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Degenerate Non-Newtonian Fluid Equation on the Half Space

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Abstract. The degenerate non-Newtonian fluid equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - \sum_{i=1}^{N} f_i(x)D_i u = g(u,x,t), (x,t) \in \mathbb{R}_+^N \times (0,T)$$

arises in several scientific fields. When $a(x)$ and p satisfy certain conditions, the existence of solution of this equation is established. When $a^{-\frac{1}{p}}(x)f_i(x) \leq c$ for $i \in \{1, 2, \cdots, N\}$, by choosing a suitable test function, the local stability of the solutions is discussed without any boundary value condition.

Contents

1. Introduction

Consider the degenerate non-Newtonian fluid equation [**2**]:

(1.1)

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) - \sum_{i=1}^{N} f_i(x)D_i u + c(x,t)u = g(x,t),\ (x,t) \in \Omega \times (0,T)$$

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with the initial condition

$$u(x,0)=u_0(x),\ x\in\Omega, \tag{1.2}$$

where $D_i=\frac{\partial}{\partial x_i}$, $a(x)\geq 0\in C(\overline{\Omega})$, and Ω is a bounded domain in $\mathbb{R}^N$. By means of a reasonable integral description, the boundary can be classified into three parts: the nondegenerate boundary, the weakly degenerate boundary and the strongly degenerate boundary. Instead of the usual boundary condition

$$u(x,t)=0,\ (x,t)\in\partial\Omega\times(0,T), \tag{1.3}$$

the new boundary value condition was reasonably formulated to establish the existence and uniqueness results [**2**]. Benedikt et al [**3, 4**] studied the equation

$$u_t=\operatorname{div}(|\nabla u|^{p-2}\nabla u)+q(x)|u|^{\alpha-1}u,\ (x,t)\in Q_T \tag{1.4}$$

with $0<\alpha<1$, and found that the uniqueness of the solution of equation (1.4) is not true. Zhan [**5**] considered the equation

$$\frac{\partial u}{\partial t}=\operatorname{div}(d^\alpha|\nabla u|^{p-2}\nabla u)+g(u,x,t),\ (x,t)\in\Omega\times(0,T), \tag{1.5}$$

and discussed the stability of solutions dependent on the initial condition (1.2), but independent of the boundary value condition (1.3), where $d(x)=\operatorname{dist}(x,\partial\Omega)$ and $\alpha>0$ is a constant. By comparing (1.4) with (1.5), the degeneracy of d^α might not only counteract the effect from the source term $g(u,x,t)$, but also take place of the boundary value condition (1.3).

In this study, we consider the spatial variable in the half space

$$\mathbb{R}^N_+=\{x\in\mathbb{R}^N: x^N>0\}$$

and generalize equation (1.1) to

$$\frac{\partial u}{\partial t}-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)-\sum_{i=1}^N f_i(x)D_iu=g(u,x,t),\ (x,t)\in Q_T=\mathbb{R}^N_+\times(0,T). \tag{1.6}$$

The basic assumption is that $a(x)\in C(\overline{\mathbb{R}^N_+})$ satisfies

$$a(x)>0,\ x\in\mathbb{R}^N_+,\ a(x)=0,\ x\in\partial\mathbb{R}^N_+. \tag{1.7}$$

In addition to the nonlinear source term $g(u,x,t)$, equation (1.6) contains a linear convection term D_iu. We follow with interest in whether the degeneracy of $a(x)$ on the boundary can also counteract the effect from the convection term.

Let us recall the definitions of weak solutions and summarize our main results.

Definition 1.1. A function $u(x,t)$ is said to be a weak solution of equation (1.6) with the initial condition (1.2), if

$$u\in L^\infty(Q_T)\bigcap L^2(Q_T),\ u_t\in L^2(Q_T),\ a(x)\,|\nabla u|^p\in L^1(Q_T), \tag{1.8}$$

and for any function $\varphi\in C_0^\infty(Q_T)$ there holds

$$\iint_{Q_T}\Big\{-u\varphi_t+a(x)\,|\nabla u|^{p-2}\,\nabla u\cdot\nabla\varphi+u\sum_{i=1}^N[f_{ix_i}(x)\varphi+f_i(x)\varphi_{x_i}]$$

$$-g(u,x,t)\varphi\Big\}dxdt=0, \tag{1.9}$$

where $f_{ix_i}=\frac{\partial f_i(x)}{\partial x_i}$.

DEFINITION 1.2. A function $u(x,t)$ is said to be a weak solution of equation (1.6) with the initial condition (1.2) and the boundary value condition (1.3) (where the boundary $\partial\Omega$ is replaced by $\partial\mathbb{R}^N$), if it satisfies (1.8)-(1.9) and (1.3) is satisfied in the sense of the trace.

For simplicity, we assume that $a(x)$, $f_i(x)$ and $g(s,x,t)$ are C^1 functions and first restrict our attention to the existence of weak solutions.

THEOREM 1.3. *Suppose that $a(x)$, $f_i(x)$ and $f_{ix_i}(x)$ are bounded functions when $p>2$ and $g(s,x,t)\in L^2(Q_T)$ for $|s|\le c$. If*

$$u_0(x)\in L^\infty(\mathbb{R}^N_+)\bigcap L^2(\mathbb{R}^N_+),\ \ a(x)|\nabla u_0(x)|^p\in L^1(\mathbb{R}^N_+), \tag{1.10}$$

$$\int_{\mathbb{R}^N_+} a(x)^{-\frac{2}{p-2}}(x)dx<\infty \ \ \text{and}\ \ \int_{\mathbb{R}^N_+} a(x)^{-\frac{1}{p-1}}(x)dx<\infty, \tag{1.11}$$

then there exists a weak solution u of equation (1.6) with initial boundary value conditions (1.2) and (1.3). If $g(u,x,t)\ge 0$ with $u_0(x)\ge 0$, then the solution u is nonnegative.

If $\Omega\subset\mathbb{R}^N$ is a bounded domain and $\int_\Omega a^{-\frac{1}{p-1}}(x)dx<\infty$, the well-posedness of equation (1.1) had been presented by Yin-Wang [**2**]. Roughly speaking, the condition $\int_{\mathbb{R}^N_+} a(x)^{-\frac{1}{p-1}}(x)dx<\infty$ can induce the weak solution $u\in W^{1,\gamma}_{loc}(\mathbb{R}^N)$ for $\gamma>1$, so that one can define the trace of u on the boundary. Although $\int_\Omega a(x)^{-\frac{2}{p-2}}(x)dx<\infty$ implies $\int_\Omega a(x)^{-\frac{1}{p-1}}(x)dx<\infty$ in a bounded domain Ω, but the domain $\mathbb{R}^N_+$ considered in our study is unbounded, so the two assumptions in (1.11) have their independent senses.

The main purpose of this paper is to study the stability of weak solutions, so we do not pay much attention on the optimal conditions to ensure the existence of weak solutions. There is one more point, which we should touch on that, if $\int_\Omega a(x)^{-\frac{1}{p-1}}(x)dx<\infty$, the uniqueness of weak solution to equation (1.6) with the usual initial-boundary value conditions (1.2)-(1.3) can be proved in a similar way as those in [**2**], but we do not plan to demonstrate our discussions on the uniqueness at this stage and will be presenting them in a subsequent work together with the variational methods [**17**] and mountain pass theorem [**18**]. Instead, here we focus on the stability of weak solutions of equation (1.6) without any boundary value condition.

