssim \|\nabla^2 f\|_{L_x^2} \tag{2.16}$$

$$\|f\|_{L_x^2} \lesssim \|\nabla^2 f\|_{L_x^2}. \tag{2.17}$$

The intrinsic and extrinsic formulations are equivalent in the following sense, see [Section 2, [**37**]].

LEMMA 2.4. *Suppose that Q is an admissible harmonic map in Definition 1.1. If $u \in \mathbf{H}^{\mathbf{k}}_{\mathbf{Q}}$ then for $k = 2, 3$*

$$\|u\|_{\mathbf{H}^k_Q} \sim \|u\|_{\mathfrak{H}^k}, \tag{2.18}$$

in the sense that there exist continuous functions $\mathcal{P}, \mathcal{Q}$ such that

$$\|u\|_{\mathbf{H}^k_Q} \le \mathcal{P}(\|u\|_{\mathfrak{H}^k}) C(R_0, \|u\|_{\mathfrak{H}^2}) \tag{2.19}$$

$$\|u\|_{\mathfrak{H}^k} \le \mathcal{Q}(\|u\|_{\mathbf{H}^{\mathbf{k}}_{\mathbf{Q}}}) C(R_0, \|u\|_{\mathbf{H}^2_Q}). \tag{2.20}$$

Lemma 2.4 and its proof imply the following corollary, by which we can view Theorem 1.1 as a small data problem in the intrinsic sense. The proof of Corollary 2.5 is presented in Section 7.

COROLLARY 2.5. If (u_0, u_1) belongs to $\mathbf{H^3} \times \mathbf{H^2}$ satisfying (1.4) then for $0 < \mu_1 \le 1, 0 < \mu_2 \le 1$

$$\|\nabla du_0\|_{L^2} + \|\nabla u_1\|_{L^2} + \|du\|_{L^2} + \|u_1\|_{L^2} \le C(R_0)\mu_2 + C(R_0)\mu_1. \tag{2.21}$$

LEMMA 2.6. *We have the decay estimates for heat equations on $\mathbb{H}^2$:*

$$\|e^{s\Delta_{\mathbb{H}^2}} f\|_{L_x^\infty} \lesssim e^{-\frac{s}{4}} s^{-1} \|f\|_{L_x^1} \tag{2.22}$$

$$\|e^{s\Delta_{\mathbb{H}^2}} f\|_{L_x^2} \lesssim e^{-\frac{s}{4}} \|f\|_{L_x^2} \tag{2.23}$$

$$\|e^{s\Delta_{\mathbb{H}^2}} f\|_{L_x^p} \lesssim s^{\frac{1}{p} - \frac{1}{r}} \|f\|_{L_x^r}, \tag{2.24}$$

$$\|e^{s\Delta_{\mathbb{H}^2}} (-\Delta_{\mathbb{H}^2})^\alpha f\|_{L_x^q} \lesssim s^{-\alpha} e^{-\delta s} \|f\|_{L_x^q}, \tag{2.25}$$

where $1 \le r \le p \le \infty$, $\alpha \in [0, 1]$, $1 < q < \infty$, $0 < \delta \ll 1$.

PROOF. (2.22) and (2.24) are known in the literature, see [**37, 8**]. (2.23) is a corollary of the spectral gap of $\frac{1}{4}$ for $-\Delta_{\mathbb{H}^2}$. The $s^{-\alpha}$ part of (2.25) follows by interpolation between the three estimates of [Lemma 2.11,[**33**]]. Thus it suffices to prove (2.25) for s large. The case of (2.25) when $\alpha = 0$ follows by directly estimating the heat kernel given in [**4**]. Since one has $e^{s\Delta}(-\Delta)^\alpha f = e^{\frac{s}{2}\Delta} e^{\frac{s}{2}\Delta} (-\Delta)^\alpha f$, by applying the exponential decay $L^p - L^p$ estimate to the first $e^{\frac{s}{2}\Delta}$ and the $s^{-\alpha}$ decay of $L^p \to (-\Delta)^\alpha L^p$ for the second $e^{\frac{s}{2}\Delta}$ proved just now, we obtain the full (2.25). □

The $\mathbb{R}^2$ version of the following lemma was proved in [Lemma 2.5,[**56**]]. We remark that the same arguments work in the $\mathbb{H}^2$ case, because the proof in [**56**] only uses the decay estimate (2.24) and the self-ajointness of $e^{t\Delta_{\mathbb{R}^2}}$, which are also satisfied by $e^{t\Delta_{\mathbb{H}^2}}$.

LEMMA 2.7. *For $f \in L_x^2$ defined on $\mathbb{H}^2$, one has*

$$\int_0^\infty \|e^{s\Delta_{\mathbb{H}^2}} f\|_{L_x^\infty}^2 ds \lesssim \|f\|_{L_x^2}^2.$$

Without confusion, we will always use Δ instead of $\Delta_{\mathbb{H}^2}$.

2.3. The Local and conditional global well-posedness. We quickly sketch the local well-posedness and conditional global well-posedness for (1.2). The local well-posedness of (1.2) for $(u_0, u_1) \in \mathbf{H^3} \times \mathbf{H^2}$ is standard by fixed point argument. Thus we present the following lemma with a rough proof.

LEMMA 2.8. *For any initial data* $(u_0, u_1) \in \mathbf{H^3} \times \mathbf{H^2}$, *there exists* $T > 0$ *depending only on* $\|(u_0, u_1)\|_{\mathbf{H^3}\times\mathbf{H^2}}$ *such that (1.2) has a unique local solution* $(u, \partial_t u) \in C([0,T]; \mathbf{H^3} \times \mathbf{H^2})$.

PROOF. In the coordinates (2.1), (1.2) can be written as the following semilinear wave equation

$$\frac{\partial^2 u^k}{\partial t^2} - \Delta u^k + \overline{\Gamma}^k_{ij} \frac{\partial u^i}{\partial t}\frac{\partial u^j}{\partial t} - h^{ij}\overline{\Gamma}^k_{mn}\frac{\partial u^m}{\partial x^i}\frac{\partial u^n}{\partial x^j} = 0. \tag{2.26}$$

Notice that $\mathbf{H^3}$ and $\mathbf{H^2}$ are embedded to L^∞ as illustrated in Remark 9.1, we can prove the local well-posedness of (2.26) by the standard contradiction mapping argument in the complete metric space $\mathbf{H^3} \times \mathbf{H^2}$ with the metric given by

$$\mathrm{dist}(u,w) = \sum_{j=1}^{2} \|u^j - w^j\|_{H^3} + \sum_{j=1}^{2} \|\partial_t u^j - \partial_t w^j\|_{H^2}.$$

Moreover we can obtain the blow-up criterion: $T_* > 0$ is the lifespan of (2.26) if and only if

$$\lim_{t\to T_*} \|(u(t,x), \partial_t u(t,x))\|_{\mathbf{H^3}\times\mathbf{H^2}} = \infty. \tag{2.27}$$

□

The conditional global well-posedness is given by the following proposition. We remark that in the flat case $M = \mathbb{R}^d$, $1 \le d \le 3$, Theorem 7.1 of Shatah, Struwe [**45**] gave a local theory for Cauchy problem in $H^2 \times H^1$.

PROPOSITION 2.9. Let $(u_0, u_1) \in \mathbf{H^3} \times \mathbf{H^2}$ be the initial data of (1.2), T_* is the maximal lifespan determined by Lemma 2.8. If the solution $(u, \partial_t u)$ satisfies uniformly for all $t \in [0, T_*)$

$$\|\nabla du\|_{L^2_x} + \|du\|_{L^2_x} + \|\nabla \partial_t u\|_{L^2_x} + \|\partial_t u\|_{L^2_x} \le C_1, \tag{2.28}$$

for some $C_1 > 0$ independent of $t \in [0, T_*)$ then $T_* = \infty$.

PROOF. By the local well-posedness in Lemma 2.8, it suffices to obtain a uniform bound for $\|(u, \partial_t u)\|_{\mathbf{H^3}\times\mathbf{H^2}}$ with respect to $t \in [0,T]$. By Lemma 2.4, it suffices to prove the intrinsic norms are uniformly bounded up to order three. We first point out a useful inequality which can be verified by integration by parts

$$\|\nabla^2 du\|^2_{L^2_x} \lesssim \|\nabla \tau(u)\|^2_{L^2_x} + \|du\|^6_{L^6_x} + \|\nabla du\|^2_{L^4_x}\|du\|^2_{L^4_x} + C(\|u\|^2_{\mathfrak{H}^2}), \tag{2.29}$$

where $\tau(u)$ denotes the tension field which in the local coordinates is written as

$$\tau(u) = \left(\Delta u^k + h^{pq}\overline{\Gamma}^k_{ij}\frac{\partial u^i}{\partial x^p}\frac{\partial u^j}{\partial x^q}\right)\frac{\partial}{\partial y^k}.$$

Thus (2.29), Gagliardo-Nirenberg inequality and Young inequality further yield

$$\|\nabla^2 du\|^2_{L^2_x} \lesssim \mathcal{P}(\|u\|^2_{\mathfrak{H}^2}) + \|\nabla \tau(u)\|^2_{L^2_x}, \tag{2.30}$$

where $\mathcal{P}(x)$ is some polynomial. Define

$$E_3(u,\partial_t u)=\frac{1}{2}\int_{\mathbb{H}^2}|\nabla\tau(u)|^2\mathrm{dvol_h}+\frac{1}{2}\int_{\mathbb{H}^2}\left|\nabla^2\partial_t u\right|^2\mathrm{dvol_h}.$$

Then integration by parts yields

$$\begin{aligned}\frac{d}{dt}E_3(u,\partial_t u)&=\int_{\mathbb{H}^2}h^{ii}\left\langle\nabla_t\nabla_i\tau(u),\nabla_i\tau(u)\right\rangle\mathrm{dvol_h}\\&\quad+\int_{\mathbb{H}^2}h^{ii}h^{jj}\left\langle\nabla_t\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u,\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u\right\rangle\mathrm{dvol_h}.\end{aligned}$$

Furthermore we have

$$\begin{aligned}&\int_{\mathbb{H}^2}h^{ii}h^{jj}\left\langle\nabla_t\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u,\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u\right\rangle\mathrm{dvol_h}\\&=\int_{\mathbb{H}^2}h^{ii}h^{jj}\left\langle\nabla_i\nabla_j\nabla_t\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u,\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u\right\rangle\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}O\big(|\nabla\partial_t u|\left|\nabla^2\partial_t u\right|\big)\mathrm{dvol_h}+O\Big(\int_{\mathbb{H}^2}|du|\,|\partial_t u|\,|\nabla\partial_t u|\left|\nabla^2\partial_t u\right|\mathrm{dvol_h}\Big)\\&+\int_{\mathbb{H}^2}O\big(|\nabla\partial_t u|\left|\nabla^2\partial_t u\right|\big)\mathrm{dvol_h}+\int_{\mathbb{H}^2}O\big(|du|^2|\partial_t u|^2\left|\nabla^2\partial_t u\right|\big)\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}O\big(|\partial_t u|^2\,|\nabla du|\left|\nabla^2\partial_t u\right|\big)\mathrm{dvol_h}\end{aligned}$$

Since u solves (1.2), $\nabla_t\partial_t u=\tau(u)$. Then by integration by parts the leading term can be expanded as

$$\begin{aligned}&\int_{\mathbb{H}^2}h^{ii}h^{jj}\left\langle\nabla_i\nabla_j\nabla_t\partial_t u,\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u\right\rangle\mathrm{dvol_h}\\&=\int_{\mathbb{H}^2}h^{ii}h^{jj}\left\langle\nabla_i\nabla_j\tau(u),\nabla_i\nabla_j\partial_t u-\Gamma^k_{ij}\nabla_k\partial_t u\right\rangle\mathrm{dvol_h}\\&=-\int_{\mathbb{H}^2}h^{ii}\left\langle\nabla_i\tau(u),\nabla_t\nabla_i\tau(u)\right\rangle\mathrm{dvol_h}+\int_{\mathbb{H}^2}O\big(|\nabla\tau(u)|\,|du|\,|\partial_t u|\,|\tau(u)|\big)\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}O\big(|\nabla\tau(u)|\left|\nabla^2 u\right||du|\,|\partial_t u|\big)\mathrm{dvol_h}+\int_{\mathbb{H}^2}O\big(|\nabla\tau(u)|\,|\partial_t u|\,|du|^2\big)\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}O\big(|\nabla\tau(u)|\,|\nabla\partial_t u|\,|du|^2\big)\mathrm{dvol_h}+\int_{\mathbb{H}^2}O\big(|\nabla\tau(u)|\,|\partial_t u|\,|du|^3\big)\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}O\big(|\nabla\partial_t u|\,|\nabla\tau(u)|\big)\mathrm{dvol_h}.\end{aligned}$$

Thus we conclude

$$\begin{aligned}&\frac{d}{dt}E_3(u,\partial_t u)\\&\le\|du\|_{L^8_x}\|\partial_t u\|_{L^4_x}\|\nabla\partial_t u\|_{L^8_x}\left\|\nabla^2\partial_t u\right\|_{L^2_x}+\|\nabla\partial_t u\|_{L^2_x}\left\|\nabla^2\partial_t u\right\|_{L^2_x}\\&+\left\|\nabla^2\partial_t u\right\|_{L^2_x}\|du\|^2_{L^8_x}\|\partial_t u\|^2_{L^8_x}+\|\nabla du\|_{L^6_x}\|\partial_t u\|^2_{L^6_x}\left\|\nabla^2\partial_t u\right\|_{L^2_x}\\&+\|\nabla\tau(u)\|_{L^2_x}\|\nabla du\|_{L^6_x}\|du\|_{L^6_x}\|\partial_t u\|_{L^6_x}+\|\nabla\tau(u)\|_{L^2_x}\|\nabla du\|_{L^4_x}\|du\|^2_{L^8_x}\\&+\|\nabla\tau(u)\|_{L^2_x}\|du\|^3_{L^{12}_x}\|\partial_t u\|_{L^4_x}+\|\nabla\tau(u)\|_{L^2_x}\|\partial_t u\|_{L^6_x}\|\tau(u)\|_{L^6_x}\|du\|_{L^6_x}\\&+\|\nabla\tau(u)\|_{L^2_x}\|\partial_t u\|_{L^6_x}\|du\|^2_{L^8_x}+\|\nabla\tau(u)\|_{L^2_x}\|\nabla\partial_t u\|_{L^2_x}+\|\nabla\tau(u)\|_{L^2_x}\left\|\nabla^2\partial_t u\right\|_{L^2_x}.\end{aligned}$$

Hence Young's inequality, Sobolev embedding and (2.29), (2.30) give

$$\frac{d}{dt}E_3(u,\partial_t u) \le CE_3(u,\partial_t u) + C.$$

where C depends only on C_1 in (2.28). Thus Gronwall shows

$$E_3(u,\partial_t u) \le e^{Ct}(E_3(u_0,u_1) + C).$$

If $T_* < \infty$ this contradicts with (2.27). □

2.4. Geometric identities related to Gauges. Let $\{e_1(t,x), e_2(t,x)\}$ be an orthonormal frame for $u^*(T\mathbb{H}^2)$. Let $\phi_\alpha = (\psi^1_\alpha, \psi^2_\alpha)$ for $\alpha = 0,1,2$ be the components of $\partial_{t,x}u$ in the frame $\{e_1, e_2\}$, i.e.,

$$\phi^j_\alpha = \langle \partial_\alpha u, e_j \rangle .$$

For given $\mathbb{R}^2$-valued function ϕ defined on $[0,T] \times \mathbb{H}^2$, associate ϕ with a tangent filed $e\phi$ on $u^*(TN)$ by

$$\phi \leftrightarrow e\phi = \sum_{j=1}^{2} \phi^j e_j, \tag{2.31}$$

The map u induces a covariant derivative on the trivial boundle $([0,T] \times \mathbb{H}^2, \mathbb{R}^2)$ defined by

$$D_\alpha \phi = \partial_\alpha \phi + [A_\alpha]\phi,$$

where the coefficient matrix is defined by

$$[A_\alpha]^k_j = \langle \nabla_\alpha e_j, e_k \rangle .$$

It is easy to check the torsion free identity

$$D_\alpha \phi_\beta = D_\beta \phi_\alpha, \tag{2.32}$$

and the commutator identity

$$e[D_\alpha, D_\beta]\phi = e(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\phi + e[A_\alpha, A_\beta]\phi = \mathbf{R}(u)(\partial_\alpha u, \partial_\beta u)(e\phi). \tag{2.33}$$

In the two dimensional case, (2.33) can be further simplified to

$$e[D_\alpha, D_\beta]\phi = e(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\phi = \mathbf{R}(u)(\partial_\alpha u, \partial_\beta u)(e\phi). \tag{2.34}$$

Remark 2.1 Sometimes in the same line, we will use both the intrinsic quantities such as $\mathbf{R}(\partial_t u, \partial_s u)$ and frame dependent quantities such as ϕ_i. This will not cause trouble by remembering the correspondence (2.31). And we define a matrix valued function $\mathbf{a} \wedge \mathbf{b}$ by

$$(\mathbf{a} \wedge \mathbf{b})\mathbf{c} = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{b}, \mathbf{c} \rangle \mathbf{a}, \tag{2.35}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors on $\mathbb{R}^2$. It is easy to see (2.35) coincide with (2.4) by letting $X = a_1e_1 + a_2e_2$, $Y = b_1e_1 + b_2e_2$, $Z = c_1e_1 + c_2e_2$. Hence (2.34) can be written as

$$[D_\alpha, D_\beta]\phi = (\phi_\alpha \wedge \phi_\beta)\phi \tag{2.36}$$

Lemma 2.10. *With the notions and notations given above, (1.2) can be written as*

$$D_t\phi_t - h^{ij}D_i\phi_j + h^{ij}\Gamma^k_{ij}\phi_k = 0 \tag{2.37}$$

PROOF. In the intrinsic formulation, (1.2) can be written as

$$\nabla_t \partial_t u - \left(\nabla_{x_i} \partial_{x_j} u - u_* (\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}) \right) h^{ij} = 0.$$

Expanding $\nabla_i \partial_j u$ and $u_*(\nabla_i \partial_j)$ by the frame $\{e_i\}_{i=1}^2$ yields

$$h^{ij} \nabla_i \partial_j u - h^{ij} u_* (\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}) = \sum\nolimits_{l=1}^{2} h^{ij} \nabla_i \left(\langle \partial_j u, e_l \rangle e_l \right) - \Gamma_{i,j}^k h^{ij} \partial_k u$$
$$= h^{ij} \left(\partial_i \psi_j^p e_p + [A_i]_l^p \psi_j^l e_p \right) - \Gamma_{i,j}^k h^{ij} \psi_k^l e_l = e h^{ij} \left(D_i \phi_j \right) - e \Gamma_{i,j}^k h^{ij} \phi_k$$

And $\nabla_t \partial_t u$ is expanded as

$$\nabla_t \partial_t u = \sum\nolimits_{l=1}^{2} \nabla_t \left(\langle \partial_t u, e_l \rangle e_l \right) = \left(\partial_t \phi_0^p e_p + [A_0]_l^p \phi_0^l e_p \right) = e \left(D_0 \phi_0 \right).$$

Hence (2.37) follows. □

3. Caloric Gauge

Denote the space $C([0,T]; \mathbf{H^3} \times \mathbf{H^2})$ by $\mathcal{X}_T$. The caloric gauge was first introduced by Tao [**56**] for the wave maps from $\mathbb{R}^{2+1}$ to $\mathbb{H}^n$. We give the definition of the caloric gauge in our setting.

DEFINITION 3.1. Let $u(t,x) : [0,T] \times \mathbb{H}^2 \to \mathbb{H}^2$ be a solution of (1.2) in $\mathcal{X}_T$. Suppose that the heat flow initiated from u_0 converges to a harmonic map $Q : \mathbb{H}^2 \to \mathbb{H}^2$. Then for a given orthonormal frame $\Xi(x) \triangleq \{\Xi_j(Q(x))\}_{j=1}^2$ which spans the tangent space $T_{Q(x)}\mathbb{H}^2$ for any $x \in \mathbb{H}^2$, by saying a caloric gauge we mean a tuple consisting of a map $\widetilde{u} : \mathbb{R}^+ \times [0,T] \times \mathbb{H}^2 \to \mathbb{H}^2$ and an orthonormal frame $\Omega \triangleq \{\Omega_j(\widetilde{u}(s,t,x))\}_{j=1}^2$ such that

$$\begin{cases} \partial_s \widetilde{u} = \tau(\widetilde{u}) \\ \nabla_s \Omega_j = 0 \\ \lim\limits_{s\to\infty} \Omega_j = \Xi_j \end{cases} \tag{3.1}$$

where the convergence of frames is defined by

$$\begin{cases} \lim\limits_{s\to\infty} \widetilde{u}(s,t,x) = Q(x) \\ \lim\limits_{s\to\infty} \langle \Omega_i(s,t,x), \Theta_j(\widetilde{u}(s,t,x)) \rangle = \langle \Xi_i(Q(x)), \Theta_j(Q(x)) \rangle \end{cases} \tag{3.2}$$

The remaining part of this section is devoted to the existence of the caloric gauge.

3.1. Warming up for the heat flows. In this subsection, we prove the estimates needed for the existence of the caloric gauge and the bounds for connection coefficients.

The equation of the heat flow is given by

$$\begin{cases} \partial_s u = \tau(u) \\ u(0,x) = v(x) \end{cases} \tag{3.3}$$

The energy density e is defined by

$$e(u) = \frac{1}{2}|du|^2.$$

The following lemma is due to Li, Tam [**36**]. (3.6), (3.7) are proved in [**37**].

LEMMA 3.2. *Given initial data $v : \mathbb{H}^2 \to \mathbb{H}^2$ with bounded energy density, suppose that $\tau(v) \in L^p_x$ for some $p > 2$ and the image of $\mathbb{H}^2$ under the map v is contained in a compact subset of $\mathbb{H}^2$. Then the heat flow equation (3.3) has a global solution u. Moreover for some $K, C > 0$, we have*

$$(\partial_s - \Delta)|du|^2 + 2|\nabla du|^2 \le K|du|^2 \tag{3.4}$$

$$(\partial_s - \Delta)|\partial_s u|^2 + 2|\nabla \partial_s u|^2 \le 0 \tag{3.5}$$

$$(\partial_s - \Delta)|\partial_s u| \le 0 \tag{3.6}$$

$$(\partial_s - \Delta)(|du|e^{-Cs}) \le 0. \tag{3.7}$$

Consider the heat flow from $\mathbb{H}^2$ to $\mathbb{H}^2$ with a parameter

$$\begin{cases} \partial_s \widetilde{u} = \tau(\widetilde{u}) \\ \widetilde{u}(s,t,x) \restriction_{s=0} = u(t,x) \end{cases} \tag{3.8}$$

We will give two types of estimates of $\nabla^k \partial_s \widetilde{u}, \nabla^k \partial_x \widetilde{u}$ in the following. One is the decay of $\|\nabla^k \partial_s \widetilde{u}\|_{L^2_x}$ as $s \to \infty$ which can be easily proved via energy arguments. The other is the global boundedness of $\|\partial_x \widetilde{u}\|_{L^\infty_x}$ away from $s = 0$ and the decay of $\|\nabla^k \partial_s \widetilde{u}\|_{L^\infty_x}$ as $s \to \infty$, both of which need additional efforts. And we will prove the decay estimates with respect to s for $\|\partial_t \widetilde{u}\|_{L^\infty_x \bigcap L^2_x}$, which is the key integrability gain to compensate the loss of decay of $\partial_x \widetilde{u}$. We start with the estimate of $\|d\widetilde{u}\|_{L^\infty_x}$ which is the cornerstone for all other estimates.

REMARK 3.3. The following inequality which can be verified by Moser iteration is known in the heat flow literature: If v is a nonnegative function satisfying

$$\partial_t v - \Delta v \le 0,$$

then for $t \ge 1$,

$$v(x,t) \le \int_{t-1}^{t} \int_{B(x,1)} v(y,s) \mathrm{dvol_y} ds.$$

Introduce the norm:

$$\begin{aligned} \|u(t,x)\|_{\mathcal{X}_T} =& \|\nabla du\|_{C([0,T];L^2_x)} + \|\nabla \partial_t u\|_{C([0,T];L^2_x)} \\ &+ \|du\|_{C([0,T];L^2_x)} + \|\partial_t u\|_{C([0,T];L^2_x)}. \end{aligned} \tag{3.9}$$

Trivial applications of Remark 3.3, (3.7) and the non-increasing of the energy along the heat flow give the bounds for $\|d\widetilde{u}\|_{L^\infty_x}$. See also [**37**] for another proof.

LEMMA 3.4. *Let $(u, \partial_t u)$ solve* (1.2) *in $\mathcal{X}_T$ (see (3.9)) with $\|u\|_{\mathcal{X}_T} \le M$. If $\widetilde{u}$ is the solution to* (3.8) *with initial data $u(t,x)$, then we have uniformly for $t \in [0,T]$, $s \in [1,\infty)$*

$$\|d\widetilde{u}(s,t,x)\|_{L^\infty_x} \lesssim \|du(t,x)\|_{L^2_x}, \tag{3.10}$$

The decay of $\|\nabla^k \partial_s \widetilde{u}\|_{L^2_x}$ follows from an energy argument and the bound of the energy density provided by (3.10).

LEMMA 3.5. *Let $(u, \partial_t u) \in \mathcal{X}_T$ with $\|u\|_{\mathcal{X}_T} \le M$, then for some universal constant $\delta > 0$ the solution $\widetilde{u}(s,t,x)$ to heat flow (3.8) satisfies*

$$\|\partial_s \widetilde{u}(s,t,x)\|_{L^2_x} \lesssim e^{-\delta s} MC(M), \text{ for } s > 0 \tag{3.11}$$

$$\|\nabla \partial_s \widetilde{u}(s,t,x)\|_{L^2_x} \lesssim e^{-\delta s} MC(M), \text{ for } s \ge 2 \tag{3.12}$$

$$\int_0^\infty \|\nabla\partial_s\widetilde{u}(s,t,x)\|^2_{L^2_x} ds \lesssim MC(M). \tag{3.13}$$

for all $t\in[0,T]$. The constant $C(M)$ grows polynomially as M grows.

