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Invariant tori for a fifth order nonlinear partial differential equation with unbounded perturbation

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Abstract. In this paper, we are concerned with small perturbation of the nonlinear partial differential equation

$$u_t = u_{5x} - \frac{5}{16}(8u_{xx}^2 + 8u_x u_{xxx})$$

under periodic boundary conditions. Using an abstract infinite dimensional KAM theorem, we obtain the existence of many two-dimensional invariant tori and thus many time quasi-periodic solutions for the above equation under sufficiently small Hamiltonian perturbation.

Contents

1. Introduction and main results

With regard to the Hamiltonian partial differential equation (HPDE)

$$\dot{w} = Aw + F(w),$$

where Aw is linear Hamiltonian vector-field with $d := \mathrm{ord}A > 0$, $F(w)$ is nonlinear Hamiltonian vector-field with $\delta := \mathrm{ord}F$, and it is analytic in the neighborhood of

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the origin $w = 0$. When $\delta \leq 0$, the vector-field F is called bounded perturbation. When $\delta > 0$, the vector-field F is called unbounded perturbation.

After Kuksin and Wayne [**9, 20**] established the earliest KAM theorem for PDEs with bounded perturbation, many authors paid attention to the existence of KAM tori for HPDEs with bounded perturbation, and plenty of achievements were obtained. We can't list all the papers in this field, here we list only two survey papers [**6, 10**].

However, fewer results of KAM theory for HPDEs with unbounded perturbation are obtained. The first KAM theorem for unbounded perturbation was established by Kuksin [**7, 8**]. In [**8**], Kuksin proved the persistence of the finite-gap solutions alongside the hierarchy of KdV equation with periodic boundary conditions under the condition $0 < \delta < d - 1$. Recently, KAM theory for unbounded perturbation has been extended to the limiting case $0 < \delta = d-1$. A new estimate for the small-denominators equation with critical unbounded variable coefficient is obtained by Liu and Yuan [**13**]. Using the new estimate, they established a KAM theorem for infinite dimensional Hamiltonian systems with $0 < \delta \leq d - 1$ in [**14**]. The readers can refer to [**2, 15, 1, 17, 18, 21**] for more results about unbounded perturbation.

For a long time the fifth order partial differential equations have been paid close attention in physics. For example, in 1987, Fuchssteiner etc. [**4**] discussed the hereditariness of recursion operators for some fifth order nonlinear partial differential equations. In 2001, Verhoeven and Muserre [**19**] extended the N-soliton solutions of the Kaup-Kupershmidt equation on a nonzero background decreasing as $(x + \frac{1}{a})^{-2}$. In 2005, Das & Popowicz [**3**] studied the properties of a nonlinearly dispersive integrable system of fifth order and its associated hierarchy in which the systems are related to the Kaup-Kupershmidt and the Sawada-Kotera equations under appropriate Miura transformation.

The problem that we address in this paper is the existence of a family of quasi-periodic solutions for small perturbation of another fifth order equation

$$u_t = u_{5x} - \frac{5}{16}(8u_{xx}^2 + 8u_x u_{xxx}) \tag{1.1}$$

subject to periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x), -\infty < t < \infty, \tag{1.2}$$

which is a approximation of the FG equation [**12**] [1]

$$u_t = u_{5x} - \frac{5}{16}(8u_{xx}^2 + 8u_x u_{xxx} + 16uu_x u_{xx} + 4u^3 u_{xxx} + 4u^3 u_x - u^4 u_x).$$

Many researchers focused on the existence of the quasi-periodic solutions of the HPDEs with higher order frequency. For example, Kappeler and Pöschel [**11**] considered the second KdV equation

$$\partial_t u = \partial_x^5 u - 10u\partial_x^3 u - 20\partial_x u \partial_x^2 u + 30u^2 \partial_x u$$

under small perturbations and proved the existence of a Cantorian branch of KAM tori and many time quasi-periodic solutions. A natural question is that whether the system (1.1) possesses quasi-periodic solutions under small Hamiltonian perturbation. In this paper, we will answer this question.

[1]Here in order to find the proper Hamiltonian function we have to omit some higher order term.

To set the stage we introduce for any integer $N > \frac{7}{2}$ the phase space

$$\mathscr{H}_0^N = \{u \in L^2(\mathbb{T}, \mathbb{R}) : \hat{u}(0) = 0, \|u\|_N^2 = \sum_{j \in \mathbb{Z} \setminus \{0\}} |j|^{2N} |\hat{u}(j)|^2 < \infty\}$$

of real valued functions on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, where

$$\hat{u}(j) = \int_0^{2\pi} u(x) e_{-j}(x) dx, e_j(x) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}jx}.$$

We endow $\mathscr{H}_0^N$ with the Poisson structure proposed by Gardner

$$\{F, G\} = \int_{\mathbb{T}} \frac{\partial F}{\partial u(x)} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial G}{\partial u(x)} \mathrm{d}x,$$

where F and G are differentiable functions on $\mathscr{H}_0^N$ with L^2-gradients in $\mathscr{H}_0^1$.

