Dynamics of PDE, Vol.15, No.4, 235-263, 2018

Optimal rate of convergence in stratified Boussinesq system

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Communicated by Dong Li, received September 27, 2016.

Abstract. We study the vortex patch problem for $2d-$stratified Navier-Stokes system. We aim at extending several results obtained in [**1, 12, 20**] for standard Euler and Navier-Stokes systems. We shall deal with smooth initial patches and establish global strong estimates uniformly with respect to the viscosity in the spirit of [**28, 39**]. This allows to prove the convergence of the viscous solutions towards the inviscid one. In the setting of a Rankine vortex, we show that the rate of convergence for the vortices is optimal in L^p space and is given by $(\mu t)^{\frac{1}{2p}}$. This generalizes the result of [**1**] obtained for L^2 space.

Contents

1. Introduction

This paper is mainly motivated by the analysis of the initial value problem for the stratified Navier-Stokes system. This system of partial differential equations governs the evolution of a viscous incompressible fluid like the atmosphere and the ocean where one should take into account the friction forces and the stratification under the Boussinesq approximation, see [**35**]. The state of the fluid is described

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2000 *Mathematics Subject Classification.* 35B65, 35Q35, 76D05.

Key words and phrases. $2d$-Stratified Boussinesq system, regular vortex patches, rate of convergence, global well-posedness, optimal rate.

The authors wish to thank Dr. Taoufik Hmidi from the University of Rennes 1 for the fruitful discussions on the subject during his visit to the University of Batna 2 in May 2016. Particular thanks to editor-in-chief professor Y. Charles Li, professor Dong Li and the referees for their careful reading and constructive advices.

by a triplet (v_μ, p_μ, ρ_μ) where $v_\mu(t,x)$ denotes the velocity field which is assumed to be incompressible and the thermodynamical variables $p_\mu(t,x)$ and $\rho_\mu(t,x)$ which are two scalar functions representing respectively the pressure and the density. The equations being solved take the form

$$(\mathrm{B}_{\mu,\kappa})\qquad \begin{cases} \partial_t v_\mu + v_\mu\cdot\nabla v_\mu - \mu\Delta v_\mu + \nabla p_\mu = \rho_\mu \vec{e}_2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\rho_\mu + v_\mu\cdot\nabla\rho_\mu - \kappa\Delta\rho_\mu = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div} v_\mu = 0,\\ (v_\mu,\rho_\mu)_{|t=0} = (v_\mu^0, \rho_\mu^0). \end{cases}$$

The two coefficients μ, κ stand respectively for the kinematic viscosity and molecular diffusivity and $\vec{e}_2 = (0,1)$. For a better understanding of the system $(\mathrm{B}_{\mu,\kappa})$ it is more convenient to write it using the vorticity-density formulation. Thus the vorticity $\omega \triangleq \partial_1 v^2 - \partial_2 v^1$ and the density satisfy the equivalent system,

$$(\mathrm{VD}_{\mu,\kappa})\qquad \begin{cases} \partial_t\omega_\mu + v_\mu\cdot\nabla\omega_\mu - \mu\Delta\omega_\mu = \partial_1\rho_\mu & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\rho_\mu + v_\mu\cdot\nabla\rho_\mu - \kappa\Delta\rho_\mu = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ v_\mu = \nabla^\perp\Delta^{-1}\omega_\mu,\\ (\rho_\mu,\omega_\mu)_{|t=0} = (\rho_\mu^0, \omega_\mu^0). \end{cases}$$

It is clear that $(\mathrm{B}_{\mu,\kappa})$ coincides with the classical incompressible Navier-Stokes system when the initial density ρ_μ^0 is identically constant. For a general review on the mathematical theory of the Navier-Stokes system we refer for instance to [**3, 32**]. We notice that the system $(\mathrm{VD}_{\mu,\kappa})$ is the subject of intensive research activities especially in the last decades. A lot of results have been obtained and we shall restrict the discussion to some of them. When the coefficients μ and κ are strictly positive, it was proved in [**7, 19**] that the system $(\mathrm{B}_{\mu,\kappa})$ admits a unique global solution for arbitrarily large data. For $\mu > 0, \kappa = 0$ the global well-posedness problem was solved independently by Chae [**8**] and Hou and Li [**29**] for smooth initial data in Sobolev spaces $H^s, s > 2$. Those results were improved by Abidi and Hmidi in [**2**] for $(v^0,\rho^0) \in B^{-1}_{\infty,1}\cap L^2 \times B^0_{2,1}$. Later, Danchin and Paicu investigated in [**15**] the global well-posedness for any initial data (v^0,ρ^0) in $L^2\times L^2$. The opposite case $\mu = 0$ and $\kappa > 0$ is also well-explored. Actually, Chae proved in [**8**] the global well-posedness for $(v^0,\rho^0)\in H^s\times H^s$ for $s > 2$ which was later improved by Hmidi and Keraani in [**22**] for critical Besov spaces, that is, $(v^0,\rho^0)\in B^{\frac{2}{p}+1}_{p,1}\times B^{-1+\frac{2}{p}}_{p,1}\cap L^r,\ r > 2$. The global existence in the framework of Yudovich solutions was accomplished in [**14**] by Danchin and Paicu for $(v^0,\rho^0)\in L^2\times L^2\cap B^{-1}_{\infty,1}$ and $\omega^0\in L^r\cap L^\infty$ with $r\geq 2$. For other connected topics we refer the reader to [**22, 24, 25, 27, 30, 34, 38**].

The main focus of the current paper is twofold. In the first part, we study the persistence regularity of the vortex patches for $(\mathrm{B}_{\mu,\kappa})$ for $\kappa = 1$, denoted simply by (B_μ). In the second part we shall deal with the strong convergence towards the limit system when the viscosity μ goes to zero. The limit system is nothing but the stratified Euler equations,

$$(\mathrm{B}_0)\qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \rho\vec{e}_2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\rho + v\cdot\nabla\rho - \Delta\rho = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div} v = 0,\\ (v,\rho)_{|t=0} = (v^0,\rho^0). \end{cases}$$

Before giving more details about our main contribution we shall review some aspects of the vortex patch problem for the viscous/inviscid incompressible fluid. Recall first the classical Navier-Stokes equations,

$$(\mathrm{NS}_\mu)\qquad \begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div} v = 0,\\ v_{|t=0} = v^0. \end{cases}$$

Notice that the incompressible Euler system (E), denoted sometimes by (NS_0), is given by

$$(\mathrm{E})\qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div} v = 0,\\ v_{|t=0} = v^0. \end{cases}$$

We point out that the global existence of classical solutions for Euler system is based on the structure of the vorticity which is transported by the flow, that is,

$$\partial_t\omega + v\cdot\nabla\omega = 0.$$

This provides an infinite family of conservation laws and in particular we get for all $p\in[1,\infty]$

$$\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p}. \tag{1.1}$$

We mention that the conservation laws (1.1) served as a suitable framework for Yudovich [**37**] to relax the classical hyperbolic theory and show that (NS_μ) and (E) are globally well-posed whenever $\omega^0\in L^1\cap L^\infty$. In this pattern, the velocity is no longer in the Lipschitz class but belongs to the log −Lipschitz space, denoted by LL[1]. It is known that with this regularity the associated flow Ψ is continuous with respect to $(t,x)-$variables and the vorticity can be recovered from its initial value according to the formula,

$$\omega(t,\Psi(t,x)) = \omega^0(x). \tag{1.2}$$

In particular, when the initial vorticity $\omega^0 = \mathbf{1}_{\Omega_0}$ is a vortex patch with Ω_0 being a regular bounded domain, then the advected vorticity remains a vortex patch relative to a domain $\Omega_t \triangleq \Psi(t,\Omega_0)$ which is homeomorphic to Ω_0. It is important to emphasize that the regularity persistence of the boundary does not follow from the general theory of Yudovich because the flow is not in general better than $C^{e^{-\alpha t}}$ where α depends on ω^0. This problem was solved by Chemin who proved in [**10**] that when the initial boundary is $C^{1+\varepsilon}$ then the boundary of the patch keeps this regularity through the time. Broadly speaking, Chemin's strategy is entirely based on the control of Lipschitz norm of the velocity by means of logarithmic estimate of $\|\omega\|_{C^\varepsilon(X)}$ with $C^\varepsilon(X)$ is an anisotropic Hölder space associated to an adequate family of vector fields that capture the conormal regularity of the velocity (see section 3.1).

The study for the viscous case was initiated by Danchin in [**12**] who proved that if $\omega^0 = \mathbf{1}_{\Omega_0}$, such that the domain Ω_0 is $C^{1+\varepsilon}$ then the velocity v_μ is Lipschitz uniformly with respect to the viscosity μ. He also showed that the transported vorticity by the viscous flow Ψ_μ remains in the class $C^{1+\varepsilon'}, \forall\varepsilon' < \varepsilon$. Note that contrary to the Hölderian regularity, there is no loss of regularity in the Besov

[1]The space LL is the set of bounded functions u such that

$$\|u\|_{LL} \triangleq \sup_{0<|x-y|<1} \frac{|u(x)-u(y)|}{|x-y|\log\frac{e}{|x-y|}}.$$

spaces $B^{\varepsilon}_{p,\infty}$, $\forall\, p < \infty$. For the borderline case $p = \infty$ Hmidi showed in [**20**] that this loss of regularity is artificial and his proof is mainly related to some smoothing effects for the transport-diffusion equation using Lagrangian coordinates. There is a large literature dealing with this subject and some connected topics and for more details we refer the reader to the papers [**5, 16, 17, 18, 20**] and the references therein.

It could be interesting to extend some of the foregoing results to the stratified Navier-Stokes system (B_μ). The investigation of this system with initial vorticity of patch type has been started recently in [**28**] for $\mu = 0$. It was proved that if the boundary of the initial patch is smooth enough then the velocity is Lipschitz for any positive time and the transported domain Ω_t preserves its initial regularity. In addition, the vorticity can be decomposed into a singular part which is a vortex patch term and a regular part, which is deeply related to the smoothing effect for density, i.e. $\omega(t) = \mathbf{1}_{\Omega_t} + \widetilde{\rho}(t)$. Later, the second author studied in [**39**] the same system but the usual dissipation operator $-\Delta$ is replaced by the critical fractional Laplacian $(-\Delta)^{\frac{1}{2}}$. He obtained sharper results compared to the incompressible Euler equations [**10, 28**] and describes the asymptotic behavior of the solutions for large time.

We are now ready to state the first main result, dealing with the global well-posedness for the system (B_μ) under a vortex patch initial data. More precisely, we have:

Theorem 1.1. *Let Ω_0 be a simply connected bounded domain such that its boundary $\partial\Omega_0$ is $C^{1+\varepsilon}$ with $0 < \varepsilon < 1$. Let $\omega^0_\mu = \mathbf{1}_{\Omega_0}$ and $\rho^0_\mu \in L^1 \cap L^\infty$ then the following assertions hold.*

(i) *The system (B_μ) admits a unique global solution (v_μ, ρ_μ) such that*

$$(v_\mu, \rho_\mu) \in L^\infty_{loc}(\mathbb{R}_+; \mathrm{Lip}) \times L^\infty_{loc}(\mathbb{R}_+; L^1 \cap L^\infty).$$

More precisely, there exists $C_0 \triangleq C(\varepsilon, \Omega_0) > 0$ such that, for all $\mu \in]0, 1[$ and for all $t \in \mathbb{R}_+$ we have

$$\|\nabla v_\mu(t)\|_{L^\infty} \le C_0 e^{C_0 t \log^2(1+t)}. \tag{1.3}$$

(ii) *The boundary of the transported domain $\Omega_\mu(t) \triangleq \Psi_\mu(t, \Omega_0)$ is $C^{1+\varepsilon}$ for every $t \ge 0$ uniformly on μ, where Ψ_μ denotes the viscous flow associated to v_μ.*

Let us give a bunch of comments about Theorem 1.1 in the following few remarks.