Proof. First we notice that (3.5), (2.22) and maximum principle yield

$$\|\partial_s\widetilde{u}(s,t,x)\|_{L^2_x} \lesssim e^{-\frac{s}{4}}\|\partial_s\widetilde{u}(0,t,x)\|_{L^2_x} \tag{3.14}$$

$$\|\partial_s\widetilde{u}(s,t,x)\|_{L^\infty_x} \lesssim s^{-1}e^{-\frac{s}{4}}\|\partial_s\widetilde{u}(0,t,x)\|_{L^2_x}. \tag{3.15}$$

We introduce three energy functionals:

$$\mathcal{E}_1(\widetilde{u}) = \frac{1}{2}\int_{\mathbb{H}^2}|\nabla\widetilde{u}|^2dx,\ \mathcal{E}_2(u) = \frac{1}{2}\int_{\mathbb{H}^2}|\partial_s\widetilde{u}|^2\mathrm{dvol_h},\ \mathcal{E}_3(u) = \frac{1}{2}\int_{\mathbb{H}^2}|\nabla\partial_s\widetilde{u}|^2\mathrm{dvol_h}.$$

By integration by parts and (3.8), we have

$$\frac{d}{ds}\mathcal{E}_1(\widetilde{u}) = -\int_{\mathbb{H}^2}|\tau(\widetilde{u})|^2\mathrm{dvol_h}.$$

Thus the energy is decreasing with respect to s and

$$\|d\widetilde{u}\|^2_{L^2_x} + \int_0^s \|\partial_s\widetilde{u}\|^2_{L^2_x} \le \mathcal{E}_1(u_0). \tag{3.16}$$

The non-positive sectional curvature assumption with integration by parts yields

$$\|\nabla d\widetilde{u}(s)\|^2_{L^2_x} \le \|\tau(\widetilde{u}(s))\|^2_{L^2_x} + \|d\widetilde{u}\|^2_{L^2_x}$$

Hence by (3.16), (3.8) we conclude

$$\|\widetilde{u}\|^2_{\mathfrak{H}^2} + \int_0^s \|\partial_s\widetilde{u}\|^2_{L^2_x} ds \lesssim \|u_0\|^2_{\mathfrak{H}^2}. \tag{3.17}$$

Again by (3.8) and integration by parts, one has

$$\begin{aligned}\frac{d}{ds}\mathcal{E}_2(\widetilde{u}) &= \int_{\mathbb{H}^2}\langle\nabla_s\partial_s\widetilde{u},\partial_s\widetilde{u}\rangle\mathrm{dvol_h} = \int_{\mathbb{H}^2}\langle\nabla_s\tau(\widetilde{u}),\partial_s\widetilde{u}\rangle\mathrm{dvol_h}\\ &\le -\int_{\mathbb{H}^2}\langle\nabla\partial_s(\widetilde{u}),\nabla\partial_s\widetilde{u}\rangle\mathrm{dvol_h} + C\int_{\mathbb{H}^2}|d\widetilde{u}|^2|\partial_s\widetilde{u}|^2\mathrm{dvol_h}.\end{aligned} \tag{3.18}$$

Integrating (3.18) with respect to s in (s_1,s_2) for any $1<s_1<s_2$, we infer from (3.15) and (3.16) that

(3.19)

$$\mathcal{E}_2(\widetilde{u}(s_2)) - \mathcal{E}_2(\widetilde{u}(s_1)) + \int_{s_1}^{s_2}\mathcal{E}_3(\widetilde{u}(s))ds \lesssim \|d\widetilde{u}\|^2_{L^\infty_s L^2_x}\int_{s_1}^{s_2}\|\partial_s\widetilde{u}\|^2_{L^\infty_x} ds \lesssim M^4e^{-\delta s_1}.$$

Then by (3.14) we have for $1<s<s_1<s_2$ and any $t\in[0,T]$

$$\int_{s_1}^{s_2}\|\nabla\partial_s\widetilde{u}(\tau,t,x)\|^2_{L^2_x}d\tau \lesssim M^2e^{-\delta s}. \tag{3.20}$$

Integration by parts and (3.8) yield

$$\begin{aligned}&\frac{d}{ds}\mathcal{E}_3(\widetilde{u}(s))\\ &\le -\int_{\mathbb{H}^2}\left(\left|\nabla^2\partial_s\widetilde{u}\right|^2 + C\,|d\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,|\partial_s\widetilde{u}|^2 + C\,|d\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,|\partial_s\widetilde{u}|^2\right)\mathrm{dvol_h}\\ &+ C\int_{\mathbb{H}^2}\left(|d\widetilde{u}|^3\,|\partial_t\widetilde{u}|\,|\nabla\partial_t\widetilde{u}| + C\left|\nabla^2\widetilde{u}\right|\,|d\widetilde{u}|\,|\partial_t\widetilde{u}|\,|\nabla\partial_t\widetilde{u}| + C|\partial_t\widetilde{u}|^2|d\widetilde{u}|^4\right)\mathrm{dvol_h}\end{aligned}$$

(3.21)
$$+C\int_{\mathbb{H}^2}\big(|\nabla\partial_t\widetilde{u}|^2|d\widetilde{u}|^2+C|d\widetilde{u}|^2\left|\nabla^2\partial_t\widetilde{u}\right||\partial_t\widetilde{u}|\big)\mathrm{dvol_h}.$$

By Hölder, (3.20), (3.15), we see for $1<s<s_1<s_2$ and any $t\in[0,T]$

$$\begin{aligned}&\int_{s_1}^{s_2}\int_{\mathbb{H}^2}|d\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,|\partial_s\widetilde{u}|^2\mathrm{dvol_h}ds\\&\lesssim\|\partial_s\widetilde{u}\|^2_{L^4_sL^\infty_x([s_1,s_2]\times\mathbb{H}^2)}\|d\widetilde{u}\|_{L^\infty_sL^2_x([s_1,s_2]\times\mathbb{H}^2)}\|\nabla\partial_s\widetilde{u}\|_{L^2_sL^2_x([s_1,s_2]\times\mathbb{H}^2)}\\&\lesssim M^4e^{-\delta s_1}.\end{aligned}$$

Similarly we have from (3.20), (3.15), (3.10) that for $1<s<s_1<s_2$ and any $t\in[0,T]$

$$\begin{aligned}&\int_{s_1}^{s_2}\int_{\mathbb{H}^2}|d\widetilde{u}|^3\,|\nabla\partial_s\widetilde{u}|\,|\partial_s\widetilde{u}|\,\mathrm{dvol_h}\mathrm{ds}\\&\lesssim\|d\widetilde{u}\|^3_{L^\infty_sL^\infty_x([s_1,s_2]\times\mathbb{H}^2)}\|\nabla\partial_s\widetilde{u}\|_{L^2_sL^2_x([s_1,s_2]\times\mathbb{H}^2)}\|\partial_s\widetilde{u}\|_{L^2_sL^2_x([s_1,s_2]\times\mathbb{H}^2)}\\&\lesssim M^5e^{-\delta s_1}.\end{aligned}$$

And similarly we obtain for $1<s<s_1<s_2$ and all $t\in[0,T]$

$$\begin{aligned}&\int_{s_1}^{s_2}\int_{\mathbb{H}^2}|\nabla d\widetilde{u}|\,|d\widetilde{u}|\,|\partial_s\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\mathrm{dvol_h}\mathrm{ds}\\&\lesssim\|d\widetilde{u}\|_{L^\infty_sL^\infty_x}\|\partial_s\widetilde{u}\|_{L^2_sL^\infty_x}\|\nabla d\widetilde{u}\|_{L^\infty_sL^2_x}\|\nabla\partial_s\widetilde{u}\|_{L^2_sL^2_x}\\&\lesssim M^4e^{-\delta s_1},\end{aligned}$$

where the integrand domains are $[s_1,s_2]\times\mathbb{H}^2$. The remaining three terms in (3.21) are easier to bound. In fact, Sobolev embedding, (3.15) and (3.17) show

$$\int_{s_1}^{s_2}\int_{\mathbb{H}^2}|\nabla\widetilde{u}|^4|\partial_s\widetilde{u}|^2\mathrm{dvol_h}\mathrm{ds}\le\|\partial_s\widetilde{u}\|^2_{L^2_sL^\infty_x}\|\nabla d\widetilde{u}\|^4_{L^\infty_sL^2_x}\le M^6e^{-\delta s_1}.$$

Similarly we obtain

$$\int_{s_1}^{s_2}\int_{\mathbb{H}^2}|d\widetilde{u}|^2|\nabla\partial_s\widetilde{u}|^2\mathrm{dvol_h}\mathrm{ds}\le\|\nabla\partial_s\widetilde{u}\|^2_{L^2_sL^2_x}\|d\widetilde{u}\|^2_{L^\infty_sL^\infty_x}\le M^4e^{-\delta s_1}.$$

The last remaining term in (3.21) is absorbed by the negative term on the left. Indeed, for sufficiently small $\eta>0$

$$\begin{aligned}&\int_{s_1}^{s_2}\int_{\mathbb{H}^2}|d\widetilde{u}|^2\left|\nabla^2\partial_s\widetilde{u}\right||\partial_s\widetilde{u}|\mathrm{dvol_h}\mathrm{ds}\\&\lesssim\eta\left\|\nabla^2\partial_s\widetilde{u}\right\|^2_{L^2_sL^2_x([s_1,s_2]\times\mathbb{H}^2)}+\eta^{-1}\|d\widetilde{u}\|^4_{L^\infty_sL^\infty_x}\|\partial_s\widetilde{u}\|^2_{L^2_sL^2_x}\\&\lesssim\eta\left\|\nabla^2\partial_s\widetilde{u}\right\|^2_{L^2_sL^2_x}+\eta^{-1}M^6e^{-\delta s_1}.\end{aligned}$$

(3.17) implies that there exists $s_0\in(1,2)$ such that

$$\int_{\mathbb{H}^2}|\nabla\partial_s u|^2(s_0,t,x)\mathrm{dvol_h}\le M^2. \tag{3.22}$$

Hence applying (2.10) and Gronwall inequality to (3.21), we have for $s>s_0$

$$\int_{\mathbb{H}^2}|\nabla\partial_s\widetilde{u}|^2(s,t,x)\mathrm{dvol_h}$$

$$\lesssim e^{-\delta(s-s_0)} \int_{\mathbb{H}^2} |\nabla\partial_s\widetilde{u}|^2(s_0,t,x)\mathrm{dvol_h} + MC(M)\int_{s_0}^{s} e^{-\delta(s-\tau)}e^{-\delta\tau}d\tau$$

$$\lesssim MC(M)\left(e^{-\delta(s-s_0)} + e^{-\delta s}(s-s_0)\right). \tag{3.23}$$

Since $s_0 \in (1,2)$, we have verified (3.12) for $s \geq 2$. (3.13) follows directly from (3.12), (3.17) and integrating (3.18) with respect to s. □

We then prove the pointwise decay of $|\nabla\partial_s\widetilde{u}|$ with respect to s. First we need the Bochner formula for high derivatives of $\widetilde{u}$ along the heat flow. The proof of following four lemmas is direct calculations with the Bochner technique. Considering that the proof is quite standard, we state the results without detailed calculations.

Lemma 3.6. *Let $\widetilde{u}$ be a solution to heat flow equation. Then $|\nabla\partial_s\widetilde{u}|^2$ satisfies*

$$\partial_s|\nabla\partial_s\widetilde{u}|^2 - \Delta|\nabla\partial_s\widetilde{u}|^2 + 2\left|\nabla^2\partial_s\widetilde{u}\right|^2 \lesssim |\nabla\partial_s\widetilde{u}|^2 + |\partial_s\widetilde{u}|\,|d\widetilde{u}|^3\,|\nabla\partial_s\widetilde{u}|$$

$$+ |\nabla\partial_s\widetilde{u}|^2|d\widetilde{u}|^2 + |\partial_s\widetilde{u}|\,|\nabla d\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,|d\widetilde{u}|\,. \tag{3.24}$$

Lemma 3.7. *Let $\widetilde{u}$ be a solution to heat flow equation, then we have*

$$\partial_s|\nabla d\widetilde{u}|^2 - \Delta|\nabla d\widetilde{u}|^2 + 2\left|\nabla^2 d\widetilde{u}\right|^2 \lesssim |\nabla d\widetilde{u}|^2 + |d\widetilde{u}|^2|\nabla d\widetilde{u}|^2$$

$$+ |d\widetilde{u}|^2 + |d\widetilde{u}|^4\,|\nabla d\widetilde{u}|\,. \tag{3.25}$$

Lemma 3.8. *Let $\widetilde{u}$ be a solution to heat flow equation. Then $|\nabla\partial_t\widetilde{u}|^2$ satisfies*

$$(\partial_s - \Delta)|\partial_t\widetilde{u}|^2 = -2|\nabla\partial_t\widetilde{u}|^2 - \mathbf{R}(\widetilde{u})(\nabla\widetilde{u},\partial_t\widetilde{u},\nabla\widetilde{u},\partial_t\widetilde{u}) \leq 0. \tag{3.26}$$

$$\partial_s|\nabla\partial_t\widetilde{u}|^2 - \Delta|\nabla\partial_t\widetilde{u}|^2 + 2\left|\nabla^2\partial_t\widetilde{u}\right|^2 \lesssim |\nabla\partial_t\widetilde{u}|^2 + |\partial_s\widetilde{u}|\,|d\widetilde{u}|^2\,|\nabla\partial_t\widetilde{u}|$$

$$+ |d\widetilde{u}|^3\,|\partial_t\widetilde{u}|\,|\nabla\partial_t\widetilde{u}| + |d\widetilde{u}|\,|\partial_t\widetilde{u}|\,|\nabla d\widetilde{u}|\,|\nabla\partial_t\widetilde{u}| + |\nabla\partial_t\widetilde{u}|^2|d\widetilde{u}|^2. \tag{3.27}$$

Lemma 3.9. *Let $\widetilde{u}$ be a solution to heat flow equation, then*

$$\partial_s|\nabla_t\partial_s\widetilde{u}|^2 - \Delta|\nabla_t\partial_s\widetilde{u}|^2 + 2|\nabla\nabla_t\partial_s\widetilde{u}|^2 \lesssim |\nabla_t\partial_s\widetilde{u}|^2 + |\nabla d\widetilde{u}|\,|\partial_t\widetilde{u}|\,|\nabla_t\partial_s\widetilde{u}|\,|\partial_s\widetilde{u}|$$

(3.28)

$$+ |\nabla_t\partial_s\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,|\partial_t\widetilde{u}|\,|d\widetilde{u}| + |\nabla\partial_t\widetilde{u}|\,|\partial_s\widetilde{u}|\,|\nabla_t\partial_s\widetilde{u}|\,|d\widetilde{u}| + |\nabla_t\partial_s\widetilde{u}|^2|d\widetilde{u}|^2.$$

We have previously seen that the bound of $\|d\widetilde{u}\|_{L^\infty}$ is useful for bounding $\|\nabla\partial_s\widetilde{u}\|_{L^2}$. In order to bound $\|\nabla\partial_s\widetilde{u}\|_{L^\infty}$, it is convenient if one has a bound for $\|\nabla d\widetilde{u}\|_{L^\infty}$ firstly.

Lemma 3.10. *If $(u(t,x),\partial_t u(t,x))$ is a solution to (1.2) with $\|u(t,x)\|_{\mathcal{X}_T} \leq M$. Then for $s \geq 2$*

$$\|\nabla d\widetilde{u}\|_{L^\infty_x} \lesssim MC(M). \tag{3.29}$$

Proof. The proof of (3.29) is also based on Remark 3.3. One can rewrite (3.25) by Young inequality in the following form

$$\partial_s|\nabla d\widetilde{u}|^2 - \Delta|\nabla d\widetilde{u}|^2 + 2\left|\nabla^2 d\widetilde{u}\right|^2 \leq C\left(1 + |d\widetilde{u}|^2\right)|\nabla d\widetilde{u}|^2 + |d\widetilde{u}|^2 + |d\widetilde{u}|^8.$$

Since for $s \geq 1$, $\|d\widetilde{u}\|_{L^\infty} \lesssim M$, $\|\partial_s\widetilde{u}\|_{L^\infty} \lesssim M$, let $r(s,t,x) = |\nabla d\widetilde{u}|^2 + M^2 + M^8$, then we have

$$\partial_s r - \Delta r \leq C\left(M^2+1\right)r.$$

Let $v = e^{-C\left(M^2+1\right)s}r$. For $s \geq 2$, it is obvious that v satisfies

$$\partial_s v - \Delta v \leq 0.$$

By Remark 3.3, we deduce for $d(x,y)\le 1$, $s\ge 2$

$$v(s,t,x)\lesssim \int_{s-1}^{s}\int_{B(x,1)} v(\tau,t,y)d\tau \mathrm{dvol_y}.$$

Thus by $\|u\|_{\mathfrak{H}^2}\le M$ and (3.17), we conclude

$$\left|\nabla d\widetilde{u}\right|^2(s,t,x)\lesssim MC(M).$$

Hence (3.29) follows. □

Now we prove the decay of $\|\nabla\partial_s\widetilde{u}\|_{L^\infty_x}$ as $s\to\infty$.

LEMMA 3.11. *If $(u,\partial_t u)$ is a solution to (1.2) in $\mathcal{X}_T$ with $\|u(t,x)\|_{\mathcal{X}_T}\le M$. Then for some universal constant $\delta>0$*

$$\|\nabla\partial_s\widetilde{u}\|_{L^\infty_x}\lesssim MC(M)e^{-\delta s},\ \text{for } s\ge 1. \tag{3.30}$$

PROOF. By (3.15), for $s\ge 1$

$$\|\partial_s\widetilde{u}\|_{L^\infty_x}\lesssim e^{-\delta s}M. \tag{3.31}$$

We can rewrite (3.24) by Young inequality as

$$\partial_s|\nabla\partial_s\widetilde{u}|^2-\Delta|\nabla\partial_s\widetilde{u}|^2\le(1+|d\widetilde{u}|^2)|\nabla\partial_s\widetilde{u}|^2+|d\widetilde{u}|^6|\partial_s\widetilde{u}|^2+|d\widetilde{u}|^2|\nabla d\widetilde{u}|^2|\partial_s\widetilde{u}|^2. \tag{3.32}$$

Let $g(s,t,x)=|d\widetilde{u}|^6|\partial_s\widetilde{u}|^2+|d\widetilde{u}|^2|\nabla d\widetilde{u}|^2|\partial_s\widetilde{u}|^2$, then by Lemma 3.4, Lemma 3.5 and (3.31), $g(s,t,x)\le C(M)Me^{-\delta s}$ for $s\ge 1$. Let $f(s,t,x)=\left|\nabla\partial_s\widetilde{u}\right|^2(s,t,x)+\frac{1}{\delta}C(M)Me^{-\delta s}$, then

$$\partial_s f-\Delta f\le C\left(M^2+1\right)f.$$

Then $\bar{v}=e^{-C(M^2+1)s}f$ satisfies

$$\partial_s\bar{v}-\Delta\bar{v}\le 0.$$

Applying Remark 3.3 to $\bar{v}$ as before implies

$$\left|\nabla\partial_s\widetilde{u}\right|^2(s,t,x)+\frac{1}{\delta}C(M)Me^{-\delta s}\le\int_{s-1}^{s}\int_{\mathbb{H}^2}\left|\nabla\partial_s\widetilde{u}\,(\tau,t,y)\right|^2\mathrm{dvol_h}\mathrm{d}\tau+C(M)Me^{-\delta s}.$$

Therefore, (3.30) follows from

$$\int_{s-1}^{s}\int_{\mathbb{H}^2}\left|\nabla\partial_s\widetilde{u}\,(\tau,t,y)\right|^2\mathrm{dvol_h}\mathrm{d}\tau\lesssim MC(M)e^{-\delta s}, \tag{3.33}$$

which arises from (3.12). □

We move to the decay for $|\partial_t\widetilde{u}|$ with respect to s.

LEMMA 3.12. *If $(u,\partial_t u)$ is a solution to (1.2) in $\mathcal{X}_T$ with $\|u(t,x)\|_{\mathcal{X}_T}\le M$. Then*

$$\|\partial_t\widetilde{u}\|_{L^2_x}\lesssim MC(M)e^{-\delta s},\ \text{for } s>0 \tag{3.34}$$

$$\|\partial_t\widetilde{u}\|_{L^\infty_x}\lesssim MC(M)e^{-\delta s},\ \text{for } s\ge 1 \tag{3.35}$$

$$\int_0^\infty\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}ds\lesssim MC(M), \tag{3.36}$$

$$\|\nabla\partial_t\widetilde{u}\|_{L^\infty_x}\lesssim MC(M)e^{-\delta s},\ \text{for } s\ge 1. \tag{3.37}$$

PROOF. The maximum principle and (2.22) imply

$$\|\partial_t\widetilde{u}(s,t,x)\|^2_{L^\infty_x} \le s^{-2}e^{-\delta s}\|\partial_t\widetilde{u}(0,t,x)\|^2_{L^2_x}. \tag{3.38}$$

Moreover further calculations with (3.26) show

$$(\partial_s-\Delta)|\partial_t\widetilde{u}|\le 0.$$

Thus maximum principle and (2.23) give

$$\|\partial_t\widetilde{u}(s,t,x)\|_{L^2_x}\lesssim e^{-\frac14 s}\|\partial_t u\|_{L^2_x}\le M. \tag{3.39}$$

Therefore, (3.34) and (3.35) follow from (3.38) and (3.39) respectively. Second, we prove (3.36) by energy arguments. Introduce the energy functionals

$$\mathcal{E}_4(\widetilde{u})=\frac12\int_{\mathbb{H}^2}|\partial_t\widetilde{u}|^2\mathrm{dvol_h},\ \mathcal{E}_5(\widetilde{u})=\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|^2\mathrm{dvol_h}.$$

Then integration by parts gives

$$\frac{d}{ds}\mathcal{E}_4(\widetilde{u})+\mathcal{E}_5(\widetilde{u})\le\int_{\mathbb{H}^2}|d\widetilde{u}|^2|\partial_t\widetilde{u}|^2\mathrm{dvol_h}. \tag{3.40}$$

Integrating this formula with respect to s in $[0.\kappa)$ with $\kappa>1$ shows

$$\int_0^\kappa\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}ds\le\|\partial_t\widetilde{u}(\kappa)\|^2_{L^2_x}+\int_0^1\|\partial_t\widetilde{u}\|^2_{L^4}\|d\widetilde{u}\|^2_{L^4}ds+\mathcal{E}_1(\widetilde{u})M\int_1^\kappa e^{-2\delta s}ds,$$

where we have used (3.34), (3.35) and Hölder. By Sobolev embedding and letting $\kappa\to\infty$, we obtain

$$\int_0^\infty\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}ds\le M^4+M^2. \tag{3.41}$$

Finally, the proof of (3.37) follows by the same arguments as (3.30) illustrated in Lemma 3.11. □

LEMMA 3.13. *Let $(u,\partial_t u)$ be a solution to (1.2) in $\mathcal{X}_T$ with $\|u(t,x)\|_{\mathcal{X}_T}\le M$. Then*

$$\left\|s^{\frac12}\nabla_t\partial_s\widetilde{u}\right\|_{L^\infty_sL^2_x}\lesssim MC(M)\text{ for }s\in[0,1]. \tag{3.42}$$

Moreover, for $s\in[1,\infty)$ and some $0<\delta\ll1$

$$\|\nabla_t\partial_s\widetilde{u}\|_{L^\infty_sL^\infty_x}\lesssim e^{-\delta s}MC(M). \tag{3.43}$$

PROOF. It is easy to see $|\nabla_t\partial_s\widetilde{u}|\le|\nabla\partial_t\widetilde{u}|+\left|h^{ii}\nabla_i\nabla_t\partial_i\widetilde{u}\right|+|\partial_t\widetilde{u}|\,|d\widetilde{u}|^2$, then

$$|\nabla_t\partial_s\widetilde{u}|\le\left|\nabla^2\partial_t\widetilde{u}\right|+|\partial_t\widetilde{u}|\,|d\widetilde{u}|^2+|\nabla\partial_t\widetilde{u}|. \tag{3.44}$$

Integration by parts gives

$$\begin{aligned}&\frac{d}{ds}\|\nabla\partial_t\widetilde{u}\|^2_{L^2}\\&\le-\int_{\mathbb{H}^2}\left|\nabla^2\partial_t\widetilde{u}\right|^2\mathrm{dvol_h}+\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|\,|\partial_t\widetilde{u}|\,|d\widetilde{u}|\,|\partial_s\widetilde{u}|\,\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}\left|\nabla^2\partial_t\widetilde{u}\right||d\widetilde{u}|^2\,|\partial_s\widetilde{u}|\,\mathrm{dvol_h}+\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|^2\mathrm{dvol_h}+\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|^2|d\widetilde{u}|^2\mathrm{dvol_h}.\end{aligned} \tag{3.45}$$

By Sobolev embedding, we obtain

$$\frac{d}{ds}\|\nabla\partial_t\widetilde{u}\|_{L^2}^2 \le C\|\nabla\partial_t\widetilde{u}\|_{L^2}^2\left(1+\|\partial_s\widetilde{u}\|_{L^\infty}^2+\|d\widetilde{u}\|_{L^\infty}^2\right)+\|\nabla d\widetilde{u}\|_{L^2}^4\|\partial_s\widetilde{u}\|_{L^\infty}^2.$$

Thus we get

$$\|\nabla\partial_t\widetilde{u}\|_{L^2}^2 \le \|\nabla\partial_t\widetilde{u}(0,t,x)\|_{L^2}^2+e^{\int_0^s V(\tau)d\tau}\int_0^s e^{-\int_0^\kappa V(\tau)d\tau}\|\nabla d\widetilde{u}\|_{L^2}^4\|\partial_s\widetilde{u}\|_{L^\infty}^2\,d\kappa,$$

where $V(s)=Cs+C\|d\widetilde{u}\|_{L^\infty}^2+C\|\partial_s\widetilde{u}\|_{L^\infty}^2$. By Lemma 2.7 and Lemma 3.2

$$\int_0^1\|d\widetilde{u}\|_{L^\infty}^2 ds+\int_0^1\|\partial_s\widetilde{u}\|_{L^\infty}^2 ds\le M^2. \tag{3.46}$$

Hence we conclude for $s\in[0,1]$,

$$\|\nabla\partial_t\widetilde{u}\|_{L^2}^2\le\|\nabla\partial_t\widetilde{u}(0,t,x)\|_{L^2}^2+e^{MC(M)s}MC(M). \tag{3.47}$$

With (3.45), we further deduce that

$$\int_0^1\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L^2}ds\lesssim MC(M). \tag{3.48}$$

Integration by parts shows,

$$\begin{aligned}
&\frac{d}{ds}\left(\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}^2 s\right)\\
&\le -s\int_{\mathbb{H}^2}\left|\nabla^3\partial_t\widetilde{u}\right|^2\mathrm{dvol_h dt}+\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}^2+s\|\partial_s\widetilde{u}\|_{L_x^\infty}\|\nabla\partial_t\widetilde{u}\|_{L_x^4}\|d\widetilde{u}\|_{L_x^4}\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}\\
&+s\|\partial_t\widetilde{u}\|_{L_x^\infty}\|\nabla\partial_s\widetilde{u}\|_{L_x^2}\|d\widetilde{u}\|_{L_x^\infty}\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}+s\|d\widetilde{u}\|_{L^8}^2\|\nabla\partial_t\widetilde{u}\|_{L_x^4}\left\|\nabla^3\partial_t\widetilde{u}\right\|_{L_x^2}\\
&+s\|d\widetilde{u}\|_{L_x^\infty}\|\nabla\partial_t\widetilde{u}\|_{L_x^2}\|\partial_s\widetilde{u}\|_{L_x^\infty}\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}+s\|d\widetilde{u}\|_{L^\infty}^2\|\nabla\partial_t\widetilde{u}\|_{L_x^2}\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}\\
&+s\|d\widetilde{u}\|_{L^\infty}^2\|\nabla d\widetilde{u}\|_{L_x^2}\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}+s\|\partial_t\widetilde{u}\|_{L_x^\infty}\|\nabla d\widetilde{u}\|_{L_x^2}\|\partial_s\widetilde{u}\|_{L_x^\infty}\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}\\
&+s\|\nabla d\widetilde{u}\|_{L_x^2}\|\partial_t\widetilde{u}\|_{L_x^\infty}\left\|\nabla^3\partial_t\widetilde{u}\right\|_{L_x^2}\|d\widetilde{u}\|_{L_x^\infty}+s\|d\widetilde{u}\|_{L^\infty}^2\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L^2}^2.
\end{aligned}$$

Then Gronwall with (3.17), (3.46) yields for all $s\in[0,1]$

$$\left\|\nabla^2\partial_t\widetilde{u}\right\|_{L_x^2}^2 s+\int_0^s\int_{\mathbb{H}^2}\left|\nabla^3\partial_t\widetilde{u}\right|^2\tau\mathrm{dvol_h}d\tau\le MC(M). \tag{3.49}$$

Thus by (3.49), (3.48), (3.47), and (3.44), we conclude

$$\left\|s^{\frac12}\nabla_t\partial_s\widetilde{u}\right\|_{L_s^\infty[0,1]L_x^2}\le MC(M). \tag{3.50}$$

(3.43) follows by the same path as Lemma 3.11 with the help of (3.28). The essential ingredient is to prove for $s_1\ge 2$

$$\int_{s_1}^{s_1+1}\|\nabla^2\partial_t\widetilde{u}\|_{L_x^2}^2 ds\lesssim MC(M)e^{-\delta s_1}. \tag{3.51}$$

The remaining proof is devoted to verifying (3.51). By (2.10) and (3.45), we obtain for any $0<c\ll 1$

$$\begin{aligned}
&\frac{d}{ds}\|\nabla\partial_t\widetilde{u}\|_{L^2}^2+c\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|^2\mathrm{dvol_h}+c\int_{\mathbb{H}^2}\left|\nabla^2\partial_t\widetilde{u}\right|^2\mathrm{dvol_h}\\
&\lesssim\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|\,|\partial_t\widetilde{u}|\,|d\widetilde{u}|\,|\partial_s\widetilde{u}|\,\mathrm{dvol_h}+\int_{\mathbb{H}^2}\left|\nabla^2\partial_t\widetilde{u}\right||d\widetilde{u}|^2\,|\partial_s\widetilde{u}|\,\mathrm{dvol_h}
\end{aligned}$$

$$+\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|^2|d\widetilde{u}|^2\mathrm{dvol_h}+\frac{1}{c}\int_{\mathbb{H}^2}|\nabla\partial_t\widetilde{u}|^2\mathrm{dvol_h}. \tag{3.52}$$

By Lemma 3.3 and (3.15), we have for $s\ge 1$

$$\|d\widetilde{u}\|_{L^\infty_x}\lesssim\|du\|_{L^2_x}\lesssim M \tag{3.53}$$

$$\|\partial_s\widetilde{u}\|_{L^\infty_x}\lesssim e^{-\delta s}\|\partial_s\widetilde{u}(0,t)\|_{L^2_x}\lesssim e^{-\delta s}M. \tag{3.54}$$

Then by Sobolev embedding and Gronwall inequality, for $s\ge 1$

$$\begin{aligned}\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}&\lesssim e^{-cs}\|\nabla\partial_t\widetilde{u}(1,t,x)\|^2_{L^2_x}+e^{-cs}\int_1^s e^{c\kappa}\|\nabla\partial_t\widetilde{u}(\kappa)\|^2_{L^2_x}d\kappa\\&+e^{-cs}\int_1^s e^{c\kappa}\|\nabla d\widetilde{u}(\kappa,t)\|^4_{L^2_x}\|\partial_s\widetilde{u}(\kappa,t)\|^2_{L^\infty_x}d\kappa.\end{aligned} \tag{3.55}$$

Hence (3.41), (3.17), (3.54) and (3.55) give for $s\in[0,\infty)$

$$\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}\le MC(M), \tag{3.56}$$

where (3.56) when $s\in[0,1]$ follows by (3.47). Integrating (3.40) with respect to s in $[s_1,s_2]$ for $1\le s_1\le s_2<\infty$ yields

$$\int_{s_1}^{s_2}\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}ds\le\|\partial_t\widetilde{u}\|^2_{L^2_x}(s_2)-\|\partial_t\widetilde{u}\|^2_{L^2_x}(s_1)+\int_{s_1}^{s_2}\|\partial_t\widetilde{u}\|^2_{L^2_x}\|d\widetilde{u}\|^2_{L^\infty_x}ds. \tag{3.57}$$

By (3.34), Lemma 3.4,

$$\int_{s_1}^{s_2}\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}ds\lesssim MC(M)e^{-\delta s_1}. \tag{3.58}$$

Thus in any interval $[s_*,s_*+1]$ there exists $s^0_*\in[s_*,s_*+1]$ such that

$$\left\|\nabla\partial_t\widetilde{u}(s^0_*)\right\|^2_{L^2_x}ds\lesssim MC(M)e^{-\delta s_*}. \tag{3.59}$$

Fix $s_*\ge 1$, applying Gronwall to (3.52) in $[s^0_*,a]$ with $a\in[s^0_*,s_*+2]$ gives

$$\begin{aligned}\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}(a,t)&\le e^{-ca}\left\|\nabla\partial_t\widetilde{u}(s^0_*,t,x)\right\|^2_{L^2_x}+e^{-ca}\int_{s^0_*}^a e^{cs}\|\nabla\partial_t\widetilde{u}(s)\|^2_{L^2_x}ds\\&+e^{-ca}\int_{s^0_*}^a e^{cs}\|\nabla d\widetilde{u}(s)\|^4_{L^2_x}\|\partial_s\widetilde{u}(s)\|^2_{L^\infty_x}ds.\end{aligned} \tag{3.60}$$

Thus by (3.34), Lemma 3.4, (3.59) and the fact that a at leat ranges over all $[s_*+1,s_*+2]$, we have for $s\ge 2$,

$$\|\nabla\partial_t\widetilde{u}\|^2_{L^2_x}\le MC(M)e^{-\delta s}. \tag{3.61}$$

Integrating (3.52) with respect to s again in $[s_1,s_1+1]$, we obtain (3.51) by (3.61) and (3.34), Lemma 3.3. Finally using maximum principle and Remark 3.3 as Lemma 3.11, we get (3.43) from (3.55), (3.61), (3.44) and Lemma 3.11. □

In the remaining part of this subsection, we consider the short time behaviors of the differential fields under the heat flow. Since the energy of the solution to the heat flow in our case will not decay to zero, we can not expect that it behaves as a solution to the linear heat equation in the large time scale. However, one can still expect that the solution to the heat flow is almost governed by the linear equation in the short time scale. We summarize these useful estimates in the following proposition.