THEOREM 1.4. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.6) with the initial values $u_0(x)$ and $v_0(x)$ respectively. If*

$$a^{-\frac{1}{p}}(x)f_i(x)\le c,\quad 1\le i\le N, \tag{1.12}$$

then there exists a constant $\beta\ge\max\left\{\frac{p}{p-1},2\right\}$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N_+} e^{-x^N}\left(x^N\right)^\beta|u(x,t)-v(x,t)|^2dx\\ \le&\int_{\mathbb{R}^N_+} e^{-x^N}\left(x^N\right)^\beta|u_0(x)-v_0(x)|^2dx,\ \ a.e.\ t\in[0,T),\end{aligned} \tag{1.13}$$

where $x=\left\{x^1,x^2,\cdots,x^{N-1},x^N\right\}$.

Inequality (1.13) is regarded as the local stability of weak solutions. According to (1.13), we know that the weak solution of equation (1.6) with the initial condition (1.2) is unique. Certainly, condition (1.12) implies that $f_i(x) = 0$ on the boundary $\partial\mathbb{R}^N_+$. From the physical point of view, $f_i(x)D_iu$ represents the convection phenomena. Theorem 1.4 tells us that the degeneracy of convection might take place of the usual boundary value condition (1.3). Actually, we have the following theorem to supplement our conclusion.

THEOREM 1.5. *Let $u(x,t)$ be the unique nonnegative bounded solution of equation (1.6) with the initial condition (1.2). If $f_i(x)$ is bounded, $|g(u,x,t)| \le c$ for a given $s \in (0,T)$, and M is a constant such that*

$$u(x,t) \le M, \ (x,t) \in \mathbb{R}^N_+ \times (s,T),$$

then there holds

$$u(x,t) \le Cd(x), \ (x,t) \in \mathbb{R}^N_+ \times (s,T), \tag{1.14}$$

where the constant C depending upon M, N, p and s, and $d(x) = dist(x, \partial\mathbb{R}^N_+) = x^N$.

Inequality (1.14) indicates that the unique solution of equation (1.6) with the initial condition (1.2) has the homogeneous boundary value, and so the usual boundary value condition (1.3) becomes redundant. Compared with the existing results in the literature, for example, see [**2**], from a technical perspective, the obstacle not only comes from the degeneracy of $a(x)$ on the boundary, but also comes from the unboundedness of the half space.

We would like to mention here that if the domain $\Omega \subset \mathbb{R}^N$ is a bounded domain, one can obtain the analogous results by a similar but simpler way, see [**6, 7, 8**]. In the past years we have been interested in and working on the problem that the solution is free from the limitation of boundary value conditions, see [**9, 10**].

The rest of the paper is organized as follows. Proof of Theorem 1.3 is presented in Section 2 and proof of Theorem 1.4 is shown in Section 3. Section 4 is dedicated to estimates near the boundary, and Section 5 is an appendix on the Fichera-Oleinik theory.

2. Proof of Theorem 1.3

In this section, we considers the initial value problem for equation (1.6). For any positive integer n, we denote by

$$x^n = (0,0,\cdots,0,n) \ \text{ and } \ B_n(x^n) = \{x \in \mathbb{R}^N_+ : |x - x^n| < n\}.$$

To prove the existence of the solution of equation (1.6), we first consider the following regularized problem

$$\begin{aligned} &u_{nt} - \operatorname{div}\left\{\left[a(x) + \frac{1}{n}\right]\left(|\nabla u_n|^2 + \frac{1}{n}\right)^{\frac{p-2}{2}} \nabla u_n\right\} - \sum_{i=1}^N f_i(x)D_iu \\ &= g(u_n,x,t), \ (x,t) \in Q_{Tn}, \\ &u_n(x,t) = 0, \ (x,t) \in \partial B_n \times (0,T), \\ &u_n(x,0) = u_{0n}(x), \ x \in B_n, \end{aligned} \tag{2.1}$$

where $B_n = B_n(x^n)$, $Q_{Tn} = B_n \times (0,T)$, $u_{0n} \in C_0^\infty(\mathbb{R}^N_+)$ and $\text{supp}u_{on} \subset B_n$. In addition, $\left(a(x) + \frac{1}{n}\right)|\nabla u_{0n}|^p \in L^1(\mathbb{R}^N_+)$ is uniformly bounded, u_{0n} converges to u_0 in $L^2(\mathbb{R}^N_+)$, and u_{0n} converges to u_0 in $W^{1,p}_0(\mathbb{R}^N_+)$. As we know [**11**], the above problem has a unique classical solution u_n and there exists a constant c only dependent on $\|u_0\|_{L^\infty(\mathbb{R}^N_+)}$ and $\|u_0\|_{L^2(\mathbb{R}^N_+)}$ such that

$$\|u_n\|_{L^\infty(Q_{Tn})} \leqslant cT \text{ and } \|u_n\|_{L^2(Q_{Tn})} \leqslant cT. \tag{2.2}$$

LEMMA 2.1. *If $\int_{\mathbb{R}^N_+} a(x)^{-\frac{2}{p-2}}(x)dx < \infty$ and $f_{ix_i}(x)$ is bounded, then there is a subsequence of u_n (here we still denote it by u_n), which converges to a weak solution u of equation (1.6) with the initial condition (1.2).*

PROOF. Since $a(x)$, $b_i(x)$ and $g(s,x,t)$ are bounded when $|s| \le c$, multiplying both sides of (2.1) by u_n and integrating it over Q_{Tn}, by (2.2) we have

$$\begin{aligned}
&\frac{1}{2}\int_{B_n} u_n^2 dx + \iint_{Q_{Tn}} \left[a(x) + \frac{1}{n}\right]\left(|\nabla u_n|^2 + \frac{1}{n}\right)^{\frac{p-2}{2}} |\nabla u_n|^2 dxdt \\
\le\ & \frac{1}{2}\int_{B_n} u_{0n}^2 dx + \iint_{Q_{Tn}} \left(\sum_{i=1}^N |f_i(x)D_i u_n||u_n| + |u_n g(u_n,x,t)|\right) dxdt \\
\leqslant\ & c,
\end{aligned} \tag{2.3}$$

which is due to

$$\begin{aligned}
&\iint_{Q_{Tn}} |f_i(x)D_i u_n||u_n| dxdt \\
=\ & \iint_{Q_{Tn}} \left|f_i(x)a^{-\frac{1}{p}}(x)a^{-\frac{1}{p}}(x)D_i u_n\right| |u_n| dxdt \\
\le\ & \iint_{Q_{Tn}} \left[c(\varepsilon)\left|f_i(x)a^{-\frac{1}{p}}(x)\right|^{\frac{p}{p-1}} + \varepsilon a(x)|\nabla u_n|^p\right] dxdt \\
\le\ & c(\varepsilon)\iint_{Q_{Tn}} \left|f_i(x)a^{-\frac{1}{p}}(x)\right|^{\frac{p}{p-1}} dxdt + \varepsilon \iint_{Q_{Tn}} a(x)|\nabla u_n|^p dxdt \\
\le\ & c(\varepsilon)\iint_{Q_{Tn}} a^{-\frac{1}{p-1}}(x) dxdt + \varepsilon \iint_{Q_{Tn}} a(x)|\nabla u_n|^p dxdt \\
\le\ & c.
\end{aligned}$$