Remark 1.2. Compared to the incompressible Navier-Stokes system, we see that a Lipschitz norm of the velocity has a logarithmic growth for large time. This is due to the logarithmic factor in the growth of the vorticity, namely we have:

$$\|\omega_\mu(t)\|_{L^\infty} \le C_0 \log^2(1+t).$$

Remark 1.3. When the viscosity μ is identically zero, we obtain the same result as in [**28**] for the stratified Euler system (B_0), that is to say:

$$\|\nabla v(t)\|_{L^\infty} \le C_0 e^{C_0 t \log^2(1+t)}. \tag{1.4}$$

Now we shall briefly outline the ideas of the proof which is done in the spirit of the pioneering work of Chemin [**10**]. In order to get a bound for the quantity

$\|\nabla v_\mu(t)\|_{L^\infty}$ we first show that the co-normal regularity of the vorticity $\partial_X \omega_\mu$ is controlled in $C^{\varepsilon-1}$, with $0<\varepsilon<1$. We then take advantage of the logarithmic estimate to derive the Lipschitz norm of the velocity, with X is a family of selected vector fields which satisfies the transport equation,

$$\partial_t X + v_\mu \cdot \nabla X = X \cdot \nabla v_\mu.$$

As it was pointed in [**12, 20**] the situation in the viscous case is more delicate than the inviscid one due to the Laplacian operator which does not commute with the family X. Actually, the evolution of the directional derivative $\partial_X \omega_\mu$ is governed by an inhomogeneous transport-diffusion equation,

$$(\partial_t + v_\mu \cdot \nabla - \mu\Delta)\partial_X \omega_\mu = -\mu[\Delta, X]\omega_\mu + \partial_X \partial_1 \rho_\mu, \tag{1.5}$$

where $[\Delta, X]$ denotes the commutator between Δ and X. Thus the difficulties reduce to understanding the terms $[\Delta, X]\omega_\mu$ and $\partial_X\partial_1\rho_\mu$ which apparently need more regularity to be well-defined than what is initially prescribed. To circumvent the problem for the first term we shall use the formalism developed in [**12, 20**] for $2d-$incompressible Navier-Stokes system. However, to deal with the second term we find more convenient to diagonalize the system written in the vorticity-density formulation and introduce the coupled function $\Gamma_\mu \triangleq (1-\mu)\omega_\mu - \partial_1\Delta^{-1}\rho_\mu$ in the spirit of [**26**]. This function satisfies the following transport-diffusion equation,

$$\partial_t \Gamma_\mu + v_\mu \cdot \nabla \Gamma_\mu - \mu\Delta\Gamma_\mu = [\partial_1\Delta^{-1}, v_\mu \cdot \nabla]\rho_\mu \triangleq H_\mu.$$

By applying the directional derivative ∂_X to the last equation we find

$$(\partial_t + v_\mu \cdot \nabla - \mu\Delta)\partial_X \Gamma_\mu = -\mu[\Delta, X]\Gamma_\mu + \partial_X H_\mu.$$

At a formal level, and this will be justified rigorously as we shall see in the proofs, we see that H_μ is of order zero with respect to ρ_μ according to the smoothing effect of the singular operator $\partial_1\Delta^{-1}$. Thus instead of manipulating $\partial_X\partial_1\rho_\mu$ in the equation (1.5) which consumes two derivatives we need just to understand $\partial_X H_\mu$ which exhibits a good behavior on ρ_μ as it was revealed in [**28**].

The second part of this paper is devoted to the inviscid limit problem which is in fact well-explored for the classical Navier-Stokes system (NS_μ). We mention that for smooth initial data the convergence towards Euler equations holds true and the rate of convergence in the energy space L^2 is bounded by μt, see [**4**] for initial data $v_0 \in H^s$ with $s>4$. In [**9**], Chemin proved a strong convergence in L^2 for Yudovich's initial data and obtained that the rate is controlled by $(\mu t)^{\frac{1}{2}e^{-Ct}}$, which degenerating in time. To obtain a better result, Constantin and Wu [**11**] had to work under vortex patch structure and they obtained $(\mu t)^{\frac{1}{2}}$. Afterwards, Abidi and Danchin [**1**] improved this result and showed that the rate of convergence is exactly $(\mu t)^{\frac{3}{4}}$ which is proved to be optimal for the Rankine vortex.

Our second main result reads as follows.

Theorem 1.4. *Let (v_μ, ρ_μ), (v, ρ), (ω_μ, ρ_μ) and (ω, ρ) be the solutions of (B_μ), (B_0), (VD_μ) and (VD_0) respectively with the same initial data such that $\omega_\mu^0 = \omega^0 = \mathbf{1}_{\Omega_0}$, where Ω_0 is a $C^{1+\varepsilon}$ simply connected bounded domain. Then for all $t \geq 0, \mu \in]0,1[$ and $p \in [2, +\infty[$ we have:*

(i) $\|v_\mu(t) - v(t)\|_{L^p} + \|\rho_\mu(t) - \rho(t)\|_{L^p} \leq C_0 e^{e^{C_0 t \log^2(2+t)}} (\mu t)^{\frac{1}{2}+\frac{1}{2p}}$.

(ii) $\|\omega_\mu(t) - \omega(t)\|_{L^p} \leq C_0 e^{e^{C_0 t \log^2(1+t)}} (\mu t)^{\frac{1}{2p}}$.

Remark 1.5. When ρ_μ^0 and ρ^0 are constants and $p = 2$ we get the result of Abidi and Danchin [**1**].

The proof of Theorem 1.4 will be done using the approach of [**1**] by combining some classical ingredients like L^p-estimates, real interpolation results and some smoothing effects for the density and the vorticity.

The last result is dedicated to prove that $(\mu t)^{\frac{1}{2p}}$ is optimal for vortices in the case of Rankine initial data.

Theorem 1.6. *We assume that ρ_μ^0 and ρ^0 being constants and $\omega_\mu^0 = \omega^0 = \mathbf{1}_{\mathbb{D}}$ with $\mathbb{D}$ the unit disc. Then there exist two positive constants C_1 and C_2 independent on μ and t, such that for $\mu t \leq 1$, and $p \in [2, +\infty[$ we have:*

$$C_1(\mu t)^{\frac{1}{2p}} \leq \|\omega_\mu(t) - \omega(t)\|_{L^p} \leq C_2(\mu t)^{\frac{1}{2p}},$$

with C_1 and C_2 depending on p.

Note that the approach that we shall propose here is different from [**1**] which is specific for $p = 2$. The proof of Abidi and Danchin uses the explicit form of Fourier transform of the Rankine vortex given through Bessel function combined with its asymptotic behavior. Nevertheless, these tools are useless for $p \neq 2$ and the alternative is to make the computations in the physical variable using the explicit structure of the heat kernel.

For the reader's convenience, we provide a brief outline of this article. Section 2, starts with few important results about the Littlewood-Paley decomposition, para-differential calculus and some functional spaces. Moreover, we state some useful technical lemmas, in particular two smoothing effects estimates for transport-diffusion equations governing respectively the density and the vorticity evolution. Section 3, mainly treats the general version of Theorem 1.1. Section 4 is divided into two parts. The first one is dedicated to the upper bound rate of convergence. The second part deals with the optimality of the rate of convergence between the vortices. We end this paper with an appendix where we give the proof of some technical propositions.

2. Tools

Before proceeding, we specify some of the notations we will constantly use during this work. We denote by C a positive constant which may be different in each occurrence but it does not depend on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$ with C independent of X and Y. The notation C_0 means a constant depend on the involved norms of the initial data.

2.1. Littlewood-Paley theory. Our results mostly rely on Fourier analysis methods based on a nonhomogeneous dyadic partition of unity with respect to the Fourier variable. The so-called Littlewood-Paley decomposition enjoying particularly "nice" properties. These properties are the basis for introducing the important scales of Besov and Hölder spaces and for their study.

Let $\chi \in \mathscr{D}(\mathbb{R}^2)$ be a reference cut-off function, monotonically decaying along rays and so that

$$\begin{cases} \chi \equiv 1 & \text{if } \|\xi\| \leq \frac{1}{2} \\ 0 \leq \chi \leq 1 & \text{if } \frac{1}{2} \leq \|\xi\| \leq 1 \\ \chi \equiv 0 & \text{if } \|\xi\| \geq 1. \end{cases}$$

Define $\varphi(\xi) \triangleq \chi(\frac{\xi}{2}) - \chi(\xi)$. We obviously check that $\varphi \geq 0$ and

$$\operatorname{supp} \varphi \subset \mathcal{C} \triangleq \{\xi \in \mathbb{R}^2 : \frac{1}{2} \leq \|\xi\| \leq 1\}.$$

Then we have the following elementary properties, see for example [**3, 10**].

PROPOSITION 2.1. *Let χ and φ be as above. Then the following assertions are hold.*

(1) *Decompositon of the unity:*

$$\forall \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1.$$

(2) *Almost orthogonality in the sense of ℓ^2:*

$$\forall \xi \in \mathbb{R}^2, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{q \geq 0} \varphi^2(2^{-q}\xi) \leq 1.$$

The Littlewood-Paley or cut-off operators are defined as follows.

DEFINITION 2.2. For every $u \in \mathcal{S}'(\mathbb{R}^2)$, setting

$$\Delta_{-1} u \triangleq \chi(\mathrm{D})u, \quad \Delta_q u \triangleq \varphi(2^{-q}\mathrm{D})u \quad \text{if } q \in \mathbb{N}, \quad S_q u \triangleq \sum_{j \leq q-1} \Delta_j u \quad \text{for } q \geq 0.$$

Some properties of Δ_q and S_q are listed in the following proposition.

PROPOSITION 2.3. *Let $u, v \in \mathcal{S}'(\mathbb{R}^2)$ we have*

(i) $|p - q| \geq 2 \Longrightarrow \Delta_p \Delta_q u \equiv 0$,
(ii) $|p - q| \geq 4 \Longrightarrow \Delta_q(S_{p-1} u \Delta_p v) \equiv 0$,
(iii) $\Delta_q, S_q : L^p \to L^p$ *uniformly with respect to q and p.*
(iv)

$$u = \sum_{q \geq -1} \Delta_q u.$$

Likewise the homogeneous operators $\dot{\Delta}_q$ and $\dot{S}_q$ are defined by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u, \quad \dot{S}_q = \sum_{j \leq q-1} \dot{\Delta}_j v. \tag{2.1}$$

Now, we will recall the definition of the Besov spaces.

DEFINITION 2.4. For $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ (resp. the homogeneous Besov space $\dot{B}^s_{p,r}$) is the set of all tempered distributions $u \in \mathcal{S}'$ (resp. $u \in \mathcal{S}'_{|\mathbf{P}}$) such that

$$\|u\|_{B^s_{p,r}} \triangleq \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r} < \infty.$$

$$\Big(\text{resp. } \|u\|_{\dot{B}^s_{p,r}} \triangleq \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < \infty\Big).$$

We have denoted by $\mathbf{P}$ the set of polynomials.

REMARK 2.5. We notice that:

(1) If $s \in \mathbb{R}_+ \backslash \mathbb{N}$, the Hölder space noted by C^s coincides with $B^s_{\infty,\infty}$.

(2) $\big(C^s, \|\cdot\|_{C^s}\big)$ is a Banach space coincides with the usual Hölder space C^s with equivalent norms,

$$\|u\|_{C^s} \lesssim \|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^s} \lesssim \|u\|_{C^s}. \tag{2.2}$$

(3) If $s \in \mathbb{N}$, the obtained space is so-called Hölder-Zygmund space and still denoted by $B^s_{\infty,\infty}$.

2.2. Paradifferential calculus. The well-known *Bony's* decomposition [**6**] enables us to split formally the product of two tempered distributions u and v into three pieces. In what follows, we shall adopt the following definition for paraproduct and remainder:

Definition 2.6. For a given $u, v \in \mathcal{S}'$ we have

$$uv = T_u v + T_v u + \mathscr{R}(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \mathscr{R}(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v \quad \text{and} \quad \widetilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

The mixed space-time spaces are stated as follows.

Definition 2.7. Let $T > 0$ and $(s, \beta, p, r) \in \mathbb{R}\times[1,\infty]^3$. We define the spaces $L^\beta_T B^s_{p,r}$ and $\widetilde{L}^\beta_T B^s_{p,r}$ respectively by:

$$L^\beta_T B^s_{p,r} \triangleq \Big\{ u : [0,T] \to \mathcal{S}'; \|u\|_{L^\beta_T B^s_{p,r}} = \big\| \big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{\ell^r} \big\|_{L^\beta_T} < \infty \Big\},$$

$$\widetilde{L}^\beta_T B^s_{p,r} \triangleq \Big\{ u : [0,T] \to \mathcal{S}'; \|u\|_{\widetilde{L}^\beta_T B^s_{p,r}} = \big(2^{qs}\|\Delta_q u\|_{L^\beta_T L^p}\big)_{\ell^r} < \infty \Big\}.$$

The relationship between these spaces is given by the following embeddings. Let $\varepsilon > 0$, then

$$\begin{cases} L^\beta_T B^s_{p,r} \hookrightarrow \widetilde{L}^\beta_T B^s_{p,r} \hookrightarrow L^\beta_T B^{s-\varepsilon}_{p,r} & \text{if } r \geq \beta, \\ L^\beta_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\beta_T B^s_{p,r} \hookrightarrow L^\beta_T B^s_{p,r} & \text{if } \beta \geq r. \end{cases} \tag{2.3}$$

Accordingly, we have the following interpolation result.