PROPOSITION 3.14. Let $u:[0,T]\times\mathbb{H}^2\to\mathbb{H}^2$ be a solution to (1.2) satisfying

$$\|(\nabla du,\nabla\partial_t u)\|_{L^2\times L^2}+\|(du,\partial_t u)\|_{L^2\times L^2}\le M.$$

If $\widetilde{u}:\mathbb{R}^+\times[0,T]\times\mathbb{H}^2\to\mathbb{H}^2$ is the solution to (3.8) with initial data $u(t,x)$, then for any $\eta>0$, it holds uniformly for $(s,t)\in(0,1)\times[0,T]$ that

$$\begin{aligned}&s^{\frac12}\|\nabla d\widetilde{u}\|_{L^\infty_x}+s^{\frac12}\|\nabla\partial_t\widetilde{u}\|_{L^\infty_x}+s\|\nabla\partial_s\widetilde{u}\|_{L^\infty_x}+s^{\frac12}\|\partial_s\widetilde{u}\|_{L^\infty_x}\\&+s^{\frac12}\|\nabla\partial_s\widetilde{u}\|_{L^2_x}+\left\|s^{\frac12}\nabla^2\partial_s\widetilde{u}\right\|_{L^2_{s,x}}+s\|\nabla_t\partial_s\widetilde{u}\|_{L^\infty_x}+s^\eta\|d\widetilde{u}\|_{L^\infty_x}\le MC(M).\end{aligned}$$

PROOF. Since $\|\nabla d\widetilde{u}\|_{L^2_x}\le M$ shown by (3.17), Sobolev embedding implies $\|d\widetilde{u}\|_{L^p_x}\le M$ for any $p\in(2,\infty)$. Then (3.7) and (2.24) yield $s^\eta\|d\widetilde{u}\|_{L^\infty_x}\le M$ for any $\eta>0$ and all $(t,s)\in[0,T]\times(0,1)$. By (3.25), one has

$$\partial_s|\nabla d\widetilde{u}|-\Delta|\nabla d\widetilde{u}|\le K|\nabla d\widetilde{u}|+|d\widetilde{u}|^2|\nabla d\widetilde{u}|+|d\widetilde{u}|+|d\widetilde{u}|^4.$$

Furthermore we obtain

$$(\partial_s-\Delta)\left(e^{-sK}e^{-\int_0^s\|d\widetilde{u}(\tau)\|^2_{L^\infty_x}d\tau}|\nabla d\widetilde{u}|\right)\le e^{-sK}e^{-\int_0^s\|d\widetilde{u}(\tau)\|^2_{L^\infty_x}d\tau}\left(|d\widetilde{u}|+|d\widetilde{u}|^4\right).$$

Then maximum principle implies for $s\in[0,2]$

$$\begin{aligned}\|\nabla d\widetilde{u}(s)\|_{L^\infty_x}&\lesssim\left\|e^{\Delta\frac{s}{2}}(e^{-\frac{sK}{2}}e^{-\int_0^{\frac{s}{2}}\|d\widetilde{u}(\tau)\|^2_{L^\infty_x}d\tau}|\nabla d\widetilde{u}|(\frac{s}{2}))\right\|_{L^\infty_x}\\&+\left\|\int_{\frac{s}{2}}^s e^{\Delta(s-\tau)}e^{-K\tau}e^{-\int_0^\tau\|d\widetilde{u}(\tau_1)\|^2_{L^\infty_x}d\tau_1}(|d\widetilde{u}|+|d\widetilde{u}|^4)(\tau)d\tau\right\|_{L^\infty_x}.\end{aligned}$$

By the smoothing effect of the heat semigroup, we obtain for $s\in[0,1]$

$$\|\nabla d\widetilde{u}(s)\|_{L^\infty_x}\lesssim s^{-\frac12}\left\|\nabla d\widetilde{u}(\frac{s}{2})\right\|_{L^2_x}+\int_{\frac{s}{2}}^s\left\||d\widetilde{u}|^4(\tau)\right\|_{L^\infty_x}+\left\||d\widetilde{u}|(\tau)\right\|_{L^\infty_x}d\tau.$$

Then Lemma 2.6 and Lemma 3.2 show for $s\in(0,1)$

$$\|\nabla d\widetilde{u}(s)\|_{L^\infty_x}\le s^{-\frac12}\left\|\nabla d\widetilde{u}(\frac{s}{2})\right\|_{L^2_x}+\int_{\frac{s}{2}}^s\tau^{-3/2}(\|du\|^4_{L^{\frac83}_x}+\|du\|_{L^2_x})d\tau.$$

Therefore by Sobolev inequality we conclude

$$\begin{aligned}\||\nabla d\widetilde{u}|(s)\|_{L^\infty_x}&\le s^{-\frac12}\sup_{t\in[0,T]}\left(\|\nabla du(t)\|^4_{L^2_x}+\|du(t)\|_{L^2_x}\right)\\&+s^{-\frac12}\sup_{s\in[0,1]}\||\nabla d\widetilde{u}|(s)\|_{L^2_x}.\end{aligned}$$

Thus by (3.17), we obtain for $s\in[0,1]$

$$s^{\frac12}\||\nabla d\widetilde{u}|(s)\|_{L^\infty_x}\le MC(M).\tag{3.62}$$

By (3.21) we have

$$\begin{aligned}&\frac{d}{ds}\left(s\mathcal{E}_3(\widetilde{u}(s))\right)\\&\lesssim\mathcal{E}_3(\widetilde{u}(s))-\int_{\mathbb{H}^2}s\left|\nabla^2\partial_s\widetilde{u}\right|^2\mathrm{dvol_h}+\int_{\mathbb{H}^2}s\left(|d\widetilde{u}||\nabla\partial_s\widetilde{u}||\partial_s\widetilde{u}|^2\right)\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}s\left(|\nabla d\widetilde{u}||d\widetilde{u}||\partial_s\widetilde{u}||\nabla\partial_s\widetilde{u}|+|\partial_s\widetilde{u}|^2|d\widetilde{u}|^4+|\nabla\partial_s\widetilde{u}|^2|d\widetilde{u}|^2\right)\mathrm{dvol_h}\\&+\int_{\mathbb{H}^2}s\left(|d\widetilde{u}|^3|\partial_s\widetilde{u}||\nabla\partial_s\widetilde{u}|+|d\widetilde{u}|^2\left|\nabla^2\partial_s\widetilde{u}\right||\partial_s\widetilde{u}|\right)\mathrm{dvol_h}.\end{aligned}\tag{3.63}$$

The terms in the right hand side can be bounded by Sobolev and Hölder as follows

$$\int_{\mathbb{H}^2} s\,|d\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,|\partial_s\widetilde{u}|^2 \mathrm{dvol_h} \le s^{\frac12}\,\|d\widetilde{u}\|_{L^\infty_x}\,\|\nabla\partial_s\widetilde{u}\|_{L^2_x} s^{\frac12}\,\|\partial_s\widetilde{u}\|^2_{L^4_x}$$

$$\int_{\mathbb{H}^2} s|d\widetilde{u}|^3\,|\partial_s\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\mathrm{dvol_h} \le s\,\|d\widetilde{u}\|^3_{L^{12}_x}\,\|\nabla\partial_s\widetilde{u}\|_{L^2_x}\|\partial_s\widetilde{u}\|_{L^4_x}$$

$$\int_{\mathbb{H}^2} s\,|\nabla d\widetilde{u}|\,|d\widetilde{u}|\,|\partial_s\widetilde{u}|\,|\nabla\partial_s\widetilde{u}|\,\mathrm{dvol_h} \le \|s\nabla d\widetilde{u}\|_{L^\infty_x}\|\nabla\partial_s\widetilde{u}\|_{L^2_x}\|d\widetilde{u}\|_{L^4_x}\|\partial_s\widetilde{u}\|_{L^4_x}$$

$$\int_{\mathbb{H}^2} s|\partial_s\widetilde{u}|^2|d\widetilde{u}|^4\mathrm{dvol_h} \le \|\partial_s\widetilde{u}\|^2_{L^2_x}\, s\,\|d\widetilde{u}\|^4_{L^\infty_x}$$

$$\int_{\mathbb{H}^2} s|\nabla\partial_s\widetilde{u}|^2|d\widetilde{u}|^2\mathrm{dvol_h} \le \|\nabla\partial_s\widetilde{u}\|^2_{L^2_x}\, s\,\|d\widetilde{u}\|^2_{L^\infty_x}.$$

The highest order term can be absorbed by the negative term, indeed we have

$$\begin{aligned}\int_{\mathbb{H}^2} s|d\widetilde{u}|^2\left|\nabla^2\partial_s\widetilde{u}\right|\,|\partial_s\widetilde{u}|\,\mathrm{dvol_h} &\le \frac{s}{2C}\int_{\mathbb{H}^2}\left|\nabla^2\partial_s\widetilde{u}\right|^2\mathrm{dvol_h} + \mathrm{C}\int_{\mathbb{H}^2}\mathrm{s}|\partial_s\widetilde{\mathrm{u}}|^2|\mathrm{d}\widetilde{\mathrm{u}}|^4\mathrm{dvol_h}\\ &\le \frac{s}{2C}\int_{\mathbb{H}^2}\left|\nabla^2\partial_s\widetilde{u}\right|^2\mathrm{dvol_h} + C\,\|\partial_s\widetilde{u}\|^2_{L^2_x}\, s\,\|d\widetilde{u}\|^4_{L^\infty_x}.\end{aligned}$$

Recall the fact $|d\widetilde{u}|\,(s) \le e^{\Delta s}\,|du|$ when $s\in[0,1]$, $|\partial_s\widetilde{u}|\,(s) \le e^{\Delta s}\,|\tau(u)|$, the terms involved above are bounded by smoothing effect

$$s\,\|d\widetilde{u}\|^2_{L^\infty_x} + s^{\frac14}\,\|\partial_s\widetilde{u}\|_{L^4_x} \le \|du\|^2_{L^2_x} + \|\tau(u)\|_{L^2_x}. \tag{3.64}$$

Thus integrating (3.63) with respect to s in $[0,s]$ with (3.62) gives for $s\in[0,1]$

$$s\mathcal{E}_3(\widetilde{u}(s)) + \int_0^s\int_{\mathbb{H}^2} s\left|\nabla^2\partial_s\widetilde{u}\right|^2\mathrm{dvol_h}ds \lesssim \int_0^s \|\nabla\partial_s\widetilde{u}\|^2_{L^2_x}\,ds'. \tag{3.65}$$

Therefore by (3.13), we conclude for $s\in[0,1]$

$$\int_{\mathbb{H}^2} s|\nabla\partial_s\widetilde{u}|^2\mathrm{dvol_h} \le MC(M). \tag{3.66}$$

By (3.24), we deduce

$$\partial_s\,|\nabla\partial_s\widetilde{u}| - \Delta\,|\nabla\partial_s\widetilde{u}| \le |\nabla\partial_s\widetilde{u}|\,|d\widetilde{u}|^2 + |\partial_s\widetilde{u}|\,|d\widetilde{u}|^3 + |\partial_s\widetilde{u}|\,|\nabla d\widetilde{u}|\,|d\widetilde{u}|\,.$$

Then as above considering the equation of $e^{-\int_0^s \|d\widetilde{u}(\tau)\|^2_{L^\infty_x}d\tau}\,|\nabla\partial_s\widetilde{u}|$, we obtain by maximum principle that

$$\begin{aligned}&\|\nabla\partial_s\widetilde{u}(s)\|_{L^\infty_x}\\ &\le s^{-\frac12}\|\nabla\partial_s\widetilde{u}(\frac{s}{2})\|_{L^2_x} + \int_{\frac{s}{2}}^s \||\partial_s\widetilde{u}|\,|d\widetilde{u}|^3(\tau)\|_{L^\infty_x} + \||\partial_s\widetilde{u}|\,|\nabla d\widetilde{u}|\,|d\widetilde{u}|\,(\tau)\|_{L^\infty_x}d\tau.\end{aligned}$$

Hence (3.62) and (3.66) give

$$\begin{aligned}\||\nabla\partial_s\widetilde{u}|\,(s)\|_{L^\infty_x} \le& s^{-1}M + \left(\sup_{s\in[0,1]} s\|\partial_s\widetilde{u}\|_{L^\infty_x}\|d\widetilde{u}\|_{L^\infty_x}\right)\int_{\frac{s}{2}}^s \tau^{-1}\,\|d\widetilde{u}\|^2_{L^\infty_x}d\tau\\ &+ \left(\sup_{s\in[0,1]} s\|\nabla d\widetilde{u}\|_{L^\infty_x}\|d\widetilde{u}\|_{L^\infty_x}\right)\int_{\frac{s}{2}}^s \tau^{-1}\|d\widetilde{u}\|_{L^\infty_x}d\tau\\ &\le s^{-1}M + s^{-1}M^2\int_{\frac{s}{2}}^s\Big(\|d\widetilde{u}\|^2_{L^\infty_x} + \|d\widetilde{u}\|_{L^\infty_x}\Big)d\tau.\end{aligned}$$

Consequently, we have by Lemma 2.7,

$$\| |\nabla\partial_s\widetilde{u}|\,(s)\|_{L^\infty_x} \le MC(M)s^{-1}. \tag{3.67}$$

By Lemma 3.8, one deduces

$$\partial_s |\nabla\partial_t\widetilde{u}| - \Delta |\nabla\partial_t\widetilde{u}| \le K |\nabla\partial_t\widetilde{u}| + |\nabla\partial_t\widetilde{u}|\,|d\widetilde{u}|^2 + |\partial_s\widetilde{u}||d\widetilde{u}|^2 + |d\widetilde{u}|^3\,|\partial_t\widetilde{u}| + |d\widetilde{u}|\,|\partial_t\widetilde{u}|\,|\nabla d\widetilde{u}|.$$

Considering the equation of $e^{-\int_0^s \left(\|d\widetilde{u}(\tau)\|^2_{L^\infty_x} - K\right)d\tau}\,|\nabla\partial_t\widetilde{u}|$, we have by maximum principle that

$$\begin{aligned}
\|\nabla\partial_t\widetilde{u}\|_{L^\infty_x} &\le s^{-\frac12}\|\nabla\partial_t\widetilde{u}\|_{L^2_x} + \int_{\frac{s}{2}}^s (s-\tau)^{-\frac12}\| |d\widetilde{u}|^3\,|\partial_t\widetilde{u}|\,(\tau)\|_{L^2_x}d\tau \\
&+ \int_{\frac{s}{2}}^s \| |d\widetilde{u}|\,|\partial_t\widetilde{u}|\,|\nabla d\widetilde{u}|\,(\tau)\|_{L^\infty_x} + \| |\partial_s\widetilde{u}||d\widetilde{u}|^2\|_{L^\infty_x}d\tau \\
&\le s^{-\frac12}\|\nabla\partial_t\widetilde{u}\|_{L^2_x} + \sup_{s\in[0,1]}\left(\|\partial_t\widetilde{u}\|_{L^4_x}\|d\widetilde{u}\|^3_{L^{12}_x}\right)\int_{\frac{s}{2}}^s (s-\tau)^{-\frac12}d\tau \\
&+ \sup_{s\in[0,1]}\left(s^{\frac12}\|\nabla d\widetilde{u}\|_{L^\infty_x}\right)\int_{\frac{s}{2}}^s \tau^{-\frac12}\|\partial_t\widetilde{u}\|_{L^\infty_x}\|d\widetilde{u}\|_{L^\infty_x}d\tau \\
&+ \sup_{s\in[0,1]}\left(s\,\|d\widetilde{u}\|^2_{L^\infty_x}\,s^{\frac12}\,\|\partial_s\widetilde{u}\|_{L^\infty_x}\right)\int_{\frac{s}{2}}^s \tau^{-\frac32}d\tau.
\end{aligned}$$

Hence we deduce by Lemma 2.7

$$\| |\nabla\partial_t u|\,(s)\|_{L^\infty_x} \le MC(M)s^{-\frac12}. \tag{3.68}$$

The bounds for $|\nabla_t\partial_s\widetilde{u}|$ follows by the same arguments as (3.62) with help of Lemma 3.13 and (3.28). □

We summarize the long time and short time behaviors as a proposition.

Proposition 3.15. Let $u:[0,T]\times\mathbb{H}^2\to\mathbb{H}^2$ be a solution to (1.2) satisfying

$$\|(\nabla du, \nabla\partial_t u)\|_{L^2\times L^2} + \|(du,\partial_t u)\|_{L^2\times L^2} \le M,$$

If $\widetilde{u}:\mathbb{R}^+\times[0,T]\times\mathbb{H}^2\to\mathbb{H}^2$ is the solution to (3.8) with initial data $u(t,x)$, then for any $\eta>0$, it holds uniformly for $t\in[0,T]$ that

$$\begin{aligned}
&\|d\widetilde{u}\|_{L^\infty_s[1,\infty)L^\infty_x} + \|\nabla d\widetilde{u}\|_{L^\infty_s[1,\infty)L^\infty_x} + \|\nabla d\widetilde{u}\|_{L^\infty_s L^2_x} + \|\nabla\partial_t\widetilde{u}\|_{L^\infty_s L^2_x} \\
&+ \|s^{\frac12}\,|\nabla d\widetilde{u}|\|_{L^\infty_s[0,1]L^\infty_x} + \|e^{\delta s}\,|\partial_s\widetilde{u}|\|_{L^\infty_s L^2_x} + \|s\,|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_s[0,1]L^\infty_x} \\
&+ \|s^{\frac12}\,|\nabla\partial_t\widetilde{u}|\|_{L^\infty_s[0,1]L^\infty_x} + \|s\,|\nabla\partial_s\widetilde{u}|\|_{L^\infty_s[0,1]L^\infty_x} + \|s^{\frac12}e^{\delta s}\,|\partial_s\widetilde{u}|\|_{L^\infty_s L^\infty_x} \\
&+ \|s^{\frac12}\,|\nabla\partial_s\widetilde{u}|\|_{L^\infty_s[0,1]L^2_x} + \|s^{\frac12}\,|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_s[0,1]L^2_x} + \|s^{\eta}d\widetilde{u}\|_{L^\infty_s(0,1)L^\infty_x} \\
&\left\|s^{\frac12}e^{\delta s}\,|\nabla\partial_t\widetilde{u}|\right\|_{L^\infty_s L^\infty_x} + \|se^{\delta s}\,|\nabla\partial_s\widetilde{u}|\|_{L^\infty_s L^\infty_x} + \|se^{\delta s}|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_s L^\infty_x} \\
&\|e^{\delta s}\,|\nabla\partial_t\widetilde{u}|\|_{L^\infty_s L^2_x} + \left\|s^{\frac12}e^{\delta s}\,|\nabla\partial_s\widetilde{u}|\right\|_{L^\infty_s L^2_x} + \left\|s^{\frac12}e^{\delta s}\nabla_t\partial_s\widetilde{u}\right\|_{L^\infty_s L^2_x} \le MC(M).
\end{aligned}$$

Lemma 3.16. *If $(u,\partial_t u)$ solves (1.2) and $\|u(t,x)\|_{\mathcal{X}_T}\le M$, then we have*

$$\left\|\nabla^2 d\widetilde{u}\right\|_{L^2_x} \le \max(s^{-\frac12},1)MC(M) \tag{3.69}$$

$$\left\|\nabla^2 d\widetilde{u}\right\|_{L^\infty_x} \le \max(s^{-1},1)MC(M) \tag{3.70}$$

$$se^{\delta' s}\|\nabla^2\partial_s\widetilde{u}\|_{L^2_x} \lesssim MC(M) \tag{3.71}$$

$$s^{\frac{3}{2}}e^{\delta' s}\|\nabla^2\partial_s\widetilde{u}\|_{L^\infty_x} \lesssim MC(M) \tag{3.72}$$

PROOF. The Bochner formula for $\left|\nabla^2 d\widetilde{u}\right|^2$ is as follows

$$\begin{aligned}\partial_s|\nabla^2 d\widetilde{u}|^2 - \Delta|\nabla^2 d\widetilde{u}|^2 + 2|\nabla^3 d\widetilde{u}|^2 &\lesssim |\nabla^2 d\widetilde{u}|^2(|d\widetilde{u}|^2+1) + |\nabla d\widetilde{u}|^2|\nabla^2 d\widetilde{u}||d\widetilde{u}| \\ &+ |d\widetilde{u}|^3|\nabla d\widetilde{u}||\nabla^2 d\widetilde{u}| + |\nabla d\widetilde{u}||\nabla^2 d\widetilde{u}|^2.\end{aligned} \tag{3.73}$$

Interpolation by parts and $\tau(\widetilde{u}) = \partial_s\widetilde{u}$ give

$$\left\|\nabla^2 d\widetilde{u}\right\|^2_{L^2_x} \lesssim \|\nabla\partial_s\widetilde{u}\|^2_{L^2_x} + \|\nabla d\widetilde{u}\|^3_{L^2_x} + \|\nabla d\widetilde{u}\|^2_{L^2_x}\|du\|^2_{L^\infty_x} \tag{3.74}$$

Then Proposition 3.15 yields (3.69). (3.73) shows $|\nabla^2 d\widetilde{u}|$ satisfies

$$\partial_s|\nabla^2 d\widetilde{u}| - \Delta|\nabla^2 d\widetilde{u}| \lesssim |\nabla^2 d\widetilde{u}|(|d\widetilde{u}|^2+1) + |\nabla d\widetilde{u}|^2|d\widetilde{u}| + |d\widetilde{u}|^3|\nabla d\widetilde{u}| + |\nabla d\widetilde{u}||\nabla^2 d\widetilde{u}|. \tag{3.75}$$

Let $f = |\nabla^2 d\widetilde{u}|e^{-\int_0^s(\|d\widetilde{u}\|^2_{L^\infty}+\|\nabla d\widetilde{u}\|_{L^\infty}+1)d\kappa}$. Then for $s\in[0,1]$, by Duhamel principle and smoothing effect, Lemma 2.7,

$$\|f(s,x)\|_{L^\infty_x} \lesssim s^{-\frac{1}{2}}\|f(\frac{s}{2},x)\|_{L^2_x} + \int_{\frac{s}{2}}^s (s-\tau)^{-\frac{1}{2}}\||\nabla d\widetilde{u}|^2|d\widetilde{u}| + |d\widetilde{u}|^3|\nabla d\widetilde{u}|\|_{L^2_x}d\tau. \tag{3.76}$$

Then (3.70) when $s\in[0,1]$ follows by Lemma 2.7 and Proposition 3.15. (3.69) gives $\|\nabla^2 d\widetilde{u}\|_{L^2_x} \le MC(M)$ for all $s\ge 1$. Meanwhile Proposition 3.15 shows $\|\nabla d\widetilde{u}\|_{L^\infty} + \|d\widetilde{u}\|_{L^\infty} \le MC(M)$ when $s\ge 1$. Then if let $Z \triangleq e^{-C_1(M)s}(e^{-C_1(M)s}|\nabla^2 d\widetilde{u}| + C_1(M))$, then $(\partial_s - \Delta)Z \le 0$. Applying Remark 3.3 to Z gives

$$\|\nabla^2 d\widetilde{u}(s,x)\|^2_{L^\infty_x} \lesssim \int_{s-1}^s \|\nabla^2 d\widetilde{u}(\tau,x)\|^2_{L^2_x}d\tau + MC(M). \tag{3.77}$$

Then (3.70) when $s\ge 1$ follows by (3.74), (3.33) and Proposition 3.15. The Bochner formula for $\left|\nabla^2\partial_s\widetilde{u}\right|^2$ is as follows

$$\begin{aligned}&\partial_s\left|\nabla^2\partial_s\widetilde{u}\right|^2 - \Delta\left|\nabla^2\partial_s\widetilde{u}\right|^2 + 2\left|\nabla^3\partial_s\widetilde{u}\right|^2 \lesssim |\nabla^2\partial_s\widetilde{u}|^2(|d\widetilde{u}|^2+1) + |\partial_s\widetilde{u}|^2|\nabla^2\partial_s\widetilde{u}||\nabla d\widetilde{u}| \\ &+ |\partial_s\widetilde{u}||d\widetilde{u}||\nabla\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}| + |\nabla^2\partial_s\widetilde{u}|^2|d\widetilde{u}||\partial_s\widetilde{u}| + |\nabla^2 d\widetilde{u}||d\widetilde{u}||\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}| \\ &+ |\nabla\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}||d\widetilde{u}||\nabla d\widetilde{u}| + |\nabla\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}||d\widetilde{u}||\nabla d\widetilde{u}| + |\nabla d\widetilde{u}||\nabla^2\partial_s\widetilde{u}|^2.\end{aligned} \tag{3.78}$$

Then one has

$$\begin{aligned}&\frac{d}{ds}\int_{\mathbb{H}^2} s^2\left|\nabla^2\partial_s\widetilde{u}\right|^2 \mathrm{dvol_h} \\ &\le \int_{\mathbb{H}^2} 2s\left|\nabla^2\partial_s\widetilde{u}\right|^2 - 2s^2\left|\nabla^3\partial_s\widetilde{u}\right|^2 + s^2\left|\nabla^2\partial_s\widetilde{u}\right|^2|d\widetilde{u}|^2\mathrm{dvol_h} \\ &+ \int_{\mathbb{H}^2} s^2|\nabla^2 d\widetilde{u}||d\widetilde{u}||\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}| + s^2|\nabla\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}||d\widetilde{u}||\nabla d\widetilde{u}|\mathrm{dvol_h} \\ &+ \int_{\mathbb{H}^2} s^2|\partial_s\widetilde{u}||d\widetilde{u}||\nabla\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}| + s^2|\nabla^2\partial_s u|^2|d\widetilde{u}||\partial_s\widetilde{u}|\mathrm{dvol_h} \\ &+ \int_{\mathbb{H}^2} s^2|\partial_s\widetilde{u}|^2|\nabla^2\partial_s\widetilde{u}||\nabla d\widetilde{u}| + s^2|\nabla^2\partial_s\widetilde{u}|^2|\nabla d\widetilde{u}|\mathrm{dvol_h}\end{aligned}$$

$$+\int_{\mathbb{H}^2} s^2|\nabla\partial_s\widetilde{u}||\nabla^2\partial_s\widetilde{u}||d\widetilde{u}||\nabla d\widetilde{u}|\mathrm{dvol_h}+\int_{\mathbb{H}^2} s^2\big|\nabla^2\partial_s\widetilde{u}\big|^2\mathrm{dvol_h}.$$