For any bounded domain $\Omega \subset \mathbb{R}^N_+$, since $p > 2$, by (1.8) and (2.3) we have

$$\int_0^T\int_\Omega |\nabla u_n|^2 dxdt \le c(\Omega)\left(\int_0^T\int_\Omega |\nabla u_n|^p dxdt\right)^{\frac{1}{p}} \le c(\Omega). \tag{2.4}$$

Multiplying both sides of (2.1) by u_{nt} and integrating it over Q_{Tn} gives

$$\begin{aligned}
&\iint_{Q_{Tn}} (u_{nt})^2 dxdt \\
=\ & \iint_{Q_{Tn}} \operatorname{div}\left[\left(a(x) + \frac{1}{n}\right)\left(|\nabla u_n|^2 + \frac{1}{n}\right)^{\frac{p-2}{2}}\right] \cdot u_{nt} dxdt \\
& + \sum_{i=1}^N \iint_{Q_{Tn}} u_{nt} f_i(x) D_i(u_n) dxdt + \iint_{Q_{Tn}} g(u_n,x,t) u_{nt} dxdt.
\end{aligned} \tag{2.5}$$

In view of $|f_i(x)| \le c$ and $\int_{\mathbb{R}^N_+} a^{-\frac{2}{p-2}}(x)dx < \infty$ for $p > 2$, we find

$$\begin{aligned}\iint_{Q_{Tn}} a^{\frac{-2}{p-2}}(x)|f_i(x)|^{\frac{2p}{p-2}}dxdt &\le c\iint_{Q_{Tn}} a^{\frac{-2}{p-2}}(x)dxdt \\ &\le c. \end{aligned} \tag{2.6}$$

Using the Hölder's inequality and (2.6) leads to

$$\begin{aligned}&\iint_{Q_{Tn}} u_{nt}f_i(x)D_iu_ndxdt \\ \leqslant\ & \frac{1}{4}\iint_{Q_{Tn}}(u_{nt})^2dxdt + c\sum_{i=1}^{N}\iint_{Q_{Tn}} f_i^2(x)|\nabla u_n|^2dxdt \\ &+\frac{1}{4}\iint_{Q_{Tn}}(u_{nt})^2dxdt \\ \le\ & \sum_{i=1}^{N}\left(\iint_{Q_{Tn}} a^{\frac{-2}{p-2}}(x)|f_i(x)|^{\frac{2p}{p-2}}dxdt\right)^{\frac{p-2}{p}}\left(\iint_{Q_{Tn}} a(x)|\nabla u_n|^pdxdt\right)^{\frac{2}{p}} \\ \le\ & \frac{1}{4}\iint_{Q_{Tn}}(u_{nt})^2dxdt + c. \end{aligned} \tag{2.7}$$

Since $g(s,x,t) \in L^2(Q_T)$ when $|s| \le c$, by (2.5) it is clear to see that

$$\left|\iint_{Q_{Tn}} g(u_n,x,t)u_{nt}dxdt\right| \le \frac{1}{4}\iint_{Q_{Tn}}(u_{nt})^2dxdt + c. \tag{2.8}$$

Combining (2.7)-(2.8), we have

$$\iint_{Q_{Tn}}(u_{nt})^2dxdt + \iint_{Q_{Tn}}\left[a(x)+\frac{1}{n}\right]\frac{d}{dt}\int_0^{|\nabla u_n(x,t)|^2+\frac{1}{n}} s^{\frac{p-2}{2}}dsdxdt \le c, \tag{2.9}$$

and thus

$$\iint_{Q_{Tn}}(u_{nt})^2dxdt \le c. \tag{2.10}$$

Let

$$\bar{u}_n = \begin{cases} u_n, & \text{if } x \in B_n, \\ 0, & \text{if } x \in \mathbb{R}^N_+ \setminus B_n. \end{cases}$$

According to (2.9)-(2.10), we have

$$\int_0^T\int_{\mathbb{R}^N_+}\left[a(x)+\frac{1}{n}\right] \mid \nabla\bar{u_n} \mid^p dxdt \le c \tag{2.11}$$

and

$$\int_0^T\int_{\mathbb{R}^N_+}(\bar{u_{nt}})^2dxdt \le c. \tag{2.12}$$

From (2.2), (2.4), (2.11) and (2.12), there exists a function u and an $n-$dimensional vector function $\overrightarrow{\zeta} = (\zeta_1, \cdots, \zeta_n)$ satisfying that

$$u \in L^\infty(Q_T),\ \ u_t \in L^2(Q_T),\ \ \left|\overrightarrow{\zeta}\right| \in L^{\frac{p}{p-1}}(Q_T),$$

and

$$\bar{u_n} \rightharpoonup *u, \quad \text{in} \quad L^\infty(Q_T),$$

$$\bar{u_n} \to u \text{ in } \quad L^2_{loc}(Q_T),$$
$$\left[a(x) + \frac{1}{n}\right] |\nabla \bar{u_n}|^{p-2} \nabla \bar{u_n} \rightharpoonup \overrightarrow{\zeta} \quad \text{in} \quad L^{\frac{p}{p-1}}(Q_T).$$

To prove that u satisfies equation (1.6), we notice that for any function $\varphi \in C_0^\infty(Q_T)$, there holds

$$\begin{aligned} &\iint_{Q_T} \Big\{ - \bar{u_n}\varphi_t + \left[a(x) + \frac{1}{n}\right] \left[|\nabla \bar{u_n}|^2 + \frac{1}{n}\right]^{\frac{p-2}{2}} \nabla \bar{u_n} \cdot \nabla \varphi \\ &+ \bar{u_n} \sum_{i=1}^{N} [f_{ix_i}(x)\varphi + f_i(x)\varphi_{x_i}] + g(\bar{u_n}, x, t)\varphi \Big\} dxdt = 0. \end{aligned} \tag{2.13}$$

Since

$$\bar{u_n} \to u \text{ in } \quad L^2_{loc}(Q_T),$$

we know $\bar{u_n} \to u$ a.e. in Q_T. Let $n \to \infty$. It follows from (2.13) that

$$\iint_{Q_T} \left\{ \frac{\partial u}{\partial t}\varphi + \vec{\varsigma} \cdot \nabla\varphi + u \sum_{i=1}^{N} [f_{ix_i}(x)\varphi + f_i(x)\varphi_{x_i}] + g(u, x, t)\varphi \right\} dxdt = 0. \tag{2.14}$$

Following [**11, 12**], we obtain

$$\iint_{Q_T} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \mathrm{d}x\mathrm{d}t = \iint_{Q_T} \overrightarrow{\zeta} \cdot \nabla \varphi \mathrm{d}x\mathrm{d}t \tag{2.15}$$

for any function $\varphi \in C_0^\infty(Q_T)$. By combining (2.14) and (2.15), we arrive at (1.9). □

LEMMA 2.2. *If $\int_{\mathbb{R}^N_+} a(x)^{-\frac{1}{p-1}}(x)dx < \infty$, and u is a weak solution of equation (1.6) with the initial condition (1.2). Then the trace of u on the boundary $\partial\mathbb{R}^N_+$ can be defined in the traditional way.*