Then the equation (1.1) can be written in the form

$$u_t = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial H}{\partial u} \right) \tag{1.3}$$

with Hamiltonian

$$H(u) = \int_{\mathbb{T}} \left(\frac{1}{2} u_{xx}^2 + \frac{5}{12} u_x^3 \right) \mathrm{d}x. \tag{1.4}$$

In (1.1), the order of nonlinearity $\delta = 3$ and the linear operator order $d = 5$ satisfy $0 < \delta < d-1$. So the perturbation belongs to the noncritical unbounded case. We will apply the KAM theory in [**11**] to prove the existence of KAM tori. The main work is to transform the Hamiltonian into its normal form up to order four to extract parameters. In this process, the first plague we meet is the difficulty of establishing the regularity of the vector field resulting from the higher order frequency after transformation. Fortunately, we conquer it by careful computation and analysis using some inequations. The second plague is to check the condition (3.6). Because of the complexity of higher order frequency, the matrixes of transformation A, B in (3.8) become very complex. To obtain the conclusion of (3.6), we find a proper matrix T to reduce A into a simple form. Even so, we only obtain the 2-dimensional invariant tori. In the past, there are many researchers to prove the existence of 2-dimensional tori. For example, Liang [**16**] discussed 1D Schrödinger equations with the nonlinearity $|u|^{2p}u$ under the periodic boundary conditions, and obtained the persistence of many 2-dimensional invariant tori for the index set $J = \{j_1, j_2\}$ with $j_2 > \sqrt{p} j_1 > 0$. Gao and Liu [**5**] considered the nonlinear wave equation

$$u_{tt} - u_{xx} + mu + u^5 = 0$$

under Dirichlet boundary conditions, and gave the existence of many 2-dimensional invariant tori for the index set $J = \{n_1, n_2\}$ with $n_1 = 1, n_2 \geq 10$. It needs to notice that the results above are about systems with bounded perturbation. Similar to above papers, we obtain the following theorem.

THEOREM 1.1. *Consider the nonlinear equation*

$$u_t = u_{5x} - \frac{5}{16}(8u_{xx}^2 + 8u_x u_{xxx}) + \varepsilon \frac{\partial K}{\partial u} \tag{1.5}$$

subject to the periodic boundary condition (1.2), where K is real analytic in a complex neighbourhood U of the origin in $\mathscr{H}_{0,\mathbb{C}}^N$ which is the complexification of

$\mathscr{H}_0^N$ and satisfies the regularity condition

$$\frac{\partial K}{\partial u} : U \to \mathscr{H}^N_{0,\mathbb{C}}, \ \left\| \frac{\partial K}{\partial u} \right\|_{N,U} = \sup_{u \in U} \left\| \frac{\partial K}{\partial u} \right\|_N \leq 1.$$

Then, for any given index set $\mathcal{J} = \{j_1 < j_2\} \subset \mathbb{Z} \setminus \{0\}$, there exists an $\varepsilon_0 > 0$ depending only on $\mathcal{J}, N$ and U, such that for $0 < \varepsilon < \varepsilon_0$, there exist

(1) a nonempty Cantor set $\Pi_\varepsilon \subset \Pi$ with $\operatorname{meas}(\Pi \setminus \Pi_\varepsilon) \to 0$ as $\varepsilon \to 0$, where Π is a compact subset of $\mathbb{R}^2$ with positive Lebesgue measure,

(2) a Lipschitz family of real analytic torus embeddings

$$\Phi : \mathbb{T}^2 \times \Pi_\varepsilon \to \mathscr{S}^2_{N+\frac{1}{2}} \cap \mathscr{H}_0^N,$$

where $\mathscr{S}^2_{N+\frac{1}{2}} = \mathbb{T}^2 \times \mathbb{R}^2 \times \ell^2_{N+\frac{1}{2}} \times \ell^2_{N+\frac{1}{2}}$, $\ell^2_{N+\frac{1}{2}}$ is the Hilbert space of all complex-valued sequences with norm (2.2),

(3) a Lipschitz map $\phi : \Pi_\varepsilon \to \mathbb{R}^2$,

such that for each $(\theta, \xi) \in \mathbb{T}^2 \times \Pi_\varepsilon$, the curve $u(t) = \Phi(\theta + \phi(\xi)t, \xi)$ is a quasi-periodic solution of equation (1.5) winding around the invariant $\Phi(\mathbb{T}^2 \times \{\xi\})$.

Moreover, each embedding is real analytic on $D(s/2) = \{|\Im\varphi| < s/2\}$, and

$$\|\Phi - \Phi_0\|^{\sup}_{r,N+\frac{1}{2},D(s/2)\times\Pi_\varepsilon} + \frac{\alpha}{M}\|\Phi - \Phi_0\|^{\operatorname{lip}}_{r,N+\frac{1}{2},D(s/2)\times\Pi_\varepsilon} \leq \frac{c\varepsilon}{\alpha},$$

$$|\phi - \omega|^{\sup}_{\Pi_\varepsilon} + \frac{\alpha}{M}|\phi - \omega|^{\operatorname{lip}}_{\Pi_\varepsilon} \leq c\varepsilon,$$

where

$$\Phi_0 : \mathbb{T}^2 \times \Pi \to \mathbb{T}^2 \times \{0\} \times \{0\} \times \{0\}, \ (\varphi, \xi) \mapsto (\varphi, 0, 0, 0)$$

is the trivial embedding for each ξ. α is a parameter which depends on ε. c is positive constant which depends on the same parameters as γ, where γ comes from the KAM theorem 4.1.