Corollary 2.8. *Let $T > 0$, $s_1 < s < s_2$ and $\zeta \in (0,1)$ such that $s = \zeta s_1 + (1-\zeta)s_2$. Then we have*

$$\|u\|_{\widetilde{L}^a_T B^s_{p,r}} \leq C\|u\|^{\zeta}_{\widetilde{L}^a_T B^{s_1}_{p,\infty}} \|u\|^{1-\zeta}_{\widetilde{L}^a_T B^{s_2}_{p,\infty}}. \tag{2.4}$$

The following *Bernstein* inequalities describe a bound on the derivatives of a function in the L^b-norm in terms of the value of the function in the L^a-norm, under the assumption that the Fourier transform of the function is compactly supported. For more details we refer [**3, 10**].

Lemma 2.9. *There exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$, for every function u and every $q \in \mathbb{N}\cup\{-1\}$, we have*

(i)

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|S_q u\|_{L^a},$$

(ii)

$$C^{-k}2^{qk}\|\Delta_q u\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \le C^k 2^{qk}\|\Delta_q u\|_{L^a}.$$

A noteworthy consequence of Bernstein inequality (i) is the following embedding:

$$B^s_{p,r} \hookrightarrow B^{\widetilde{s}}_{\widetilde{p},\widetilde{r}} \quad \text{whenever } \widetilde{p} \ge p,$$

with

$$\widetilde{s} < s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) \quad \text{or} \quad \widetilde{s} = s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) \quad \text{and} \quad \widetilde{r} \le r.$$

2.3. Useful results. This paragraph is reserved to some useful properties freely used throughout this article. The most results concerning the system (VD_μ) rely strongly on a priori estimates in Besov spaces for the transport-diffusion equation:

$$(\mathrm{TD}_\mu) \qquad \begin{cases} \partial_t a + v \cdot \nabla a - \mu \Delta a = f \\ a_{|t=0} = a^0. \end{cases}$$

We start by the persistence of Besov regularity for (TD_μ), whose proof may be found for example in [**3**].

Proposition 2.10. *Let $(s,r,p) \in]-1,1[\times[1,\infty]^2$ and v be a smooth divergence free vector-field. We assume that $a^0 \in B^s_{p,r}$ and $f \in L^1_{loc}(\mathbb{R}_+; B^s_{p,r})$. Then for every smooth solution a of (TD_μ) and $t \ge 0$ we have*

$$\|a(t)\|_{B^s_{p,r}} \le Ce^{CV(t)}\bigg(\|a^0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}} d\tau\bigg),$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$

and C a constant which depends only on s and not on the viscosity. For the limit case

$$s = -1, r = \infty \text{ and } p \in [1,\infty] \quad \text{or} \quad s = 1, r = 1 \text{ and } p \in [1,\infty]$$

the above estimate remains true despite we change $V(t)$ by $Z(t) \overset{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}$. In addition if $a = \mathrm{curl}\ v$, then the above estimate holds true for all $s \in [1,+\infty[$.

Next, we state the maximal smoothing effect result for (TD_μ) in mixed time-space spaces, whose proof was developped in [**21**].

Proposition 2.11. *Let $s \in]-1,1[$, $(p_1,p_2,r) \in [1,+\infty]^3$ and v be a divergence free vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{Lip})$. Then for every smooth solution a of (TD_μ) we have*

$$(2.5) \quad \mu^{\frac{1}{r}}\|a\|_{\widetilde{L}^r_t B^{s+\frac{2}{r}}_{p_1,p_2}} \le Ce^{CV(t)}(1+\mu t)^{\frac{1}{r}}\Big(\|a^0\|_{B^s_{p_1,p_2}} + \|f\|_{L^1_t B^s_{p_1,p_2}}\Big), \quad \forall t \in \mathbb{R}_+.$$

The asymptotic behavior in L^p−norm with $p \in [2,\infty]$ of every (ω_μ, ρ_μ) solution of (VD_μ) is given by the following proposition. To be precise we have:

Proposition 2.12. *Let (ω_μ, ρ_μ) be a smooth solution of (VD_μ) such that $\rho_0 \in L^1 \cap L^p$ and $\omega_0 \in L^2 \cap L^p$ with $p \in [2,\infty]$. Then for $t \ge 0$,*

$$\|\omega_\mu(t)\|_{L^p} + \|\nabla \rho_\mu\|_{L^1_t L^p} \le C_0 \log^{2-\frac{2}{p}}(1+t).$$

REMARK 2.13. This property has been recently accomplished by [**28**] for Stratified Euler equations (B_μ), with $\mu = 0$. We point out that the proof of such estimate remains available in our case with minor modifications due to the laplacien term, which has a good sign.

We end this paragraph by the Calderón-Zygmund estimate which constitutes a deep statement of harmonic analysis.

PROPOSITION 2.14. *Let $p \in]1, \infty[$ and v be a divergence-free vector field which its vorticity $\omega \in L^p$. Then $\nabla v \in L^p$ and*

$$\|\nabla v\|_{L^p} \le c \frac{p^2}{p-1} \|\omega\|_{L^p}, \tag{2.6}$$

with c being a universal constant.

3. Smooth vortex patch problem

In this section we will give a detailed proof for the first main result stated in Theorem 1.1. We will inspire the general ideas from Chemin's result, we then follow the argument performed more recently by [**28, 39**] for Stratified Euler system. For this aim, we will state the general framework study of the vortex patch problem.

3.1. Vortex patch tool box. Before entering into details of the proof of the Theorem 1.1, we will state few important ingredients concerning the study of vortex patch problem. We will start with the concept of an admissible family of vector fields and some related properties, from which we will derive the notion of anisotropic Hölder space. At the end, we state the so-called *stationnary logarithmic estimate* which is the key step to prove that the velocity is a Lipschitz function.

DEFINITION 3.1. Let $\varepsilon \in]0, 1[$. A family of vector fields $X = (X_\lambda)_{\lambda\in\Lambda}$ is said to be admissible if and only if the following assertions hold.

- Regularity:

$$\forall \lambda \in \Lambda \quad X_\lambda, \mathrm{div} X_\lambda \in C^\epsilon.$$

- Non-degeneray:

$$I(X) \triangleq \inf_{x\in\mathbb{R}^d} \sup_{\lambda\in\Lambda} \big|X_\lambda(x)\big| > 0. \tag{3.1}$$

Setting

$$\widetilde{\|} X_\lambda\|_{C^\varepsilon} \triangleq \|X_\lambda\|_{C^\varepsilon} + \|\mathrm{div} X_\lambda\|_{C^\varepsilon}. \tag{3.2}$$

DEFINITION 3.2. Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be an admissible family. The action of each factor X_λ on $u \in L^\infty$ is defined as the directional derivative of u along X_λ by the formula,

$$\partial_{X_\lambda} u = \mathrm{div}(u X_\lambda) - u \mathrm{div} X_\lambda.$$

The anisotropic Hölder spaces, denoted by $C^\varepsilon(X)$ are defined below.

DEFINITION 3.3. Let $\varepsilon \in]0, 1[$ and X be an admissible family of vector fields. We say that $u \in C^\varepsilon(X)$ if and only if:

- $u \in L^\infty$ and satisfies

$$\forall\, \lambda \in \Lambda,\ \partial_{X_\lambda} u \in C^{\varepsilon-1}, \quad \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} < +\infty.$$

- $C^\varepsilon(X)$ is a normed space with

$$\|u\|_{C^\varepsilon(X)} \triangleq \frac{1}{I(X)}\Big(\|u\|_{L^\infty} \sup_{\lambda\in\Lambda} \widetilde{\|}X_\lambda\|_{C^\varepsilon} + \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}}\Big).$$

Now, let us take an initial family of vector-field $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ and define its time evolution $X_t = \big(X_{t,\lambda})_{\lambda\in\Lambda}$ by

$$X_{t,\lambda}(x) \triangleq (X_{0,\lambda}\Psi)(t, \Psi^{-1}(t,x)), \tag{3.3}$$

that is X_t is the vector-field X_0 transported by the flow Ψ associated to v. From this definition the evolution family X_t satisfies the following transport equation.

Proposition 3.4. *Let v be a Lipschitzian vector-field, Ψ its flow and $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ is the family defined by* (3.3). *Then the following equation holds true.*

$$\begin{cases} (\partial_t + v\cdot\nabla)X_{t,\lambda} = \partial_{X_{t,\lambda}} v & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ X_{t,\lambda|t=0} = X_{0,\lambda}. \end{cases} \tag{3.4}$$

To prove the Theorem 1.1, we state the following stationnary logarithmic estimate initially introduced by Chemin [**10**]. More precisely,

Theorem 3.5. *Let $\varepsilon \in]0,1[$ and $X = (X_\lambda)_{\lambda\in\Lambda}$ be a family of vector fields as in Definition 3.1. Let v be a divergence-free vector field such that its vorticity ω belongs to $L^2 \cap C^\varepsilon(X)$. Then there exists a constant C depending only on ε, such that*

$$\|\nabla v\|_{L^\infty} \le C\Bigg(\|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log\Bigg(e + \frac{\|\omega\|_{C^\varepsilon(X)}}{\|\omega\|_{L^\infty}}\Bigg)\Bigg). \tag{3.5}$$

We shall now make precise to the boundary regularity and the tangent space used in the proof of Theorem 1.1.

Definition 3.6. Let $\varepsilon > 0$.

(1) A closed curve Σ is said to be $C^{1+\varepsilon}-$regular if there exists $f \in C^{1+\varepsilon}(\mathbb{R}^2)$ such that Σ is locally a zero set of f, i.e., there exists a neighborhood V of Σ such that

$$\Sigma = f^{-1}\{0\} \cap V, \quad \nabla f(x) \neq 0 \quad \forall x \in V. \tag{3.6}$$

(2) A vector field X with $C^\varepsilon-$regularity is said to be tangent to Σ if $X\cdot\nabla f_{|\Sigma} = 0$. The set of such vector fields will be denoted by $\mathcal{T}^\varepsilon_\Sigma$.

Given a compact curve Σ of the class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$. The co-normal space C^ε_Σ associated to Σ is defined by

$$C^\varepsilon_\Sigma \triangleq \big\{u \in L^\infty(\mathbb{R}^2); \forall X \in \mathcal{T}^\varepsilon_\Sigma,\ (\mathrm{div}X = 0) \Rightarrow \mathrm{div}(Xu) \in C^{\varepsilon-1}\big\}.$$

The following Danchin's result stated in [**13**], showing that C^ε_Σ contains the characteristic function of a bounded open domain surruonded by the curve Σ. More generally we have:

Proposition 3.7. *Let Ω_0 be a $C^{1+\varepsilon}-$bounded domain, with $0 < \varepsilon < 1$ and $f \in C^\varepsilon(\mathbb{R}^2)$, then we have:*

$$f\mathbf{1}_{\Omega_0} \in C^\varepsilon_\Sigma.$$

According to the previous proposition, we strive to give a general version of the Theorem 1.1 which allows to deal with more general structures than the vortex patches. Thus we have:

THEOREM 3.8. *Let $0<\varepsilon<1, X_0$ be a family of admissible vector fields and v^0_μ be a free-divergence vector field such that $\omega^0_\mu\in L^2\cap C^\varepsilon(X_0)$. Let $\rho^0_\mu\in L^1\cap L^\infty$, then for $\mu\in]0,1[$ the system (B_μ) admits a unique global solution $(v_\mu,\rho_\mu)\in L^\infty_{loc}(\mathbb{R}_+;Lip)\times L^\infty(\mathbb{R}_+;L^1\cap L^\infty)$. More precisely:*

$$\|\nabla v_\mu(t)\|_{L^\infty}\le C_0 e^{C_0 t\log^2(1+t)}. \tag{3.7}$$

Furthermore, we have:

$$\|\omega_\mu(t)\|_{C^\varepsilon(X_t)}+\|\partial_{X_t}\psi_\mu(t)\|_{C^\varepsilon}\le C_0 e^{\exp\{C_0 t\log^2(2+t)\}}.$$