PROOF. If we denote $Q_{\Omega T} = \Omega \times (0, T)$, then

$$\begin{aligned} \iint_{Q_{\Omega T}} |\nabla u|\, dxdt &= \iint_{\left\{(x,t)\in Q_{\Omega T}; a^{\frac{1}{p-1}} |\nabla u| \leqslant 1\right\}} |\nabla u|\, dxdt \\ &\quad + \iint_{\left\{(x,t)\in Q_{\Omega T}; a^{\frac{1}{p-1}} |\nabla u| > 1\right\}} |\nabla u|\, dxdt \\ &\leqslant \iint_{Q_{\Omega T}} a^{-\frac{1}{p-1}} dxdt + \iint_{Q_{\Omega T}} a\, |\nabla u|^p\, dxdt \\ &\leqslant c. \end{aligned}$$

This is due to the assumption that $\int_{\mathbb{R}^N_+} a(x)^{-\frac{1}{p-1}}(x)dx \leq c$, i.e.

$$\iint_{Q_{\Omega T}} |\nabla u| \mathrm{d}x\mathrm{d}t \leqslant c + c(\Omega).$$

Hence, ∇u is uniformly bounded in $L^1(Q_{\Omega T})$ and u has the trace on the boundary $\partial\Omega$. In particular, for the arbitrary Ω, we can define the trace of u on $\partial\mathbb{R}^N_+$. □

Proof of Theorem 1.3 follows from Lemmas 2.1 and 2.2 immediately.

3. Proof of Theorem 1.4

Proof of Theorem 1.4. Let u and v be the two weak solutions of equation (1.6) with the different initial values $u_0(x)$ and $v_0(x)$ respectively, and

$$u_0(x) \in L^\infty(Q_T)\bigcap L^2(Q_T), \quad v_0(x) \in L^\infty(Q_T)\bigcap L^2(Q_T),$$

$$a(x)|\nabla u_0|^p \in L^1(Q_T), \quad a(x)|\nabla v_0|^p \in L^1(Q_T).$$

For a small positive constant $\lambda > 0$, let

$$A_\lambda = \{x \in \mathbb{R}^N_+ : x^N < \lambda\}, \ \ d_\lambda = \mathrm{dist}(x, A_\lambda).$$

Then

$$d_\lambda = \begin{cases} 0, & \text{if } x \in A_\lambda, \\ x^N - \lambda, & \text{if } \ x \in \mathbb{R}^N_+ \setminus A_\lambda. \end{cases}$$

Choose the constant $\beta \geq \frac{p}{p-1}$, and let $\chi_{[\tau,s]}$ be the characteristic function on $[\tau, s]$, and u_ε and v_ε be the mollified functions of the solutions u and v respectively. Take $0 \leq \phi_n \in C_0^\infty(\mathbb{R}^N_+)$ and $\phi_n \leq e^{-x^N}$ such that

$$\lim_{n\to\infty} \phi_n(x) = e^{-x^N}. \tag{3.1}$$

Denote that $\Omega_n = supp\phi_n$ is the support set of ϕ_n.

For any fixed $\tau, s \in [0, T]$, we may choose $\chi_{[\tau,s]}(u_\varepsilon - v_\varepsilon)\phi_n(x)d_\lambda^\beta$ as a test function in the weak solution formula (1.9). Denote by $Q_{\tau s} = \mathbb{R}^N_+ \times [\tau, s]$, and we then have

$$\begin{aligned} &\iint_{Q_{\tau s}} (u_\varepsilon - v_\varepsilon)\phi_n(x)d_\lambda^\beta \frac{\partial(u-v)}{\partial t} dxdt \\ = \ & -\iint_{Q_{\tau s}} a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla \left[(u_\varepsilon - v_\varepsilon)\phi_n(x)d_\lambda^\beta\right] dxdt \\ & -\iint_{Q_{\tau s}} \left\{ \left[(u-v)\sum_{i=1}^N f_{ix_i}(x) + (g(u,x,t) - g(v,x,t))\right]\right. \\ & \left. \cdot(u_\varepsilon - v_\varepsilon)\phi_n(x)d_\lambda^\beta \right\} dxdt - \iint_{Q_{\tau s}} (u-v)\sum_{i=1}^N f_i(x)\left[(u_\varepsilon - v_\varepsilon)\phi_n(x)d_\lambda^\beta\right]_{x_i} dxdt. \end{aligned} \tag{3.2}$$

For any given bounded domain $\Omega \subset \mathbb{R}^N_+$, we know that $\nabla u \in L^p(Q_{\Omega T})$ and $\nabla v \in L^p(Q_{\Omega T})$, where $Q_{\Omega T} = \Omega \times (0, T)$. In view of the definitions of the mollified functions u_ε and v_ε, we have

$$u_\varepsilon \in L^\infty(Q_T), \quad v_\varepsilon \in L^\infty(Q_T), \tag{3.3}$$

$$\|\nabla u_\varepsilon\|_{p,\Omega} \leq \|\nabla u\|_{p,\Omega}, \quad \|\nabla v_\varepsilon\|_{p,\Omega} \leq \|\nabla v\|_{p,\Omega}, \tag{3.4}$$

and

$$\lim_{\varepsilon\to 0} \|\nabla u_\varepsilon - \nabla u\|_{p,\Omega} = 0. \tag{3.5}$$

By (3.3)-(3.5) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\tau^s\int_{\Omega_n} a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla\left[(u_\varepsilon-v_\varepsilon)\phi_n(x)d_\lambda^\beta\right]dxdt\\
=&\int_\tau^s\int_{\Omega_n} a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\left[\phi_n(x)(u-v)d_\lambda^\beta\right]dxdt\\
=&\int_\tau^s\int_{\Omega_n} a(x)\phi_n(x)d_\lambda^\beta(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)dxdt\\
&+\int_\tau^s\int_{\Omega_n} a(x)(u-v)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\left[\phi_n(x)d_\lambda^\beta\right]dxdt. \qquad (3.6)
\end{aligned}$$

Thus, it gives

$$\int_\tau^s\int_{\Omega_n} a(x)\phi_n(x)d_\lambda^\beta(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)dxdt\geq 0.$$