2. The normal form

In this section, we will normalize the Hamiltonian up to order four. To this end, we need some preparations.

Writting

$$u(t,x) = \sum_{j\neq 0} \gamma_j q_j(t) e_j(x), \tag{2.1}$$

where $\gamma_j = \sqrt{|j|}$. The coordinates are taken from the Hilbert space $\ell^2_{N+\frac{1}{2}}$ of all complex-valued sequences $(q_j)_{j\neq 0}$ with

$$\|q\|^2_{N+\frac{1}{2}} = \sum_{j\neq 0} |j|^{2N+1}|q_j|^2 < \infty, \ q_{-j} = \bar{q}_j. \tag{2.2}$$

Now (1.3) can be written as

$$\dot{q}_j = \mathrm{i}\sigma_j \frac{\partial H}{\partial q_{-j}}, \ \sigma_j = \begin{cases} 1, & j \geq 1, \\ -1, & j \leq -1 \end{cases} \tag{2.3}$$

with the Hamiltonian

$$H(q) = \sum_{j\geq 1} j^5 |q_j|^2 - \frac{5}{12\sqrt{2\pi}} \sum_{j+k+l=0} jkl\gamma_j\gamma_k\gamma_l q_j q_k q_l = \Lambda + G, \tag{2.4}$$

and the corresponding symplectic structure is

$$-\mathrm{i}\sum_{j\geq 1}\mathrm{d}q_j\wedge\mathrm{d}q_{-j}. \tag{2.5}$$

The associated Hamilton vector field with Hamiltonian $H(q)$ is given by

$$X_H=\mathrm{i}\sigma_j\sum_{j\neq 0}\frac{\partial H}{\partial q_{-j}}\frac{\partial}{\partial q_j}.$$

LEMMA 2.1. *The Hamiltonian vector field X_G is real analytic as a map from $\ell^2_{N+\frac{1}{2}}$ into $\ell^2_{N-\frac{5}{2}}$ for each $N>\frac{7}{2}$. Moreover, $\|X_G\|_{N-\frac{5}{2}}=O(\|q\|^2_{N+\frac{1}{2}})$.*

PROOF. Since

$$G(q)=-\frac{5}{12\sqrt{2\pi}}\sum_{j+k+l=0}jkl\gamma_j\gamma_k\gamma_l q_j q_k q_l,$$

hence,

$$\left|\frac{\partial G}{\partial q_{-j}}\right|\leq\frac{5}{12\sqrt{2\pi}}|j|\gamma_j\sum_{k+l=j}|kl|\gamma_k\gamma_l|q_kq_l|=\frac{5}{12\sqrt{2\pi}}|j|\gamma_j g_j.$$

Defining $w=(w_j)_j=(|j|\gamma_j|q_j|)_j$, $g=(g_j)$, then $g_j=(w*w)_j$, consequently $g=w*w$. For $q\in\ell^2_{N+\frac{1}{2}}$, we have $w\in\ell^2_{N-1}$. Hence we have

$$\|g\|_{N-1}=\|w*w\|_{N-1}\leq C\|w\|^2_{N-1}\leq C\|q\|^2_{N+\frac{1}{2}},$$

and therefore

$$\|\partial_q G\|_{N-\frac{5}{2}}\leq C\|g\|_{N-1}\leq C\|q\|^2_{N+\frac{1}{2}}.$$

The proof is completed. □

LEMMA 2.2. *Suppose a_1,a_2,b_1,b_2 are nonzero integers, and $\frac{b_1}{a_1},\frac{b_2}{a_2}$ are fractions in lowest terms. Then if $\frac{b_1}{a_1}\pm\frac{b_2}{a_2}$ is a integer, we have $a_1=a_2$.*

PROOF. If $\frac{b_1}{a_1}\pm\frac{b_2}{a_2}$ equals the integer m, we obtain

$$a_2b_1+a_1b_2=ma_1a_2.$$

Obviously, $a_1|a_1b_2$, then $a_1|a_2b_1$. Meanwhile, $\frac{b_1}{a_1}$ is a fraction in lowest terms, then $a_1|a_2$. Similarly, we have $a_2|a_1$. Consequently, $a_1=a_2$. □

LEMMA 2.3. *Suppose j,l,k are nonzero integers with $j+k+l=0$. Then*

$$j^5+l^5+k^5=5jkl(j^2+l^2+jl)\neq 0, \tag{2.6}$$

and

$$j^2+l^2+jl\geq\frac{1}{2}\max\{j^2,k^2,l^2\}. \tag{2.7}$$

PROOF. If $j+k+l=0$, then $k=-j-l$, and

$$\begin{aligned}j^5+k^5+l^5&=j^5+l^5-(j+l)^5=-5j^4l-10j^3l^2-10j^2l^3-5jl^4\\&=5jkl(j^2+l^2+jl)\neq 0.\end{aligned}$$