Integrating the above formula in $s\in[s_1,\tau]$ with any $0<s_1<\tau<2$, by Sobolev embedding, Gagliardo-Nirenberg and Young inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{H}^2}\tau^2\big|\nabla^2\partial_s\widetilde{u}\big|^2(\tau,t)\mathrm{dvol_h}-\int_{\mathbb{H}^2}s_1^2\big|\nabla^2\partial_s\widetilde{u}\big|^2(s_1,t)\mathrm{dvol_h}\\
&\lesssim\int_{s_1}^{\tau}\int_{\mathbb{H}^2} s\big|\nabla^2\partial_s\widetilde{u}\big|^2-s^2\big|\nabla^3\partial_s\widetilde{u}\big|^2+\|d\widetilde{u}\|^2_{L^\infty_sL^4_x}s^2\big|\nabla^2\partial_s\widetilde{u}\big|^2\mathrm{dvol_h}ds\\
&+\int_{s_1}^{\tau}\int_{\mathbb{H}^2} s^3|d\widetilde{u}|^2|\partial_s\widetilde{u}|^2\big|\nabla^2 d\widetilde{u}\big|^2+s^3|\nabla\partial_s\widetilde{u}|^2|d\widetilde{u}|^2|\nabla d\widetilde{u}|^2\mathrm{dvol_h}ds\\
&+\int_{s_1}^{\tau}\int_{\mathbb{H}^2} s^3|\partial_s\widetilde{u}|^2|d\widetilde{u}|^2|\nabla\partial_s\widetilde{u}|^2+s^2|d\widetilde{u}|^2|\partial_s\widetilde{u}|^2\mathrm{dvol_h}ds\\
&+\int_{s_1}^{\tau}\int_{\mathbb{H}^2} s^3|\nabla d\widetilde{u}|^2|\partial_s\widetilde{u}|^2+s^2|\nabla d\widetilde{u}|^2\mathrm{dvol_h}ds\\
&+\int_{s_1}^{\tau}\int_{\mathbb{H}^2} s^3|\nabla\partial_s\widetilde{u}|^2|d\widetilde{u}|^2|\nabla d\widetilde{u}|^2\mathrm{dvol_h}ds.
\end{aligned}$$

Thus letting $s_1\to0$, for $\tau\in(0,2)$, we deduce from (3.65), (3.70) and Proposition 3.15 that

$$\|s\nabla^2\partial_s\widetilde{u}\|_{L^2_x}\lesssim MC(M),$$

from which (3.71) when $s\in(0,1)$ follows. Integrating (3.78) with respect to x in $\mathbb{H}^2$, one obtains by (2.10) and Proposition 3.15 especially the L^∞_x bounds for $|d\widetilde{u}|+|\nabla d\widetilde{u}|$ that for $s\ge1$ and any $0<c\ll1$

$$\begin{aligned}
&\frac{d}{ds}\big\|\nabla^2\partial_s\widetilde{u}\big\|^2_{L^2_x}+c\big\|\nabla^2\partial_s\widetilde{u}\big\|^2_{L^2_x}\lesssim\|\partial_s\widetilde{u}\|_{L^\infty_x}\|\nabla\partial_s\widetilde{u}\|_{L^2_x}\big\|\nabla^2\partial_s\widetilde{u}\big\|_{L^2_x}+\frac{1}{c}\big\|\nabla^2\partial_s\widetilde{u}\big\|^2_{L^2_x}\\
&+\|\partial_s\widetilde{u}\|_{L^2_x}\big\|\nabla^2\partial_s\widetilde{u}\big\|_{L^2_x}+\|\nabla\partial_s\widetilde{u}\|_{L^2_x}\big\|\nabla^2\partial_s\widetilde{u}\big\|_{L^2_x}+\big\|\nabla^2\partial_s\widetilde{u}\big\|_{L^2_x}\|\partial_s\widetilde{u}\|^2_{L^4_x}.
\end{aligned}\tag{3.79}$$

Meanwhile integrating (3.63) with respect to s in (s',∞), we obtain from the exponential decay of $|\partial_s\widetilde{u}|+|\nabla\partial_s\widetilde{u}|$ in Proposition 3.15 that for $s'\ge1$

$$\int_{s'}^{\infty}\|\nabla^2\partial_s\widetilde{u}\|_{L^2_x}d\tau\lesssim e^{-\delta s'}MC(M).\tag{3.80}$$

Hence Gronwall inequality gives if choosing $0<c<\delta$ then for $s\ge1$ one has

$$\begin{aligned}
\big\|\nabla^2\partial_s\widetilde{u}(s)\big\|^2_{L^2_x}&\le e^{-cs}MC(M)+e^{-cs}\int_1^s e^{c\tau}\|\nabla^2\partial_s\widetilde{u}\|^2_{L^2_x}d\tau+e^{-cs}\int_1^s e^{c\tau}e^{-\delta\tau}d\tau\\
&\lesssim MC(M).
\end{aligned}\tag{3.81}$$

Applying Gronwall inequality to (3.79) again in $(\frac{s}{2},s)$, we deduce from (3.81) that

$$\big\|\nabla^2\partial_s\widetilde{u}(s)\big\|^2_{L^2_x}\le e^{-\frac{c}{2}s}MC(M)+e^{-cs}\int_{\frac{s}{2}}^s e^{c\tau}\|\nabla^2\partial_s\widetilde{u}\|^2_{L^2_x}d\tau+e^{-cs}\int_{\frac{s}{2}}^s e^{c\tau}e^{-\delta\tau}d\tau.\tag{3.82}$$

Thus (3.71) follows by (3.80). Finally (3.72) follows by (3.71) and applying Remark 3.3 to (3.78) as before. ☐

3.2. The existence of caloric gauge. As a preparation for the existence of the caloric gauge, we prove that the heat flows initiated from $u(t,x)$ with different t converge to the same harmonic map as u_0.

LEMMA 3.17. *If $(u,\partial_t u)$ is a solution to (1.2) in $\mathcal{X}_T$, then there exists a harmonic map $\widetilde{Q}$ such that as $s\to\infty$,*

$$\lim_{s\to\infty}\sup_{(x,t)\in\mathbb{H}^2\times[0,T]} dist_{\mathbb{H}^2}(\widetilde{u}(s,x,t),\widetilde{Q}(x))=0.$$

PROOF. The global existence of $\widetilde{u}$ is due to Lemma 3.2, the embedding $\mathbf{H^2}\hookrightarrow L^\infty$, $\mathbf{H^1}\hookrightarrow L^p$ for $p\in[2,\infty)$ and diamagnetic inequality. Then (3.5), maximum principle and (2.22) show

$$\|\partial_s\widetilde{u}(s,t,x)\|^2_{L^\infty_x}\le s^{-1}e^{-\frac14 s}\int_{\mathbb{H}^2}|\partial_s\widetilde{u}(0,t,x)|^2\mathrm{dvol_h}. \tag{3.83}$$

Thus (3.8) yields

$$\sup_{(x,t)\in\mathbb{H}^2\times[0,T]}|\partial_s\widetilde{u}(s,t,x)|\le s^{-\frac12}e^{-\frac18 s}\int_{\mathbb{H}^2}|\partial_t u(t,x)|^2\mathrm{dvol_h}\le Cs^{-1}e^{-\frac18 s}.$$

Therefore for any $1<s_0<s_1<\infty$ it holds

$$d_{\mathbb{H}^2}(\widetilde{u}(s_0,t,x),\widetilde{u}(s_1,t,x))\lesssim\int_{s_0}^{s_1}e^{-\frac18 s}ds,$$

which implies $\widetilde{u}(s,t,x)$ converges to some map $\widetilde{Q}(t,x)$ uniformly on $(t,x)\in[0,T]\times\mathbb{H}^2$. By [Theorem 5.2,[**36**]], for any fixed t, $\widetilde{Q}(t,x)$ is a harmonic map form $\mathbb{H}^2\to\mathbb{H}^2$. It suffices to verify $\widetilde{Q}(t,x)$ is indeed independent of t. By (3.26), maximum principle and (2.22),

$$\sup_{x\in\mathbb{H}^2}|\partial_t\widetilde{u}(s,t,x)|^2\le s^{-1}e^{-\frac14 s}\int_{\mathbb{H}^2}|\partial_t\widetilde{u}(0,t,x)|^2\mathrm{dvol_h}. \tag{3.84}$$

As a consequence, for $0\le t_1<t_2\le T$ one has

$$d_{\mathbb{H}^2}(\widetilde{u}(s,t_1,x),\widetilde{u}(s,t_2,x))\le\int_{t_1}^{t_2}|\partial_t\widetilde{u}(s,t,x)|dt\le Cs^{-\frac12}e^{-s/8}(t_2-t_1).$$

Let $s\to\infty$, we get $d_{\mathbb{H}^2}(\widetilde{Q}(t_1,x),\widetilde{Q}(t_2,x))=0$, thus finishing the proof. □

LEMMA 3.18. *Let Q be an admissible harmonic map in Definition 1.1, and μ_1,μ_2 be sufficiently small. If $(u,\partial_t u)$ is a solution to (1.2) in $\mathcal{X}_T$, then $\widetilde{u}(s,t,x)$ uniformly converges to Q as $s\to\infty$.*

PROOF. By Lemma 3.17, it suffices to prove $Q=\widetilde{Q}$. In the coordinate (2.1), the harmonic map equation can be written as

$$\Delta\widetilde{Q}^l+h^{ij}\overline{\Gamma}^l_{pq}\frac{\partial\widetilde{Q}^p}{\partial x_i}\frac{\partial\widetilde{Q}^q}{\partial x_j}=0 \tag{3.85}$$

$$\Delta Q^l+h^{ij}\overline{\Gamma}^l_{pq}\frac{\partial Q^p}{\partial x_i}\frac{\partial Q^q}{\partial x_j}=0. \tag{3.86}$$

Denote the heat flow initiated from u_0 by $U(s,x)$, then by (2.10),

$$\begin{aligned}&\|U^1(s,x)-Q^1(x)\|_{L^2}+\|U^2(s,x)-Q^2(x)\|_{L^2}\\&\lesssim\|\nabla(U^1-Q^1)\|_{L^2}+\|\nabla(U^2-Q^2)\|_{L^2}.\end{aligned}$$

[Lemma 2.3,[**37**]] shows that for $k=1,2, l=1,2$,

$$\|\nabla^l U^k\|_{L^2} \lesssim C(\|U\|_{\mathfrak{H}^2}, R_0, \|Q\|_{\mathfrak{H}^2})\|\nabla^{l-1} dU\|_{L^2}.$$

By energy arguments, one obtains $\|\nabla dU\|_{L^2} \le C(\|\nabla du_0\|_{L^2}, \|du_0\|_{L^2})$ and the energy decreases along the heat flow, see (3.17). Thus we have by Sobolev embedding and Corollary 2.5 that

$$\begin{aligned} &\|U^1(s,x) - Q^1(x)\|_{L^2} + \|U^2(s,x) - Q^2(x)\|_{L^2} + \|dU\|_{L^2} \\ &\le C(R_0)\mu_2 + C(R_0)\mu_1 \end{aligned} \tag{3.87}$$

$$\|U^1(s,x)\|_{L^\infty} + \|U^2(s,x)\|_{L^\infty} \le C(R_0). \tag{3.88}$$

Hence letting $s \to \infty$, we have for some constant $C(R_0)$

$$\|\widetilde{Q}^1\|_{L^\infty} + \|\widetilde{Q}^2\|_{L^\infty} \le C,\ \|\nabla \widetilde{Q}^1\|_{L^2} + \|\nabla \widetilde{Q}^2\|_{L^2} \le \mu_1 C(R_0) \tag{3.89}$$

Multiplying the difference between (3.85) and (3.86) with $-Q^l + \widetilde{Q}^l$, we have by integration by parts that

$$\begin{aligned} \left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|_{L^2} &\le \left\langle h^{ij}\left(\overline{\Gamma}^l_{pq}(Q) - \overline{\Gamma}^l_{pq}(\widetilde{Q})\right)\frac{\partial Q^p}{\partial x_i}\frac{\partial Q^q}{\partial x_j}, -Q^l + \widetilde{Q}^l\right\rangle \\ &+ \left\langle h^{ij}\overline{\Gamma}^l_{pq}(\widetilde{Q})\left(\frac{\partial Q^p}{\partial x_i} - \frac{\partial \widetilde{Q}^p}{\partial x_i}\right)\frac{\partial Q^q}{\partial x_j}, -Q^l + \widetilde{Q}^l\right\rangle \\ &+ \left\langle h^{ij}\overline{\Gamma}^l_{pq}(\widetilde{Q})\frac{\partial \widetilde{Q}^p}{\partial x_i}\left(\frac{\partial Q^q}{\partial x_i} - \frac{\partial \widetilde{Q}^q}{\partial x_j}\right), -Q^l + \widetilde{Q}^l\right\rangle. \end{aligned}$$

Thus using the explicit formula for $\overline{\Gamma}^l_{pq}$, by (3.87), (3.88), (3.89) we get

$$\begin{aligned} &\left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|^2_{L^2} \\ &\lesssim \left(\left\|Q^l - \widetilde{Q}^l\right\|^2_{L^2} + \left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|^2_{L^2}\right)\left(\sum_{k=1}^2 \left\|\nabla \widetilde{Q}^k\right\|^2_{L^2} + \|\nabla Q^k\|^2_{L^2}\right). \end{aligned}$$

Therefore, we conclude for some constant $C(R_0)$ which is independent of μ_1, μ_2 provided $0 \le \mu_1, \mu_2 \le 1$

$$\begin{aligned} &\sum_{l=1}^2 \left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|^2_{L^2} \\ &\le C(R_0)\left(\|dQ\|^2_{L^2} + \left\|d\widetilde{Q}\right\|^2_{L^2}\right)\left(\sum_{l=1}^2 \left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|^2_{L^2} + \left\|Q^l - \widetilde{Q}^l\right\|^2_{L^2}\right). \end{aligned}$$

Let μ_1, μ_2 be sufficiently small, (2.10) gives

$$\begin{aligned} &\sum_{l=1}^2 \left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|^2_{L^2} + \left\|Q^l - \widetilde{Q}^l\right\|^2_{L^2} \\ &\le (\mu_1 + \mu_2)\left(\sum_{l=1}^2 \left\|\nabla\left(Q^l - \widetilde{Q}^l\right)\right\|^2_{L^2} + \left\|Q^l - \widetilde{Q}^l\right\|^2_{L^2}\right). \end{aligned}$$

Hence $\widetilde{Q} = Q$. □

Now we are ready to prove the existence of the caloric gauge in Definition 3.1.

PROPOSITION 3.19. Given any solution $(u, \partial_t u)$ of (1.2) in $\mathcal{X}_T$ with $(u_0, u_1) \in \mathbf{H}^3_{\mathbf{Q}} \times \mathbf{H}^2_{\mathbf{Q}}$. For any fixed frame $\Xi \triangleq \{\Xi_1(Q(x)), \Xi_2(Q(x))\}$, there exists a unique corresponding caloric gauge defined in Definition 3.1.

PROOF. We first show the existence part. Choose an arbitrary orthonormal frame $E_0(t,x) \triangleq \{\mathsf{e}_i(t,x)\}_{i=1}^2$ such that $E_0(t,x)$ spans the tangent space $T_{u(t,x)}\mathbb{H}^2$ for each $(t,x) \in [0,T] \times \mathbb{H}^2$. The desired frame does exist, in fact we have a global orthonormal frame for $\mathbb{H}^2$ defined by (2.3). Then evolving (3.8) with initial data $u(t,x)$, we have from Lemma 3.18 that $\widetilde{u}(s,t,x)$ converges to Q uniformly for $(t,x) \in [0,T] \times \mathbb{H}^2$ as $s \to \infty$. Meanwhile, we evolve E_0 in s according to

$$\begin{cases} \nabla_s \Omega_i(s,t,x) = 0 \\ \Omega_i(s,t,x) \restriction_{s=0} = \mathsf{e}_i(t,x) \end{cases} \tag{3.90}$$

Denote the evolved frame as $E_s \triangleq \{\Omega_i(s,t,x)\}_{i=1}^2$. We claim that there exists some orthonormal frame $E_\infty \triangleq \{\mathsf{e}_i(\infty,t,x)\}_{i=1}^2$ which spans $T_{Q(x)}\mathbb{H}^2$ for each $(t,x) \in [0,T] \times \mathbb{H}^2$ such that

$$\lim_{s\to\infty} \Omega_i(s,t,x) = \mathsf{e}_i(\infty,t,x). \tag{3.91}$$

Indeed, by the definition of the convergence of frames given in (3.2) and the fact $\widetilde{u}(s,t,x)$ converges to $Q(x)$, it suffices to show for some scalar function $c_i : [0,T] \times \mathbb{H}^2 \to \mathbb{R}$

$$\lim_{s\to\infty} \langle \Omega_i(s,t,x), \Theta_i(\widetilde{u}(s,t,x))\rangle = c_i(t,x). \tag{3.92}$$

By direct calculations,

$$|\nabla_s \Theta_i(\widetilde{u}(s,t,x))| \lesssim |\partial_s \widetilde{u}|\,.$$

then (3.83) and $\nabla_s \Omega = 0$ imply that for $s > 1$

$$\left|\partial_s \langle \Omega_i(s,t,x), \Theta_i(\widetilde{u}(s,t,x))\rangle\right| \lesssim M e^{-\delta s}.$$

Hence (3.92) holds for some $c_i(t,x)$, thus verifying (3.91). It remains to adjust the initial frame E_0 to make the limit frame E_∞ coincide with the given frame Ξ. This can be achieved by the gauge transform invariance illustrated in Section 2.1. Indeed, since for any $U : [0,T] \times \mathbb{H}^2 \to SO(2)$, and the solution $\widetilde{u}(s,t,x)$ to (3.8), one has $\nabla_s U(t,x)\Omega(s,t,x) = U(t,x)\nabla_s\Omega(s,t,x)$, then the following gauge symmetry holds

$$\begin{aligned} E_0 \triangleq \{\mathsf{e}_i(t,x)\}_{i=1}^2 &\mapsto E'_0 \triangleq \{U(t,x)\mathsf{e}_i(t,x)\}_{i=1}^2 \\ E_s \triangleq \{\Omega_i(s,t,x)\}_{i=1}^2 &\mapsto E'_s \triangleq \{U(t,x)\Omega_i(s,t,x)\}_{i=1}^2 \,. \end{aligned}$$

Therefore choosing $U(t,x)$ such that $U(t,x)E_\infty = \Xi$, where E_∞ is the limit frame obtained by (3.91), suffices for our purpose. The uniqueness of the gauge follows from the identity

$$\frac{d}{ds}\langle \Phi_1 - \Phi_2, \Phi_1 - \Phi_2\rangle = 0,$$

where (Φ_1) and (Φ_2) are two caloric gauges satisfying (3.1). ☐

3.3. Expressions for the connection coefficients. The following lemma gives the expressions for the connection coefficients matrix $A_{x,t}$ by differential fields. The proof of Lemma 3.20 is almost the same as [Lemma 3.6,[**37**]], thus we omit it.

LEMMA 3.20. *Suppose that $\Omega(s,t,x)$ is the caloric gauge constructed in Proposition 3.19, then we have for $i=1,2$*

$$\lim_{s\to\infty}[A_i]^j_k(s,t,x)=\langle\nabla_i\Xi_k(x),\Xi_j(x)\rangle \tag{3.93}$$

$$\lim_{s\to\infty}A_t(s,t,x)=0 \tag{3.94}$$

Particularly let $\Xi(x)=\Theta(Q(x))$ in Proposition 3.19, denote A_i^∞ the limit coefficient matrix, i.e., $[A_i^\infty]^k_j=\langle\nabla_i\Xi_k(Q(x)),\Xi_j(Q(x))\rangle$, then we have for $i=1,2$, $s>0$,

(3.95)
$$A_i(s,t,x)\sqrt{h^{ii}(x)}=\int_s^\infty\sqrt{h^{ii}(x)}\mathbf{R}(\widetilde{u}(\kappa))\left(\partial_s\widetilde{u}(\kappa),\partial_i\widetilde{u}(\kappa)\right)d\kappa+\sqrt{h^{ii}(x)}A_i^\infty.$$

(3.96)
$$A_t(s,t,x)=\int_s^\infty\phi_s\wedge\phi_t d\kappa,$$

REMARK 3.21. For convenience, we rewrite (3.95) as $A_i(s,t,x)=A_i^\infty(s,t,x)+A_i^{con}(s,t,x)$, where A_i^∞ denotes the limit part, and A_i^{con} denotes the controllable part, i.e.,

$$A_i^{con}=\int_s^\infty\phi_s\wedge\phi_i d\kappa.$$

Similarly, we split ϕ_i into $\phi_i=\phi_i^\infty+\phi_i^{con}$, where $\phi_i^{con}=\int_s^\infty\partial_s\phi_i d\kappa$, and

$$\phi_i^\infty=\left(\langle\partial_iQ(x),\Xi_1(Q(x))\rangle,\langle\partial_iQ(x),\Xi_2(Q(x))\rangle\right)^t.$$

4. Derivation of the master equation for the heat tension field

Recall that the heat tension filed ϕ_s satisfies

$$\phi_s=h^{ij}D_i\phi_j-h^{ij}\Gamma^k_{ij}\phi_k. \tag{4.1}$$

And we define the wave tension filed as Tao by

$$\mathfrak{W}=D_t\phi_t-h^{ij}D_i\phi_j+h^{ij}\Gamma^k_{ij}\phi_k. \tag{4.2}$$

In fact (4.1) is the gauged equation for the heat flow equation, and (4.2) is the gauged equation for the wave map (1.2), see Lemma 2.7. The evolution of ϕ_s with respect to t is given by the following lemma.

LEMMA 4.1. *The heat tension field ϕ_s satisfies*

$$\begin{aligned}D_tD_t\phi_s-h^{ij}D_iD_j\phi_s+h^{ij}\Gamma^k_{ij}D_k\phi_s=\partial_s\mathfrak{W}&+h^{ij}\mathbf{R}(\partial_s\widetilde{u},\partial_i\widetilde{u})\left(\partial_j\widetilde{u}\right)\\&+\mathbf{R}(\partial_t\widetilde{u},\partial_s\widetilde{u})\left(\partial_t\widetilde{u}\right).\end{aligned}\tag{4.3}$$

PROOF. By the torsion free identity and the commutator identity, we have

$$\begin{aligned}D_tD_t\phi_s&=D_tD_s\phi_t=D_sD_t\phi_t+\mathbf{R}(\partial_t\widetilde{u},\partial_s\widetilde{u})\left(\partial_t\widetilde{u}\right)\\&=D_s\left(\mathfrak{W}+h^{ij}D_i\phi_j-h^{ij}\Gamma^k_{ij}\phi_k\right)+\mathbf{R}(\partial_t\widetilde{u},\partial_s\widetilde{u})\left(\partial_t\widetilde{u}\right)\\&=\partial_s\mathfrak{W}+h^{ij}D_sD_i\phi_j-h^{ij}\Gamma^k_{ij}D_s\phi_k+\mathbf{R}(\partial_t\widetilde{u},\partial_s\widetilde{u})\left(\partial_t\widetilde{u}\right)\\&=\partial_s\mathfrak{W}+h^{ij}D_iD_j\phi_s-h^{ij}\Gamma^k_{ij}D_k\phi_s+h^{ij}\mathbf{R}(\partial_s\widetilde{u},\partial_i\widetilde{u})\left(\partial_j\widetilde{u}\right)+\mathbf{R}(\partial_t\widetilde{u},\partial_s\widetilde{u})\left(\partial_t\widetilde{u}\right).\end{aligned}$$

Thus (4.3) is verified. □

The evolution of $\mathfrak{W}$ with respect to s is given by the following lemma.

LEMMA 4.2. *Under orthogonal coordinates, the wave tension field* $\mathfrak{W}$ *satisfies*

$$\partial_s\mathfrak{W} = \Delta\mathfrak{W} + 2h^{ii}A_i\partial_i\mathfrak{W} + h^{ii}A_iA_i\mathfrak{W} + h^{ii}\partial_iA_i\mathfrak{W} - h^{ii}\Gamma^k_{ii}A_k\mathfrak{W} + h^{ii}\left(\mathfrak{W}\wedge\phi_i\right)\phi_i + 3h^{ii}(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\nabla_t\partial_i\widetilde{u}.$$

PROOF. In the following calculations, we always use the convention in Remark 2.1. By $\mathfrak{W} = D_t\phi_t - \phi_s$, we have from commutator equality that

$$\partial_s\mathfrak{W} = D_s(D_t\phi_t - \phi_s) = D_tD_t\phi_s - D_s\phi_s + \mathbf{R}(\partial_s\widetilde{u}, \partial_t\widetilde{u})\left(\partial_t\widetilde{u}\right).$$

Further applications of the torsion free identity and commutator identity show

$$\begin{aligned}
&D_tD_t\phi_s - D_s\phi_s\\
&= D_tD_t\left(h^{ij}D_i\phi_j - h^{ij}\Gamma^k_{ij}\phi_k\right) - D_s\left(h^{ij}D_i\phi_j - h^{ij}\Gamma^k_{ij}\phi_k\right)\\
&= h^{ij}D_tD_tD_i\phi_j - h^{ij}\Gamma^k_{ij}D_tD_t\phi_k - \left(h^{ij}D_sD_i\phi_j - h^{ij}\Gamma^k_{ij}D_s\phi_k\right)\\
&= h^{ij}D_t\left(D_iD_j\phi_t + \mathbf{R}(\partial_t\widetilde{u}, \partial_i\widetilde{u})(\partial_j\widetilde{u})\right) - h^{ij}\Gamma^k_{ij}\left(D_kD_t\phi_t + \mathbf{R}(\partial_t\widetilde{u}, \partial_k\widetilde{u})(\partial_t\widetilde{u})\right)\\
&- \left(h^{ij}D_iD_j\phi_s - h^{ij}\Gamma^k_{ij}D_k\phi_s + h^{ij}\mathbf{R}(\partial_s\widetilde{u}, \partial_i\widetilde{u})(\partial_j\widetilde{u})\right)\\
&= h^{ij}D_tD_iD_j\phi_t - h^{ij}\Gamma^k_{ij}D_kD_t\phi_t - h^{ij}D_iD_j\phi_s + h^{ij}\Gamma^k_{ij}D_k\phi_s - h^{ij}\mathbf{R}(\partial_s\widetilde{u}, \partial_i\widetilde{u})(\partial_j\widetilde{u})\\
&+ h^{ij}\nabla_t\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_i\widetilde{u})(\partial_j\widetilde{u})\right) - h^{ij}\Gamma^k_{ij}\mathbf{R}(\partial_t\widetilde{u}, \partial_k\widetilde{u})\left(\partial_t\widetilde{u}\right).
\end{aligned}$$

The leading term can be written as

$$\begin{aligned}
h^{ij}D_tD_iD_j\phi_t &= h^{ij}D_iD_tD_j\phi_t + h^{ij}\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_i\widetilde{u})e\left(D_j\phi_t\right)\right)\\
&= h^{ij}D_iD_jD_t\phi_t + h^{ij}\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_i\widetilde{u})\nabla_j\partial_t\widetilde{u}\right) + h^{ij}\nabla_i\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_j\widetilde{u})\partial_t\widetilde{u}\right).
\end{aligned}$$

Thus we conclude as

$$\begin{aligned}
\partial_s\mathfrak{W} = &h^{ij}D_iD_j(D_t\phi_t - \phi_s) - h^{ij}\Gamma^k_{ij}D_k(D_t\phi_t - \phi_s) + h^{ij}\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_i\widetilde{u})\nabla_j\partial_t\widetilde{u}\right)\\
&+ h^{ij}\nabla_i\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_j\widetilde{u})\partial_t\widetilde{u}\right) - h^{ij}\mathbf{R}(\partial_s\widetilde{u}, \partial_i\widetilde{u})(\partial_j\widetilde{u}) + h^{ij}\nabla_t\left(\mathbf{R}(\partial_t\widetilde{u}, \partial_i\widetilde{u})(\partial_j\widetilde{u})\right)\\
&- h^{ij}\Gamma^k_{ij}\mathbf{R}(\partial_t\widetilde{u}, \partial_k\widetilde{u})\left(\partial_t\widetilde{u}\right) + \mathbf{R}(\partial_s\widetilde{u}, \partial_t\widetilde{u})\partial_t\widetilde{u}.
\end{aligned}$$