Since $x\in\Omega_n$, $\phi_n(x)d_\lambda^\beta>0$, $|\nabla x^N|=1$, and

$$|\nabla d_\lambda|=\begin{cases}0, & \text{if } x\in A_\lambda,\\ 1, & \text{if } x\in\mathbb{R}_+^N\setminus A_\lambda,\end{cases}$$

we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\tau^s\int_{\Omega_n}(u-v)a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla[\phi_n(x)d_\lambda^\beta]dxdt\right|\\
\leq&\lim_{n\to\infty}\int_\tau^s\int_{\Omega_n}|u-v|a(x)(|\nabla u|^{p-1}+|\nabla v|^{p-1})\left|\nabla\left[\phi_n(x)d_\lambda^\beta\right]\right|dxdt\\
\leq& c\lim_{n\to\infty}\left(\int_\tau^s\int_{\Omega_n}a(x)\left(|\nabla u|^p+|\nabla v|^p\right)dxdt\right)^{\frac{p-1}{p}}\\
&\cdot\left(\int_\tau^s\int_{\Omega_n}a(x)|\nabla\left[\phi_n(x)(x^N-\lambda)^\beta\right]|^p|u-v|^pdxdt\right)^{\frac{1}{p}}\\
\leq& c\left(\int_\tau^s\int_{\mathbb{R}_+^N}a(x)\left(|\nabla u|^p+|\nabla v|^p\right)dxdt\right)^{\frac{p-1}{p}}\\
&\cdot\left(\int_\tau^s\int_{\mathbb{R}_+^N}a(x)(e^{-x^N})^p(x^N-\lambda)^{p(\beta-1)}[(x^N-\lambda)^p+1]|u-v|^pdxdt\right)^{\frac{1}{p}}\\
\leq& c\left(\int_\tau^s\int_{\mathbb{R}_+^N}(e^{-x^N})^p(x^N-\lambda)^{p(\beta-1)}\left[(x^N-\lambda)^p+1\right]a(x)|u-v|^pdxdt\right)^{\frac{1}{p}}, \qquad (3.7)
\end{aligned}$$

and further obtain

$$\begin{aligned}
\lim_{\lambda\to 0}\lim_{n\to\infty}&\left|\int_\tau^s\int_{\Omega_n}(u-v)a(x)(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\right.\\
&\left.\cdot\nabla[\phi_n(x)d_\lambda^\beta]dxdt\right|\\
\leq& c\left(\int_\tau^s\int_{\mathbb{R}_+^N}(e^{-x^N})^p(x^N)^{p(\beta-1)}\left[(x^N)^p+1\right]|u-v|^pdxdt\right)^{\frac{1}{p}}. \qquad (3.8)
\end{aligned}$$

If $p \geq 2$, since $u, v \in L^\infty(Q_T)$, then

$$\left(\int_\tau^s \int_{\mathbb{R}_+^N} (e^{-x^N})^p (x^N)^{p(\beta-1)} \left[(x^N)^p + 1\right] |u-v|^p dxdt\right)^{\frac{1}{p}}$$

$$\leq c\left(\int_\tau^s \int_{\mathbb{R}_+^N} (e^{-x^N})^p (x^N)^{p(\beta-1)} \left[(x^N)^p + 1\right] |u-v|^2 dxdt\right)^{\frac{1}{p}}. \tag{3.9}$$

In view of $\beta > \frac{p}{p-1}$, when $0 < x^N < 1$, it has

$$\begin{aligned} & (e^{-x^N})^p (x^N)^{p(\beta-1)} \left[(x^N)^p + 1\right] \\ \leq \ & e^{-x^N} (x^N)^\beta (e^{-x^N})^{p-1} \left[(x^N)^p + 1\right] \\ \leq \ & ce^{-x^N} \left(x^N\right)^\beta, \end{aligned} \tag{3.10}$$

because of the fact that (3.1) implies $(e^{-x^N})^{p-1}\left[(x^N)^p + 1\right] \leq c$.

When $x^N \geq 1$, we find

$$\begin{aligned} & (e^{-x^N})^p (x^N)^{p(\beta-1)} \left[(x^N)^p + 1\right] \\ = \ & e^{-x^N} (x^N)^\beta (e^{-x^N})^{p-1}(x)(x^N)^{p(\beta-1)-\beta} \left[(x^N)^p + 1\right] \\ \leq \ & ce^{-x^N} \left(x^N\right)^\beta, \end{aligned} \tag{3.11}$$

due to the fact that (3.1) also implies $(e^{-x^N})^{p-1}(x^N)^{p(\beta-1)-\beta}\left[(x^N)^p + 1\right] \leq c$.

Thus, in the case of $p \geq 2$, based on (3.8)-(3.11) we have

$$\begin{aligned} & \left(\int_\tau^s \int_{\mathbb{R}_+^N} (e^{-x^N})^p (x^N)^{p(\beta-1)} \left[(x^N)^p + 1\right] |u-v|^p dxdt\right)^{\frac{1}{p}} \\ \leq \ & c\left(\int_\tau^s \int_{\mathbb{R}_+^N} (e^{-x^N})^p \left(x^N\right)^{p(\beta-1)} [(x^N)^p + 1]|u-v|^2 dxdt\right)^{\frac{1}{p}} \\ \leq \ & \left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta |u-v|^2 dxdt\right)^{\frac{1}{p}}. \end{aligned} \tag{3.12}$$

In case of $1 < p < 2$, by (3.1) we find

$$\int_\tau^s \int_{\mathbb{R}_+^N} (e^{-x^N})^p [(x^N)^{p(\beta-1)-\frac{\beta}{2}}[(x^N)^p + 1]]^{\frac{2}{2-p}} dxdt \leq c. \tag{3.13}$$

Using the generalized Hölder inequality [**19**], we further get

$$\left(\int_\tau^s\int_{\mathbb{R}^N_+}(e^{-x^N})^p(x^N)^{p(\beta-1)}[(x^N)^p+1]|u-v|^p|u-v|^p dxdt\right)^{\frac{1}{p}}$$

$$\leq c\left(\int_\tau^s\int_{\mathbb{R}^N_+}(e^{-x^N})^p(x^N)^{\beta}|u-v|^2 dxdt\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_\tau^s\int_{\mathbb{R}^N_+}(e^{-x^N})^p[(x^N)^{p(\beta-1)-\frac{\beta}{2}}[(x^N)^p+1]]^{\frac{2}{2-p}} dxdt\right)^{\frac{2-p}{2}}$$

$$\leq c\left(\int_\tau^s\int_{\mathbb{R}^N_+}e^{-x^N}(x^N)^{\beta}|u-v|^2 dxdt\right)^{\frac{1}{2}}. \tag{3.14}$$

We then deduce that

$$\lim_{\varepsilon\to 0}\int_\tau^s\int_{\Omega_n}f_i(x)(u-v)[(u_\varepsilon-v_\varepsilon)\phi_n(x)d_\lambda^\beta]_{x_i}dxdt$$

$$= \int_\tau^s\int_{\Omega_n}f_i(x)(u-v)[(u-v)\phi_n(x)d_\lambda^\beta]_{x_i}dxdt$$

$$= \int_\tau^s\int_{\Omega_n}f_i(x)(u-v)^2\phi_n(x)(d_\lambda^\beta)_{x_i}dxdt$$

$$-\int_\tau^s\int_{\Omega_n}f_i(x)(u-v)^2\phi_{nx_i}(x)d_\lambda^\beta\delta_i^N dxdt$$

$$+\int_\tau^s\int_{\Omega_n}f_i(x)(u-v)\phi_n(x)(u-v)_{x_i}d_\lambda^\beta dxdt. \tag{3.15}$$