Meanwhile, we know

$$j^2+l^2+jl\geq\frac{1}{2}(j^2+l^2)+\frac{1}{2}(j+l)^2\geq\frac{1}{2}\max\{j^2,l^2\},$$

$$j^2+l^2+jl=k^2+l^2+kl\geq\frac{1}{2}(l^2+k^2)+\frac{1}{2}(l+k)^2\geq\frac{1}{2}k^2,$$

namely,

$$j^2+l^2+jl\geq\frac{1}{2}\max\{j^2,k^2,l^2\}.$$

□

LEMMA 2.4. *Suppose $j,l,m,n\in\mathbb{Z}\setminus\{0\}$, and define*

$$\Delta=\{(j,l,m,n)\in\mathbb{Z}^4\setminus\{0\}|j+l+m+n=0\},$$
$$\Delta_1=\{(j,l,m,n)\in\Delta|j+l,j+m,j+n\neq0\}.$$

Then if $(j,l,m,n)\in\Delta_1$, we have

$$\left|\frac{(m^2+n^2+mn)(j^5+l^5+m^5+n^5)}{(j+l)^2jl}\right|\geq\frac{5}{2}\max\{|j|,|l|,|m|,|n|\}.$$

PROOF. If $(j,l,m,n)\in\Delta_1$, then $n=-j-l-m$, and

$$\begin{aligned}j^5+l^5+m^5+n^5&=j^5+l^5+m^5-(j+l+m)^5\\&=-5(j+l)(j+m)(j+n)(j^2+l^2+m^2+jm+jl+lm)\\&=\frac{5}{2}(j+l)(j+m)(j+n)(j^2+l^2+m^2+n^2)\neq0.\end{aligned}$$

Furthermore, it is easy to see that

$$\frac{1}{2}(j^2+l^2+m^2+n^2)\geq|jl|, \tag{2.8}$$

and

$$m^2+n^2+mn\geq\frac{1}{2}m^2+\frac{1}{2}n^2+mn=\frac{1}{2}(m+n)^2=\frac{1}{2}(j+l)^2. \tag{2.9}$$

In what follows, we wil1 prove that

$$|(j+l)(j+m)(j+n)|>\frac{1}{2}\max\{|j|,|l|,|m|,|n|\}. \tag{2.10}$$

Without loss of generality, we assume that $|j|=\max\{|j|,|l|,|m|,|n|\}$. (2.10) can be divided into two cases to prove.

Case1. There are three components have the same sign, then

$$|j|=|l|+|m|+|n|,$$

so we have

$$\begin{aligned}|(j+l)(j+m)(j+n)|&=|(j+l)(l+n)(l+m)|\geq|(l+m)(l+n)|\\&\geq\frac{2|l|+|n|+|m|}{2}>\frac{|j|}{2}.\end{aligned}$$

Case2. There are two components have the same sign, suppose l and j have the same sign, then we have

$$|(j+l)(j+m)(j+n)|\geq|j+l|\geq|j|>\frac{|j|}{2}.$$

In view of (2.8) (2.9) and (2.10), we can get

$$\left|\frac{(m^2+n^2+mn)(j^5+l^5+m^5+n^5)}{(j+l)^2jl}\right|\geq\frac{5}{2}\max\{|j|,|l|,|m|,|n|\}.$$

□

LEMMA 2.5. *There exists a real analytic symplectic coordinate transformation* Φ *defined in a neighborhood of the origin of* $\ell^2_{N+\frac{1}{2}}$, *which transforms the Hamiltonian (2.4) into its normal form up to order four. That is*

$$H \circ \Phi = \Lambda - B + \bar{P} \tag{2.11}$$

with

$$B = \frac{5}{32\pi} \sum_{j \neq l} \frac{\sigma_{jl} j^3 l^3}{j^2 + l^2 + jl} |q_j|^2 |q_l|^2 + \frac{5}{48\pi} \sum_{j \neq 0} j^4 |q_j|^4,$$

$$\|\partial_q \bar{P}\|_{N-\frac{1}{2}} = O(\|q\|^4_{N+\frac{1}{2}}). \tag{2.12}$$

PROOF. (1) The first step is to eliminate the three order term G of q.