PROOF. The most difficult point in the proof is to estimate suitably the quantity ω_μ in $C^\varepsilon(X_t)$ norm. For this aim, we shall use the following coupled function Γ_μ defined by $\Gamma_\mu=(1-\mu)\omega_\mu-\mathcal{L}\rho_\mu$, with $\mathcal{L}=\partial_1\Delta^{-1}$. After few computations, we obtain that Γ_μ evolves the following inhomogenous transport-diffusion equation:

$$(\partial_t+v_\mu\cdot\nabla-\mu\Delta)\Gamma_\mu=[\mathcal{L},v_\mu\cdot\nabla]\rho_\mu. \tag{3.8}$$

To simplify the presentation in what follows, we temporarily drop the viscosity parameter μ.
By virtue of (3.4) of the Proposition 3.4, one can check that the quantity $\partial_{X_{t,\lambda}}\Gamma$ satisfies the equation,

$$(\partial_t+v\cdot\nabla-\mu\Delta)\partial_{X_{t,\lambda}}\Gamma=X_{t,\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}-\mu[\Delta,X_{t,\lambda}]\Gamma. \tag{3.9}$$

According to [**12, 20**], the commutator $[\Delta,X_{t,\lambda}]$ can be decomposed as the sum of two terms in the following way:

$$\mu[\Delta,X_{t,\lambda}]\Gamma=F+\mu G,$$

with

$$F\triangleq 2\mu T_{\nabla X^i_{t,\lambda}}\partial_i\nabla\Gamma+2\mu T_{\partial_i\nabla\Gamma}\nabla X^i_{t,\lambda}+\mu T_{\Delta X^i_{t,\lambda}}\partial_i\Gamma+\mu T_{\partial_i\Gamma}\Delta X^i_{t,\lambda}.$$

and

$$G\triangleq 2\mathscr{R}(\nabla X^i_{t,\lambda},\partial_i\nabla\Gamma)+\mathscr{R}(\Delta X^i_{t,\lambda},\partial_i\Gamma).$$

Here, we have used Enstein's convention for the summation over the repeated indices. Thus the equation (3.9) takes the following form,

$$(\partial_t+v\cdot\nabla-\mu\Delta)\partial_{X_{t,\lambda}}\Gamma=X_{t,\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}-(F+\mu G),$$

Applying Theorem 3.38 page 162 in [**3**], one gets

$$\begin{aligned}\big\|\partial_{X_\lambda}\Gamma\big\|_{L^\infty_t C^{\varepsilon-1}} \quad\le\quad & Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\Gamma^0\|_{C^{\varepsilon-1}}+\|\partial_{X_\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_t C^{\varepsilon-1}}\\ & +(1+\mu t)\|F\|_{L^\infty_t C^{\varepsilon-3}}+\mu\|G\|_{\widetilde{L}^1_t C^{\varepsilon-1}}\Big).\end{aligned} \tag{3.10}$$

Recall from [**3, 20**] the following two inequalities

$$\|F\|_{L^\infty_t C^{\varepsilon-3}}\le C\|\Gamma\|_{L^\infty_t L^\infty}\|X_\lambda\|_{L^\infty_t C^\varepsilon}.$$

and

$$\|G\|_{\widetilde{L}^1_t C^{\varepsilon-1}}\le C\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}\|X_\lambda\|_{L^\infty_t C^\varepsilon}.$$

Combining with (3.10), one finds

(3.11)

$$\begin{aligned}\big\|\partial_{X_\lambda}\Gamma\big\|_{L^\infty_t C^{\varepsilon-1}} &\le Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\Gamma^0\|_{C^{\varepsilon-1}}+\|\partial_{X_\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_t C^{\varepsilon-1}}\\ &\quad+(1+\mu t)\|\Gamma\|_{L^\infty_t L^\infty}\|X_\lambda\|_{L^\infty_t C^\varepsilon}+\mu\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}\|X_\lambda\|_{L^\infty_t C^\varepsilon}\Big).\end{aligned}$$

• **Estimate of** $\|\partial_{X_{0,\lambda}}\Gamma^0\big\|_{C^{\varepsilon-1}}$. From the definition of the function Γ we have:

$$\big\|\partial_{X_{0,\lambda}}\Gamma^0\big\|_{C^{\varepsilon-1}}\le\big\|\partial_{X_{0,\lambda}}\omega^0\big\|_{C^{\varepsilon-1}}+\big\|\partial_{X_{0,\lambda}}\mathcal{L}\rho^0\big\|_{C^{\varepsilon-1}}. \tag{3.12}$$

On the one hand, from Definition 3.3 we write

$$\big\|\partial_{X_{0,\lambda}}\omega^0\big\|_{C^{\varepsilon-1}}\lesssim\|\omega^0\|_{C^\varepsilon(X_0)}. \tag{3.13}$$

On the other hand, employing the fact C^ε is an algebra, then we obtain the general result

$$\begin{aligned}\|\partial_{X_\lambda}u\|_{C^{\varepsilon-1}} &\le \|\mathrm{div}(uX_\lambda)\|_{C^{\varepsilon-1}}+\|u\ \mathrm{div}X_\lambda\|_{C^{\varepsilon-1}}\\ &\lesssim \|uX_\lambda\|_{C^\varepsilon}+\|u\ \mathrm{div}X_\lambda\|_{L^\infty}\\ &\lesssim \|u\|_{C^\varepsilon}\widetilde{\|}X_\lambda\|_{C^\varepsilon}.\end{aligned} \tag{3.14}$$

Consequently

$$\big\|\partial_{X_{0,\lambda}}\mathcal{L}\rho^0\big\|_{C^{\varepsilon-1}}\lesssim\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}\|\mathcal{L}\rho^0\|_{C^\varepsilon}.$$

Concerning $\|\mathcal{L}\rho^0\|_{C^\varepsilon}$, using the fact that $\mathcal{L}$ is of order -1. Then Bernstein's inequality yields for $p\ge\frac{2}{1-\varepsilon}$,

$$\begin{aligned}\|\mathcal{L}\rho^0\|_{C^\varepsilon} &\le \|\mathcal{L}\rho^0\|_{L^\infty}+\sup_{q\in\mathbb{N}}2^{q\varepsilon}\|\Delta_q\mathcal{L}\rho^0\|_{L^\infty}\\ &\lesssim \|\mathcal{L}\rho^0\|_{L^\infty}+\sup_{q\in\mathbb{N}}2^{q(\varepsilon-1+2/p)}\|\Delta_q\rho^0\|_{L^p}.\end{aligned} \tag{3.15}$$

Furtheremore, $\mathcal{L}$ have a non local structure, i.e.,

$$\mathcal{L}\rho(t,x)\triangleq\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x_1-y_1)}{|x-y|^2}\rho(t,y)dy,$$

and so

$$|\mathcal{L}\rho(t,x)|\le\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|\rho(t,y)|}{|x-y|}dy=\Big(\frac{1}{2\pi|\cdot|}\star|\rho(t,\cdot)|\Big)(x).$$

Applying the convolution product properties and $\|\rho(t)\|_{L^1\cap L^\infty}\le\|\rho^0\|_{L^1\cap L^\infty}$, we obtain

$$\begin{aligned}\|\mathcal{L}\rho(t)\|_{L^\infty} &\lesssim \|\rho(t)\|_{L^1\cap L^\infty}\\ &\lesssim \|\rho^0\|_{L^1\cap L^\infty}.\end{aligned} \tag{3.16}$$

Putting together (3.15) and (3.16). Then in view of $\Delta_q:L^p\to L^p$ is continuous and $L^p=[L^1,L^\infty]_{\frac{1}{p}}$, we deduce

$$\|\mathcal{L}\rho^0\|_{C^\varepsilon}\le\|\rho^0\|_{L^1\cap L^\infty}.$$

Therefore

$$\big\|\partial_{X_{0,\lambda}}\mathcal{L}\rho^0\big\|_{C^{\varepsilon-1}}\le C_0\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon} \tag{3.17}$$

More generally for $t>0$

$$\big\|\partial_{X_{t,\lambda}}\mathcal{L}\rho(t)\big\|_{C^{\varepsilon-1}}\le C_0\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}, \tag{3.18}$$

Inserting (3.13) and (3.17) in (3.12) to get

$$\big\|\partial_{X_{0,\lambda}}\Gamma^0\big\|_{C^{\varepsilon-1}}\le C_0\big(1+\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}\big). \tag{3.19}$$

• **Estimate of** $\|\partial_{X_\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}}$. To estimate this term we write again in view of (3.14),

$$\|\partial_{X_\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}}\lesssim C\widetilde{\|}X_{t,\lambda}\|_{L^\infty_tC^\varepsilon}\Big\|[\mathcal{L},v\cdot\nabla]\rho\Big\|_{L^1_tC^\varepsilon}.$$

Then in accordance with the Proposition 4.4 stated in appendix, the last estimate becomes

$$\|\partial_{X_{t,\lambda}}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}}\le C_0\widetilde{\|}X_{t,\lambda}\|_{L^\infty_tC^\varepsilon}t. \tag{3.20}$$

• **Estimate of** $\|\Gamma\|_{L^\infty_tL^\infty}$**.** By definition we have for $\mu\in]0,1[$,

$$\|\Gamma\|_{L^\infty_tL^\infty}\le\|\omega\|_{L^\infty_tL^\infty}+\|\mathcal{L}\rho\|_{L^\infty_tL^\infty}. \tag{3.21}$$

Thanks to the Proposition 2.12 we have for $t>0$,

$$\|\omega(t)\|_{L^\infty}\le C_0\log^2(2+t).$$

Note that the term $\|\mathcal{L}\rho\|_{L^\infty_tL^\infty}$ will be done exactly as in (3.16). Then in view of the last estimate, (3.21) takes the form

$$\|\Gamma\|_{L^\infty_tL^\infty}\le C_0\log^2(2+t). \tag{3.22}$$

• **Estimate of** $\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}$. Applying the maximal smoothing effect (2.5) to the equation (3.8), it happens

$$\mu\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\le Ce^{CV(t)}(1+\mu t)\Big(\|\Gamma^0\|_{B^0_{\infty,\infty}}+\big\|[\mathcal{L},v\cdot\nabla]\rho\big\|_{L^1_tB^0_{\infty,\infty}}\Big).$$

Using the fact $L^\infty\hookrightarrow B^0_{\infty,\infty}$ and $C^\varepsilon\hookrightarrow B^0_{\infty,\infty}$ for $\varepsilon>0$, it follows

$$\mu\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\le Ce^{CV(t)}(1+\mu t)\Big(\|\Gamma^0\|_{L^\infty}+\big\|[\mathcal{L},v\cdot\nabla]\rho\big\|_{L^1_tC^\varepsilon}\Big),$$

and, in turn, using once more the Proposition 4.4, we get

$$\mu\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\le Ce^{CV(t)}(1+\mu t)\big(\|\Gamma^0\|_{L^\infty}+C_0t\big), \tag{3.23}$$

For $\|\Gamma^0\|_{L^\infty}$, applying the same argument as in (3.16), we deduce

$$\|\Gamma^0\|_{L^\infty}\le\|\omega^0\|_{L^\infty}+\|\rho^0\|_{L^1\cap L^\infty}$$

together with (3.23), it holds that for $\mu\in]0,1[$

$$\mu\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\le C_0e^{CV(t)}(1+t)^2. \tag{3.24}$$

Plugging (3.19), (3.20), (3.22), (3.24) in (3.11), then after few computations we obtain for $\mu\in]0,1[$

$$\big\|\partial_{X_\lambda}\Gamma\big\|_{L^\infty_tC^{\varepsilon-1}}\le C_0e^{CV(t)}(1+t^2)\log^2(2+t)\big(1+\widetilde{\|}X_\lambda\|_{L^\infty_tC^\varepsilon}\big). \tag{3.25}$$

But,

$$\big\|\partial_{X_{t,\lambda}}\omega(t)\big\|_{C^{\varepsilon-1}}\le\big\|\partial_{X_{t,\lambda}}\mathcal{L}\rho(t)\big\|_{C^{\varepsilon-1}}+\big\|\partial_{X_{t,\lambda}}\Gamma(t)\big\|_{C^{\varepsilon-1}}$$

combined with (3.16), (3.18) and (3.25) we get

(3.26)

$$\begin{aligned}\big\|\partial_{X_{t,\lambda}}\omega(t)\big\|_{C^{\varepsilon-1}}&\le C_0e^{CV(t)}(1+t^2)\log^2(2+t)\big(1+\widetilde{\|}X_\lambda\|_{L^\infty_tC^\varepsilon}\big)+C_0\widetilde{\|}X_\lambda\|_{L^\infty_tC^\varepsilon}\\&\le C_0e^{CV(t)}(1+t^2)\log^2(2+t)\big(1+\widetilde{\|}X_\lambda\|_{L^\infty_tC^\varepsilon}\big).\end{aligned}$$