As for the first term of the right hand of (3.15), it is straightforward to see that

$$\lim_{\lambda\to 0}\lim_{n\to\infty}\left|\int_\tau^s\int_{\Omega_n}f_i(x)(u-v)^2\phi_n(x)(d_\lambda^\beta)_{x_i}dxdt\right|$$

$$\leq c\lim_{\lambda\to 0}\lim_{n\to\infty}\int_\tau^s\int_{\Omega_n}|f_i(x)(u-v)^2|\phi_n(x)(x^N-\lambda)^{\beta-1}|d_{\lambda x_i}|dx$$

$$\leq c\lim_{\lambda\to 0}\int_\tau^s\int_{\mathbb{R}^N_+}|u-v|e^{-x^N}(x^N-\lambda)^{\beta-1}dx$$

$$= c\left(\int_\tau^s\int_{\mathbb{R}^N_+}e^{-x^N}(x^N)^{\beta-1}|u-v|dx\right)$$

$$\leq c\left(\int_\tau^s\int_{\mathbb{R}^N_+}e^{-x^N}(x^N)^{\beta}|u-v|^2dxdt\right)^{\frac{1}{2}}\left(\int_\tau^s\int_{\mathbb{R}^N_+}e^{-x^N}|x^N|^{\beta-2}dxdt\right)^{\frac{1}{2}}$$

$$\leq c\left(\int_\tau^s\int_{\mathbb{R}^N_+}e^{-x^N}(x^N)^{\beta}|u-v|^2dxdt\right)^{\frac{1}{2}}, \tag{3.16}$$

because of $\beta \geq 2$ and

$$\int_\tau^s\int_{\mathbb{R}^N_+}e^{-x^N}(x^N)^{\beta-2}dxdt \leq c.$$

As for the second term of the right hand of (3.15), since $u, v \in L^\infty(Q_T)$ and $|f_i(x)| \le c$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{n\to\infty}\left|\int_\tau^s\int_{\Omega_n} f_i(x)(u-v)^2\phi_{nx_i}(x)d_\lambda^\beta\delta_i^N dxdt\right| \\
=\ & \int_\tau^s\int_{\mathbb{R}_+^N} f_i(x)(u-v)^2(e^{-x^N})_{x_i}(x^N)^\beta\delta_i^N dxdt| \\
\le\ & c\left(\int_\tau^s\int_{\mathbb{R}_+^N} e^{-x^N}(x^N)^\beta|u-v|^2dxdt\right)^{\frac{1}{2}}\left(\int_\tau^s\int_{\mathbb{R}_+^N} e^{-x^N}|x^N|^\beta dxdt\right)^{\frac{1}{2}} \\
\le\ & c\left(\int_\tau^s\int_{\mathbb{R}_+^N} e^{-x^N}(x^N)^\beta|u-v|^2dxdt\right)^{\frac{1}{2}}, \qquad (3.17)
\end{aligned}$$

based on the inequality:

$$\int_\tau^s\int_{\mathbb{R}_+^N}(e^{-x^N})(x^N)^\beta dxdt \le c.$$

As for the third term of the right hand of (3.15), we get

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{n\to\infty}\left|\int_\tau^s\int_{\Omega_n} f_i(x)(u-v)(u-v)_{x_i}\phi_n(x)d_\lambda^\beta dxdt\right| \\
\le\ & \left(\int_\tau^s\int_{\mathbb{R}_+^N}\left[(u-v)a^{-\frac{1}{p}}(x)f_i(x)e^{-x^N}(x^N)^\beta\right]^{p'}dxdt\right)^{\frac{1}{p'}} \\
&\cdot\left(\int_\tau^s\int_{\mathbb{R}_+^N} a(x)(|\nabla u|^p+|\nabla v|^p)dxdt\right)^{\frac{1}{p}} \\
\le\ & c\left(\int_\tau^s\int_{\mathbb{R}_+^N}[(u-v)a^{-\frac{1}{p}}(x)f_i(x)e^{-x^N}(x^N)^\beta]^{p'}dxdt\right)^{\frac{1}{p'}}, \qquad (3.18)
\end{aligned}$$

where $p' = \frac{p}{p-1}$.

In case of $p > 2$, it is easy to see that $1 < p' < 2$. According to (1.12), $a^{-\frac{1}{p}}(x)f_i(x) \le c$ holds. It follows from the Hölder inequality that

$$\begin{aligned}
&\left(\int_\tau^s\int_{\mathbb{R}_+^N}[(u-v)a^{-\frac{1}{p}}(x)f_i(x)e^{-x^N}(x^N)^\beta]^{p'}dxdt\right)^{\frac{1}{p'}} \\
\le\ & c\left(\int_\tau^s\int_{\mathbb{R}_+^N}(e^{-x^N})^{p'}[a^{-\frac{1}{p}}(x)f_i(x)(x^N)^{\frac{\beta}{2}}]^{\frac{2}{2-p'}}dxdt\right)^{\frac{2-p'}{2}} \\
&\cdot\left(\int_\tau^s\int_{\mathbb{R}_+^N}(e^{-x^N})^{p'}(x^N)^\beta|u-v|^2dxdt\right)^{\frac{1}{2}} \\
\le\ & \left(\int_\tau^s\int_{\mathbb{R}_+^N}(e^{-x^N})^{p'}(x^N)^\beta|u-v|^2dxdt\right)^{\frac{1}{2}}. \qquad (3.19)
\end{aligned}$$

If $1 < p \leq 2$, then $p' \geq 2$. Using the inequality $a^{-\frac{1}{p}}(x)f_i(x) \leq c$ again yields

$$\begin{aligned}
&\left(\int_\tau^s \int_{\mathbb{R}_+^N} [(u-v)a^{-\frac{1}{p}}(x)f_i(x)e^{-x^N}(x^N)^\beta]^{p'} dxdt\right)^{\frac{1}{p'}} \\
\leq\ & c\left(\int_\tau^s \int_{\mathbb{R}_+^N} [a^{-\frac{1}{p}}(x)f_i(x)e^{-x^N}(x^N)^\beta]^{p'} (u-v)^2 dxdt\right)^{\frac{1}{p'}} \\
\leq\ & c\left(\int_\tau^s \int_{\mathbb{R}_+^N} [e^{-x^N}(x^N)^\beta]^{p'-1}\phi_n(x)(x^N)^\beta (u-v)^2 dxdt\right)^{\frac{1}{p'}} \\
\leq\ & c\left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N}(x^N)^\beta (u-v)^2 dxdt\right)^{\frac{1}{p'}}. \qquad (3.20)
\end{aligned}$$

Note that $g(s,x,t) \in L^2(Q_T)$ when $|s| \leq c$. It gives

$$\begin{aligned}
& -\lim_{\varepsilon\to 0}\int_\tau^s \int_{\Omega_n} \left\{ \left[(u-v)\sum_{i=1}^N f_{ix_i}(x) + (g(u,x,t)-g(v,x,t))\right]\right. \\
& \left.\cdot(u_\varepsilon - v_\varepsilon)\phi_n(x)d_\lambda^\beta\right\} dxdt \\
=\ & -\int_\tau^s \int_{\Omega_n} \left\{ \left[(u-v)\sum_{i=1}^N f_{ix_i}(x) + (g(u,x,t)-g(v,x,t))\right]\right. \\
& \left.\cdot(u-v)\phi_n(x)d_\lambda^\beta\right\} dxdt.
\end{aligned}$$