Let the transformation $\Phi_1 = X^1_{F^3}$ be the time-1 map of the flow of the Hamiltonian vector field X_{F^3}, and then

$$\begin{aligned} H_1 &= H \circ \Phi_1 = H \circ X^1_{F^3} \\ &= \Lambda + \{\Lambda, F^3\} + G + \int_0^1 (1-t)\{\{\Lambda, F^3\}, F^3\} \circ X^t_{F^3} dt + \int_0^1 \{G, F^3\} \circ X^t_{F^3} dt. \end{aligned}$$

To solve $\{\Lambda, F^3\} + G = 0$, we make the ansatz

$$F^3 = \sum_{j,k,l \neq 0} F^3_{jkl} q_j q_k q_l,$$

then from (2.6) and (2.7), it is justified to define F^3 by setting

$$F^3_{jkl} = \begin{cases} \frac{-5}{12\sqrt{2\pi}} \frac{\gamma_j \gamma_k \gamma_l}{j^2 + l^2 + jl}, & j + k + l = 0, j, k, l \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} H_1 &= \Lambda + \int_0^1 t\{G, F^3\} \circ X^t_{F^3} dt \\ &= \Lambda + \frac{1}{2}\{G, F^3\} + \frac{1}{2} \int_0^1 (1 - t^2)\{\{G, F^3\}, F^3\} \circ X^t_{F^3} dt \\ &= \Lambda + \frac{1}{2}\{G, F^3\} + P_1, \end{aligned}$$

where

$$P_1 = \frac{1}{2} \int_0^1 (1 - t^2)\{\{G, F^3\}, F^3\} \circ X^t_{F^3} dt = O(q^5),$$

$$\frac{1}{2}\{G, F^3\} = \frac{-5}{64\pi} \sum_{\substack{j+l+m+n=0, \\ j+l \neq 0}} \frac{(j+l)^2 kl}{m^2 + n^2 + mn} \gamma_j \gamma_l \gamma_m \gamma_n q_j q_l q_m q_n.$$

Moreover the $j-$th element of gradient $\partial_q F^3$ explicitly reads

$$\frac{\partial F^3}{\partial q_{-j}} = \sum_{k+l=j} \left(F^3_{(-j)kl} + F^3_{k(-j)l} + F^3_{kl(-j)} \right) q_k q_l.$$

Then from (2.7), we get the estimate

$$\left| \frac{\partial F^3}{\partial q_{-j}} \right| \leq \frac{5}{2\sqrt{2\pi}\gamma_j^3} \sum_{k+l=j} \gamma_k \gamma_l |q_k||q_l|,$$

therefore we obtain

$$\|\partial_q F^3\|_{N+\frac{3}{2}} = O(\|q\|^2_{N+\frac{1}{2}}).$$

(2) The second step is to normalize the four order term $\frac{1}{2}\{G, F^3\}$.

$\frac{1}{2}\{G, F^3\}$ can be written as

$$\frac{1}{2}\{G, F^3\} = -B - Q,$$

where

$$B = \frac{5}{32\pi}\sum_{j\neq l}\frac{(j+l)^2 jl}{j^2+l^2+jl}|jl||q_j|^2|q_l|^2 + \frac{5}{48\pi}\sum_{j\neq 0} j^4|q_j|^4,$$

$$Q = \frac{5}{64\pi}\sum_{(j,l,m,n)\in\Delta_1}\frac{(j+l)^2 jl}{m^2+n^2+mn}\gamma_j\gamma_l\gamma_m\gamma_n q_j q_l q_m q_n.$$

Then

$$H_1 = \Lambda - B - Q + P_1.$$

Notice that

$$\sum_{j\neq l}\frac{(j+l)^2 jl}{j^2+l^2+jl}|jl||q_j|^2|q_l|^2 = \sum_{j\neq l}\left[\sigma_{jl} j^2 l^2 |q_j|^2|q_l|^2 + \frac{\sigma_{jl} j^3 l^3}{j^2+l^2+jl}|q_j|^2|q_l|^2\right],$$

and

$$\sum_{j\neq l}\sigma_{jl} j^2 l^2 |q_j|^2|q_l|^2 = 0,$$

where $\sigma_{jl} = \sigma_j\sigma_l$, then we have

$$B = \frac{5}{32\pi}\sum_{j\neq l}\frac{\sigma_{jl} j^3 l^3}{j^2+l^2+jl}|q_j|^2|q_l|^2 + \frac{5}{48\pi}\sum_{j\neq 0} j^4|q_j|^4.$$

It remains to eliminate Q by another coordinate transformation $\Phi_2 = X^1_{F^4}$. Then

$$\begin{aligned} H_2 =& H_1\circ\Phi_2 = H_1\circ X^1_{F^4} \\ =&\Lambda + \{\Lambda, F^4\} - B - Q + \int_0^1 (1-t)\{\{\Lambda, F^4\}, F^4\}\circ X^t_{F^4} dt \\ &+ \int_0^1 \{-B-Q, F^4\}\circ X^t_{F^4} dt + P_1\circ X^1_{F^4}. \end{aligned}$$