The term $\widetilde{\|}X_\lambda\|_{L^\infty_t C^\varepsilon}$ may be bounded by taking advantage of (3.4) and the Proposition 2.10, we thus have

$$\|X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\Big(\|X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\|X_{\tau,\lambda}v(\tau)\|_{C^\varepsilon}d\tau\Big). \tag{3.27}$$

According to [**3, 10**], the quantity $\|\partial_{X_{t,\lambda}}v\|_{C^\varepsilon}$ satisfies,

$$\|\partial_{X_{t,\lambda}}v\|_{C^\varepsilon} \le C\big(\|\partial_{X_{t,\lambda}}\omega\|_{C^{\varepsilon-1}} + \|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon}\|\omega(t)\|_{L^\infty} + \|X_{t,\lambda}\|_{C^\varepsilon}\|\nabla v(t)\|_{L^\infty}\big).$$

Plug the last estimate in (3.27) to obtain

$$\begin{aligned}\|X_{t,\lambda}\|_{C^\varepsilon} \quad \le \quad & Ce^{CV(t)}\Big(\|X_{0,\lambda}\|_{C^\varepsilon} + C\int_0^t e^{-CV(\tau)}\Big(\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}} \\ & +\|\mathrm{div}X_{\tau,\lambda}\|_{C^\varepsilon}\|\omega(\tau)\|_{L^\infty} + \|X_{\tau,\lambda}\|_{C^\varepsilon}\|\nabla v(\tau)\|_{L^\infty}\Big)\Big)d\tau.\end{aligned} \tag{3.28}$$

To conclude, it is enough to treat the term $\mathrm{div}X_{t,\lambda}$. To do this, we apply "div" to (3.4) and using the fact $\mathrm{div}v = 0$, we deduce that $\mathrm{div}X_{t,\lambda}$ evolves the equation

$$(\partial_t + v\cdot\nabla)\mathrm{div}X_{t,\lambda} = 0.$$

Again the Proposition 2.10 gives

$$\|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\|\mathrm{div}X_{0,\lambda}\|_{C^\varepsilon}. \tag{3.29}$$

Combining (3.28) and (3.29), then (3.2) allows us to write

$$\begin{aligned}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \quad \le \quad & Ce^{CV(t)}\Big(\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}\big(1+\|\omega\|_{L^1_tL^\infty}\big) \\ & +C\int_0^t e^{-CV(\tau)}\big(\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}} + \|X_{\tau,\lambda}\|_{C^\varepsilon}\|\nabla v(\tau)\|_{L^\infty}\big)d\tau\Big).\end{aligned}$$

Then, the Proposition 2.12 implies

$$\begin{aligned}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \quad \le \quad & C_0e^{CV(t)}\Big(\log^2(2+t) + C\int_0^t e^{-CV(\tau)}\big(\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}} \\ & +\|X_{\tau,\lambda}\|_{C^\varepsilon}\|\nabla v(\tau)\|_{L^\infty}\big)d\tau\Big),\end{aligned}$$

Gronwall's inequality asserts that

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \quad \le \quad C_0e^{CV(t)}\Big(\log^2(2+t) + C\int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}d\tau\Big),$$

combined with (3.26), it holds for $t>0$

$$\begin{aligned}e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \quad \le \quad & C_0(1+t)(1+t^2)\log^2(2+t) \\ & +C_0C\int_0^t(1+\tau^2)\log^2(2+\tau)e^{-CV(\tau)}\widetilde{\|}X_{\tau,\lambda}\|_{L^\infty_\tau C^\varepsilon}d\tau.\end{aligned}$$

Setting $\phi_1(t) = C_0(1+t)(1+t^2)\log^2(2+t)$ and $\phi_2(t) = C_0C(1+t^2)\log^2(2+t)$, then for $t>0$ the last estimate becomes

$$e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \le \phi_1(t) + \int_0^t \phi_2(\tau)e^{-CV(\tau)}\widetilde{\|}X_{\tau,\lambda}\|_{L^\infty_\tau C^\varepsilon}d\tau.$$

Again Gronwall's inequality gives

$$e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\le\phi_1(t)+\int_0^t\phi_1(\tau)\phi_2(\tau)e^{\int_\tau^t\phi_2(\tau')d\tau'}d\tau.$$

After a few computations we shall have for $t>0$

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\le C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)},$$

accordingly (3.26) becomes

$$\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\le C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)}.$$

Putting together the last two estimates, we end up with

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}+\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\le C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)},\quad\forall\lambda\in\Lambda.\tag{3.30}$$

On the other hand, according to the Definition 3.3, we recall that:

$$\|\omega(t)\|_{C^\varepsilon(X_t)}=\frac{1}{I(X_t)}\Big(\|\omega\|_{L^\infty}\sup_{\lambda\in\Lambda}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}+\sup_{\lambda\in\Lambda}\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\Big).\tag{3.31}$$

The required estimate for ω in $C^\varepsilon(X_t)$ norm follows by showing that X_t defined in (3.3) is a non degenerate family, that is to say, $I(X_t)>0$. For that purpose, we derive $X_{t,\lambda}\circ\Psi(t,x)\triangleq\partial_{X_{0,\lambda}}\Psi(t,x)$ with respect to time and using the fact

$$\begin{cases}\frac{\partial}{\partial t}\Psi(t,x)=v(t,\Psi(t,x))\\ \Psi(0,x)=x,\end{cases}$$

it follows

$$\begin{cases}\partial_t\partial_{X_{0,\lambda}}\Psi(t,x)=\nabla v(t,\Psi(t,x))\partial_{X_{0,\lambda}}\Psi(t,x)\\ \partial_{X_{0,\lambda}}\Psi(0,x)=X_{0,\lambda}.\end{cases}$$

One deduce that this equation is a time reversibile. Thus Gronwall's inequality asserts that

$$|X_{0,\lambda}(x)|^{\pm1}\le|\partial_{X_{0,\lambda}}\Psi(t,x)|e^{V(t)}.$$

In accordance with (3.1) and (3.3), one has

$$I(X_t)\ge I(X_0)e^{-V(t)}>0.\tag{3.32}$$

Consequently, (3.30), (3.31) and (3.32) leading to

$$\|\omega(t)\|_{C^\varepsilon(X_t)}\le C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)}.\tag{3.33}$$

Now, we are in position to apply the logarithmic estimate (3.5). By virtue of (3.33), the Proposition (2.12), the increasing of the function $(0,\infty)\ni\zeta\mapsto\zeta\log(e+a/\zeta)$ and the decreasing of $(0,\infty)\ni\zeta\mapsto\log(e+a/\zeta)$, it holds

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty}&\le C_0\left(\log(2+t)+\log^2(2+t)\log\left(e+\frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{\log^2(2+t)}\right)\right)\\ &\le C_0\Big(\log(2+t)+t^3\log^4(2+t)+\log^2(2+t)\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big).\end{aligned}$$

Hence, the growth of the exponential function and Gronwall's inequality yield

$$\|\nabla v(t)\|_{L^\infty}\le C_0e^{C_0t\log^2(2+t)},\tag{3.34}$$

combining this estimate with (3.33), we get

$$\|\omega(t)\|_{C^\varepsilon(X_t)}\le C_0e^{\exp C_0t\log^2(2+t)}.$$

To finalize, let us estimate $\partial_{X_{t,\lambda}}\Psi(t)$. First, we employ that $\partial_{X_{0,\lambda}}\Psi(t) = X_{t,\lambda}\circ\Psi(t)$ for every $\lambda\in\Lambda$, then by virtue of (2.2) we thus have

$$\begin{aligned}\|X_{t,\lambda}\circ\Psi(t)\|_{C^\varepsilon} &\leq \|X_{t,\lambda}\|_{C^\varepsilon}\|\nabla\Psi(t)\|^{\varepsilon}_{L^\infty}\\ &\leq \|X_{t,\lambda}\|_{C^\varepsilon}e^{CV(t)} \quad \forall\lambda\in\Lambda.\end{aligned}$$

Here we have used the classical estimate $e^{-CV(t)}\leq\|\nabla\Psi^{\pm1}(t)\|_{L^\infty}\leq e^{CV(t)}$. Hence, (3.34) ensures that

$$\|X_{t,\lambda}\circ\Psi(t)\|_{C^\varepsilon}\leq C_0e^{\exp C_0 t\log^2(2+t)}, \tag{3.35}$$

this concludes the proof. □

3.2. Proof of Theorem 1.1. The proof of the Theorem 1.1 requires two principal steps:

(1) The velocity vector fields is a Lipschitz function globally in time, which immediately follows from Theorem 3.8.
(2) The persistence of Hölderian regularity in time of the transported patch, i.e., $\partial\Omega_t$ is a simple curve with $C^{1+\varepsilon}$−regularity given by the following scheme:
  (2.i) Fabricate an initial admissible family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$, which enables us to show that $\mathbf{1}_{\Omega_0}\in C^\varepsilon(X_0)$ and parametrize its boundary $\partial\Omega_0$ by a simple curve.
  (2.ii) The regularity of evolution family $X_t = (X_{t,\lambda})_{\lambda\in\{0,1\}}$ and the boundary $\partial\Omega_t$, with $\Omega_t = \Psi(t,\Omega_0)$.

(2.i) Since $\partial\Omega_0$ is a curve of the class $C^{1+\varepsilon}$. Consequently, (1) of the Definition 3.6 ensures the existence of a local chart (f_0, V_0), with V_0 is a neighborhood of $\partial\Omega_0$ such that

$$\begin{cases} f_0\in C^{1+\varepsilon}(\mathbb{R}^2), \quad \nabla f_0(x)\neq 0 \quad \text{on } V_0\\ \partial\Omega_0 = f_0^{-1}(\{0\})\cap V_0,\end{cases}$$

On the other hand, let $\chi\in\mathscr{D}(\mathbb{R}^2), 0\leq\chi\leq 1$ and

$$\operatorname{supp}\chi\subset V_0, \quad \chi(x)=1 \quad \forall x\in W_0,$$

where W_0 is a small neighborhood of $\partial\Omega_0$ such that $W_0\Subset V_0$. Next, define for every $x\in\mathbb{R}^2$ the family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ by:

$$X_{0,0}(x) = \nabla^\perp f_0(x) \quad \text{and} \quad X_{0,1}(x) = (1-\chi(x))\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.36}$$

It is worthwhile to examine the admissibility of the family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$. First, we obviously check that $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is non-degenerate, and that each component $X_{0,\lambda}$ and its divergence are in $C^\varepsilon(\mathbb{R}^2)$, then according to Definition 3.1, we conclude that $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is an admissible family.
Second, $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is a tangential family (see, (2)-Definition 3.6) with respect to $\Sigma = \partial\Omega_0$, i.e.,

$$X_{0,\lambda}\in\mathcal{T}^\varepsilon_{|\Sigma}, \quad \forall\lambda\in\{0,1\}.$$

Indeed, for the component $X_{0,0}$, clearly we have:

$$X_{0,0}(x)\cdot\nabla f_0(x) = \nabla^\perp f_0(x)\cdot\nabla f_0(x) = 0, \quad \forall x\in\partial\Omega_0,$$

while for the component $X_{0,1}$, using the fact $\chi \equiv 1$ on W_0, we immediately obtain

$$\begin{aligned} X_{0,1}(x)\cdot\nabla f_0(x) &= (1-\chi(x))\partial_1 f_0(x)\\ &= 0. \end{aligned}$$

(2.ii) For every $\lambda \in \{0,1\}$ and $x \in \mathbb{R}^2$, we set $X_{t,\lambda}(x) = \big(\partial_{X_{0,\lambda}}\Psi\big)\big(t,\Psi^{-1}(t,x)\big)$. Using the same argument as in (3.29), (3.30) and (3.32), we infer that (X_t) still remains non-degenerate for every $t \geq 0$, and that each $X_{t,\lambda}$ still has components and divergence in C^ε. This means that $X_t = (X_{t,\lambda})_{\lambda\in\{0,1\}}$ is an admissible family for all $t \geq 0$.