Since $f_{ix_i}(x)$ is bounded, $g(s,x,t)$ is Lipschitz, and $u,\ v \in L^\infty(Q_T)$, we have

$$\begin{aligned}
& \lim_{\lambda\to 0}\lim_{n\to\infty}\int_\tau^s \int_{\Omega_n} \left\{ \left[(u-v)\sum_{i=1}^N f_{ix_i}(x) + (g(u,x,t)-g(v,x,t))\right]\right. \\
& \left.\cdot(u-v)\phi_n(x)d_\lambda^\beta\right\} dxdt \\
=\ & \int_\tau^s \int_{\mathbb{R}_+^N} \left\{ \left[(u-v)\sum_{i=1}^N f_{ix_i}(x) + (g(u,x,t)-g(v,x,t))\right]\right. \\
& \left.\cdot(u-v)e^{-x^N}(x^N)^\beta\right\} dxdt \\
\leq\ & c\int_\tau^s \int_{\mathbb{R}_+^N} |(u-v)e^{-x^N}(x^N)^\beta| dxdt \\
\leq\ & c\left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N}(x^N)^\beta |u-v|^2 dxdt\right)^{\frac{1}{2}} \left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N}(x^N)^\beta dxdt\right)^{\frac{1}{2}} \\
\leq\ & c\left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N}(x^N)^\beta |u-v|^2 dxdt\right)^{\frac{1}{2}}. \qquad (3.21)
\end{aligned}$$

By (3.6)-(3.21), letting $\varepsilon \to 0$, $n \to \infty$ and $\lambda \to 0$ in (3.2) yields

$$\begin{aligned} &\int_\tau^s \int_{\mathbb{R}_+^N} (u-v) e^{-x^N} (x^N)^\beta \frac{\partial (u-v)}{\partial t} dxdt \\ \leq\ & c \left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta |u(x,t) - v(x,t)|^2 dxdt \right)^q, \end{aligned} \tag{3.22}$$

where $q < 1$.

We then have

$$\begin{aligned} &\int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta [u(x,s) - v(x,s)]^2 dx \\ &- \int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta [u(x,\tau) - v(x,\tau)]^2 dx \\ \leq\ & c \left(\int_\tau^s \int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta |u(x,t) - v(x,t)|^2 dxdt \right)^q, \end{aligned} \tag{3.23}$$

where $q < 1$, which implies

$$\begin{aligned} &\int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta \mid u(x,s) - v(x,s) \mid^2 dx \\ \leq\ & \int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta \mid u(x,\tau) - v(x,\tau) \mid^2 dx. \end{aligned}$$

Because of the arbitrariness of τ, we obtain

$$\begin{aligned} &\int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta \mid u(x,s) - v(x,s) \mid^2 dx \\ \leq\ & \int_{\mathbb{R}_+^N} e^{-x^N} (x^N)^\beta \mid u_0 - v_0 \mid^2 dx. \end{aligned}$$

Consequently, the proof is completed. □

4. Estimates near the boundary

LEMMA 4.1. *If $p > \max\left\{1, \frac{2N}{N+2}\right\}$, u is the generalized solution of equation (1.1) in Q_T, and g is a suitable smooth function, then u_{x_j} $(j = 1, 2, \cdots, N)$ is locally Hölder continuous in Q_T.*

If $a(x) \equiv 1$, the equation becomes the type of an evolutionary $p-$Laplacian equation. Since $a(x)$ is only equal to zero on the boundary, Lemma 4.1 can be proved in a similar way as shown in [**1**], so we omit the details of the proof here. We would like to mention that in the case of $a(x) = [dist(x, \partial\mathbb{R}_+^N)]^\alpha$ with $\alpha > 0$, $f_i = 0$, $c(x,t) = 0$ and $g = 0$, Lemma 4.1 had been proved in [**15**].

PROOF OF THEOREM 1.5. Since u is the unique nonnegative bounded solution of equation (1.6) with the initial condition (1.2), u can be regarded as the limit of

u_n which is the solution of the following problem:

$$\frac{\partial u}{\partial t}-\operatorname{div}\left[\left(a(x)+\frac{1}{n}\right)\left(|\nabla u|^2+\frac{1}{n}\right)^{\frac{p-2}{2}}\nabla u\right]$$

$$-\sum_{i=1}^{N}f_i(x)D_iu-g(u,x,t)=0,\quad (x,t)\in B_n^+\times(0,T), \tag{4.1}$$

$$u(x,0)=u_{0,n}(x),x\in B_n^+, \tag{4.2}$$

$$u(x,t)=0,(x,t)\in\partial B_n^+\times(0,T), \tag{4.3}$$

where $u_{0,n}(x)$ is the smoothly mollified function of $u_0(x)$ and $B_n^+=\{x\in\mathbb{R}_+^N:|x|<n\}$.

For $(x_0,t_0)\in\mathbb{R}_+^N\times(s,T)$, let

$$x_0=(x^1,x^2,\cdots,x^{N-1},x^N)\ \text{ and }\ x_{N0}=(x^1,x^2,\cdots,x^{N-1},0).$$

Then, $(x_{N0},t_0)\in\partial\mathbb{R}_+^N\times(s,T)$. We may assume that $(x_{N0},t_0)\equiv(0,0)$. Let $y=(0,\ldots,0,-1)$, and denote the set $\aleph_k$ by

$$\aleph_k=\left\{(x,t):\ x_N>0,\ 1<|x-y|<1+\frac{1}{k},\ -s_n\le t\le 0\right\},$$

where s_n tends to zero when $n\to\infty$ such that

$$\lim_{n\to\infty}ns_n=0.$$

Assume $(x_0,t_0)\in\ \aleph_k$, i.e.

$$1<d(x_0,y)=x^N+1<1+\frac{1}{k}.$$

We now consider the problem

$$\frac{\partial v}{\partial t}-\operatorname{div}\left[\left(a(x)+\frac{1}{n}\right)\left(|\nabla v|^2+\frac{1}{n}\right)^{\frac{p-2}{2}}\nabla v\right]$$

$$-\sum_{i=1}^{N}f_i(x)D_iv-g(v,x,t)=0,\ \ (x,t)\in\aleph_k, \tag{4.4}$$

$$v(x,-s_n)=u_n(x,-s_n),\ \ x\in B_k^+, \tag{4.5}$$

$$v(x,t)=u_n(x,t)-\frac{1}{n},\ \ (x,t)\in\partial B_k^+\times[-s_n,0], \tag{4.6}$$

where u_n is the solution of the problem (4.1)-(4.3), $0<s_n<s<T$, s_nn is small enough, and

$$B_k^+=\left\{x:\ x_N>0,1<|x-y|<1+\frac{1}{k}\right\}.$$

By the comparison theorem (p119, [**16**]), we have

$$v\le u_n. \tag{4.7}$$

Define

$$\eta_k(x,t)=e^{-k(|x-y|-1)}e^t, \tag{4.8}$$

and

$$\Psi_k=CM(1-\eta_k(x,t))+\gamma t,\ \ (x,t)\in\aleph_k, \tag{4.9}$$

where the constants γ and C are to be determined so that $v \leq \Psi_k$ on the parabolic boundary of $\aleph_k$.