Defining

$$F^4 = \sum_{j,l,m,n\neq 0} F^4_{jlmn} q_j q_l q_m q_n$$

with coefficients

$$\mathrm{i}F^4_{jlmn} = \begin{cases} \frac{5}{64\pi}\frac{(j+l)^2 jl\gamma_j\gamma_l\gamma_m\gamma_n}{(m^2+n^2+mn)(j^5+l^5+m^5+n^5)}, & (j,l,m,n)\in\Delta_1, \\ 0, & \text{otherwise}, \end{cases}$$

we see that

$$\{\Lambda, F^4\} = Q.$$

Then we have

$$\begin{aligned} H_2 =&\Lambda - B + \int_0^1 (1-t)\{\{\Lambda, F^4\}, F^4\} \circ X_{F^4}^t dt \\ &+ \int_0^1 \{-B-Q, F^4\} \circ X_{F^4}^t dt + P_1 \circ X_{F^4}^1 = \Lambda - B + \bar{P} \end{aligned}$$

with

$$\bar{P} = \int_0^1 (1-t)\{\{\Lambda, F^4\}, F^4\} \circ X_{F^4}^t dt + \int_0^1 \{-B-Q, F^4\} \circ X_{F^4}^t dt + P_1 \circ X_{F^4}^1.$$

In the following we need to establish the regularity of the vector field X_{F^4} . From Lemma 2.4, we know

$$\left| \frac{(m^2+n^2+mn)(j^5+l^5+m^5+n^5)}{(j+l)^2 jl} \right| \geq \frac{5}{2} \max\{|j|, |l|, |m|, |n|\}.$$

Hence

$$|F_{jlmn}^4| \leq \frac{1}{16\pi} \cdot \frac{\gamma_j \gamma_l \gamma_m \gamma_n}{\max\{|j|, |l|, |m|, |n|\}}.$$

Moreover, the $j-$th element of gradient $\partial_q F^4$ explicitly reads

$$\frac{\partial F^4}{\partial q_{-j}} = \sum_{l+m+n=j} \left(F_{(-j)lmn}^4 + F_{l(-j)mn}^4 + F_{lm(-j)n}^4 + F_{lmn(-j)}^4 \right) q_l q_m q_n.$$

Thus, we get the estimate

$$\left| \frac{\partial F^4}{\partial q_{-j}} \right| \leq \frac{1}{16\pi\gamma_j} \sum_{l+m+n=j} \gamma_l \gamma_m \gamma_n |q_l||q_m||q_n| := \frac{1}{16\pi\gamma_j} r_j,$$

where r_j stands for the sum

$$\sum_{l+m+n=j} \gamma_l \gamma_m \gamma_n |q_l||q_m||q_n|.$$

Defining $w = (w_j)_j = (\gamma_j |q_j|)_j$, $r = (r_j)_j$, then $r_j = (w * w * w)_j$, consequently $r = w * w * w$. For $q \in \ell^2_{N+\frac{1}{2}}$, we have $w \in \ell^2_N$. Hence we have

$$\|r\|_N = \|w * w * w\|_N \leq C\|w\|_N^3 \leq C\|q\|_{N+\frac{1}{2}}^3,$$

and therefore

$$\|\partial_q F^4\|_{N+\frac{1}{2}} \leq C\|r\|_N \leq C\|q\|_{N+\frac{1}{2}}^3.$$

Namely,

$$\|\partial_q F^4\|_{N+\frac{1}{2}} = O(\|q\|_{N+\frac{1}{2}}^3).$$

Let $\Phi = \Phi_1 \circ \Phi_2$, then Φ transforms the Hamiltonian function (2.4) into (2.11). The proof of Lemma 2.5 is completed. □

3. The proof of Theorem 1.1

From the transformation Φ in Lemma 2.5, we get the new Hamiltonian

$$H_* = H_2 + \varepsilon K \circ \Phi = \Lambda - B + \bar{P} + \varepsilon K \circ \Phi$$

of equation (1.5), where Λ is real analytic in the neighbourhood V of the origin in $\ell^2_{N+\frac{1}{2}}$, $K \circ \Phi$ satisfies

$$\|X_{K\circ\Phi}\|_{N+\frac{1}{2},V} = \|D\Phi^{-1} X_K \circ \Phi\|_{N+\frac{1}{2},V} \leq C.$$

We introduce symplectic polar and real coordinates $(\varphi, y, z, \bar{z})$ by setting

$$\Psi : \begin{cases} q_{j_b} = \sqrt{\xi_b + y_b}\, e^{-i\varphi_b}, q_{-j_b} = \sqrt{\xi_b + y_b}\, e^{i\varphi_b}, & b = 1, 2, \\ q_j = z_j, q_{-j} = \bar{z}_j, & j \in \mathbb{Z}_* = \mathbb{Z} \setminus \mathcal{J}, \end{cases} \tag{3.1}$$

where

$$\xi = (\xi_1, \xi_2) \in \Pi \subset \mathbb{R}^2, \tag{3.2}$$

and here Π is a compact subset of $\mathbb{R}^2$ with positive Lebesgue measure. Then

$$\Lambda = \frac{1}{2} \sum_{1\leq b\leq 2} \sigma_{j_b} j_b^5 (\xi_b + y_b) + \frac{1}{2} \sum_{j\in\mathbb{Z}_*} \sigma_j j^5 z_j \bar{z}_j,$$