Now, we will parametrize the boundary $\partial\Omega_0$. To do this, let $x_0 \in \partial\Omega_0$ and define the curve γ^0 by the following ordinary differential equation

$$\begin{cases} \partial_\sigma\gamma^0(\sigma) = X_{0,0}(\gamma^0(\sigma))\\ \gamma^0(0) = x_0. \end{cases}$$

By classical arguments we can see that γ^0 belongs to $C^{1+\varepsilon}(\mathbb{R},\mathbb{R}^2)$. A natural way to define the evolution parametrization of $\partial\Omega_t$ is to set for every $t \geq 0$,

$$\gamma(t,\sigma) \triangleq \Psi(t,\gamma^0(\sigma)).$$

Clearly that $\gamma(t,\cdot)$ is the transported of γ^0 by the flow Ψ. By applying the criterion differentiation with respect to σ, we readily get

$$\partial_\sigma\gamma(t,\sigma) = \big(\partial_{X_{0,0}}\Psi\big)(t,\,\gamma^0(\sigma)).$$

On the other hand, $\partial_{X_{0,0}}\Psi \equiv X_{0,0}\circ\Psi$, thus we have from estimate 3.35 of the Theorem 3.8 that $\partial_{X_{0,0}}\psi \in L^\infty_{loc}(\mathbb{R}_+;\, C^\varepsilon)$, accordingly $\gamma(t)$ belongs to $L^\infty_{loc}(\mathbb{R}_+;\, C^{1+\varepsilon})$. This tells us the regularity persistence of the boundary $\partial\Omega_t$ and so the proof of the Theorem 1.1 is accomplished.

4. The rate convergence

4.1. General statement. In this paragraph we are interested in the rate convergence between (v_μ,ρ_μ) and (v,ρ), the solutions of (B_μ) and (B_0). To be precise, we will provide a more general version of the Theorem 1.4. For this purpose, we state the following auxiliary result which shows that any vortex patch with smooth bounded domain belongs to $\dot{B}^{\frac{1}{p}}_{p,\infty}$.

PROPOSITION 4.1. *Let Ω_0 be a $C^{1+\varepsilon}-$bounded domain, with $0<\varepsilon<1$, then the function $\mathbf{1}_{\Omega_0}$ belongs to $\dot{B}^{\frac{1}{p}}_{p,\infty}$, with $p \in [2,\infty[$.*

PROOF. We follow the formalism performed in [**33**] with more details. Since Ω_0 is $C^{1+\varepsilon}-$bounded domain, then in view of $C^{1+\varepsilon} \hookrightarrow Lip$ we deduce that $\mathbf{1}_{\Omega_0} \in L^\infty \cap BV$, with BV[2] is the Banach space of functions with bounded variation. By means of the Proposition 4.5 stated in the appendix, we have

$$BV \hookrightarrow \dot{B}^1_{1,\infty}$$

[2] BV is the space of functions of bounded variations defined by

$$BV(\mathbb{R}^2) \triangleq \Big\{u \in L^1(\mathbb{R}^2) : \forall i = 1,\ldots,2,\ \exists\lambda_i \in \mathscr{M}_b(\mathbb{R}^2,\mathbb{R}^2);\int_{\mathbb{R}^2} u\frac{\partial\varphi}{\partial x_i}dx = -\int_{\mathbb{R}^2}\varphi d\lambda_i \quad \forall\varphi\in\mathscr{D}(\mathbb{R}^2)\Big\}$$

equipped with the norm

$$\|u\|_{BV} \triangleq \|u\|_{L^1} + |Du|(\mathbb{R}^2),$$

where $|Du|(\mathbb{R}^2)$ is the total variation of measure Du.

In particular for $q \in \mathbb{Z}$

$$\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^1} \lesssim 2^{-q}\|\mathbf{1}_{\Omega_0}\|_{BV},$$

combined with $L^p = [L^1, L^\infty]_{\frac{1}{p}}$, we deduce

$$\begin{aligned}\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^p} &\lesssim \|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^1}^{\frac{1}{p}}\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^\infty}^{1-\frac{1}{p}} \\ &\lesssim 2^{-\frac{q}{p}}\|\mathbf{1}_{\Omega_0}\|_{BV}\|\mathbf{1}_{\Omega_0}\|_{L^\infty} \\ &\lesssim 2^{-\frac{q}{p}}\|\mathbf{1}_{\Omega_0}\|_{L^\infty\cap BV}.\end{aligned}$$

Here we have used the fact that $\dot{\Delta}_q$ maps continuously L^∞ into it self. Thus we obtain for $q \in \mathbb{Z}$

$$2^{-\frac{q}{p}}\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^p} \leq C\|\mathbf{1}_{\Omega_0}\|_{L^\infty\cap BV}.$$

Taking the supremum over $q \in \mathbb{Z}$, we finally obtain that $\mathbf{1}_{\Omega_0} \in \dot{B}^{\frac{1}{p}}_{p,\infty}$ □

Now, we state the general version of the Theorem 1.4. Roughly speaking we have:

THEOREM 4.2. *Let (v_μ, ρ_μ) and (v, ρ) be the solutions of (B_μ) and (B_0) respectively with (v^0_μ, ρ^0_μ) and (v^0, ρ^0) their initial data. Let ω^0_μ, ω^0 be their vorticties with $\omega^0_\mu \in L^\infty \cap B^{\frac{1}{p}}_{p,\infty}$, $\omega^0 \in L^1 \cap L^\infty$ and $\rho^0, \rho^0_\mu \in L^1 \cap Ł^p$. Setting $\Pi(t) = \|v_\mu - v\|_{L^p} + \|\rho_\mu - \rho\|_{L^p}$, then we have the following rate of convergence.*

$$\Pi(t) \leq Ce^{C(t+V_\mu(t)+V(t))}\Big(\Pi(0)+(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)\big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\rho^0_\mu\|_{L^p}\big)\Big) \quad p \in [2, \infty[,$$

where

$$V_\mu(t) = \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau, \quad V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

The proof of the previous Theorem requires the following interpolation result.

PROPOSITION 4.3. *Let $(p, r, \eta) \in [1, \infty]^2 \times]-1, 1[$ and v_μ be a free divergence vector field depicted by the Biot-Savart law $v_\mu = \Delta^{-1}\nabla^\perp\omega_\mu$, i.e.,*

$$v_\mu(t, x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega_\mu(t, y)dy.$$

Then the following estimate holds true.

$$\|\Delta v_\mu\|_{L^r_t L^p} \leq C\|\omega_\mu\|^{\frac{1+\eta}{2}}_{\widetilde{L}^r_t B^\eta_{p,\infty}}\|\omega_\mu\|^{\frac{1-\eta}{2}}_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}}.$$

PROOF. To prove this estimate, let $N \in \mathbb{N}$ be a parameter that will be chosen judiciously later. The fact $\Delta v_\mu = \nabla^\perp\omega_\mu$, interpolation in frequency and Bernstein inequality enable us to write

(4.1)

$$\begin{aligned}\|\Delta v_\mu\|_{L^r_t L^p} &\leq \sum_{q\leq N}\|\Delta_q\nabla^\perp\omega_\mu\|_{L^r_t L^p} + \sum_{q>N}\|\Delta_q\nabla^\perp\omega_\mu\|_{L^r_t L^p} \\ &\leq \sum_{q\leq N} 2^{q(1-\eta)}2^{q\eta}\|\Delta_q\omega_\mu\|_{L^r_t L^p} + \sum_{q>N} 2^{q(-1-\eta)}2^{q(2+\eta)}\|\Delta_q\omega_\mu\|_{L^r_t L^p} \\ &\leq 2^{N(1-\eta)}\|\omega_\mu\|_{\widetilde{L}^r_t B^\eta_{p,\infty}} + 2^{-N(1+\eta)}\|\omega_\mu\|_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}}.\end{aligned}$$

Now, we choose N such that

$$2^{N(1-\eta)}\|\omega_\mu\|_{\widetilde{L}^r_t B^{\eta}_{p,\infty}} \approx 2^{-N(1+\eta)}\|\omega_\mu\|_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}},$$

whence

$$2^{2N} \approx \frac{\|\omega_\mu\|_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}}}{\|\omega_\mu\|_{\widetilde{L}^r_t B^{\eta}_{p,\infty}}}. \tag{4.2}$$

Inserting (4.2) in (4.1), we obtain the desired estimate and so the proof is completed. □

Proof of the Theorem 4.2. We set $U = v_\mu - v, \Theta = \rho_\mu - \rho$ and $P = p_\mu - p$. We intend to estimate the quantity $\|U\|_{L^p} + \|\Theta\|_{L^p}$. To do this, making few computations we discover that U and Θ evolve the nonlinear equations,

$$(\widetilde{B}_\mu)\quad \begin{cases} \partial_t U + (v_\mu\cdot\nabla)U - \mu\Delta v_\mu + \nabla P = \Theta e_2 - (U\cdot\nabla)v & (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\Theta + (v_\mu\cdot\nabla)\Theta - \Delta\Theta = -U\cdot\nabla\rho & (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div}U = 0, \\ U_{|t=0} = U_0, \quad \Theta_{|t=0} = \Theta_0. \end{cases}$$

• **Estimate of $\|U(t)\|_{L^p}$.** Multiply the first equation in $(\widetilde{B}_\mu)$ by $U|U|^{p-2}$, and integrating by parts over the space variables $\mathbb{R}^2$, then in view of $\mathrm{div}v_\mu = \mathrm{div}v = 0$, it follows

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|U(t)\|^p_{L^p} &\le \int_{\mathbb{R}^2}\big|\nabla P\cdot U|U|^{p-2}\big|dx + \mu\int_{\mathbb{R}^2}\big|\Delta v_\mu\cdot U|U|^{p-2}\big|dx\\ &\quad+\int_{\mathbb{R}^2}\big|(U\cdot\nabla)v\cdot U|U|^{p-2}\big|dx + \int_{\mathbb{R}^2}\big|\Theta e_2\cdot U|U|^{p-2}\big|dx.\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|U(t)\|^p_{L^p} &\le \|\nabla P(t)\|_{L^p}\|U(t)\|^{p-1}_{L^p} + \mu\|\Delta v_\mu(t)\|_{L^p}\|U(t)\|^{p-1}_{L^p}\\ &\quad+\|\nabla v\|_{L^\infty}\|U(t)\|^p_{L^p} + \|\Theta(t)\|_{L^p}\|U(t)\|^{p-1}_{L^p},\end{aligned}$$

so, integrating in time over $[0,t]$, we obtain

$$\begin{aligned}\|U(t)\|_{L^p} &\le \|U_0\|_{L^p} + \int_0^t\|\nabla P(\tau)\|_{L^p}d\tau + \mu\int_0^t\|\Delta v_\mu(\tau)\|_{L^p}d\tau\\ &\quad+\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|U(\tau)\|_{L^p}d\tau + \int_0^t\|\Theta(\tau)\|_{L^p}d\tau.\end{aligned} \tag{4.3}$$

Concerning the term $\|\nabla P\|_{L^p}$, applying the "div" operator to the first equation of $(\widetilde{B}_\mu)$, one finds after easy algebraic computations

$$-\Delta P = \mathrm{div}\big(U\cdot\nabla(v_\mu + v)\big) + \partial_2\Theta,$$

then we have

$$-\nabla P = \nabla\Delta^{-1}\mathrm{div}\big(U\cdot\nabla(v_\mu+v)\big) + \nabla\Delta^{-1}\partial_2\Theta.$$

The boundedness of Riesz transform on L^p, $p\in]1,\infty[$ into it self leading to

$$\|\nabla P\|_{L^p} \lesssim \|U\|_{L^p}\big(\|\nabla v_\mu\|_{L^\infty} + \|\nabla v\|_{L^\infty}\big) + \|\Theta\|_{L^p}.$$

Inserting the above estimate into (4.3), we deduce that

$$\|U(t)\|_{L^p} \lesssim \|U_0\|_{L^p} + \int_0^t \|U(\tau)\|_{L^p}\big(\|\nabla v_\mu(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{L^\infty}\big)d\tau + \mu\int_0^t \|\Delta v_\mu(\tau)\|_{L^p}d\tau + \int_0^t \|\Theta(\tau)\|_{L^p}d\tau. \tag{4.4}$$

• **Estimate of** $\|\Theta\|_{L^1_tL^p}$. Multiply the second equation in $(\widetilde{B}_\mu)$ by $\Theta|\Theta|^{p-2}$, and integrating by parts over $\mathbb{R}^2$. Then by virtue of $\mathrm{div} v_\mu = \mathrm{div} v = 0$, it happens

$$\frac{1}{p}\frac{d}{dt}\|\Theta(t)\|^p_{L^p} + (p-1)\int_{\mathbb{R}^2}|\nabla\Theta(t)|^2|\Theta(t)|^{p-2}dx \le \int_{\mathbb{R}^2}\big|U(t)\cdot\nabla\rho(t)\big||\Theta(t)|^{p-1}dx,$$

owing to Hölder's inequality, we shall have

$$\frac{1}{p}\frac{d}{dt}\|\Theta(t)\|^p_{L^p} + (p-1)\int_{\mathbb{R}^2}|\nabla\Theta(t)|^2|\Theta(t)|^{p-2}dx \le \|U(t)\|_{L^p}\|\nabla\rho(t)\|_{L^\infty}\|\Theta(t)\|^{p-1}_{L^p}.$$