Using $\mathfrak{W} = D_t\phi_t - \phi_s$ and (2.5) yields

$$\begin{aligned}
\partial_s\mathfrak{W} = &\Delta\mathfrak{W} + 2h^{ii}A_i\partial_i\mathfrak{W} + h^{ii}A_iA_i\mathfrak{W} + h^{ii}\partial_iA_i\mathfrak{W} - h^{ij}\Gamma^k_{ij}A_k\mathfrak{W}\\
&+ \left\{-h^{ii}\left(\partial_s\widetilde{u}\wedge\partial_i\widetilde{u}\right)\partial_i\widetilde{u} + h^{ii}(\nabla_t\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\partial_i\widetilde{u}\right\}\\
&+ h^{ii}(\partial_t\widetilde{u}\wedge\nabla_t\partial_i\widetilde{u})\partial_i\widetilde{u} + h^{ii}(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\nabla_t\partial_i\widetilde{u}\\
&+ h^{ii}(\nabla_i\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\partial_t\widetilde{u} + h^{ii}(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\nabla_i\partial_t\widetilde{u}\\
&+ \left\{h^{ii}(\partial_t\widetilde{u}\wedge\nabla_i\partial_i\widetilde{u})\partial_t\widetilde{u} - h^{ii}\Gamma^k_{ii}(\partial_t\widetilde{u}\wedge\partial_k\widetilde{u})\partial_t\widetilde{u} + (\partial_s\widetilde{u}\wedge\partial_t\widetilde{u})\partial_t\widetilde{u}\right\}.
\end{aligned} \tag{4.4}$$

Recalling the facts that $\mathfrak{W}$ is the gauged field for $\nabla_t\partial_t\widetilde{u} - \tau(\widetilde{u})$ and $\partial_s\widetilde{u} = \tau(\widetilde{u})$, we have

$$\begin{aligned}
-h^{ii}\left(\partial_s\widetilde{u}\wedge\partial_i\widetilde{u}\right)\partial_i\widetilde{u} + h^{ii}(\nabla_t\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\partial_i\widetilde{u} &= h^{ii}\left(\left(\nabla_t\partial_t\widetilde{u} - \partial_s\widetilde{u}\right)\wedge\partial_i\widetilde{u}\right)\partial_i\widetilde{u}\\
&= h^{ii}\left(\mathfrak{W}\wedge\phi_i\right)\phi_i.
\end{aligned} \tag{4.5}$$

Meanwhile, $\partial_s\widetilde{u} = \tau(\widetilde{u})$ also implies

$$h^{ii}(\partial_t\widetilde{u}\wedge\nabla_i\partial_i\widetilde{u})\partial_t\widetilde{u} - h^{ii}\Gamma^k_{ii}(\partial_t\widetilde{u}\wedge\partial_k\widetilde{u})\partial_t\widetilde{u} + (\partial_s\widetilde{u}\wedge\partial_t\widetilde{u})\partial_t\widetilde{u}$$

$$= h^{ii}\left(\partial_t\widetilde{u}\wedge(\tau(\widetilde{u})-\partial_s\widetilde{u})\right)\partial_t\widetilde{u}=0. \tag{4.6}$$

Bianchi identity gives

$$\begin{aligned} h^{ii}(\partial_t\widetilde{u}\wedge\nabla_t\partial_i\widetilde{u})\partial_i\widetilde{u}+h^{ii}(\nabla_i\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\partial_t\widetilde{u} &= -h^{ii}(\partial_i\widetilde{u}\wedge\partial_t\widetilde{u})\nabla_t\partial_i\widetilde{u} \\ &= h^{ii}(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\nabla_t\partial_i\widetilde{u}. \end{aligned} \tag{4.7}$$

By (4.5), (4.6) and (4.7), (4.4) can be further simplified as

$$\begin{aligned} \partial_s\mathfrak{W} =&\Delta\mathfrak{W}+2h^{ii}A_i\partial_i\mathfrak{W}+h^{ii}A_iA_i\mathfrak{W}+h^{ii}\partial_iA_i\mathfrak{W}-h^{ii}\Gamma^k_{ii}A_k\mathfrak{W} \\ &+h^{ii}\left(\mathfrak{W}\wedge\phi_i\right)\phi_i+3h^{ii}(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\nabla_t\partial_i\widetilde{u}. \end{aligned}$$

□

LEMMA 4.3. *Let Q be an admissible harmonic map in Definition 1.1. Fix the frame Ξ in Remark 3.21 by taking $\Xi(Q(x))=\Theta(Q(x))$ given by (2.1). Recall the definitions of A_i^∞ in Lemma 3.20. Then*

$$|A_i^\infty|\lesssim|dQ|,|\sqrt{h^{ii}}\phi_i^\infty|\lesssim|dQ| \tag{4.8}$$

$$|h^{ii}\left(\partial_iA_i^\infty-\Gamma^k_{ii}A_k^\infty\right)|\lesssim|dQ|^2. \tag{4.9}$$

PROOF. Recall the definition

$$[A_i^\infty]^j_k=\langle\nabla_i\Theta_k,\Theta_j\rangle,\Theta_1=e^{Q^2(x)}\frac{\partial}{\partial y_1},\Theta_2=\frac{\partial}{\partial y_2}.$$

Since A_i is skew-symmetric, it suffices to consider the $[A_i]^1_2$ terms. Direct calculation gives

$$\begin{aligned} [A_1^\infty]^1_2=\langle\nabla_1\Theta_2,\Theta_1\rangle=e^{Q^2(x)}\frac{\partial Q^k}{\partial x_1}\left\langle\nabla_{\frac{\partial}{\partial y_k}}\frac{\partial}{\partial y_2},\frac{\partial}{\partial y_1}\right\rangle=e^{-Q^2(x)}\frac{\partial Q^k}{\partial x_1}\overline{\Gamma}^1_{k2} \\ =-e^{-Q^2(x)}\frac{\partial Q^1}{\partial x_1}, \end{aligned}$$

and similarly we obtain

$$[A_1^\infty]^2_1=e^{-Q^2(x)}\frac{\partial Q^1}{\partial x_1};\ [A_2^\infty]^2_1=-[A_2^\infty]^1_2=e^{-Q^2(x)}\frac{\partial Q^1}{\partial x_2}.$$

Thus one has

$$\begin{aligned} &h^{ii}\left(\partial_i[A_i^\infty]^1_2-\Gamma^k_{ii}[A_k^\infty]^1_2\right) \\ &=-\left(\frac{\partial^2Q^1}{\partial {x_2}^2}e^{2x_2}+\frac{\partial^2Q^1}{\partial {x_1}^2}-\frac{\partial Q^1}{\partial x_1}\frac{\partial Q^2}{\partial x_1}e^{2x_2}-\frac{\partial Q^1}{\partial x_2}\frac{\partial Q^2}{\partial x_2}-\frac{\partial Q^1}{\partial x_2}\right)e^{-Q^2(x)} \end{aligned} \tag{4.10}$$

Writing the harmonic map equation for Q in the coordinate (2.1) shows for $l=1,2$

$$h^{ii}\frac{\partial^2Q^l}{\partial {x_i}^2}-h^{ii}\Gamma^k_{ii}\partial_kQ^l+h^{ii}\overline{\Gamma}^l_{pq}\frac{\partial Q^p}{\partial x_i}\frac{\partial Q^q}{\partial x_i}=0.$$

Let $l=1$ in the above equation, we have

$$e^{2x_2}\frac{\partial^2Q^1}{\partial {x_1}^2}+\frac{\partial^2Q^1}{\partial {x_2}^2}-\frac{\partial Q^1}{\partial x_2}-2e^{2x_2}\frac{\partial Q^1}{\partial x_1}\frac{\partial Q^2}{\partial x_1}-2\frac{\partial Q^1}{\partial x_2}\frac{\partial Q^2}{\partial x_2}=0,$$

which combined with (4.10) yields

$$h^{ii}\left(\partial_iA_i^\infty-\Gamma^k_{ii}A_k^\infty\right)=\left(e^{2x_2}\frac{\partial Q^1}{\partial x_1}\frac{\partial Q^2}{\partial x_1}+\frac{\partial Q^1}{\partial x_2}\frac{\partial Q^2}{\partial x_2}\right)e^{-Q^2(x)}. \tag{4.11}$$

Writing the energy density in coordinates (2.1), we obtain

$$\begin{aligned}|dQ|^2 &= h^{ij}\left\langle \frac{\partial Q^k}{\partial x_i}\frac{\partial}{\partial y_k}, \frac{\partial Q^k}{\partial x_j}\frac{\partial}{\partial y_k}\right\rangle \\ &= e^{2x_2}\left|\frac{\partial Q^1}{\partial x_1}\right|^2 e^{-2Q_2} + e^{2x_2}\left|\frac{\partial Q^2}{\partial x_1}\right|^2 + \left|\frac{\partial Q^1}{\partial x_2}\right|^2 e^{-2Q_2} + \left|\frac{\partial Q^2}{\partial x_2}\right|^2.\end{aligned}$$

Thus (4.9) follows by (4.11) and Young inequality. (4.8) is much easier and follows immediately by the same arguments. □

Now we separate the main term in the equation of ϕ_s. Recall the limit of $A_{s,t,x}$ given in (3.93), (3.94), one can easily see the main term of (4.3) is a magnetic wave equation. Precisely, we have the following lemma.

LEMMA 4.4. *Fix the frame Ξ in Proposition 3.3 by letting $\Xi_i(x) = \Theta_i(Q(x))$, $i = 1, 2$. Then the heat tension filed ϕ_s satisfies*

$$\begin{aligned}&(\partial_t^2 - \Delta)\phi_s + W\phi_s \\ &= -2A_t\partial_t\phi_s - A_tA_t\phi_s - \partial_tA_t\phi_s + \partial_s w + \mathbf{R}(\partial_t\widetilde{u}, \partial_s\widetilde{u})(\partial_t\widetilde{u}) + 2h^{ii}A_i^{con}\partial_i\phi_s \\ &+ h^{ii}A_i^{con}A_i^{\infty}\phi_s + h^{ii}A_i^{\infty}A_i^{con}\phi_s + h^{ii}A_i^{con}A_i^{con}\phi_s + h^{ii}\left(\partial_iA_i^{con} - \Gamma_{ii}^kA_k^{con}\right)\phi_s \\ &+ h^{ii}\left(\phi_s \wedge \phi_i^{\infty}\right)\phi_i^{con} + h^{ii}\left(\phi_s \wedge \phi_i^{con}\right)\phi_i^{\infty} + h^{ii}\left(\phi_s \wedge \phi_i^{con}\right)\phi_i^{con},\end{aligned}$$

where A_x^{∞}, A_x^{con} are defined in Remark 3.21, and W is given by

(4.12)
$$W\varphi = -2h^{ii}A_i^{\infty}\partial_i\varphi - h^{ii}A_i^{\infty}A_i^{\infty}\varphi - h^{ii}\left(\varphi \wedge \phi_i^{\infty}\right)\phi_i^{\infty} - h^{ii}(\partial_iA_i^{\infty} - \Gamma_{ii}^kA_k^{\infty}).$$

Furthermore, $-\Delta + W$ is a self-adjoint operator in $L^2(\mathbb{H}^2; \mathbb{C}^2)$. And it is strictly positive if $0 < \mu_1 \ll 1$.

PROOF. By (4.3), expanding $D_{x,t}$ as $\partial_{t,x} + A_{t,x}$ implies

$$\begin{aligned}&\partial_t^2\phi_s - \Delta\phi_s \\ &= -2A_t\partial_t\phi_s - A_tA_t\phi_s - \partial_tA_t\phi_s + h^{ii}A_iA_i\phi_s + h^{ii}\left(\partial_iA_i - \Gamma_{ii}^kA_k\right)\phi_s \\ &+ 2h^{ii}A_i\partial_i\phi_s + \partial_s\mathfrak{W} + h^{ii}\mathbf{R}(\partial_s\widetilde{u}, \partial_i\widetilde{u})(\partial_i\widetilde{u}) + \mathbf{R}(\partial_t\widetilde{u}, \partial_s\widetilde{u})(\partial_t\widetilde{u}).\end{aligned} \tag{4.13}$$

By Remark 3.21, $A_i = A_i^{\infty} + A_i^{con}$, $\phi_i = \phi_i^{\infty} + \phi_i^{con}$. Then fixing Ξ to be $(\Theta_1(Q), \Theta_2(Q))$, we have (4.13) reduces to

$$\begin{aligned}&\partial_t^2\phi_s - \Delta\phi_s - 2h^{ii}A_i^{\infty}\partial_i\phi_s - h^{ii}A_i^{\infty}A_i^{\infty}\phi_s - h^{ii}\left(\phi_s \wedge \phi_i^{\infty}\right)\phi_i^{\infty} - h^{ii}(\partial_iA_i^{\infty} - \Gamma_{ii}^kA_k^{\infty})\phi_s \\ &= -2A_t\partial_t\phi_s - A_tA_t\phi_s - \partial_tA_t\phi_s + \partial_s w + (\phi_t \wedge \phi_s)\phi_t + h^{ii}A_i^{con}\partial_i\phi_s + h^{ii}A_i^{con}A_i^{\infty}\phi_s \\ &+ h^{ii}A_i^{\infty}A_i^{con}\phi_s + h^{ii}A_i^{con}A_i^{con}\phi_s + h^{ii}\left(\partial_iA_i^{con} - \Gamma_{ii}^kA_k^{con}\right)\phi_s + h^{ii}\left(\phi_s \wedge \phi_i^{\infty}\right)\phi_i^{con} \\ &+ h^{ii}\left(\phi_s \wedge \phi_i^{con}\right)\phi_i^{\infty} + h^{ii}\left(\phi_s \wedge \phi_i^{con}\right)\phi_i^{con}.\end{aligned}$$

Then from the non-negativeness of the sectional curvature for the target $N = \mathbb{H}^2$ and the skew-symmetry of the connection matrix A_i^{∞}, we have W is a nonnegative symmetric operator in $L^2(\mathbb{H}^2; \mathbb{C}^2)$ by direct calculations, see Lemma 7.3 in Section 7. The self-adjointness of W follows from Kato's perturbation theorem. In fact, there exists a self-adjoint realization denote by $((\Delta_{col}), D(\Delta_{col}))$ of $(\Delta, C_c^{\infty}(\mathbb{H}^2, \mathbb{C}^2))$. It is known that $D(\Delta_{col})$ consists of functions $f \in L^2$ whose Laplacian Δf in distribution sense belong to L^2, see for instance [**50**]. Write W as

$W = V_1 + V_2\nabla$, then V_1 and V_2 are of exponential decay as $d(x,0) \to \infty$ by Lemma 4.3 and Definition 1.1. For any fixed $\varepsilon > 0$, take $R > 0$ sufficiently large such that

$$\|V_1(x)\|_{L^\infty_{d(x,0)\geq R}} \leq \varepsilon,\ \|V_2(x)\|_{L^\infty_{d(x,0)\geq R}} \leq \varepsilon,$$

then for any $f \in C_c^\infty(\mathbb{H}^2, \mathbb{C}^2)$,

$$\|V_1(x)f + V_2\nabla f\|_{L^2_{d(x,0)\geq R}} \leq \varepsilon\|f\|_{L^2} + \varepsilon\|\nabla f\|_{L^2}. \tag{4.14}$$

For this R, the compactness of Sobolev embedding in bounded domains implies there exists $C(\varepsilon, R)$ such that

$$\|V_1(x)f + V_2\nabla f\|_{L^2_{d(x,0)\leq R}} \leq C(\varepsilon, R)\|f\|_{L^2} + \varepsilon\|\Delta f\|_{L^2}. \tag{4.15}$$

Hence by (4.14) and (4.15), one has for any $\varepsilon > 0$ there exists $C(\varepsilon)$ such that

$$\|V_1(x)f + V_2\nabla f\|_{L^2} \leq C(\varepsilon)\|f\|_{L^2} + \varepsilon\|\Delta f\|_{L^2}. \tag{4.16}$$

Since $C_c^\infty(\mathbb{H}^2, \mathbb{C}^2)$ is a core of Δ_{col}, Kato's compact perturbation theorem shows $-\Delta + W$ is self-adjoint in L^2 with domain $D(\Delta_{col})$. □

5. Bootstrap for the heat tension filed

5.1. Strichartz estimates for wave equation with magnetic potential. Theorem 5.2 and Remark 5.5 of Anker, Pierfelice [**2**] obtained the Strichartz estimates for linear wave/Klein-Gordon equation: Let $((p,q),(\widetilde{p},\widetilde{q}))$ be a $(\sigma,\tilde{\sigma})$ admissible couple, i.e.,

$$\left\{(p^{-1},q^{-1}) \in (0,\frac{1}{2}] \times (0,\frac{1}{2}) : \frac{1}{p} > \frac{1}{2}(\frac{1}{2} - \frac{1}{q})\right\} \cup \left\{\left(0,\frac{1}{2}\right)\right\}$$
$$\sigma \geq \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \tilde{\sigma} \geq \frac{3}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right).$$

If u solves $\partial_t^2 u - \Delta u = g$ with initial data (f_0, f_1), then

$$\left\|\widetilde{D}_x^{-\sigma+\frac{1}{2}}u\right\|_{L_t^pL_x^q} + \left\|\widetilde{D}_x^{-\sigma-\frac{1}{2}}\partial_t u\right\|_{L_t^pL_x^q} \lesssim \left\|\widetilde{D}_x^{\frac{1}{2}}f_0\right\|_{L^2} + \left\|\widetilde{D}_x^{-\frac{1}{2}}f_1\right\|_{L^2} + \left\|\widetilde{D}_x^{\tilde{\sigma}-\frac{1}{2}}g\right\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}}.$$

where $\widetilde{D} = (-\Delta - \frac{1}{4} + \kappa^2)$ for some $\kappa > \frac{1}{2}$.

Let $\rho(x) = e^{-d(x,0)}$. The endpoint and non-endpoint Strichartz estimates for magnetic wave equations in the small potential case were obtained in the first author's work [Corollary. 1.1. Proposition 3.1 [**38**]]. We recall this for reader's convenience. Consider the magnetic wave equation on $\mathbb{H}^2$,

$$\begin{cases} \partial_t^2 f - \Delta f + B_0(x)f + \sum_{i=1}^2 h^{ii}B_i(x)\partial_i f = F \\ f(0,x) = f_0(x), \partial_t f(0,x) = f_1(x) \end{cases} \tag{5.1}$$

LEMMA 5.1 ([**38**]). *Assume that B_0, B_1, B_2 in (5.1) satisfy for some $\varrho > 0$*

$$\|B_0\|_{L^2\cap e^{-r\varrho}L^\infty} + \sum_{i=1}^2 \|\sqrt{h^{ii}}B_i\|_{L^2\cap e^{-r\varrho}L^\infty} \leq \mu_1. \tag{5.2}$$

And assume that the Schrödinger operator $H = -\Delta + B_0 + h^{ii}B_i\partial_i$ is symmetric. If $0 < \mu_1 \ll 1$, u solves (5.1), then for any $0 < \sigma \ll \varrho$, $p \in (2,6)$

$$\|\rho^\sigma \nabla f\|_{L_t^2L_x^2} + \left\|(-\Delta)^{\frac{1}{4}}f\right\|_{L_t^2L_x^p} + \|\partial_t f\|_{L_t^\infty L_x^2} + \|\nabla f\|_{L_t^\infty L_x^2}$$
$$\lesssim \|\nabla f_0\|_{L^2} + \|f_1\|_{L^2} + \|F\|_{L_t^1L_x^2}.$$

Hence by Lemma 4.3, Lemma 4.4 and Lemma 5.1, we have:

PROPOSITION 5.2. Let W be defined above and $0<\mu_1\ll 1$, $0<\sigma\ll\varrho\ll 1$, then we have the weighted and endpoint Strichartz estimates for the magnetic wave equation: If f solves the equation

$$\begin{cases} \partial_t^2 f-\Delta f+Wf=F \\ f(0,x)=f_0, \partial_t f(0,x)=f_1 \end{cases}$$

then it holds for any $p\in(2,6)$, $0<\sigma\ll\varrho$

$$\begin{aligned} &\left\| |D|^{\frac12} f\right\|_{L_t^2 L_x^p}+\|\rho^\sigma\nabla f\|_{L_t^2L_x^2}+\|\partial_t f\|_{L_t^\infty L_x^2}+\|\nabla f\|_{L_t^\infty L_x^2}+\|\rho^\sigma\nabla f\|_{L_t^2L_x^2} \\ &\lesssim \|\nabla f_0\|_{L^2}+\|f_1\|_{L^2}+\|F\|_{L_t^1L_x^2}. \end{aligned} \tag{5.3}$$

REMARK 5.3. For all $\sigma\in\mathbb{R}$, $p\in(1,\infty)$, $\|\widetilde{D}^\sigma f\|_p$ is equivalent to $\|(-\Delta)^{\sigma/2}f\|_p$. Tataru [**58**] shows for all $p\in(1,\infty)$, $\|\Delta f\|_p$ is equivalent to $\|\nabla^2 f\|_p+\|\nabla f\|_p+\|f\|_p$.

5.2. Setting of Bootstrap. We fix the constants $\mu_1,\varepsilon_1,\varrho,\sigma$ to be

$$0<\mu_2<\mu_1\ll\varepsilon_1\ll 1,\ 0<\sigma\ll\varrho\ll 1. \tag{5.4}$$

Let $L>0$ be sufficiently large say $L=100$. Define $\omega:\mathbb{R}^+\to\mathbb{R}^+$ and $a:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\omega(s)=\begin{cases} s^{\frac12} & \text{when } 0\le s\le 1 \\ s^L & \text{when } s\ge 1 \end{cases}, a(s)=\begin{cases} s^{\frac34} & \text{when } 0\le s\le 1 \\ s^L & \text{when } s\ge 1 \end{cases}$$

PROPOSITION 5.4. Assume that $\mathcal{A}$ is the set of $T\in[0,T_*)$ such that for any $2<q<6+2\gamma$, $p\in(2,6)$ with some fixed $0<\gamma\ll 1$,

$$\begin{aligned} &\|(du,\partial_t u)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}+\|(\nabla\partial_t u,\nabla du)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)} \\ &\qquad +\|\partial_t u\|_{L_t^2L_x^q([0,T]\times\mathbb{H}^2)}\le\varepsilon_1. \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\left\|\omega(s)|D|^{-\frac12}\partial_t\phi_s\right\|_{L_s^\infty L_t^2L_x^p}+\|\omega(s)\partial_t\phi_s\|_{L_s^\infty L_t^\infty L_x^2} \\ &\quad +\|\omega(s)\nabla\phi_s\|_{L_s^\infty L_t^\infty L_x^2}+\left\|\omega(s)|D|^{\frac12}\phi_s\right\|_{L_s^\infty L_t^2L_x^p}\le\varepsilon_1. \end{aligned} \tag{5.6}$$

Then for all $T\in\mathcal{A}$ we have

$$\begin{aligned} &\left\|\omega(s)|D|^{-\frac12}\partial_t\phi_s\right\|_{L_s^\infty L_t^2L_x^p([0,T]\times\mathbb{H}^2)}+\left\|\omega(s)|D|^{\frac12}\phi_s\right\|_{L_s^\infty L_t^2L_x^p([0,T]\times\mathbb{H}^2)} \\ &+\|\omega(s)\partial_t\phi_s\|_{L_s^\infty L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}+\|\omega(s)\nabla\phi_s\|_{L_s^\infty L_t^\infty L_x^2([0,T]\times H^2)}\le\varepsilon_1^2. \end{aligned} \tag{5.7}$$

and for any $r\in(2,6+2\gamma]$ it holds that

$$\|(du,\partial_t u)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}+\|(\nabla\partial_t u,\nabla du)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}\le\varepsilon_1^2 \tag{5.8}$$

$$\|\partial_t u\|_{L_t^2L_x^r([0,T]\times\mathbb{H}^2)}\le\varepsilon_1^2. \tag{5.9}$$

Moreover we have

$$\begin{aligned} &\|du\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}+\|\partial_t u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}+\|\nabla du\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)} \\ &\qquad +\|\nabla\partial_t u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)}+\|\partial_t u\|_{L_t^2L_x^6([0,T]\times\mathbb{H}^2)}\le\varepsilon_1^2. \end{aligned} \tag{5.10}$$

The proof of Proposition 5.4 will be divided into several lemmas below. (5.7) is proved in Proposition 5.11. (5.8), (5.9) and (5.10) are proved in Proposition 5.13 and Corollary 5.15 respectively.

The bootstrap programm we apply here is based on the design of [**33, 55**]. The essential refinement is we add a spacetime bound $\|\partial_t u\|_{L^2_t L^p_x}$ to the primitive bootstrap assumption. The most important original ingredient in this part is we use the weighted Strichartz estimates in Section 5.1 to control the one order derivative terms of ϕ_s.

Proposition 5.5. Assume (5.5) holds, then we have for any $\eta>0$

$$\big\|A_t\big\|_{L^\infty_t L^\infty_x}\le \varepsilon_1 \tag{5.11}$$

$$\big\|h^{ii}\partial_i A_i(s)\big\|_{L^\infty_t L^\infty_x}\le \varepsilon_1\max(1,s^{-\eta}) \tag{5.12}$$

$$\big\|\sqrt{h^{ii}}\partial_t A_i(s)\big\|_{L^\infty_t L^\infty_x}\le \varepsilon_1 s^{-\frac12} \tag{5.13}$$

$$\big\|\sqrt{h^{ii}}A_i(s)\big\|_{L^\infty_t L^\infty_x}\le \varepsilon_1. \tag{5.14}$$

Proof. By the commutator identity and the facts $|\partial_t\widetilde{u}|\le e^{s\Delta}|\partial_t u|$, $|\partial_s\widetilde{u}|\le e^{s\Delta}|\partial_s u|$, (5.11) is bounded by Lemma 2.7,

$$\begin{aligned}\|A_t\|_{L^\infty_t L^\infty_x}&\le \Big\|\int_s^\infty \big\|\phi_t\big\|_{L^\infty_x}\big\|\phi_s\big\|_{L^\infty_x}d\kappa\Big\|_{L^\infty_t}\le \sup_{t\in[0,T]}\big\|\phi_t\big\|_{L^2_s L^\infty_x}\big\|\phi_s\big\|_{L^2_s L^\infty_x}\\&\le \sup_{t\in[0,T]}\big\|\partial_t u\big\|_{L^2_x}\big\|\partial_s u\big\|_{L^2_x}\le\varepsilon_1.\end{aligned}$$

By the commutator identity,

$$\begin{aligned}&\big\|\sqrt{h^{ii}}\partial_t A_i\big\|_{L^\infty_t L^\infty_x}\le\int_s^\infty \big\|\sqrt{h^{ii}}\partial_t\left(\phi_i\wedge\phi_s\right)\big\|_{L^\infty_t L^\infty_x}d\kappa\\&\le\int_s^\infty \big\|\sqrt{h^{ii}}\partial_t\phi_i\big\|_{L^\infty_t L^\infty_x}\big\|\phi_s\big\|_{L^\infty_t L^\infty_x}d\kappa+\int_s^\infty \big\|\sqrt{h^{ii}}\phi_i\big\|_{L^\infty_t L^\infty_x}\big\|\partial_t\phi_s\big\|_{L^\infty_t L^\infty_x}d\kappa.\end{aligned}$$

Using the relation between the induced derivative $D_{i,t}$ and the covariant derivative on $u^*(TN)$, one obtains $|\sqrt{h^{ii}}\partial_t\phi_i|\le|\nabla\partial_t\widetilde{u}|+|\sqrt{h^{ii}}A_t\phi_i|+|\sqrt{h^{ii}}A_i\phi_t|$ and similarly $|\partial_t\phi_s|\le|\nabla_t\partial_s\widetilde{u}|+|A_t\phi_s|$. Hence it suffices to prove

$$\int_s^\infty \||d\widetilde{u}|\,|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_t L^\infty_x}d\kappa+\int_s^\infty \||\partial_s\widetilde{u}|\,|\nabla\partial_t\widetilde{u}|\|_{L^\infty_t L^\infty_x}d\kappa\le\varepsilon_1 s^{-\frac12} \tag{5.15}$$

$$\int_s^\infty \|\sqrt{h^{ii}}A_i\phi_t\phi_s\|_{L^\infty_t L^\infty_x}d\kappa+\int_s^\infty \|\sqrt{h^{ii}}A_t\phi_i\phi_s\|_{L^\infty_t L^\infty_x}d\kappa\le\varepsilon_1 s^{-\frac12} \tag{5.16}$$

For $s\in(0,1]$, Proposition 3.15 and $|d\widetilde{u}|\le e^{s\Delta}|du|$ give

$$\||d\widetilde{u}|\,|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_t L^\infty_x}+\||\partial_s\widetilde{u}|\,|\nabla\partial_t\widetilde{u}|\|_{L^\infty_t L^\infty_x}\le\varepsilon_1 s^{-\frac12}s^{-1}+\varepsilon_1 s^{-\frac12}s^{-1}. \tag{5.17}$$

For $s\ge1$, we have by Proposition 3.15

$$\||d\widetilde{u}|\,|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_t L^\infty_x}+\||\partial_s\widetilde{u}|\,|\nabla\partial_t\widetilde{u}|\|_{L^\infty_t L^\infty_x}\le\varepsilon_1 e^{-\delta s}. \tag{5.18}$$

Therefore (5.18) and (5.17) yield for all $s\in(0,\infty)$

$$\||d\widetilde{u}|\,|\nabla_t\partial_s\widetilde{u}|\|_{L^\infty_t L^\infty_x}+\||\partial_s\widetilde{u}|\,|\nabla\partial_t\widetilde{u}|\|_{L^\infty_t L^\infty_x}\le\varepsilon_1 s^{-3/2}.$$

Hence we obtain (5.15). (5.16) and (5.14) can be proved similarly. By (2.2) and direct calculations similar to Lemma 4.3,

$$|h^{ii}\partial_i A_i^{\infty}| \lesssim |\nabla dQ| + |dQ|. \tag{5.19}$$

And the same route as (5.13) shows for any $\eta > 0$

$$|h^{ii}\partial_i A_i^{con}| \le \varepsilon_1 s^{-\eta}. \tag{5.20}$$

Thus (5.12) follows by (5.20), (5.19) □

LEMMA 5.6. *Assume (5.5) and (5.6) hold, then we have*

$$\left\| \sqrt{h^{pp}} |\partial_p (h^{ii}\partial_i A_i(s))| \right\|_{L_t^\infty L_x^\infty} \le \varepsilon_1 \max(s^{-1}, 1) \tag{5.21}$$

PROOF. By Remark 3.21, it suffices to bound A^{∞} and A^{con} part separately. Direct calculations as Lemma 4.3 and (2.2) yield the bound for the A^{∞} part is

$$\sqrt{h^{pp}} |\partial_p (h^{ii}\partial_i A_i^{\infty}(s))| \le |\nabla^2 dQ| + |\nabla dQ| + |dQ|.$$

Thus (1.3) shows the A^{∞} part is bounded by

$$\|\sqrt{h^{pp}} |\partial_p (h^{ii}\partial_i A_i^{\infty}(s))| \|_{L_x^\infty} \le \varepsilon_1.$$

By (2.2) and direct calculations,

$$\begin{aligned}
&\sqrt{h^{pp}} |\partial_p (h^{ii}\partial_i (\phi_i \wedge \phi_s)(s))| \\
&\le |\nabla^2 \partial_s \widetilde{u}||d\widetilde{u}| + \sqrt{h^{ii}h^{pp}} |A_i A_p||\partial_s \widetilde{u}||d\widetilde{u}| + |\partial_s \widetilde{u}||\nabla^2 d\widetilde{u}| + |\nabla \partial_s \widetilde{u}||\nabla d\widetilde{u}| \\
&+ \sqrt{h^{ii}} |A_i||\nabla \partial_s \widetilde{u}||d\widetilde{u}| + \sqrt{h^{ii}} |A_i||\partial_s \widetilde{u}||\nabla d\widetilde{u}| + \sqrt{h^{ii}} |A_i|\|\nabla \partial_s \widetilde{u}||d\widetilde{u}| \\
&+ \sqrt{h^{pp}h^{ii}} |\partial_p A_i||\nabla \partial_s \widetilde{u}||d\widetilde{u}| + \sqrt{h^{pp}h^{ii}} |\partial_p A_i||\partial_s \widetilde{u}||\nabla d\widetilde{u}|.
\end{aligned}$$

Thus the A^{con} part follows by Lemma 3.16 and interpolation. □

PROPOSITION 5.7. Suppose that (5.5), (5.6) hold. Then we have for $p \in (2, 6)$

$$\left\| a(s) \|\partial_t \phi_s\|_{L_t^2 L_x^p} \right\|_{L_s^\infty} \le \varepsilon_1 \tag{5.22}$$

$$\left\| a(s) \|\nabla \phi_s\|_{L_t^2 L_x^p} \right\|_{L_s^\infty} \le \varepsilon_1. \tag{5.23}$$

Generally we have for $\theta \in [0, 2]$

$$\left\| \omega_\theta(s) (-\Delta)^\theta \phi_s \right\|_{L_s^\infty L_t^2 L_x^p} \le \varepsilon_1 \tag{5.24}$$

$$\left\| \omega_1(s) |D| \partial_t \phi_s \right\|_{L_s^\infty L_t^2 L_x^p} \le \varepsilon_1, \tag{5.25}$$

where $\omega_\theta(s) = s^{\theta + \frac{1}{4}}$ when $s \in [0, 1]$ and $\omega_\theta(s) = s^L$ when $s \ge 1$.