$$B = \frac{5}{32\pi} \left(\frac{2\sigma_{j_1}\sigma_{j_2} j_1^3 j_2^3}{j_1^2 + j_2^2 + j_1 j_2} (\xi_1 + y_1)(\xi_2 + y_2) + 2 \sum_{\substack{1\leq b\leq 2,\\ j\in\mathbb{Z}_*}} \frac{\sigma_{j_b}\sigma_j j_b^3 j^3}{j^2 + j_b^2 + j j_b} (\xi_b + y_b) z_j \bar{z}_j \right.$$
$$\left. + \sum_{j,j'\in\mathbb{Z}_*} \frac{\sigma_j \sigma_{j'} j^3 j'^3}{j^2 + j'^2 + jj'} z_j \bar{z}_j z_{j'} \bar{z}_{j'} \right) + \frac{5}{96\pi} \left(\sum_{1\leq b\leq 2} j_b^4 (\xi_b + y_b)^2 + \sum_{j\in\mathbb{Z}_*} j^4 (z_j \bar{z}_j)^2 \right).$$

Thus the new Hamiltonian, still denoted by H, up to a constant depending on ξ, is given by

$$H = N + P = \sum_{1\leq b\leq 2} \sigma_{j_b} \omega_b y_b + \sum_{j\in\mathbb{Z}_*} \sigma_j \Omega_j z_j \bar{z}_j + \bar{Q} + \bar{P} + \varepsilon K \circ \Phi$$

with symplectic structure

$$\sum_{1\leq b\leq 2} \sigma_{j_b} \mathrm{d}y_b \wedge \mathrm{d}\varphi_b - \mathrm{i} \sum_{j\in\mathbb{Z}_*} \sigma_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j,$$

where

$$\omega_b = \frac{1}{2} j_b^5 - \frac{5}{48\pi} \sigma_{j_b} j_b^4 \xi_b - \frac{5}{16\pi} \sum_{1\leq k\neq b\leq 2} \frac{\sigma_{j_k} j_k^3 j_b^3}{j_k^2 + j_b^2 + j_k j_b} \xi_k, \tag{3.3}$$

$$\Omega_j = \frac{1}{2} j^5 - \frac{5}{16\pi} \sum_{1\leq b\leq 2, j\in\mathbb{Z}_*} \frac{\sigma_{j_b} j_b^3 j^3}{j^2 + j_b^2 + j j_b} \xi_b, \tag{3.4}$$

$$\bar{Q} = \frac{5}{32\pi} \left(\frac{2\sigma_{j_1}\sigma_{j_2} j_1^3 j_2^3}{j_1^2 + j_2^2 + j_1 j_2} y_1 y_2 + 2 \sum_{\substack{1\leq b\leq 2,\\ j\in\mathbb{Z}_*}} \frac{\sigma_{j_b}\sigma_j j_b^3 j^3}{j^2 + j_b^2 + j j_b} y_b z_j \bar{z}_j \right.$$
$$\left. + \sum_{j,j'\in\mathbb{Z}_*} \frac{\sigma_j \sigma_{j'} j^3 j'^3}{j^2 + j'^2 + jj'} z_j \bar{z}_j z_{j'} \bar{z}_{j'} \right) + \frac{5}{96\pi} \left(\sum_{1\leq b\leq 2} j_b^4 y_b^2 + \sum_{j\in\mathbb{Z}_*} j^4 (z_j \bar{z}_j)^2 \right).$$

Now consider the phase space domain

$$D(s,r) : |\Im\varphi| < s, |y| < r^2, \|z\|_{N+\frac{1}{2}} + \|\bar{z}\|_{N+\frac{1}{2}} < r. \tag{3.5}$$

We will adopt a lot of notations and definitions from [**11**], which including the phase space, weighted norm for the Hamilton vector field, etc.. More definitions are presented in Appendix.

In the following we check the assumption A, B, and C of the KAM Theorem 4.1 in Appendix.

Regarding Ω as an infinite dimensional column vector with its index $j \in \mathbb{Z}_*$, from (3.4), we know

$$\Omega_j(\xi) = \bar{\Omega}_j + \tilde{\Omega}_j(\xi),$$

where $\bar{\Omega}_j = \frac{1}{2}j^5$ is independent of ξ. Furthermore, from (3.4), we get

$$|\Omega_j|_\Pi^{\text{lip}} \le \frac{5}{16\pi} \sum_{\substack{1\le b\le 2,\\ j\in\mathbb{Z}_*}} \frac{|j_b^3 j^3|}{j^2 + j_b^2 + jj_b} \le \frac{5}{12\pi} \max\{|j_1|^3|j|, |j_2|^3|j|\}.$$

Thus,

$$|\Omega|_{-3,\Pi}^{\text{lip}} = \sup_{j\in\mathbb{Z}_*} j^{-3}|\Omega_j|_\Pi^{\text{lip}} \le \frac{5}{12\pi}\max\{|j_1|^3, |j_2|^3\} := M_1.$$