Since the second term of the left-hand side has a non-negative sign, one obtains

$$\frac{d}{dt}\|\Theta(t)\|_{L^p} \lesssim \|U(t)\|_{L^p}\|\nabla\rho(t)\|_{L^\infty}.$$

Integrating in time over $[0,t]$, we get

$$\|\Theta(t)\|_{L^p} \lesssim \|\Theta_0\|_{L^p} + \int_0^t \|U(\tau)\|_{L^p}\|\nabla\rho(\tau)\|_{L^\infty}d\tau. \tag{4.5}$$

Putting together (4.4) and (4.5), we readily get

$$\begin{aligned}\|U(t)\|_{L^p} + \|\Theta(t)\|_{L^p} \lesssim\ & \|U_0\|_{L^p} + \|\Theta_0\|_{L^p} + \int_0^t \|U(\tau)\|_{L^p}\big(\|\nabla v_\mu(\tau)\|_{L^\infty} \\ & + \|\nabla v(\tau)\|_{L^\infty}\big)d\tau + \mu\int_0^t \|\Delta v_\mu(\tau)\|_{L^p}d\tau \\ & + \int_0^t \|\Theta(\tau)\|_{L^p}d\tau + \int_0^t \|U(\tau)\|_{L^p}\|\nabla\rho(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

Since $\Pi(t) \triangleq \|U(t)\|_{L^p} + \|\Theta(t)\|_{L^p}$, then after few caculations we find that

$$\begin{aligned}\Pi(t) \lesssim\ & \Pi(0) + \int_0^t \big(1 + \|\nabla v_\mu(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{L^\infty} + \|\nabla\rho(\tau)\|_{L^\infty}\big)\Pi(\tau)d\tau \\ & + \mu\int_0^t \|\Delta v_\mu(\tau)\|_{L^p}d\tau.\end{aligned}$$

Using Gronwall's inequality, we can write

$$\Pi(t) \lesssim e^{Ct}e^{C(V_\mu(t)+V(t)+\|\nabla\rho\|_{L^1_tL^\infty})}\big(\Pi(0) + \mu\|\Delta v_\mu\|_{L^1_tL^p}\big). \tag{4.6}$$

We now turn to the estimate of the principal term $\mu\|\Delta v_\mu\|_{L^1_tL^p}$ which provides the desired rate of convergence. Take in Proposition 4.3 $\eta = \frac{1}{p}$ and $r = 1$ to obtain

$$\mu\|\Delta v_\mu\|_{L^1_tL^p} \le \underbrace{\mu\|\omega_\mu\|^{\frac12+\frac{1}{2p}}_{\widetilde{L}^1_tB^{\frac1p}_{p,\infty}}}_{\mathrm{I}}\underbrace{\|\omega_\mu\|^{\frac12-\frac{1}{2p}}_{\widetilde{L}^1_tB^{2+\frac1p}_{p,\infty}}}_{\mathrm{II}}. \tag{4.7}$$

For the term I, applying the Hölder inequality in the time variable, we deduce that

$$\mathrm{I} \le \mu t^{\frac12+\frac{1}{2p}}\|\omega_\mu\|^{\frac12+\frac{1}{2p}}_{\widetilde{L}^\infty_tB^{\frac1p}_{p,\infty}}.$$

Put $r=\infty, s=\frac{1}{p}, p_1=p, p_2=\infty$, Proposition 2.11 tell us

$$\|\omega_\mu\|_{\widetilde{L}^\infty_t B^{\frac{1}{p}}_{p,\infty}} \leq Ce^{CV_\mu(t)}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).$$

Hence

$$\mathrm{I}\leq Ce^{CV_\mu(t)}\mu t^{\frac{1}{2}+\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big)^{\frac{1}{2}+\frac{1}{2p}}.$$

Concerning II, a new use of the Proposition 2.11 gives for $r=1, s=\frac{1}{p}, p_1=p, p_2=\infty$ the following

$$\|\omega_\mu\|_{\widetilde{L}^1_t B^{2+\frac{1}{p}}_{p,\infty}} \leq Ce^{CV_\mu(t)}\mu^{-1}(1+\mu t)\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).$$

Accordingly, we infer

$$\mathrm{II}\leq Ce^{CV_\mu(t)}\mu^{\frac{1}{2p}-\frac{1}{2}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big)^{\frac{1}{2}-\frac{1}{2p}}.$$

Combining I and II, (4.7) becomes

$$\mu\|\Delta v_\mu\|_{L^1_t L^p}\lesssim Ce^{CV_\mu(t)}(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).$$

By means of the embedding $\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}=L^1_t B^{\frac{1}{p}}_{p,1}\hookrightarrow L^1_t B^{\frac{1}{p}}_{p,\infty}$, the last estimate takes the form

$$\mu\|\Delta v_\mu\|_{L^1_t L^p}\lesssim Ce^{CV_\mu(t)}(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}\Big)$$

together with (4.6), one obtains

$$\begin{aligned}\Pi(t) \;\lesssim\; & Ce^{C(t+V_\mu(t)+V(t)+\|\nabla\rho\|_{L^1_t L^\infty})}\Big(\Pi(0)+(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}\\ & +\|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}\Big)\Big).\end{aligned}$$

For the term $\|\nabla\rho\|_{L^1_t L^\infty}$ in the exponential, applying the Propositon 2.12 the last estimate takes the form

(4.8)

$$\Pi(t)\lesssim Ce^{C(t+V_\mu(t)+V(t))}\Big(\Pi(0)+(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}\Big)\Big).$$

To end the proof of our claim, let us estimate $\|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}$. Note that ∇ maps continuously $B^{1+\frac{1}{p}}_{p,1}$ into $B^{\frac{1}{p}}_{p,1}$, then the Proposition 2.11 combined with $L^p\hookrightarrow B^{\frac{1}{p}-1}_{p,1}$ gives for $p>1$

$$\begin{aligned}\|\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}+1}_{p,1}} &\leq Ce^{CV_\mu(t)}(1+t)\|\rho^0_\mu\|_{B^{\frac{1}{p}-1}_{p,1}}\\ &\leq Ce^{CV_\mu(t)}(1+t)\|\rho^0_\mu\|_{L^p}.\end{aligned}\tag{4.9}$$

Plugging (4.9) in (4.8), we find that

(4.10)

$$\Pi(t)\lesssim Ce^{C(t+V_\mu(t)+V(t))}\Big(\Pi(0)+(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\rho^0_\mu\|_{L^p}\Big)\Big).$$

Hence the proof of the Theorem 4.2 is accomplished. □

4.2. Proof of Theorem 1.4. **(i)** Substituting (1.3) and (1.4) into (4.10) and the fact $\mathbf{1}_{\Omega_0} \in B^{\frac{1}{p}}_{p,\infty}$, it happens for $\mu \in]0,1[$

$$\Pi(t) \lesssim C_0 e^{e^{C_0 t \log^2(1+t)}} (\mu t)^{\frac{1}{2}+\frac{1}{2p}}.$$

(ii) To estimate $\omega_\mu - \omega$ in L^p-norm, using the definition of ω_μ and ω we shall have

$$\|\omega_\mu(t) - \omega(t)\|_{L^p} \leq \|\nabla(v_\mu(t) - v(t))\|_{L^p}$$

combined with $B^0_{p,1} \hookrightarrow L^p$ and Bernstein inequality leads to

$$\|\omega_\mu(t) - \omega(t)\|_{L^p} \lesssim \|v_\mu(t) - v(t)\|_{B^1_{p,1}}. \tag{4.11}$$

On the other hand, let N be a fixed number that will be chosen later. Again Bernstein's inequality leading to

$$\begin{aligned} &\|v_\mu(t) - v(t)\|_{B^1_{p,1}} \\ \leq\ & \sum_{q\leq N} 2^q \|\Delta_q(v_\mu(t) - v(t))\|_{L^p} + \sum_{q>N} 2^{-\frac{q}{p}} 2^{\frac{q}{p}} \|\Delta_q \nabla(v_\mu(t) - v(t))\|_{L^p} \\ \lesssim\ & 2^N \|v_\mu(t) - v(t)\|_{L^p} + \sup_{q\geq -1} 2^{\frac{q}{p}} \|\omega_\mu(t) - \omega(t)\|_{L^p} \sum_{q>N} 2^{-\frac{q}{p}} \\ \lesssim\ & 2^N \|v_\mu(t) - v(t)\|_{L^p} + 2^{-\frac{N}{p}} \|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}. \end{aligned} \tag{4.12}$$

In the second line we have used the fact

$$\|\Delta_q \nabla(v_\mu(t) - v(t))\|_{L^p} \approx \|\Delta_q(\omega_\mu(t) - \omega(t))\|_{L^p}, \quad \forall q \in \mathbb{N}.$$

Taking

$$2^{N(1+\frac{1}{p})} \approx \frac{\|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}}{\|v_\mu(t) - v(t)\|_{L^p}}.$$

Then (4.11) and (4.12) lead us

$$\|v_\mu(t) - v(t)\|_{B^1_{p,1}} \lesssim \|v_\mu(t) - v(t)\|_{L^p}^{\frac{1}{p+1}} \|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}^{\frac{p}{1+p}},$$

whence (4.11) yields

$$\|\omega_\mu(t) - \omega(t)\|_{L^p} \leq \|v_\mu(t) - v(t)\|_{L^p}^{\frac{1}{p+1}} \|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}^{\frac{p}{p+1}},$$

in accordance with the Theorem 1.4, it holds

$$\|\omega_\mu(t) - \omega(t)\|_{L^p} \leq C_0 e^{e^{C_0 t \log^2(2+t)}} (\mu t)^{\frac{1}{2p}} (1 + \mu t) \|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}.$$

To finalize, let us estimate $\|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}$. To do this, using the persistence of Besov spaces explicitly formulated in the Proposition 2.10, one gets

$$\begin{aligned} \|\omega_\mu(t) - \omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}} &\leq \|\omega_\mu(t)\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}} \\ &\leq Ce^{C(V_\mu(t)+V(t))} \Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\omega^0\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}} \\ &\qquad + \|\nabla\rho\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}} \Big). \end{aligned}$$

The last two terms of the right-hand side stem from (4.9). Then thanks to (1.3) and (1.4), we end up with

$$\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}} \leq C_0 e^{e^{C_0 t\log^2(2+t)}}(\mu t)^{\frac{1}{2p}}(1+\mu t).$$

This achieves the proof of the aimed estimate.

4.3. Optimality of the rate of convergence. In this paragraph we shall give the proof of Theorem 1.6 by showing that $(\mu t)^{\frac{1}{2p}}$ is optimal in L^p norm in the case of a circular vortex patch and ρ^0_μ and ρ^0 are constant densities.