By a direct calculation, we have

(4.10)

$$
\begin{aligned}
&\Psi_{kt} - \operatorname{div}\left[\left(a(x)+\frac{1}{n}\right)(|\nabla\Psi_k|^2+\frac{1}{n})^{\frac{p-2}{2}}\nabla\Psi_k\right] - \sum_{i=1}^{N} f_i(x)\Psi_{kx_i} - g(\Psi_k,x,t)\\
=&-CMe^{-k(|x-y|-1)}e^t + \gamma\\
&-\sum_{i=1}^{N} a_{x_i}\left[\left(|\nabla\Psi_k|^2+\frac{1}{n}\right)^{\frac{p-2}{2}}\Psi_{k,x_i}\right] - \left(a(x)+\frac{1}{n}\right)\sum_{i=1}^{N}\left[\left(|\nabla\Psi_k|^2+\frac{1}{n}\right)^{\frac{p-2}{2}}\Psi_{k,x_i}\right]_{x_i}\\
&-\sum_{i=1}^{N} f_i(x)\Psi_{kx_i} - g(\Psi_k,x,t)\\
=&-CMe^{-k(|x-y|-1)}e^t + \gamma - \sum_{i=1}^{N} a_{x_i}\left[\left(|kCM\eta_k|^2+\frac{1}{n}\right)^{\frac{p-2}{2}} kCM\eta_k\frac{x_i-y_i}{|x-y|}\right]\\
&-\left(a(x)+\frac{1}{n}\right)\left\{-G_nk^2CM\eta_k + \left[(kCM\eta_k)^2+\frac{1}{n}\right]^{\frac{p-2}{2}} kCM\eta_K\left(-k+\frac{N-1}{|x-y|}\right)\right\}\\
&-kCM\eta_k\sum_{i=1}^{N} f_i(x)\frac{x_i-y_i}{|x-y|} - g(\Psi_k,x,t)\\
\geq&-CMe^{-k(|x-y|-1)}e^t + \gamma - |\nabla a|\left(|kCM\eta_k|^2+\frac{1}{n}\right)^{\frac{p}{2}}\\
&+\left(a(x)+\frac{1}{n}\right)G_nk^2CM\eta_k + [k-(N-1)]\left(a(x)+\frac{1}{n}\right)\left(|kCM\eta_k|^2+\frac{1}{n}\right)^{\frac{p-2}{2}}\\
&-\sum_{i=1}^{N}|f_i(x)|kCM\eta_k - g(\Psi_k,x,t).
\end{aligned}
$$

Since $f_i(x)$ is bounded and $|g(u,x,t)| \leq M$, if we choose γ to be sufficiently large, it follows from (4.10) that

(4.11)

$$
\Psi_{k,t} - \operatorname{div}\left[\left(a(x)+\frac{1}{n}\right)\left(|\nabla\Psi_k|^2+\frac{1}{n}\right)^{\frac{p-2}{2}}\nabla\Psi_k\right] - \sum_{i=1}^{N} f_i(x)\Psi_{kx_i} - g(\Psi_k,x,t) \geq 0,
$$

where γ is dependent on C, M, p, k and s.

By the comparison principle, one can see that the solution v of the problem (4.4)-(4.6) satisfies $v \leq \Psi_k$ in $\aleph_k$. In particular, $\forall 0 < x_N < \frac{1}{k}$, by virtue of Lemma 4.1 we have

$$
\begin{aligned}
u(0,0,\ldots,0_{N-1},x_N,0) &= \lim_{n\to\infty} v(0,0,\ldots,0_{N-1},x_N,0)\\
&\leq \Psi_k(0,0,\ldots,0_{N-1},x_N,0)\\
&= CM(1-e^{-kx_N}) \leq kCMx_N.
\end{aligned}
$$

Thus, it gives

$$
u(x,t) \leq kMdist(x,\partial\mathbb{R}^N_+) \tag{4.12}
$$

for all $x \in \mathbb{R}^N_+$ such that $d(x) = dist\left(x, \partial\mathbb{R}^N_+\right) \le \frac{1}{k}$.

On the other hand, if $dist(x, \partial\mathbb{R}^N_+) > \frac{1}{k}$, we have

$$u(x,t) \le M \le kCMdist(x, \partial\mathbb{R}^N_+). \tag{4.13}$$

Consequently, by combining (4.12) and (4.13), we arrive at (1.14) which holds in both cases. □

5. Appendix: Fichera-Oleinik theory

Consider the linear degenerate equation

$$\frac{\partial u}{\partial t} - \mathrm{div}(a(x)\nabla u) - \sum_{i=1}^{N} f_i(x)D_i u + c(x,t)u = g(x,t), \tag{5.1}$$

which is a particular case of equation (1.1) (where $p = 2$). Rewrite it as

$$\frac{\partial u}{\partial t} - a(x)\Delta u - \sum_{i=1}^{N}(a_{x_i}(x) + f_i(x))D_i u + c(x,t)u = g(x,t). \tag{5.2}$$

According to the Fichera-Oleinik theory [**13, 14**], besides the initial value condition (1.2), since $a(x)\mid_{\partial\mathbb{R}^N_+} = 0$, the partial boundary, where we should impose the boundary value condition, is

$$\Sigma_p = \left\{x \in \partial\mathbb{R}^N_+ : \sum_{i=1}^{N} f_i(x)n_i(x) < 0\right\} = \left\{x \in \partial\mathbb{R}^N_+ : f_N(x) < 0\right\}, \tag{5.3}$$

where $\overrightarrow{n} = \{n_i\} = \{0, \cdots, 0, 1\}$ is the inner normal vector of $\mathbb{R}^N_+$. In particular, if $f_N(x) \ge 0$ and $x \in \partial\mathbb{R}^N_+$, then

$$\Sigma_p = \emptyset.$$

This implies that, to obtain the well-posedness of solutions of equation (5.1), no any boundary value condition is necessary.

Consider equation (1.1) or (1.6) and rewrite it as

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} \alpha^{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \beta_i(x,t)\frac{\partial u}{\partial x_i} + g(u.x.t), \tag{5.4}$$

which may be regarded as a "linear" degenerate parabolic equation, where

$$\alpha^{ij} = a(x)|\nabla u|^{p-2}\left(\delta_{ij} + (p-2)|\nabla u|^{-2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)$$

$$\beta_i = \frac{\partial a(x)}{\partial x_i}|\nabla u|^{p-2} + f_i(x),$$

and for equation (1.1),

$$g(u,x,t) = c(x,t)u + g(x,t).$$

By the Fichera-Oleinik theory, besides the initial value, the part of the boundary:

$$\sum\nolimits_p = \left\{x \in \partial\Omega : \sum_{i=1}^{N}(\beta_i - \alpha^{ij}_{x_i})n_i(x) < 0\right\}, \tag{5.5}$$

in which we should give the boundary value, becomes very complicated. This is a very interesting point. One may usually think that the boundary value condition matching a nonlinear parabolic equation (1.1) is more complicated than the one matching the linear degenerate parabolic equation (5.1). So, Yin-Wang [**2**] had made great efforts to deal with the boundary value condition of such equations. However, the obtained results in [**3, 4**] and Theorem 4 in the present paper, show that the fact may be beyond one's expectation, and the boundary value condition matching a nonlinear parabolic equation (1.1) may be simpler, even that there is not any boundary value condition is needed in some special cases. We wish to point out that since equation (1.1) (or (5.4)) is actually a nonlinear equation, it generally only has weak solutions. The Fichera-Oleinik theory may not be applied in this case, so (5.5) only supplies an experience, but can not be used in a straightforward way.

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