PROOF. By (7.13) and Duhamel principle we have

$$\|(-\Delta)^{\frac{1}{2}} \phi_s(s)\|_{L_t^2 L_x^p} \le \|(-\Delta)^{\frac{1}{2}} e^{\frac{s}{2}\Delta} \phi_s(\frac{s}{2})\|_{L_t^2 L_x^p}$$

$$+ \Big\| \int_{\frac{s}{2}}^{s} (-\Delta)^{\frac{1}{2}} e^{(s-\tau)\Delta} h^{ii} A_i \partial_i \phi_s(\tau) d\tau \Big\|_{L_t^2 L_x^p} \tag{5.26}$$

$$+ \Big\| \int_{\frac{s}{2}}^{s} (-\Delta)^{\frac{1}{2}} e^{(s-\tau)\Delta} G(\tau) d\tau \Big\|_{L_t^2 L_x^p}. \tag{5.27}$$

where $G(\tau) = h^{ii}\left(\partial_i A_i\right)\phi_s - h^{ii}\Gamma^k_{ii}A_k\phi_s + h^{ii}A_iA_i\phi_s + h^{ii}\left(\phi_s \wedge \phi_i\right)\phi_i$. For (5.26), the smoothing effect and (5.14) show

$$\begin{aligned}
&s^{\frac{3}{4}}\Big\|\int_{\frac{s}{2}}^{s}(-\Delta)^{\frac{1}{2}}e^{(s-\tau)\Delta}h^{ii}A_i\partial_i\phi_s(\tau)d\tau\Big\|_{L^2_tL^p_x}\\
&\lesssim s^{\frac{3}{4}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\big\|h^{ii}A_i\partial_i\phi_s(\tau)\big\|_{L^2_tL^p_x}d\tau\\
&\lesssim s^{\frac{3}{4}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\|\nabla\phi_s(\tau)\|_{L^2_tL^p_x}\big\|\sqrt{h^{ii}}A_i\big\|_{L^\infty_tL^\infty_x}d\tau\\
&\lesssim s^{\frac{3}{4}}\varepsilon_1\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\|\nabla\phi_s(\tau)\|_{L^2_tL^p_x}d\tau.
\end{aligned}$$

Thus we conclude when $s \in [0,1]$

$$\begin{aligned}
&s^{\frac{3}{4}}\Big\|\int_{\frac{s}{2}}^{s}(-\Delta)^{\frac{1}{2}}e^{(s-\tau)\Delta}h^{ii}A_i\partial_i\phi_s(\tau)d\tau\Big\|_{L^2_tL^p_x}\\
&\le \varepsilon_1\Big\|s^{\frac{3}{4}}\|\nabla\phi_s(s)\|_{L^2_tL^p_x}\Big\|_{L^\infty_s}.
\end{aligned}\tag{5.28}$$

Similarly we have for (5.27) that

$$\begin{aligned}
&s^{\frac{3}{4}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\|G(\tau)\|_{L^2_tL^p_x}d\tau\\
&\le s^{\frac{3}{4}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\big\|h^{ii}\partial_iA_i\big\|_{L^\infty_tL^\infty_x}\|\phi_s\|_{L^2_tL^p_x}d\tau + s^{\frac{3}{4}}\int_{\frac{s}{2}}^{s}\|A_2\|_{L^\infty_tL^\infty_x}\|\phi_s\|_{L^2_tL^p_x}d\tau\\
&+ s^{\frac{3}{4}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\Big(\big\|h^{ii}A_iA_i\big\|_{L^\infty_tL^\infty_x} + \big\|h^{ii}\phi_i\phi_i\big\|_{L^\infty_tL^\infty_x}\Big)\|\phi_s\|_{L^2_tL^p_x}d\tau.
\end{aligned}$$

Thus by Proposition 5.5 and Proposition 3.15, we have for all $s \in [0,1]$

$$(5.27) \lesssim \Big\|s^{\frac{1}{2}}\|\phi_s(s)\|_{L^2_tL^p_x}\Big\|_{L^\infty_s}.\tag{5.29}$$

For $s \ge 1$, we also have by Duhamel principle

$$\begin{aligned}
&s^L\big\|(-\Delta)^{\frac{1}{2}}\phi_s(s)\big\|_{L^2_tL^p_x}\\
&\le s^L\big\|(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}\phi_s(\frac{s}{2})\big\|_{L^2_tL^6_x} + s^L\Big\|\int_{\frac{s}{2}}^{s}(-\Delta)^{\frac{1}{2}}e^{(s-\tau)\Delta}G_1(\tau)d\tau\Big\|_{L^2_tL^p_x},
\end{aligned}$$

where G_1 is the inhomogeneous term. The linear term is bounded by

$$s^L\big\|(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}\phi_s(\frac{s}{2})\big\|_{L^2_tL^p_x} \le s^Le^{-\frac{1}{16}s}\big\|\phi_s(\frac{s}{2})\big\|_{L^2_tL^p_x}.$$

By Proposition 5.5 and smoothing effect, the first term in G_1 is bounded as

$$\begin{aligned}
&s^L\Big\|\int_{\frac{s}{2}}^{s}(-\Delta)^{\frac{1}{2}}e^{(s-\tau)\Delta}h^{ii}A_i\partial_i\phi_s(\tau)d\tau\Big\|_{L^2_tL^p_x}\\
&\le s^L\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}e^{-\delta(s-\tau)}\big\|\nabla\phi_s(\tau)\big\|_{L^2_tL^p_x}\big\|\sqrt{h^{ii}}A_i\big\|_{L^\infty_tL^\infty_x}d\tau\\
&\le \varepsilon_1 s^L\int_{\frac{s}{2}}^{s}e^{-\delta(s-\tau)}\tau^{-L}(s-\tau)^{-\frac{1}{2}}\big\|\tau^L\nabla\phi_s(\tau)\big\|_{L^2_tL^p_x}d\tau.
\end{aligned}$$

The other terms in G_1 can be estimated similarly, thus we obtain for $s \ge 1$

(5.30)

$$s^L\Big\|(-\Delta)^{\frac{1}{2}}\phi_s(s)\Big\|_{L^2_tL^p_x} \le \varepsilon_1\Big\|s^L\|\nabla\phi_s(s)\|_{L^2_tL^p_x}\Big\|_{L^\infty_s(s\ge1)} + \Big\|s^L\|\phi_s(\tau)\|_{L^2_tL^p_x}\Big\|_{L^\infty_s(s\ge1)}.$$

Combing (5.26), (5.27), with (5.30) gives corresponding estimates in (5.23) for $\nabla\phi_s$. It suffices to prove the remaining estimates in (5.23) for $\partial_t\phi_s$. Denote the inhomogeneous term in (7.15) by G_3, then Duhamel principle gives

$$s^{\frac{3}{4}}\|\partial_t\phi_s(s)\|_{L^2_tL^p_x} \le s^{\frac{3}{4}}\big\|e^{\Delta\frac{s}{2}}\partial_t\phi_s(\frac{s}{2})\big\|_{L^2_tL^p_x} + s^{\frac{3}{4}}\Big\|\int_{\frac{s}{2}}^s e^{\Delta(s-\tau)}G_3(\tau)d\tau\Big\|_{L^2_tL^p_x}.$$

The first term of G_3 is bounded by

$$s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}h^{ii}\left(\partial_tA_i\right)\partial_i\phi_s(\tau)\big\|_{L^2_tL^p_x}d\tau \le s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|\sqrt{h^{ii}}\partial_tA_i\big\|_{L^\infty_tL^\infty_x}\big\|\nabla\phi_s\big\|_{L^2_tL^p_x}d\tau.$$

This is acceptable by Proposition 5.5. The second term in G_3 is bounded as

$$\begin{aligned}
&s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}2h^{ii}A_i\partial_i\partial_t\phi_s(\tau)\big\|_{L^2_tL^p_x}d\tau\\
&\le s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}\sqrt{h^{ii}}\partial_i\left(\sqrt{h^{ii}}A_i\partial_t\phi_s\right)\big\|_{L^2_tL^p_x}d\tau + s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}h^{ii}\partial_iA_i\partial_t\phi_s\big\|_{L^2_tL^p_x}d\tau\\
&\triangleq I + II.
\end{aligned}\tag{5.31}$$

I is bounded by the smoothing effect, boundedness of Riesz transform and Proposition 5.5

$$\begin{aligned}
I &\le s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|(-\Delta)^{\frac{1}{2}}e^{\Delta(s-\tau)}\big(\sqrt{h^{ii}}A_i\partial_t\phi_s\big)\big\|_{L^2_tL^p_x}d\tau\\
&\le s^{\frac{3}{4}}\int_{\frac{s}{2}}^s (s-\tau)^{-\frac{1}{2}}\big\|\sqrt{h^{ii}}A_i\big\|_{L^\infty_tL^\infty_x}\big\|\partial_t\phi_s\big\|_{L^2_tL^p_x}d\tau\\
&\le s^{\frac{3}{4}}\int_{\frac{s}{2}}^s (s-\tau)^{-\frac{1}{2}}\varepsilon_1\big\|\partial_t\phi_s\big\|_{L^2_tL^p_x}d\tau.
\end{aligned}$$

II is estimated as the first term of G_3 above. The third term of G_3 is bounded as

$$\begin{aligned}
&s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}h^{ii}\left(\partial_i\partial_tA_i\right)\phi_s\big\|_{L^2_tL^p_x}d\tau\\
&\le s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}\sqrt{h^{ii}}\partial_i\big(\sqrt{h^{ii}}\partial_tA_i\phi_s\big)\big\|_{L^2_tL^p_x}d\tau\\
&+ s^{\frac{3}{4}}\int_{\frac{s}{2}}^s \big\|e^{\Delta(s-\tau)}h^{ii}\partial_tA_i\partial_i\phi_s\big\|_{L^2_tL^p_x}d\tau.
\end{aligned}\tag{5.32}$$

The remaining arguments are almost the same as I and II. And the rest nine terms in G_3 can be estimated as above as well. Hence the desired estimates in (5.22) for $\partial_t\phi_s$ when $s \in (0,1]$ is verified. It suffices to prove (5.22) for $\partial_t\phi_s$ when $s \ge 1$. The proof for this part is exactly close to the estimates of $\nabla\phi_s$ when $s \ge 1$ and that of I, II. (5.25) follows by the same arguments as (5.22) by applying smoothing effect of the heat semigroup. By interpolation, in order to verify (5.24), it suffices to prove

$$\|\omega_1(s)(-\Delta)\phi_s\|_{L_s^\infty L_t^2 L_x^p} \le \varepsilon_1. \tag{5.33}$$

By (7.13), Duhamel principle and the smoothing effect we have

$$\|(-\Delta)\phi_s(s)\|_{L_t^2 L_x^p} \le s^{-1} e^{-\frac{\delta}{2}s} \|\phi_s(\frac{s}{2})\|_{L_t^2 L_x^p}$$
$$+ \int_{\frac{s}{2}}^{s} (s-\tau)^{-\frac{1}{2}} e^{-\delta(s-\tau)} \big(\|\nabla(h^{ii} A_i \partial_i \phi_s)\|_{L_t^2 L_x^p} + \|\nabla G\|_{L_t^2 L_x^p}\big) d\tau.$$

Then by Lemma 5.6, Proposition 5.5, (5.23), (5.5), (5.6), one obtains

$$\|\omega_1(s)(-\Delta)\phi_s\|_{L_s^\infty L_t^2 L_x^p} \le \varepsilon_1 \left\|\omega_1(s)\nabla^2\phi_s\right\|_{L_s^\infty L_t^2 L_x^p} + \varepsilon_1.$$

Thus (5.24) follows by Remark 5.3. □

LEMMA 5.8. *Assume that (5.5), (5.6) hold, then for* $q \in (2, 6+2\gamma]$

$$\|\phi_t(s)\|_{L_s^\infty L_t^2 L_x^q} \le \varepsilon_1 \tag{5.34}$$

$$\|A_t\|_{L_t^1 L_x^\infty} \le \varepsilon_1^2 \tag{5.35}$$

PROOF. First notice that ϕ_t satisfies $(\partial_s - \Delta)|\phi_t| \le 0$, thus for any fixed (t, s, x) one has the pointwise estimate

$$|\phi_t(s,t,x)| \le |\phi_t(0,t,x)| = |\partial_t u(t,x)|.$$

Hence (5.34) follows by (5.5). From commutator identity we have

$$\|A_t\|_{L_t^1 L_x^\infty} \le \int_0^\infty \|\partial_t u\|_{L_t^2 L_x^\infty} \|\partial_s u\|_{L_t^2 L_x^\infty} ds. \tag{5.36}$$

Sobolev inequality implies for p_* slightly less than 6

$$\|\phi_s\|_{L_x^\infty} \le \||D|^{\frac{1}{2}}\phi_s\|_{L_x^{p_*}}. \tag{5.37}$$

And since $|\partial_t \widetilde{u}|$ satisfies $(\partial_s - \Delta)|\partial_t \widetilde{u}| \le 0$, then

$$\|\phi_t(s)\|_{L_x^\infty} \lesssim s^{-1/p_*} e^{-\delta s} \|\phi(\frac{s}{2})\|_{L_x^{p_*}}. \tag{5.38}$$

By (5.38), (5.37) and (5.6),

$$\int_0^1 \|\phi_t(s)\|_{L_t^2 L_x^\infty} \|\phi_s(s)\|_{L_t^2 L_x^\infty} \lesssim \int_0^1 s^{-\frac{1}{2}-\frac{1}{p_*}} \|\phi_t(\frac{s}{2})\|_{L_t^2 L_x^{p_*}} s^{\frac{1}{2}} \||D|^{\frac{1}{2}}\phi_s(s)\|_{L_t^2 L_x^{p_*}} ds. \tag{5.39}$$

$$\int_1^\infty \|\phi_t(s)\|_{L_t^2 L_x^\infty} \|\phi_s(s)\|_{L_t^2 L_x^\infty} \lesssim \int_1^\infty s^{-4L} \|\phi_t(\frac{s}{2})\|_{L_t^2 L_x^{p_*}} \||D|^{\frac{1}{2}}\phi_s(s)\|_{L_t^2 L_x^{p_*}} ds. \tag{5.40}$$

Thus (5.35) is obtained by (5.5) and (5.6). □

LEMMA 5.9. *Assume that (5.5) and (5.6) hold, then for* $p \in (2, 6+2\gamma]$ *with* $0 < \gamma \ll 1$, ϕ_t *satisfies*

$$\|\omega(s)|D|\phi_t(s)\|_{L_s^\infty L_t^2 L_x^p} \le \varepsilon_1 \tag{5.41}$$

$$\left\|\omega_{\frac{3}{4}}(s)\Delta\phi_t(s)\right\|_{L_s^\infty L_t^2 L_x^p} \le \varepsilon_1 \tag{5.42}$$

PROOF. By Duhamel principle and (7.14)

$$s^{\frac{1}{2}}\big\|(-\Delta)^{\frac{1}{2}}\phi_t(s)\big\|_{L^2_tL^p_x} \le s^{\frac{1}{2}}\big\|(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}\phi_t(\frac{s}{2})\big\|_{L^2_tL^p_x} + s^{\frac{1}{2}}\int_{\frac{s}{2}}^{s}\big\|(-\Delta)^{\frac{1}{2}}e^{(s-\tau)\Delta}\mathcal{G}(\tau)\big\|_{L^2_tL^p_x}d\tau,$$

where $\mathcal{G}$ denotes the inhomogeneous terms. By smoothing effect and Proposition 5.5, the first term in $\mathcal{G}$ is bounded by

$$\begin{aligned}
&s^{\frac{1}{2}}\int_{\frac{s}{2}}^{s}\big\|(-\Delta)^{\frac{1}{2}}e^{(s-\tau)\Delta}h^{ii}A_i\partial_i\phi_t\big\|_{L^2_tL^p_x}d\tau \\
&\le s^{\frac{1}{2}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\big\|\nabla\phi_t\big\|_{L^2_tL^p_x}\big\|\sqrt{h^{ii}}A_i\big\|_{L^\infty_tL^\infty_x}d\tau \\
&\le \varepsilon_1 s^{\frac{1}{2}}\int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}\big\|\nabla\phi_t\big\|_{L^2_tL^p_x}d\tau.
\end{aligned}$$

The large time estimates follow by the same route. Similar estimates for the rest terms in $\mathcal{G}$ and (5.34) yield (5.41). By Duhamel principle and smoothing effect, we have

$$\|\Delta\phi_t\|_{L^2_tL^p_x} \lesssim s^{-\frac{1}{2}}e^{-\delta\frac{s}{2}}\|\nabla\phi_t\|_{L^2_tL^p_x} + \int_{\frac{s}{2}}^{s}(s-\tau)^{-\frac{1}{2}}e^{-\delta(s-\tau)}\|\nabla\mathcal{G}\|_{L^2_tL^p_x}d\tau.$$

Then Lemma 5.6, Proposition 5.5, (5.41), (5.5), (5.6) give

$$\|\omega_{\frac{3}{4}}(s)\Delta\phi_t\|_{L^\infty_sL^2_tL^p_x} \lesssim \epsilon_1 + \epsilon_1\|\omega_{\frac{3}{4}}(s)\nabla^2\phi_t\|_{L^\infty_sL^2_tL^p_x}$$

Thus (5.42) follows by Remark 5.3. □

LEMMA 5.10. *Suppose that (5.5) and (5.6) hold, then the wave map tension field satisfies*

$$\Big\|s^{-\frac{1}{2}}\mathfrak{W}(s)\Big\|_{L^\infty_sL^1_tL^2_x} \le \varepsilon_1^2 \tag{5.43}$$

$$\|\nabla\mathfrak{W}(s)\|_{L^\infty_sL^1_tL^2_x} \le \varepsilon_1^2 \tag{5.44}$$

$$\Big\|s^{\frac{1}{2}}\Delta\mathfrak{W}(s)\Big\|_{L^\infty_sL^1_tL^2_x} \le \varepsilon_1^2 \tag{5.45}$$

$$\|\omega(s)\partial_s\mathfrak{W}(s)\|_{L^\infty_sL^1_tL^2_x} \le \varepsilon_1^2. \tag{5.46}$$

PROOF. Recall the equation for $\mathfrak{W}$ evolving along s:

$$\begin{aligned}
\partial_s\mathfrak{W} &= \Delta\mathfrak{W} + 2h^{ii}A_i\partial_i\mathfrak{W} + h^{ii}A_iA_i\mathfrak{W} + h^{ii}\partial_iA_i\mathfrak{W} - h^{ii}\Gamma^k_{ii}A_k\mathfrak{W} + h^{ii}\left(\mathfrak{W}\wedge\phi_i\right)\phi_i \\
&\quad + 3h^{ii}(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u})\nabla_t\partial_i\widetilde{u}.
\end{aligned} \tag{5.47}$$

Since $\mathfrak{W}(0,s,x)$=0 for all $(s,x)\in\mathbb{R}^+\times\mathbb{H}^2$, Duhamel principle gives

$$(-\Delta)^k\mathfrak{W}(s,t,x) = \int_0^s e^{(s-\tau)\Delta}(-\Delta)^kG_2(\tau)d\tau,$$

where G_2 denotes the inhomogeneous term.

Step One. In this step, we consider short time behavior, and all the integrand domain of L^∞_s is restricted in $s\in[0,1]$. By (5.14), (5.12),

$$\int_0^s\big\|h^{ii}A_iA_i\mathfrak{W}\big\|_{L^1_tL^2_x}d\kappa \le \int_0^s\big\|h^{ii}A_iA_i\big\|_{L^\infty_tL^\infty_x}\|\mathfrak{W}\|_{L^1_tL^2_x}d\kappa \le s^{\frac{3}{2}}\varepsilon_1^2\Big\|\mathfrak{W}s^{-\frac{1}{2}}\Big\|_{L^\infty_sL^1_tL^2_x}$$

$$\int_0^s \left\|h^{ii}\partial_i A_i \mathfrak{W}\right\|_{L_t^1 L_x^2} d\kappa \leq \int_0^s \left\|h^{ii}\partial_i A_i\right\|_{L_t^\infty L_x^\infty} \|\mathfrak{W}\|_{L_t^1 L_x^2} d\kappa \leq s^{\frac{1}{2}}\varepsilon_1^2 \left\|\mathfrak{W} s^{-\frac{1}{2}}\right\|_{L_s^\infty L_t^1 L_x^2}.$$

By Proposition 3.15,

(5.48)

$$\int_0^s \left\|h^{ii}\left(\mathfrak{W}\wedge\phi_i\right)\phi_i\right\|_{L_t^1 L_x^2} d\kappa \leq \int_0^s \left\|h^{ii}\phi_i\phi_i\right\|_{L_t^\infty L_x^\infty} \|\mathfrak{W}\|_{L_t^1 L_x^2} d\kappa \leq s\varepsilon_1^2 \left\|\mathfrak{W} s^{-\frac{1}{2}}\right\|_{L_s^\infty L_t^1 L_x^2}.$$

By (5.41), (5.14) and Proposition 3.15,

$$\begin{aligned}
&\int_0^s \left\|h^{ii}\left(\partial_t\widetilde{u}\wedge\partial_i\widetilde{u}\right)\nabla_i\partial_t\widetilde{u}\right\|_{L_t^1 L_x^2} d\kappa \\
&\leq \int_0^s \|d\widetilde{u}\|_{L_t^\infty L_x^6} \|\nabla\partial_t\widetilde{u}\|_{L_t^2 L_x^6} \|\partial_t\widetilde{u}\|_{L_t^2 L_x^6} d\kappa \\
&\leq \int_0^s \|d\widetilde{u}\|_{L_t^\infty L_x^6} \|\nabla\phi_t\|_{L_t^2 L_x^6} \|\partial_t\widetilde{u}\|_{L_t^2 L_x^6} d\kappa + \int_0^s \|d\widetilde{u}\|_{L_t^\infty L_x^6} \left\|\sqrt{h^{ii}} A_i\phi_t\right\|_{L_t^2 L_x^6} \|\partial_t\widetilde{u}\|_{L_t^2 L_x^6} d\kappa \\
&\leq s^{\frac{1}{2}}\varepsilon_1^2.
\end{aligned}$$

By the smoothing effect and the boundedness of Riesz transform, we have

$$\begin{aligned}
&\int_0^s \left\|e^{(s-\kappa)\Delta} h^{ii} A_i \partial_i \mathfrak{W}\right\|_{L_t^1 L_x^2} d\kappa \\
&\leq \int_0^s \left\|e^{(s-\kappa)\Delta} h^{ii}\partial_i\left(A_i\mathfrak{W}\right)\right\|_{L_t^1 L_x^2} d\kappa + \int_0^s \left\|e^{(s-\kappa)\Delta} h^{ii}\partial_i A_i \mathfrak{W}\right\|_{L_t^1 L_x^2} d\kappa \\
&\leq \int_0^s (s-\kappa)^{-\frac{1}{2}} \left\|\sqrt{h^{ii}} A_i \mathfrak{W}\right\|_{L_t^1 L_x^2} d\kappa + \int_0^s \left\|h^{ii}\partial_i A_i \mathfrak{W}\right\|_{L_t^1 L_x^2} d\kappa \\
&\leq s^{\frac{1}{2}}\varepsilon_1^2 \left\|\mathfrak{W} s^{-\frac{1}{2}}\right\|_{L_s^\infty L_t^1 L_x^2}.
\end{aligned}$$

Hence we conclude (5.43) for $s\in[0,1]$ by choosing ε_1 sufficiently small. In order to prove (5.44), we use the following Duhamel principle instead to apply (5.43),

$$(-\Delta)^{\frac{1}{2}}\mathfrak{W}(s) = (-\Delta)^{\frac{1}{2}} e^{\frac{s}{2}\Delta}\mathfrak{W}(\frac{s}{2}) + \int_{\frac{s}{2}}^s (-\Delta)^{\frac{1}{2}} e^{(s-\tau)\Delta} G_2(\tau) d\tau.$$

Then (5.44) follows by (5.43) and the smoothing effect. Again by Duhamel principle and the smoothing effect,

$$\|(-\Delta)\mathfrak{W}(s)\|_{L_x^2} \leq \|(-\Delta) e^{\frac{s}{2}\Delta}\mathfrak{W}(\frac{s}{2})\|_{L_x^2} + \int_{\frac{s}{2}}^s (s-\tau)^{-\frac{1}{2}} e^{-\delta(s-\tau)} \|(-\Delta)^{\frac{1}{2}} G_2(\tau)\|_{L_x^2} d\tau.$$

Thus Lemma 5.6, Proposition 5.5, (5.5), (5.6), Remark 5.3 and Lemma 5.9 give (5.45) for $s\in[0,1]$. For $s\in[0,1]$, (5.46) now arises from (5.43)-(5.45).

Step Two. We prove (5.43)-(5.45) for $s\geq 1$. This can be easily obtained by the same arguments as above with the help of s^{-L} decay in the long time case.