Proof of Theorem 1.6. Since the initial data $\omega^0_\mu = \omega^0 = \mathbf{1}_{\mathbb{D}}$ are radial then this structure is preserved in the evolution and thus

$$v_\mu\cdot\nabla\omega_\mu, \quad v\cdot\nabla\omega = 0.$$

Therefore the equation of ω_μ (resp. ω) takes the following form

$$\partial_t\omega_\mu - \mu\Delta\omega_\mu = 0, \quad \partial_t\omega = 0.$$

Recall that the solutions of the above equations are given by

$$\omega_\mu(t,x) = K_{\mu t}\star\omega^0_\mu(x), \quad \omega(t,x)=\omega^0(x), \tag{4.13}$$

where K_t is the heat kernel defined by

$$K_t(x) \triangleq \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$$

and satisfies

$$\int_{\mathbb{R}^2} K_t(x)dx = 1.$$

On the other hand, setting $W(t,x)=\omega_\mu(t,x)-\omega(t,x)$. Then in view of (4.13), we have

$$W(t,x) = \int_{\mathbb{R}^2} K_{\mu t}(x-y)[\mathbf{1}_{\mathbb{D}}(y)-\mathbf{1}_{\mathbb{D}}(x)]dy.$$

For $|x|<1$ we have

$$\begin{aligned} W(t,x) &= \int_{\{|y|\geq 1\}} K_{\mu t}(x-y)dy \\ &= \frac{1}{4\pi\mu t}\int_{\{|y|\geq 1\}} e^{-\frac{|x-y|^2}{4\mu t}}dy. \end{aligned}$$

Introduce $Z(t,x)=W(t,\sqrt{\mu t}x)$ and make the change of variables $y=\sqrt{\mu t}z$, one gets

$$Z(t,x) = \frac{1}{4\pi}\int_{\{|z|\geq\frac{1}{\sqrt{\mu t}}\}} e^{-\frac{|x-z|^2}{4}}dz, \quad |x|\leq\frac{1}{\sqrt{\mu t}}. \tag{4.14}$$

Let $\mu t\leq 1$, then

$$\begin{aligned} \|W(t)\|_{L^p(\mathbb{R}^2)} &\geq \|W(t)\|_{L^p(1-\sqrt{\mu t}\leq|x|\leq 1)} \\ &\geq (\mu t)^{\frac{1}{p}}\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})}. \end{aligned} \tag{4.15}$$

Now, our task is to prove the following requirement,

$$\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})} \geq C_2(\mu t)^{-\frac{1}{2p}}. \tag{4.16}$$

For this purpose, we plug the identity $|x-z|^2 \triangleq |x|^2 + |z|^2 - 2\langle x, z\rangle$ into (4.14),

$$Z(t,x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}} \int_{\{|z| \geq \frac{1}{\sqrt{\mu t}}\}} e^{-\frac{|z|^2}{4} + \frac{1}{2}\langle x, z\rangle} dz.$$

By rotation invariance, the above equation becomes

$$\begin{aligned} Z(t,x) &= \frac{1}{4\pi} e^{-\frac{|x|^2}{4}} \int_0^{2\pi} \int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{r^2}{4} + \frac{1}{2} r|x|\cos\theta} r dr d\theta \\ &\geq \frac{1}{4\pi} e^{-\frac{|x|^2}{4}} \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{r^2}{4} + \frac{1}{2} r|x|\cos\theta} r dr d\theta. \end{aligned}$$

Since $\cos\theta \geq 1 - \frac{\theta^2}{2}$ for $\theta \geq 0$, then we find

$$\begin{aligned} |Z(t,x)| &\geq \frac{1}{4\pi} \int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{|x|^2}{4} - \frac{r^2}{4} + \frac{r|x|}{2}} \left(\int_0^{\frac{\pi}{2}} e^{-\frac{1}{4} r|x|\theta^2} d\theta \right) r dr \\ &= \frac{1}{4\pi} \int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{1}{4}(|x|-r)^2} \left(\int_0^{\frac{\pi}{2}} e^{-\frac{1}{4} r|x|\theta^2} d\theta \right) r dr. \end{aligned}$$

Here, we have used Fubini's theorem. For the second integral of the right-hand side, using the change of variables $\alpha = \frac{1}{2}\sqrt{r|x|}\theta$, we get

$$|Z(t,x)| \geq \frac{1}{2\pi} \int_{\frac{1}{\sqrt{\mu t}}}^{\frac{2}{\sqrt{\mu t}}} e^{-\frac{1}{4}(|x|-r)^2} \left(\int_0^{\sqrt{r|x|}\frac{\pi}{4}} e^{-\alpha^2} \frac{d\alpha}{\sqrt{r|x|}} \right) r dr. \tag{4.17}$$

Since $r|x| \geq \frac{1}{\sqrt{\mu t}}\left(\frac{1}{\sqrt{\mu t}} - 1\right) \approx \frac{1}{\mu t} \geq 1$, then we obtain that

$$\int_0^{\sqrt{r|x|}\frac{\pi}{4}} e^{-\alpha^2} \frac{d\alpha}{\sqrt{r|x|}} \geq \frac{1}{\sqrt{r|x|}} \int_0^{\frac{\pi}{4}} e^{-\alpha^2} d\alpha = \frac{c}{\sqrt{r|x|}}.$$

Consequently for $\frac{1}{\sqrt{\mu t}} - 1 \leq |x| \leq \frac{1}{\sqrt{\mu t}}$, the formula (4.17) takes the following form

$$|Z(t,x)| \geq C \int_{\frac{1}{\sqrt{\mu t}}}^{\frac{2}{\sqrt{\mu t}}} e^{-\frac{1}{4}(|x|-r)^2} \sqrt{\frac{r}{|x|}} dr.$$

But, $\frac{r}{|x|} \geq \frac{1}{\sqrt{\mu t}}\sqrt{\mu t} = 1$ and hence

$$|Z(t,x)| \geq C \int_{\frac{1}{\sqrt{\mu t}}}^{\frac{2}{\sqrt{\mu t}}} e^{-\frac{1}{4}(|x|-r)^2} dr.$$

Making the change of variables $k = r - |x|$, we readily get

$$|Z(t,x)| \geq C \int_{\frac{1}{\sqrt{\mu t}} - |x|}^{\frac{2}{\sqrt{\mu t}} - |x|} e^{-\frac{1}{4}k^2} dk.$$

However, $\frac{1}{\sqrt{\mu t}} - |x| \leq 1$ and $\frac{2}{\sqrt{\mu t}} - |x| \geq \frac{1}{\sqrt{\mu t}}$. This leads to

$$|Z(t,x)| \geq C \int_1^{\frac{1}{\sqrt{\mu t}}} e^{-\frac{1}{4}k^2} dk \geq C > 0.$$

Therefore, for $\frac{1}{\sqrt{\mu t}} - 1 \leq |x| \leq \frac{1}{\sqrt{\mu t}}$, it follows

$$|Z(t,x)| \geq C. \tag{4.18}$$

Taking the L^p-norm for (4.18) over the annulus $\frac{1}{\sqrt{\mu t}}-1\leq |x|\leq \frac{1}{\sqrt{\mu t}}$, it holds

$$\begin{aligned}\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})} &\geq C\Big[\mathscr{L}\Big(\frac{1}{\sqrt{\mu t}}-1\leq |x|\leq \frac{1}{\sqrt{\mu t}}\Big)\Big]^{\frac{1}{p}}\\ &\geq C\Big[\pi\Big(\frac{2}{\sqrt{\mu t}}-1\Big)\Big]^{\frac{1}{p}}\\ &\geq \widetilde{C}(\mu t)^{-\frac{1}{2p}},\end{aligned}$$

where $\mathscr{L}$ is the Lebesgue measure over $\mathbb{R}^2$. Hence,

$$\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})}\geq C_1(\mu t)^{-\frac{1}{2p}}.$$

This leads to the desired estimate stated in (4.16). Combining the last estimate with (4.15), we end up with

$$\|W(t)\|_{L^p(\mathbb{R}^2)}\geq C_1(\mu t)^{\frac{1}{2p}}.$$

Now, the proof is completed. □

Appendix

This section cares with the detailed proof of two Propositions 4.4, 4.5 which are used respectively during the proof of Theorem 3.8 and Proposition 4.1.

PROPOSITION 4.4. *Let $\varepsilon\in]0,1[,\rho$ be a smooth function and v be a smooth divergence-free vector field on $\mathbb{R}^2$ with vorticity ω. Assume that $v\in L^2,\omega\in L^2\cap L^\infty$ and $\rho\in L^2\cap L^p$, with $p>\frac{2}{1-\varepsilon}$. Then the following statement holds true,*

$$\big\|[\mathcal{L},v\cdot\nabla]\rho\big\|_{C^\varepsilon}\leq C_0.$$

PROOF. Recall from [**28**] the following commutator estimate,

$$\big\|[\mathcal{L},v\cdot\nabla]\rho\big\|_{C^\varepsilon}\lesssim \|v\|_{L^2}\|\rho\|_{L^2}+\|\omega\|_{L^2\cap L^\infty}\|\rho\|_{L^p},\quad p>\frac{2}{1-\varepsilon}. \tag{4.19}$$

Let us estimate the first term of the right-hand side of (4.19). To do this, we apply the energy estimate for the velocity equation, we shall have

$$\|v(t)\|_{L^2}\leq \|v_0\|_{L^2}+\int_0^t\|\rho(\tau)\|_{L^2}d\tau.$$

A new use of [**28**] gives

$$(1+t)^{\frac{1}{2}}\|\rho(t)\|_{L^2}\lesssim \|\rho_0\|_{L^1\cap L^2}$$

thus we obtain

$$\|v(t)\|_{L^2}\leq C_0(1+t)^{\frac{1}{2}}.$$

Combining the last two estimates, we readily get

$$\|v(t)\|_{L^2}\|\rho(t)\|_{L^2}\leq C_0. \tag{4.20}$$

An usual interpolation inequality between the Lebesgue spaces yields for $p\in [2,+\infty[$

$$\begin{aligned}\|\rho(t)\|_{L^p} &\leq \|\rho(t)\|_{L^2}^{\frac{2}{p}}\|\rho_0\|_{L^\infty}^{1-\frac{2}{p}}\\ &\leq C_0(1+t)^{-\frac{1}{p}}.\end{aligned} \tag{4.21}$$

Here we have used the maximum principle for the density equation. Putting together (4.20), (4.21) and Proposition 2.12, we finally get

$$\begin{aligned}\big\|\big[\mathcal{L}, v\cdot\nabla\big]\rho\big\|_{C^\epsilon} &\le C_0 + C_0(1+t)^{-\frac{1}{p}}\log^2(2+t)\\ &\le C_0.\end{aligned}$$

This completes the proof. □

For the reader's convenience we state the following classical result.

PROPOSITION 4.5. *The following Sobolev embedding is hold.*

$$BV \hookrightarrow \dot{B}^1_{1,\infty}.$$

PROOF. According to [**31, 36**] the equivalent norm to $\dot{B}^s_{p,r}$ is defined for $\ell \in \mathbb{N}^*$, $0 < s < \ell$ and $(p,r)\in[1,\infty]^2$ by

$$|||u|||_{\dot{B}^s_{p,r}} \triangleq \left(\int_{\mathbb{R}^N} |h|^{-sr}\|\mathbf{\Delta}^\ell_h f(x)\|^r_{L^p}\frac{dh}{|h|^N}\right)^{\frac{1}{r}}.$$

Here the difference operators $\mathbf{\Delta}^\ell_h$ are given by

$$\mathbf{\Delta}^1_h = \mathbf{\Delta}_h, \quad \mathbf{\Delta}^{\ell+1}_h = \mathbf{\Delta}_h\circ\mathbf{\Delta}^\ell_h \quad \forall \ell\in\mathbb{N}^*,$$

where $\mathbf{\Delta}_h$ is defined for every $u\in\mathcal{S}'(\mathbb{R}^N)$ and $h\in\mathbb{R}^N$ by

$$\mathbf{\Delta}_h u(x) \triangleq u(x+h) - u(x).$$

From (2.1), we have for $q\in\mathbb{Z}$ and $x\in\mathbb{R}^2$

$$\dot{\Delta}_q u(x) = 2^{2q}\int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(2^q(x-y))u(y)dy,$$

with $\mathscr{F}^{-1}\varphi$ denotes the inverse Fourier of φ. As $\varphi(0)=0$ then

$$\dot{\Delta}_q u(x) = 2^{2q}\int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(2^q(x-y))\big(u(y)-u(x)\big)dy.$$

So, by making a change of variable $z = 2^d(x-y)$, we obtain

$$\begin{aligned}\dot{\Delta}_q u(x) &= 2^{2q}\int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(2^q(x-y))(u(y)-u(x))dy\\ &= \int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(z)(u(x-2^{-q}z)-u(x))dz\\ &= \int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(z)\mathbf{\Delta}_h u(x)dz, \quad h=-2^{-q}z.\end{aligned}$$

Fubini's theorem implies

$$\|\dot{\Delta}_q u\|_{L^1} \le \int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(z)\|\mathbf{\Delta}_h u\|_{L^1}dz. \tag{4.22}$$

We recall from Theorem 13.48 page 415 in [**31**] the following result

$$\begin{aligned}\|\mathbf{\Delta}_h u\|_{L^1} &\le |h||Du|(\mathbb{R}^2)\\ &= 2^{-q}|z||Du|(\mathbb{R}^2).\end{aligned}$$

Consequently

$$\|\mathbf{\Delta}_h u\|_{L^1} \le 2^{-q}|z|\|u\|_{BV}.$$

Inserting the last estimate in (4.22), we get for $q \in \mathbb{Z}$

$$\|\dot{\Delta}_q u(x)\|_{L^1} \leq 2^{-q}\|u\|_{BV} \int_{\mathbb{R}^2} \mathscr{F}^{-1}\varphi(z)|z|dz.$$

By taking the supremum over $q \in \mathbb{Z}$, we obtain the aimed estimate. □

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