Step Three. We prove the large time behavior. The Duhamel principle we use is also

$$(-\Delta)^k\mathfrak{W}(s) = (-\Delta)^k e^{\frac{s}{2}\Delta}\mathfrak{W}(\frac{s}{2}) + \int_{\frac{s}{2}}^s (-\Delta)^k e^{(s-\tau)\Delta} G_2(\tau) d\tau.$$

Let $s\geq 1$, applying smoothing effect we obtain

$$s^L\|\mathfrak{W}(s)\|_{L_t^1 L_x^2} \leq s^L e^{-s/8}\left\|\mathfrak{W}(\frac{s}{2})\right\|_{L_t^1 L_x^2} + s^L\int_{\frac{s}{2}}^s e^{-(s-\tau)/8}\|G_2(\tau)\|_{L_t^1 L_x^2} d\tau.$$

Then by Hausdorff-Young and (5.43)-(5.45), for $s \geq 1$

$$\|\mathfrak{W}\|_{L^1_t L^2_x} \leq \varepsilon_1^2 s^{-L}. \tag{5.49}$$

Similarly, we have for $s \in [1, \infty)$

$$\|\nabla \mathfrak{W}\|_{L^1_t L^2_x} + \|\Delta \mathfrak{W}\|_{L^1_t L^2_x} \leq \varepsilon_1^2 s^{-L}. \tag{5.50}$$

Thus the longtime part of (5.46) now results from (5.49), (5.50). □

LEMMA 5.11. *Suppose that (5.5) and (5.6) hold, then for* $0 < \gamma \ll 1$

$$\left\| s^{-\frac{1}{2}} \mathfrak{W}(s) \right\|_{L^\infty_s L^2_t L^{3+\gamma}_x} + \|\omega(s)\mathfrak{W}(s)\|_{L^\infty_s L^2_t L^{3+\gamma}_x} \leq \varepsilon_1 \tag{5.51}$$

$$\|\omega(s)\partial_t \phi_t(s)\|_{L^\infty_s L^2_t L^{3+\gamma}_x} \leq \varepsilon_1. \tag{5.52}$$

$$\|\partial_t A_t(s)\|_{L^\infty_s L^2_t L^{3+\gamma}_x} \leq \varepsilon_1. \tag{5.53}$$

PROOF. (5.52) is a direct corollary of (5.51). In fact, the definition of the wave map tension field gives

$$D_t \phi_t = \phi_s + \mathfrak{W}(s).$$

Hence $\partial_t \phi_t$ is bounded by $|\phi_s| + |A_t \phi_t| + |\mathfrak{W}|$, then (5.52) follows by (5.6), (5.51), (5.34) and (5.11). (5.51) follows by the same arguments as (5.46). The only difference is to use

$$\left\| h^{ii} \left(\partial_t \widetilde{u} \wedge \partial_i \widetilde{u} \right) \nabla_i \partial_t \widetilde{u} \right\|_{L^2_t L^{3+\gamma}_x} \leq \|\nabla \partial_t \widetilde{u}\|_{L^2_t L^{6+2\gamma}_x} \|\partial_t \widetilde{u}\|_{L^\infty_t L^{12+4\gamma}_x} \|d\widetilde{u}\|_{L^\infty_t L^{12+4\gamma}_x},$$

where the term $\|\partial_t \widetilde{u}\|_{L^\infty_t L^{12+4\gamma}_x} \|d\widetilde{u}\|_{L^\infty_t L^{12+4\gamma}_x}$ is bounded by Soboelv embedding and Proposition 3.15. It remains to prove (5.53). By the definition of D_t and A_t, we have

$$\begin{aligned} |\partial_t A_t(s)| &\leq \int_s^\infty |\partial_t \phi_t| \, |\phi_s| d\kappa + \int_s^\infty |\partial_t \phi_s| \, |\phi_t| \, d\kappa \\ &\leq \int_s^\infty |D_t \phi_t| \, |\phi_s| d\kappa + \int_s^\infty |A_t| \, |\phi_s| \, d\kappa + \int_s^\infty |\partial_t \phi_s| \, |\phi_t| \, d\kappa, \end{aligned}$$

By $\mathfrak{W} = D_t \phi_t - \phi_s$ and Hölder,

$$\begin{aligned} &\left\| \int_s^\infty |D_t \phi_t| \, |\phi_s| d\kappa \right\|_{L^2_t L^{3+\gamma}_x} \\ &\leq \int_s^\infty \|w\|_{L^2_t L^{3+\gamma}_x} \|\partial_s \widetilde{u}\|_{L^\infty_t L^\infty_x} d\kappa + \int_s^\infty \|\phi_s\|_{L^\infty_t L^{6+2\gamma}_x} \|\phi_s\|_{L^2_t L^{6+2\gamma}_x} d\kappa. \end{aligned} \tag{5.54}$$

Since $\|\phi_s\|_{L^{6+2\gamma}_x} \leq \||D|^{\frac{1}{2}} \phi_s\|_{L^p_x}$ for $p \in (4, 6)$, then (5.54) is acceptable by Proposition 3.15 and Proposition 5.7. Again by Hölder and Sobolev embedding, for $\frac{1}{m} + \frac{1}{4} = \frac{1}{3+\gamma}$

$$\begin{aligned} \left\| \int_s^\infty |\partial_t \phi_s| \, |\phi_t| \, d\kappa \right\|_{L^2_t L^{3+\gamma}_x} &\leq \int_s^\infty \|\partial_s \phi_t\|_{L^2_t L^4_x} \|\phi_t\|_{L^\infty_t L^m_x} d\kappa \\ &\leq \int_s^\infty \|\partial_s \phi_t\|_{L^2_t L^4_x} \|\nabla \partial_t \widetilde{u}\|_{L^2_x} d\kappa. \end{aligned}$$

Since $|\partial_s \phi_t| \leq |\partial_t \phi_s| + |A_t \phi_s|$, this is also acceptable by Proposition 3.15, Proposition 5.7, (5.35), and (5.11). Thus (5.53) follows. □

PROPOSITION 5.12. Suppose that (5.5) and (5.6) hold. Then we have for $p \in (2,6)$

$$\begin{aligned} &\left\| \omega(s)|D|^{-\frac{1}{2}}\partial_t\phi_s \right\|_{L_s^\infty L_t^2 L_x^p([0,T]\times\mathbb{H}^2)} + \left\| \omega(s)|D|^{\frac{1}{2}}\phi_s \right\|_{L_s^\infty L_t^2 L_x^p([0,T]\times\mathbb{H}^2)} \\ &+ \|\omega(s)\partial_t\phi_s\|_{L_s^\infty L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)} + \|\omega(s)\nabla\phi_s\|_{L_s^\infty L_t^\infty L_x^2([0,T]\times H^2)} \le \varepsilon_1^2. \end{aligned} \tag{5.55}$$

Proof By Lemma 4.4 and Proposition 5.2, we obtain for any $p \in (2,6)$

$$\begin{aligned} &\omega(s)\|\partial_t\phi_s\|_{L_t^\infty L_x^2} + \omega(s)\|\nabla\phi_s\|_{L_t^\infty L_x^2} + \omega(s)\left\| |\nabla|^{\frac{1}{2}}\phi_s \right\|_{L_t^2 L_x^p} \\ &+ \omega(s)\left\| |D|^{-\frac{1}{2}}\partial_t\phi_s \right\|_{L_t^2 L_x^p} + \omega(s)\|\rho^\sigma\nabla\phi_s\|_{L_t^2 L_x^2} \\ &\lesssim \omega(s)\|\partial_t\phi_s(0,s,x)\|_{L_x^2} + \omega(s)\|\nabla\phi_s(0,s,x)\|_{L_x^2} + \omega(s)\|G_4\|_{L_t^1 L_x^2}. \end{aligned} \tag{5.56}$$

where G_4 denotes the inhomogeneous term. First, the $\phi_s(0,s,x)$ term is acceptable by Proposition 3.15, $\mu_2 \ll \varepsilon_1$ and

$$|\nabla_{t,x}\phi_s(0,s,x)| \le |\nabla_{t,x}\partial_s U| + \sqrt{h^{\gamma\gamma}}|A_\gamma||\partial_s U|,$$

where $U(s,x)$ is the heat flow initiated from u_0. Second, the three terms involved with A_t are bounded by

$$\begin{aligned} \omega(s)\|A_t\partial_t\phi_s\|_{L_t^1 L_x^2} &\le \|A_t\|_{L_t^1 L_x^\infty}\omega(s)\|\partial_t\phi_s\|_{L_t^\infty L_x^2} \\ \omega(s)\|A_tA_t\phi_s\|_{L_t^1 L_x^2} &\le \|A_t\|_{L_t^1 L_x^\infty}\|A_t\|_{L_t^\infty L_x^\infty}\omega(s)\|\phi_s\|_{L_t^\infty L_x^2} \\ \omega(s)\|\partial_tA_t\phi_s\|_{L_t^1 L_x^2} &\le \|\partial_tA_t\|_{L_t^2 L_x^{3+\gamma}}\omega(s)\|\phi_s\|_{L_t^2 L_x^k}, \end{aligned}$$

where $\frac{1}{k} + \frac{1}{3+\gamma} = \frac{1}{2}$, and $k \in (2,6)$. They are admissible by (5.11), (5.35) and (5.53). The $\partial_t\widetilde{u}$ term is bounded by

$$\omega(s)\|\mathbf{R}(\partial_t\widetilde{u},\partial_s\widetilde{u})(\partial_t\widetilde{u})\|_{L_t^1 L_x^2} \le \|\partial_t\widetilde{u}\|_{L_t^2 L_x^{6+2\gamma}}\|\partial_t\widetilde{u}\|_{L_t^\infty L_x^{6+2\gamma}}\omega(s)\|\phi_s\|_{L_t^2 L_x^k},$$

where $\frac{1}{k} + \frac{1}{3+\gamma} = \frac{1}{2}$, and $k \in (2,6)$. The $\partial_s\mathfrak{W}$ term is bounded by (5.46). The A_i^{con} terms should be dealt with separately. We present the estimates for these terms as a lemma.

LEMMA 5.13 (Continuation of Proof of Proposition 5.12). *Under the assumption of Proposition 5.12, we have*

$$\omega(s)\left\|h^{ii}A_i^{con}\partial_i\phi_s\right\|_{L_t^1 L_x^2} \le \varepsilon_1\omega(s)\|\rho^\sigma\nabla\phi_s\|_{L_t^2 L_x^2} + \varepsilon_1^2 \tag{5.57}$$

$$\omega(s)\left\|h^{ii}A_i^{con}A_i^\infty\phi_s\right\|_{L_t^1 L_x^2} \le \varepsilon_1^2 \tag{5.58}$$

$$\omega(s)\left\|h^{ii}A_i^{con}A_i^{con}\phi_s\right\|_{L_t^1 L_x^2} \le \varepsilon_1^2 \tag{5.59}$$

$$\omega(s)\left\|h^{ii}\partial_iA_i^{con}\phi_s\right\|_{L_t^1 L_x^2} \le \varepsilon_1^2 \tag{5.60}$$

$$\omega(s)\left\|h^{ii}\Gamma_{ii}^kA_k^{con}\phi_s\right\|_{L_t^1 L_x^2} \le \varepsilon_1^2. \tag{5.61}$$

PROOF. Expanding ϕ_i as $\phi_i^\infty + \int_s^\infty \partial_s\phi_i d\kappa$ yields

$$A_i^{con} = \int_s^\infty \phi_i \wedge \phi_s d\kappa = \int_s^\infty \left(\int_\kappa^\infty \partial_s\phi_i(\tau)d\tau + \phi_i^\infty \right) \wedge \phi_s(\kappa)d\kappa. \tag{5.62}$$

Hence we get

$$\omega(s)\left\|h^{ii}A_i^{con}\partial_i\phi_s\right\|_{L_t^1 L_x^2}$$

$$\leq \omega(s)\Big\|h^{ii}(\int_s^\infty \phi_i^\infty \wedge \phi_s(\kappa)d\kappa)\partial_i\phi_s\Big\|_{L^1_tL^2_x}$$
$$+\,\omega(s)\Big\|h^{ii}\left(\int_s^\infty \phi_s(\kappa)\wedge\left(\int_\kappa^\infty \partial_s\phi_i(\tau)d\tau\right)d\kappa\right)\partial_i\phi_s\Big\|_{L^1_tL^2_x}$$
$$\triangleq B_1+B_2$$

The B_1 term is bounded by

$$B_1 \lesssim \omega(s)\|\rho^\sigma\nabla\phi_s\|_{L^2_tL^2_x}\Big\|\int_s^\infty \rho^{-\sigma}\phi_i^\infty\sqrt{h^{ii}}\phi_s(\kappa)d\kappa\Big\|_{L^2_tL^\infty_x}$$
$$\leq \omega(s)\|\rho^\sigma\nabla\phi_s\|_{L^2_tL^2_x}\Big\|\rho^{-\sigma}\phi_i^\infty\sqrt{h^{ii}}\Big\|_{L^\infty_x}\int_s^\infty \|\phi_s(\kappa)\|_{L^2_tL^\infty_x}d\kappa$$
$$\lesssim \omega(s)\|\rho^\sigma\nabla\phi_s\|_{L^2_tL^2_x}\Big\|\rho^{-\sigma}\sqrt{h^{ii}}\phi_i^\infty\Big\|_{L^\infty_x}\Big\|a(s)\|\nabla\phi_s(s)\|_{L^2_tL^4_x}\Big\|_{L^\infty_s}, \tag{5.63}$$

where we have used the Sobolev embedding in the last step. Hence Proposition 5.7 gives an acceptable bound,

$$B_1 \leq C\mu_1\varepsilon_1\omega(s)\|\rho^\sigma\nabla\phi_s\|_{L^2_tL^2_x}.$$

The B_2 term is bounded by

$$B_2 \lesssim \omega(s)\|\nabla\phi_s\|_{L^\infty_tL^2_x}\int_s^\infty \|\phi_s(\kappa)\|_{L^2_tL^\infty_x}\left(\int_\kappa^\infty \|\nabla\phi_s(\tau)\|_{L^2_tL^\infty_x}d\tau\right)d\kappa.$$

Meanwhile, Sobolev embedding and Proposition 5.7 give when $\tau\in(0,1)$

$$\|\nabla\phi_s(\tau)\|_{L^2_tL^\infty_x} \leq \left(\tau^{\frac{3}{4}}\|\nabla\phi_s(\tau)\|_{L^2_tL^5_x}\right)^{3/5}\left(\tau^{5/4}\big\|\nabla^2\phi_s(\tau)\big\|_{L^2_tL^5_x}\right)^{2/5}\tau^{-\frac{1}{2}-9/20}$$
$$\leq \varepsilon_1\tau^{-19/20},$$

and when $\tau\in[1,\infty)$

$$\|\nabla\phi_s(\tau)\|_{L^2_tL^\infty_x} \leq \left(\tau^{L}\|\nabla\phi_s(\tau)\|_{L^2_tL^5_x}\right)^{3/5}\left(\tau^{L}\big\|\nabla^2\phi_s(\tau)\big\|_{L^2_tL^5_x}\right)^{2/5}\tau^{-L}\leq\varepsilon_1\tau^{-L}.$$

Similarly we deduce by Sobolev embedding $\|f\|_{L^\infty}\leq\||D|^{\frac{1}{2}}f\|_{L^5}$ that

$$\|\phi_s(\tau)\|_{L^2_tL^\infty_x}\leq\varepsilon_1\tau^{-\frac{1}{2}},\ \text{when } \tau\in(0,1);\ \|\phi_s(\tau)\|_{L^2_tL^\infty_x}\leq\varepsilon_1\tau^{-L},\ \text{when } \tau\in[1,\infty).$$

Therefore we conclude

$$B_2 \leq \varepsilon_1^2\omega(s)\|\nabla\phi_s\|_{L^\infty_tL^2_x}. \tag{5.64}$$

Proposition 5.7 together with (5.63), (5.64) yields (5.57). Next we prove (5.58). Hölder yields

$$\omega(s)\|h^{ii}A_i^{con}A_i^\infty\phi_s\|_{L^1_tL^2_x}\leq\|\sqrt{h^{ii}}A_i^{con}\|_{L^2_tL_x^{\frac{10}{3}}}\omega(s)\|\phi_s\|_{L^2_tL^5_x}.$$

Using the expression $A_i^{con}=\int_s^\infty\phi_i\wedge\phi_s d\kappa$, we obtain

$$\Big\|\sqrt{h^{ii}}A_i^{con}\Big\|_{L^2_tL_x^{\frac{10}{3}}}\lesssim\Big\|\int_s^\infty\sqrt{h^{ii}}\phi_i\wedge\phi_s d\kappa\Big\|_{L^2_tL_x^{\frac{10}{3}}}\leq\|d\widetilde{u}\|_{L^\infty_tL^{10}_x}\int_s^\infty\|\phi_s\|_{L^2_tL^5_x}d\kappa$$
$$\lesssim\|\nabla d\widetilde{u}\|_{L^\infty_tL^2_x}\|\omega(s)\|\phi_s(s)\|_{L^2_tL^5_x}\|_{L^\infty_s}. \tag{5.65}$$

Therefore Proposition 5.7 gives (5.58). Third, we verify (5.59). Hölder yields

$$\omega(s)\|h^{ii}A_i^{con}A_i^{con}\phi_s\|_{L_t^1L_x^2} \le \|\sqrt{h^{ii}}A_i^{con}\|_{L_t^2L_x^{\frac{10}{3}}}\|\sqrt{h^{ii}}A_i^{con}\|_{L_t^\infty L_x^\infty}\omega(s)\|\phi_s\|_{L_t^2L_x^5}.$$

The term $\|\sqrt{h^{ii}}A_i^{con}\|_{L_t^2L_x^{\frac{10}{3}}}$ has been estimated in (5.65). The $\|\sqrt{h^{ii}}A_i^{con}\|_{L_t^\infty L_x^\infty}$ term is bounded by

$$\|\sqrt{h^{ii}}A_i^{con}\|_{L_t^\infty L_x^\infty} \lesssim \left\|\int_s^\infty \|d\widetilde{u}\|_{L_x^\infty}\|\phi_s\|_{L_x^\infty}d\kappa\right\|_{L_t^\infty}.$$

This is acceptable by Proposition 3.15 and Lemma 2.7. Forth, we prove (5.60). Hölder yields

$$\omega(s)\left\|h^{ii}\left(\partial_iA_i^{con}\right)\phi_s\right\|_{L_t^1L_x^2} \le \left\|h^{ii}\partial_iA_i^{con}\right\|_{L_t^2L_x^4}\omega(s)\|\phi_s\|_{L_t^2L_x^4}.$$

The $h^{ii}\partial_iA_i$ term is bounded by

$$\begin{aligned}
\left\|h^{ii}\partial_iA_i^{con}\right\|_{L_t^2L_x^4} &= \left\|\int_s^\infty h^{ii}\partial_i\phi_i\phi_s d\kappa + \int_s^\infty h^{ii}\phi_i\partial_i\phi_s d\kappa\right\|_{L_t^2L_x^4}\\
&\le \int_s^\infty \left\|h^{ii}\partial_i\phi_i\right\|_{L_t^\infty L_x^{20}}\|\phi_s\|_{L_t^2L_x^5}d\kappa + \int_s^\infty \|d\widetilde{u}\|_{L_t^\infty L_x^\infty}\|\nabla\phi_s\|_{L_t^2L_x^4}d\kappa\\
&\le \int_s^\infty \left(\|\nabla d\widetilde{u}\|_{L_t^\infty L_x^{20}} + \left\|h^{ii}A_i\phi_i\right\|_{L_t^\infty L_x^{20}}\right)\|\phi_s\|_{L_t^2L_x^5}d\kappa\\
&+ \int_s^\infty \|d\widetilde{u}\|_{L_t^\infty L_x^\infty}\|\nabla\phi_s\|_{L_t^2L_x^4}d\kappa.
\end{aligned}$$

Thus this is acceptable by Proposition 5.5 and interpolation between the $\|\nabla d\widetilde{u}\|_{L^\infty}$ bound and the $\|\nabla d\widetilde{u}\|_{L^2}$ bound in Proposition 3.15. Finally we notice that (5.61) is a consequence of (5.65) and

$$\omega(s)\left\|h^{ii}\Gamma_{ii}^kA_k^{con}\phi_s\right\|_{L_t^1L_x^2} \le \|A_2^{con}\|_{L_t^2L_x^{\frac{10}{3}}}\omega(s)\|\phi_s\|_{L_t^2L_x^5}.$$

□

■

Proposition 5.7 with Proposition 5.12 yields

PROPOSITION 5.14. Assume that the solution to (1.2) satisfies (5.6) and (5.5), then for any $p\in(2,6)$, $\theta\in[0,2]$

$$\|\omega(s)\nabla\phi_s\|_{L_s^\infty L_t^\infty L_x^2} + \left\|\omega(s)|D|^{\frac{1}{2}}\phi_s\right\|_{L_s^\infty L_t^2L_x^p} \le \varepsilon_1^2$$
$$\|\omega_1(s)|D|\partial_t\phi_s\|_{L_s^\infty L_t^2L_x^p} + \left\|\omega_\theta(s)(-\Delta)^\theta\phi_s\right\|_{L_s^\infty L_t^2L_x^p} \le \varepsilon_1^2.$$

5.3. Close all the bootstrap.

LEMMA 5.15. *Assume that the solution to (1.2) satisfies (5.6) and (5.5), then for any $p\in(2,6+2\gamma]$*

$$\|(du,\partial_tu)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)} + \|(\nabla\partial_tu,\nabla du)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)} \le \varepsilon_1^2 \tag{5.66}$$

$$\|\partial_tu\|_{L_t^2L_x^p([0,T]\times\mathbb{H}^2)} \le \varepsilon_1^2. \tag{5.67}$$

PROOF. First we prove (5.67). By $D_s\phi_t = D_t\phi_s$, $A_s = 0$, one has

$$\|\phi_t(0,t,x)\|_{L_t^2L_x^p} \le \left\|\int_0^\infty |\partial_s\phi_t|ds\right\|_{L_t^2L_x^p} \le \|\partial_t\phi_s\|_{L_s^1L_t^2L_x^p} + \|A_t\phi_s\|_{L_s^1L_t^2L_x^p}. \tag{5.68}$$

Sobolev embedding gives

$$\|\partial_t\phi_s\|_{L_x^{6+2\gamma}} + \|\phi_s\|_{L_x^{6+2\gamma}} \le \left\|(-\Delta)^{\vartheta}\partial_t\phi_s\right\|_{L_x^{6-\eta}} + \left\|(-\Delta)^{\vartheta}\phi_s\right\|_{L_x^{6-\eta}},$$

where $\frac{\vartheta}{2} = \frac{1}{6-\eta} - \frac{1}{6+2\gamma}$, $0<\eta\ll 1, 0<\gamma\ll 1$. Thus (5.67) follows by Proposition 5.14 and Proposition 5.5. Second, we prove (5.66). By Remark 3.21, $\phi_i(0,t,x) = \phi_i^\infty + \int_0^\infty \partial_s\phi_i d\kappa$. Since $|d\widetilde{u}| \le \sqrt{h^{ii}}|\phi_i|$, $\|\sqrt{h^{ii}}\phi_i^\infty\|_{L_x^2} \le \|dQ\|_{L^2} \le \mu_1$, it suffices to verify for any $t,x \in [0,T]\times\mathbb{H}^2$

$$\int_0^\infty \|\sqrt{h^{ii}}\partial_s\phi_i\|_{L_x^2} d\kappa \le \varepsilon_1^2.$$

This is acceptable by Proposition 5.5, Proposition 5.14 and $|\sqrt{h^{ii}}\partial_s\phi_i| \le |\nabla\phi_s| + \sqrt{h^{ii}}|A_i||\phi_s|$. Recalling (7.12) for the equation of ϕ_s evolving along the heat flow, we have by integration by parts,

$$\begin{aligned}\frac{d}{ds}\|\tau(\widetilde{u})\|_{L_x^2}^2 &= \frac{d}{ds}\langle\phi_s,\phi_s\rangle = 2\langle D_s\phi_s,\phi_s\rangle\\ &= 2h^{ii}\left\langle D_iD_i\phi_s - \Gamma_{ii}^k D_k\phi_s,\phi_s\right\rangle + \left\langle h^{ij}(\phi_s\wedge\phi_i)\phi_j,\phi_s\right\rangle\\ &= -2h^{ii}\langle D_i\phi_s, D_i\phi_s\rangle + \left\langle h^{ij}(\phi_s\wedge\phi_i)\phi_j,\phi_s\right\rangle.\end{aligned}$$

Hence $\|\partial_s\widetilde{u}\|_{L_x^2} \le e^{-\delta s}$ shows

$$\begin{aligned}&\|\tau(\widetilde{u}(0,t,x))\|_{L_x^2}^2 \lesssim \int_0^\infty h^{ii}\langle D_i\phi_s, D_i\phi_s\rangle ds\\ &\lesssim \int_0^\infty \langle\nabla\phi_s,\nabla\phi_s\rangle ds + \int_0^\infty h^{ii}\langle A_i\phi_s, A_i\phi_s\rangle ds + \int_0^\infty |d\widetilde{u}|^2|\phi_s|^2 ds. \qquad (5.69)\end{aligned}$$

The nonnegative sectional curvature property of $N=\mathbb{H}^2$ with integration by parts implies

$$\|\nabla d\widetilde{u}\|_{L_x^2}^2 \lesssim \|\tau(\widetilde{u})\|_{L_x^2}^2 + \|d\widetilde{u}\|_{L_x^2}^2.$$

Hence (5.69) gives

$$\begin{aligned}&\|\nabla d\widetilde{u}(0,t,x)\|_{L_x^2}^2 \qquad (5.70)\\ &\lesssim \int_0^\infty \langle\nabla\phi_s,\nabla\phi_s\rangle ds + \int_0^\infty |d\widetilde{u}|^2|\phi_s|^2 ds + \int_0^\infty h^{ii}\langle A_i\phi_s, A_i\phi_s\rangle ds + \|d\widetilde{u}(0,t,x)\|_{L_x^2}^2.\end{aligned}$$

Since the $|d\widetilde{u}|$ term has been estimated, by Proposition 5.14 , Proposition 5.5 and (5.70),

$$\|\nabla d\widetilde{u}\|_{L_t^\infty L_x^2([0,T]\times\mathbb{H}^2)} \le \varepsilon_1^2.$$

Finally we prove the desired estimates for $|\nabla\partial_t\widetilde{u}|$. Integration by parts yields,

$$\begin{aligned}&\frac{d}{ds}\|\nabla\partial_t\widetilde{u}\|_{L^2}^2 = \frac{d}{ds}h^{ii}\langle D_i\phi_t, D_i\phi_t\rangle = 2h^{ii}\langle D_sD_i\phi_t, D_i\phi_t\rangle\\ &= 2h^{ii}\langle D_iD_t\phi_s, D_i\phi_t\rangle + 2h^{ii}\langle(\phi_s\wedge\phi_i)\phi_t, D_i\phi_t\rangle\\ &= -2h^{ii}\langle D_t\phi_s, D_iD_i\phi_t\rangle + 2\langle D_t\phi_s, D_2\phi_t\rangle + 2h^{ii}\langle(\phi_s\wedge\phi_i)\phi_t, D_i\phi_t\rangle\end{aligned}$$

$$= -2\left\langle D_t\phi_s, h^{ii}D_iD_i\phi_t - h^{ii}\Gamma^k_{ii}D_k\phi_t\right\rangle + 2h^{ii}\left\langle(\phi_s\wedge\phi_i)\phi_t, D_i\phi_t\right\rangle.$$

Recall (7.14), the parabolic equation of ϕ_t along heat flow, then

$$\frac{d}{ds}\|\nabla\partial_t\widetilde{u}\|^2_{L^2} = -2\langle D_t\phi_s, D_s\phi_t\rangle + 2h^{ii}\langle(\phi_s\wedge\phi_i)\phi_t, D_i\phi_t\rangle + 2h^{ii}\langle D_t\phi_s, (\phi_t\wedge\phi_i)\phi_i\rangle.$$

Hence we conclude,

$$\begin{aligned}&\|\nabla\partial_t\widetilde{u}(0,t,x)\|^2_{L^2}\\ &\lesssim \int_0^\infty \langle\partial_t\phi_s, \partial_t\phi_s\rangle\, d\kappa + \int_0^\infty \langle A_t\phi_s, A_t\phi_s\rangle\, d\kappa\\ &+ \int_0^\infty \|\phi_s\|_{L^2_x}\|d\widetilde{u}\|_{L^\infty_x}\|\partial_t\widetilde{u}\|_{L^\infty_x}\|\nabla\partial_t\widetilde{u}\|_{L^2_x}d\kappa + \int_0^\infty \|\partial_t\widetilde{u}\|_{L^2_x}\|d\widetilde{u}\|^2_{L^\infty_x}\|D_t\phi_s\|_{L^2_x}d\kappa.\end{aligned}$$

Thus by Proposition 3.15, Proposition 5.14 and Proposition 5.5, we have

$$\|\nabla\partial_t\widetilde{u}(0,t,x)\|^2_{L^2}\le\varepsilon_1^4.$$

Therefore, we have proved all estimates in (5.66) and (5.67). □

We summarize what we have proved in the following corollary.

Corollary 5.16. Assume $(-T^*, T_*)$ is the lifespan of solution to (1.2). And let μ_1, μ_2 be sufficiently small, then we have

$$\begin{aligned}\|du\|_{L^\infty_t L^2_x([0,T_*]\times\mathbb{H}^2)} &+ \|\partial_t u\|_{L^\infty_t L^2_x([0,T_*]\times\mathbb{H}^2)} + \|\nabla du\|_{L^\infty_t L^2_x([0,T_*]\times\mathbb{H}^2)}\\ &+ \|\nabla\partial_t u\|_{L^\infty_t L^2_x([0,T_*]\times\mathbb{H}^2)} + \|\partial_t u\|_{L^2_t L^6_x([0,T_*]\times\mathbb{H}^2)}\le\varepsilon_1^2.\end{aligned}$$

Thus by Proposition 2.9, we have $(u, \partial_t u)$ is a global solution to (1.2).

6. Proof of Theorem 1.1

Finally, we prove Theorem 1.1 based on Proposition 5.14.

Proposition 6.1. Let u be the solution to (1.2) in $\mathcal{X}_{[0,\infty)}$. Then as $t\to\infty$, $u(t,x)$ converges to a harmonic map, namely

$$\lim_{t\to\infty}\lim_{x\in\mathbb{H}^2}\mathrm{dist}_{\mathbb{H}^2}(u(t,x), Q(x)) = 0,$$

where $Q(x):\mathbb{H}^2\to\mathbb{H}^2$ is the unperturbed harmonic map.

Proof. For $u(t,x)$, by Proposition 3.19, we have the corresponding heat flow converges to some harmonic map uniformly for $x\in\mathbb{H}^2$. Then by the definition of the distance on complete manifolds, we have

$$\mathrm{dist}_{\mathbb{H}^2}(u(t,x), Q(x))\le\int_0^\infty\|\partial_s\widetilde{u}\|_{L^\infty_x}ds. \tag{6.1}$$

For any $T>0$, $\mu>0$, since $|\partial_s\widetilde{u}|$ satisfies $(\partial_s-\Delta)|\partial_s\widetilde{u}|\le 0$, one has

(6.2)

$$\int_T^\infty\|\partial_s\widetilde{u}(s,t,x)\|_{L^\infty_x}ds\lesssim\int_T^\infty e^{-\frac{1}{8}s}\|\tau(u(t,x))\|_{L^2_x}ds\lesssim e^{-T/8}\|\nabla du(t,x)\|_{L^2_x}$$

$$\begin{aligned}\int_0^\mu\|\partial_s\widetilde{u}(s,t,x)\|_{L^\infty_x}ds &\lesssim\int_0^\mu\left\|e^{s\Delta_{\mathbb{H}^2}}\tau(u(t,x))\right\|_{L^\infty_x}ds\le\int_0^\mu s^{-\frac{1}{2}}\|\nabla du(t,x)\|_{L^2_x}ds\\ &\lesssim\mu^{\frac{1}{2}}\|\nabla du(t,x)\|_{L^2_x}\end{aligned}\tag{6.3}$$

Similarly, we have

$$\begin{aligned}\int_{\mu}^{T}\|\partial_s\widetilde{u}(s,t,x)\|_{L_x^\infty}ds &\lesssim \int_{\mu}^{T}\left\|e^{(s-\frac{\mu}{2})\Delta_{\mathbb{H}^2}}\partial_s\widetilde{u}(\frac{\mu}{2},t,x)\right\|_{L_x^\infty}ds\\ &\lesssim \int_{\mu}^{T}(s-\frac{\mu}{2})^{-\frac14}\left\|\partial_s\widetilde{u}(\frac{\mu}{2},t,x)\right\|_{L_x^4}ds\\ &\lesssim \mu^{-\frac14}\int_{\mu}^{T}\left\|\phi_s(\frac{\mu}{2},t,x)\right\|_{L_x^4}ds. \qquad (6.4)\end{aligned}$$

Therefore it suffices to prove for a fixed $\mu>0$

$$\lim_{t\to\infty}\|\phi_s(\mu)\|_{L_x^4}=0. \qquad (6.5)$$

Proposition 5.14 implies $\mu^{\frac12}\|\phi_s(\mu)\|_{L_t^2L_x^4}+\mu^{\frac12}\|\partial_t\phi_s(\mu)\|_{L_t^2L_x^4}<\infty$, thus for any $\epsilon>0$ there exists a T_0 such that

$$\|\phi_s(\mu)\|_{L_t^2L_x^4([T_0,\infty)\times\mathbb{H}^2)}+\|\partial_t\phi_s(\mu)\|_{L_t^2L_x^4([T_0,\infty)\times\mathbb{H}^2)}<\epsilon. \qquad (6.6)$$

Particularly, for any interval $[a,a+1]$ of length one with $a\ge T_0$, there exists some $t_a\in[a,a+1]$ such that

$$\|\phi_s(\mu,t_a)\|_{L_x^4}\le\epsilon/2. \qquad (6.7)$$

Then by fundamental theorem of calculus for any $t'\in[a,a+1]$

$$\Big|\|\phi_s(\mu,t')\|_{L_x^4}-\|\phi_s(\mu,t_a)\|_{L_x^4}\Big|\le\int_{t_a}^{t'}\Big|\partial_t\|\phi_s(\mu,t)\|_{L_x^4}\Big|dt. \qquad (6.8)$$

Since $|\partial_t\|\phi_s(\mu,t)\|_{L_x^4}|\le\|\partial_t\phi_s(\mu,t)\|_{L_x^4}$, by Hölder, (6.8) and (6.6) show

$$\Big|\|\phi_s(\mu,t')\|_{L_x^4}-\|\phi_s(\mu,t_a)\|_{L_x^4}\Big|\le\|\partial_t\phi_s(\mu,t)\|_{L_t^2L_x^4}(t'-a)^{\frac12}\le\|\partial_t\phi_s(\mu,t)\|_{L_t^2L_x^4}.$$

Thus we have by (6.7) that for any $t\in[a,a+1]$,

$$\|\phi_s(\mu,t)\|_{L_x^4}\le\epsilon.$$

Since a is arbitrary chosen, we obtain (6.5). Therefore, Theorem 1.1 is proved, □

7. Proof of remaining lemmas and claims

We first collect some useful inequalities for the harmonic maps.

LEMMA 7.1. *Suppose that Q is an admissible harmonic map in Theorem 1.1. If $0<\mu_1\ll 1$, then*

$$\|\nabla dQ\|_{L^2}\lesssim\mu_1 \qquad (7.1)$$

$$\|\nabla^2 dQ\|_{L^2}\lesssim\mu_1. \qquad (7.2)$$

PROOF. By integration by parts and the non-positive sectional curvature of $N=\mathbb{H}^2$,

$$\begin{aligned}\|\nabla dQ\|_{L^2}^2&\lesssim\|dQ\|_{L^2}^2+\|\tau(Q)\|_{L^2}^2\\ \|\nabla^2 dQ\|_{L^2}^2&\lesssim\|\nabla\tau(Q)\|_{L^2}^2+\|\nabla dQ\|_{L^2}^3+\|\nabla dQ\|_{L^4}^2\|dQ\|_{L^4}^2+\|dQ\|_{L^2}^6.\end{aligned}$$

Hence by $\tau(Q)=0$, we have (7.1). And then (7.2) follows from (1.3), Gagliardo-Nirenberg inequality and Sobolev embedding. □

Now we prove Corollary 2.1.

LEMMA 7.2. *Fix $R_0 > 0$, let $0 < \mu_1, \mu_2 \ll \mu_3 \ll 1$, then the initial data (u_0, u_1) in Theorem 1.1 satisfy*

$$\|du_0\|_{L^2} + \|u_1\|_{L^2} + \|\nabla du_0\|_{L^2} + \|\nabla u_1\|_{L^2} \leq \mu_3. \tag{7.3}$$

PROOF. First by (7.1), the harmonic map Q satisfies

$$\|\nabla dQ\|_{L^2} + \|dQ\|_{L^2} \leq \mu_1. \tag{7.4}$$

By (1.4) and Sobolev embedding,

$$\|u_0^k - Q^k\|_{L^\infty} \lesssim \|u_0^k - Q^k\|_{H^2} \leq \mu_2. \tag{7.5}$$

Hence $|u_0^1| + |u_0^2| \lesssim R_0 + \mu_2$. Then choosing $R = CR_0 + C\mu_2$ in [Lemma 2.3,[**37**]], we have

$$\|du_0\|_{L^2} + \|\nabla du_0\|_{L^2} \leq Ce^{8(CR_0+C\mu_2)}\big(\|\nabla^2 u_0^k\|_{L^2} + \|\nabla^2 u_0^k\|_{L^2}^2\big). \tag{7.6}$$

Again by [Lemma 2.3,[**37**]] and (7.4),

$$\|\nabla^2 Q^k\|_{L^2} \leq Ce^{8(R_0)}\big(\|\nabla dQ\|_{L^2} + \|\nabla dQ\|_{L^2}^2\big) \leq Ce^{8(R_0)}\mu_1. \tag{7.7}$$

Therefore, (1.4), (7.7) and (7.6) give

$$\|du_0\|_{L^2} + \|\nabla du_0\|_{L^2} \leq Ce^{8(CR_0+C\mu_2)}(\mu_1 + \mu_2) \tag{7.8}$$

Let μ_1 and μ_2 be sufficiently small depending on R_0, we obtain

$$\|du_0\|_{L^2} + \|\nabla du_0\|_{L^2} \leq \mu_3. \tag{7.9}$$

□

LEMMA 7.3. *Let W be the magnetic operator defined in Lemma 4.4 as*

(7.10)

$$W\varphi = -2h^{ii}A_i^\infty \partial_i \varphi - h^{ii}A_i^\infty A_i^\infty \varphi - h^{ii}\left(\varphi \wedge \phi_i^\infty\right)\phi_i^\infty - h^{ii}(\partial_i A_i^\infty - \Gamma_{ii}^k A_k^\infty),$$

Then W is symmetric with domain $C_c^\infty(\mathbb{H}^2, \mathbb{C}^2)$. And $-\Delta + W$ is strictly positive if μ_1 is sufficiently small.

PROOF. Since we work with complex valued functions here, the wedge operator $\wedge$ should be first extended to the complex number field by taking the inner product in (2.35) to be the complex inner product. By the explicit formula for Γ_{ii}^k and h^{ii}, one has

$$h^{ii}\Gamma_{ii}^k A_k^\infty = h^{11}\Gamma_{11}^2 A_2^\infty = A_2^\infty. \tag{7.11}$$

It is easy to see by the non-positiveness and symmetry of the sectional curvature that $\varphi \longmapsto -h^{ii}\left(\varphi \wedge \phi_i^\infty\right)\phi_i^\infty$ is a non-negative and symmetric operator on $L^2(\mathbb{H}^2, \mathbb{C}^2)$. And by the skew-symmetry of A_i^∞, $\varphi \longmapsto -h^{ii}\left(\varphi \wedge A_i^\infty\right)A_i^\infty$ is a non-negative and symmetric symmetric operator on $L^2(\mathbb{H}^2, \mathbb{C}^2)$. We claim that

$$\varphi \longmapsto 2h^{ii}A_i^\infty \partial_i \varphi + h^{ii}(\partial_i A_i^\infty - \Gamma_{ii}^k A_k^\infty)$$

is a symmetric operator on $L^2(\mathbb{H}^2, \mathbb{C}^2)$ as well. Indeed, by the skew-symmetry of A_i^∞, $\partial_i A_i^\infty$, integration by parts and (7.11),

$$\begin{aligned}
&\big\langle 2h^{ii}A_i^\infty \partial_i f + h^{ii}(\partial_i A_i^\infty - \Gamma_{ii}^k A_k^\infty)f, g\big\rangle \\
&= \big\langle 2h^{ii}A_i^\infty \partial_i f + h^{ii}\partial_i A_i^\infty f - A_2^\infty f, g\big\rangle \\
&= \big\langle h^{ii}\partial_i A_i^\infty f - A_2^\infty f, g\big\rangle - \big\langle 2h^{ii}\partial_i A_i^\infty f, g\big\rangle - \big\langle 2h^{ii}A_i^\infty f, \partial_i g\big\rangle + \big\langle 2h^{22}A_2^\infty f, g\big\rangle \\
&= \big\langle -h^{ii}\partial_i A_i^\infty f + A_2^\infty f, g\big\rangle - \big\langle 2h^{ii}A_i^\infty f, \partial_i g\big\rangle
\end{aligned}$$

$$= \langle f, h^{ii}\partial_i A_i^{\infty} g - A_2^{\infty} g \rangle + \langle f, 2h^{ii} A_i^{\infty} \partial_i g \rangle$$
$$= \langle f, 2h^{ii} A_i^{\infty} \partial_i g + h^{ii}\partial_i A_i^{\infty} g - A_2^{\infty} g \rangle .$$

It remains to prove $-\Delta + W$ is positive. Since we have shown $\varphi \longmapsto -h^{ii}\left(\varphi \wedge \phi_i^{\infty}\right)\phi_i^{\infty}$ and $\varphi \longmapsto -h^{ii}\left(\varphi \wedge A_i^{\infty}\right) A_i^{\infty}$ are nonnegative, it suffices to prove for some $\delta > 0$

$$\left\langle -\Delta f + 2h^{ii} A_i^{\infty} \partial_i f + h^{ii}(\partial_i A_i^{\infty} - \Gamma_{ii}^k A_k^{\infty}) f, f \right\rangle \geq \delta \langle f, f \rangle .$$

By the skew-symmetry of A_i^{∞} and $\partial_i A_i^{\infty}$, it reduces to

$$\left\langle -\Delta f + 2h^{ii} A_i^{\infty} \partial_i f, f \right\rangle \geq \delta \langle f, f \rangle .$$

Hölder, (2.10) and (4.3) imply for some universal constant $c > 0$

$$\begin{aligned}
&\left\langle -\Delta f + 2h^{ii} A_i^{\infty} \partial_i f, f \right\rangle \geq \|\nabla f\|_2^2 - 2\left\| \sqrt{h^{ii}} A_i^{\infty} \right\|_{\infty} \|\nabla f\|_2 \|f\|_2 \\
&\geq \frac{1}{2}\|\nabla f\|_2^2 + c\|f\|_2^2 - 2\left\| \sqrt{h^{ii}} A_i^{\infty} \right\|_{\infty} \|\nabla f\|_2 \|f\|_2 \\
&\geq \frac{1}{2}\|\nabla f\|_2^2 + c\|f\|_2^2 - 2\mu_1 \|\nabla f\|_2 \|f\|_2 .
\end{aligned}$$

Let μ_1 be sufficiently small, then

$$\left\langle -\Delta f + 2h^{ii} A_i^{\infty} \partial_i f, f \right\rangle \geq \delta \langle f, f \rangle .$$

□

Recall the equation of the tension field ϕ_s:

LEMMA 7.4. *The evolution of differential fields and the heat tension filed along the heat flow are given by the following:*

$$\partial_s \phi_s = h^{ii} D_i D_i \phi_s - h^{ii}\Gamma_{ii}^k D_k \phi_s + h^{ii}(\phi_s \wedge \phi_i)\phi_i \tag{7.12}$$

$$\begin{aligned}
\partial_s \phi_s - \Delta \phi_s &= 2h^{ii} A_i \partial_i \phi_s + h^{ii}\left(\partial_i A_i\right)\phi_s - h^{ii}\Gamma_{ii}^k A_k \phi_s + h^{ii} A_i A_i \phi_s \\
&\quad + h^{ii}\left(\phi_s \wedge \phi_i\right)\phi_i
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
\partial_s \phi_t - \Delta \phi_t &= 2h^{ii} A_i \partial_i \phi_t + h^{ii} A_i A_i \phi_t + h^{ii}\partial_i A_i \phi_t - h^{ii}\Gamma_{ii}^k A_k \phi_t \\
&\quad + h^{ii}\left(\phi_t \wedge \phi_i\right)\phi_i .
\end{aligned} \tag{7.14}$$

$$\begin{aligned}
\partial_s \partial_t \phi_s &= \Delta \partial_t \phi_s + 2h^{ii}\left(\partial_t A_i\right)\partial_i \phi_s + 2h^{ii} A_i \partial_i \partial_t \phi_s + h^{ii}\left(\partial_i \partial_t A_i\right)\phi_s \\
&\quad + h^{ii}\left(\partial_i A_i\right)\partial_t \phi_s - h^{ii}\Gamma_{ii}^k\left(\partial_t A_k\right)\phi_s - h^{ii}\Gamma_{ii}^k A_k \partial_t \phi_s + h^{ii}\left(\partial_t A_i\right) A_i \phi_s \\
&\quad + h^{ii} A_i\left(\partial_t A_i\right)\phi_s + h^{ii} A_i A_i \partial_t \phi_s + h^{ii}\left(\partial_t \phi_s \wedge \phi_i\right)\phi_i + h^{ii}\left(\phi_s \wedge \partial_t \phi_i\right)\phi_i \\
&\quad + h^{ii}\left(\phi_s \wedge \phi_i\right)\partial_t \phi_i .
\end{aligned} \tag{7.15}$$

PROOF. Recall that we use the orthogonal coordinates (2.1) throughout the paper. Recall the equation of ϕ_s:

$$\phi_s = h^{ii} D_i \phi_i - h^{ii}\Gamma_{ii}^k \phi_k . \tag{7.16}$$

Applying D_s to (7.16) yields

$$\begin{aligned}
D_s \phi_s &= h^{ii} D_s D_i \phi_i - h^{ii}\Gamma_{ii}^k D_s \phi_k = h^{ii} D_i D_i \phi_s - h^{ii}\Gamma_{ii}^k D_k \phi_s + h^{ii}\left(\phi_s \wedge \phi_i\right)\phi_i \\
&= \Delta \phi_s + 2h^{ii} A_i \partial_i \phi_s + h^{ii}\left(\partial_i A_i\right)\phi_s - h^{ii}\Gamma_{ii}^k A_k \phi_s + h^{ii} A_i A_i \phi_s + h^{ii}\left(\phi_s \wedge \phi_i\right)\phi_i .
\end{aligned}$$

The tension free identity and commutator identity give

$$D_s \phi_t = D_t \phi_s = D_t\left(h^{ii} D_i \phi_j - h^{ii}\Gamma_{ii}^k \phi_k\right) = h^{ii} D_t D_i \phi_i - h^{ii}\Gamma_{ii}^k D_t \phi_k$$

$$= h^{ii} D_i D_i \phi_t - h^{ii} \Gamma_{ii}^k D_t \phi_k + h^{ii} \left(\partial_t u \wedge \partial_i u \right) \partial_i u.$$

Therefore the differential filed ϕ_t satisfies

$$\partial_s \phi_t - \Delta \phi_t = 2 h^{ii} A_i \partial_i \phi_t + h^{ii} A_i A_i \phi_t + h^{ii} \partial_i A_i \phi_t - h^{ii} \Gamma_{ii}^k A_k \phi_t + h^{ii} \left(\phi_t \wedge \phi_i \right) \phi_i.$$

Applying ∂_t to (7.13) gives (7.15). □

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References

[1] L. Andersson, N. Gudapati and J. Szeftel. Global regularity for the 2+1 dimensional equivariant Einstein-wave map system. *Ann. PDE,* **3***(2017), online print https://doi.org/10.1007/s40818-017-0030-z.*

[2] *J.P. Anker and V. Pierfelice. Wave and Klein-Gordon equations on hyperbolic spaces. Anal. PDE,* **7***(2014), 953-995.*

[3] *J.P. Anker, V. Pierfelice and M. Vallarino. The wave equation on hyperbolic spaces. J. Differential Equations,* **252***(2012), 5613-5661.*

[4] *E.B. Davies, N. Mandouvalos. Heat kernel bounds on hyperbolic space and Kleinian groups. Proceedings of the London Mathematical Society,* **3***(1988), 182-208.*

[5] *W.O. Bray.* Aspects of harmonic analysis on real hyperbolic space. *In Fourier analysis (Orono, ME, 1992), volume 157 of Lecture Notes in Pure and Appl. Math., pages 77-102. Dekker, New York, 1994.*

[6] *E. Chiodaroli, J. Krieger and J. Luhrmann.* Concentration Compactness for Critical Radial Wave Maps. *arXiv preprint 2016.*

[7] *D. Christodoulou and A.S. Tahvildar-Zadeh. On the regularity of spherically symmetric wave maps. Comm. Pure Appl. Math.,* **46***(1993), 1041-1091.*

[8] *M. Cowling, S. Giulini, S. Meda. L^p-L^q estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces. I. Duke Math. J.,* **72***(1993), 109-150.*

[9] *R. Cote. On the soliton resolution for equivariant wave maps to the sphere. Comm. Pure Appl. Math.,* **68***(2010), 1946-2004.*

[10] *P. D'Ancona and V. Georgiev. On the continuity of the solution operator to the wave map system. Comm. Pure Appl. Math.,* **57***(2004), 357-383.*

[11] *T. Duyckaerts, H. Jia, C. Kenig and F. Merle. Universality of blow up profile for small blow up solutions to the energy critical wave map equation. Int. Math. Res. Not.,* **2017***(2016), 1-65.*

[12] *M.B. Erdogan, M. Goldberg, W. Schlag. Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in R^3. J. Euro. Math. Soc.,* **10***(2008), 507-531.*

[13] *M. Gell-Mann and M. L'evy. The axial vector current in beta decay. Nuovo Cimento (10),* **16***(1960), 705-726.*

[14] *P. Hartman. On homotopic harmonic maps. Canadian Journal of Mathematics.* **19***(1967), 637-687.*

[15] *S. Helgason. Differential geometry, Lie groups, and symmetric spaces. Academic press, 1979.*

[16] *E. Hebey. Sobolev spaces on Riemannian manifolds, Volume 1635 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996.*

[17] *A. D. Ionescu and S. Klainerman. On the global stability of the wave-map equation in kerr spaces with small angular momentum. Ann. PDE,* **1***(2015), online print https://doi.org/10.1007/s40818-015-0001-1.*

[18] *A. Ionescu, B. Pausader and G. Staffilani. On the global well-posedness of energy-critical Schrödinger equations in curved spaces. Anal. PDE,* **5***(2012), 705-746.*

[19] *H. Jia and C. Kenig. Asymptotic decomposition for semilinear wave and equivariant wave map equations. arXiv preprint, 2015.*

[20] *C. Kenig, A. Lawrie, W. Schlag. Relaxation of wave maps exterior to a ball to harmonic maps for all data. Geom. Funct. Anal.,* **24***(2014), 610-647.*

[21] *C. Kenig, A. Lawrie, B.P. Liu, W. Schlag. Stable soliton resolution for exterior wave maps in all equivariance classes. Adv. Math.,* **285***(2015), 235-300.*

[22] *S. Klainerman and M. Machedon. Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math.,* **46***(1993), 1221-1268.*

[23] *S. Klainerman and M. Machedon. Smoothing estimates for null forms and applications. Duke Math. J.,* **81***(1995), 99-133.*

[24] *S. Klainerman and I. Rodnianski. On the global regularity of wave maps in the critical Sobolev norm. Int. Math. Res. Not.,* **2001***(2001), 655-677.*

[25] *S. Klainerman and S. Selberg. Remark on the optimal regularity for equations of wave maps type. Comm. Partial Differential Equations,* **22***(1997), 901-918.*

[26] *J. Krieger. Global regularity of wave maps from R^{3+1} to surfaces. Comm. Math. Phys.,* **238***(2003), 333-366.*

[27] *J. Krieger. Global regularity of wave maps from R^{2+1} to H^2. Small energy. Comm. Math. Phys.,* **250***(2004), 507-580.*

[28] *J. Krieger and W. Schlag. Concentration Compactness for critical wave maps. EMS Monographs. European Mathematical Society, Zürich, 2012.*

[29] *J. Krieger, W. Schlag and D. Tataru. Renormalization and blow up for charge one equivariant wave critical wave maps. Invent. Math.,* **171***(2008), 543-615.*

[30] *A. Lawrie. The Cauchy problem for wave maps on a curved background. Calculus of Variations and Partial Differential Equations.* **45***(2012): 505-548.*

[31] *A. Lawrie, S.J. Oh and S. Shahshahani. Profile decompositions for wave equations on hyperbolic space with applications. Math. Ann.,* **365***(2016), 707-803.*

[32] *A. Lawrie, S.J. Oh and S. Shahshahani. Gap eigenvalues and asymptotic dynamics of geometric wave equations on hyperbolic space. J. Funct. Anal.,* **271** *(2015), 3111-3161.*

[33] *A. Lawrie, S.J. Oh and S. Shahshahani. The Cauchy problem for wave maps on hyperbolic space in dimensions $d \geq 4$. Int. Math. Res. Not.,* **2018***(2016), 19542051.*

[34] *A. Lawrie, S.J. Oh, and S. Shahshahani. Stability of stationary equivariant wave maps from the hyperbolic plane. Amer. J. Math.,* **139** *(2017), 1085-1147.*

[35] *A. Lawrie, S.J. Oh, and S. Shahshahani. Equivariant wave maps on the hyperbolic plane with large energy. arxiv preprint 2015, to appear in Math. Res. Lett.*

[36] *P. Li, F. Tam. The heat equation and harmonic maps of complete manifolds. Invent. Math.,* **105***(1991), 1-46.*

[37] *Z. Li, L. Zhao. Convergence to harmonic maps for the Landau-Lifshitz flows on two dimensional hyperbolic spaces. arXiv preprint 2016, to appear in Discrete and Continuous Dynamical Systems-A vol. 39, no. 1.*

[38] *Z. Li. Endpoint Strichartz estimates for magnetic wave equations on two dimensional hyperbolic spaces. arxiv preprint 1708.04143v1.*

[39] *N.S. Manton and P. Sutcliffe, Topological Solitons, Cambridge University Press, 2004.*

[40] *A. Nahmod, A. Stefanov, and K. Uhlenbeck. On the well-posedness of the wave map problem in high dimensions. Comm. Anal. Geom.,* **11***(2003), 49-83.*

[41] *P. Raphael, I. Rodnianski. Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems. Publ. Math. Inst. Hautes 'Etudes Sci., 2012, 1-122.*

[42] *C. Rodriguez. Soliton resolution for equivariant wave maps on a wormhole: I-II. arXiv preprint 2017.*

[43] *I. Rodnianski and J. Sterbenz. On the formation of singularities in the critical O(3) σ-model. Ann. Math.,* **172***(2010), 187-242.*

[44] *J. Shatah and M. Struwe. On the Cauchy problem for wave maps. Int. Math. Res. Not.,* **2002***(2002), 555-571.*

[45] *J. Shatah, M. Struwe. Geometric wave equations. American Mathematical Soc., 1998.*

[46] *J. Shatah, A.S. Tahvildar-Zadeh. On the stability of stationary wave maps. Comm. Math. Phys.,* **185***(1997), 231-256.*

[47] *J. Shatah and A. S. Tahvildar-Zadeh. On the Cauchy problem for equivariant wave maps. Comm. Pure Appl. Math.,* **47** *(1994), 719-754.*
[48] *J. Sterbenz and D. Tataru. Energy dispersed large data wave maps in 2 + 1 dimensions. Comm. Math. Phys.,* **298** *(2010), 139-230.*
[49] *J. Sterbenz and D. Tataru. Regularity of wave maps in 2+1 dimensions. Comm. Math. Phys., (2010), 231-264.*
[50] *R.S. Strichartz. Analysis of the Laplacian on the complete Riemannian manifold. J. Funct. Anal.,* **52** *(1983), 48-79.*
[51] *M. Struwe. Equivariant wave maps in two space dimensions. Comm. Pure Appl. Math.,* **56** *(2003), 815-823.*
[52] *T. Tao. Ill-posedness for one-dimensional wave maps at the critical regularity. Amer. J. Math.,* **122** *(2000), 451-463.*
[53] *T. Tao. Global regularity of wave maps, I: small critical Sobolev norm in high dimension. Int. Math. Res. Not.,* **2001** *(2001), 299-328.*
[54] *T. Tao. Global Regularity of Wave Maps II. Small Energy in Two Dimensions. Comm. Math. Phys.,* **224** *(2001), 443-544.*
[55] *T. Tao. Global regularity of wave maps III-VII. arXiv preprint 2008-2009.*
[56] *T. Tao. Global regularity of wave maps IV. arXiv preprint 2008-2009.*
[57] *D. Tataru. On global existence and scattering for the wave maps equation. Amer. J. Math.,* **123** *(2001), 37-77.*
[58] *D. Tataru. Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation. Trans. Amer. Math. Soc.,* **353** *(2001), 795-807.*
[59] *D. Tataru. Rough solutions for the wave maps equation. Amer. J. Math.,* **127** *(2005), 293-377.*

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