It means that assumption A is fulfilled with $d = 5, \delta = 3$ and

$$M_1 = \frac{5}{12\pi}\max\{|j_1|^3, |j_2|^3\}.$$

In view of (3.3), we know that $\xi \mapsto \omega$ is an affine transformation from Π to $\mathbb{R}^2$. Noting that

$$\omega = \breve{\omega} - \frac{5}{16\pi}A\xi,$$

where

$$\breve{\omega} = \begin{pmatrix} j_1^5 \\ j_2^5 \end{pmatrix}, A = \begin{pmatrix} \sigma_{j_1}\frac{j_1^4}{3} & \sigma_{j_2}\frac{j_1^3 j_2^3}{j_1^2+j_2^2+j_1j_2} \\ \sigma_{j_1}\frac{j_1^3j_2^3}{j_1^2+j_2^2+j_1j_2} & \sigma_{j_2}\frac{j_2^4}{3} \end{pmatrix}, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

then we get that

$$\det A = \sigma_{j_1j_2}\frac{j_1^2j_2^2(j_1-j_2)^2(j_1^2+j_2^2+4j_1j_2)}{(j_1^2+j_2^2+j_1j_2)^2} \neq 0.$$

Therefore, the real map $\xi \mapsto \omega$ is a lipeomorphism between Π and its image. This implies that the first part of assumption B is fulfilled with positive M_2 and L only depend on the set $\mathcal{J}$.

In what follows, we will check the second part of assumption B. Writting

$$\Omega = \bar{\Omega} - \frac{5}{16\pi}B\xi,$$

where $\bar{\Omega}$ is an infinite dimensional column vector and its $j-$th element $\bar{\Omega}_j = \frac{1}{2}j^5$, B is a $-\infty \times 2$ matrix with its $j-$row $B_j = \left(\frac{\sigma_{j_1}j_1^3j^3}{j_1^2+j^2+j_1j}, \frac{\sigma_{j_2}j_2^3j^3}{j_2^2+j^2+j_2j}\right)$, and regarding k and l as two-dimensional and infinite dimensional row vector respectively, we have to check for every $k = (k_1, k_2) \in \mathbb{Z}^2$ and $1 \le |l| \le 2$ with $l \in \mathbb{Z}^\infty$,

$$\text{meas}\{\xi \in \Pi : \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\} = 0. \tag{3.6}$$

Let

$$\mathfrak{g}(\xi) = \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle$$
$$= \langle k, \breve{\omega}\rangle + \langle l, \bar{\Omega}\rangle + \langle k, -\frac{5}{16\pi}A\xi\rangle + \langle l, -\frac{5}{16\pi}B\xi\rangle.$$

For the condition (3.6) we have to check that

$$\langle k, \breve{\omega}\rangle + \langle l, \bar{\Omega}\rangle \neq 0 \text{ or } kA + lB \neq 0. \tag{3.7}$$

Suppose that

$$kA + lB = 0, \tag{3.8}$$

and multiply the matrix $T = \mathrm{diag}(\sigma_{j_1}\cdot\frac{1}{j_1^3}, \sigma_{j_2}\cdot\frac{1}{j_2^3})$ from the right-hand side of (3.8) and we can obtain

$$k\tilde{A} + l\tilde{B} = 0, \tag{3.9}$$

where

$$\tilde{A} = \begin{pmatrix} \frac{j_1}{3} & \frac{j_1^3}{j_1^2+j_2^2+j_1j_2} \\ \frac{j_2^3}{j_1^2+j_2^2+j_1j_2} & \frac{j_2}{3} \end{pmatrix},$$

$$\tilde{B} = (\tilde{B}_j)_{j\in\mathbb{Z}\setminus\mathcal{J}},\ \tilde{B}_j = (\frac{j^3}{j_1^2+j^2+j_1j}, \frac{j^3}{j_2^2+j^2+j_2j}).$$

Using a series of elementary transformation, we get

$$\tilde{A}^{-1} = \frac{9(j_1^2+j_2^2+j_1j_2)^2}{j_1j_2(j_1-j_2)^2(j_1^2+j_2^2+4j_1j_2)} \begin{pmatrix} \frac{j_2}{3} & -\frac{j_1^3}{j_1^2+j_2^2+j_1j_2} \\ -\frac{j_2^3}{j_1^2+j_2^2+j_1j_2} & \frac{j_1}{3} \end{pmatrix}.$$

By simple calculation, we have

$$\tilde{B}_j\tilde{A}^{-1} = (b_{j1}, b_{j2}),$$

where

$$b_{j1} = \frac{3j^3(j-j_2)(j_1^2+j_2^2+j_1j_2)(j_2^2+2j_2j+j_1j+2j_1j_2)}{j_1(j_1^2+j^2+j_1j)(j_2^2+j^2+j_2j)(j_1-j_2)(j_1^2+j_2^2+4j_1j_2)}, \tag{3.10}$$

$$b_{j2} = \frac{3j^3(j-j_1)(j_1^2+j_2^2+j_1j_2)(j_1^2+2j_1j+j_2j+2j_1j_2)}{j_2(j_1^2+j^2+j_1j)(j_2^2+j^2+j_2j)(j_1-j_2)(j_1^2+j_2^2+4j_1j_2)}. \tag{3.11}$$

Therefore, if $l_j = \pm1$ or $l_j = \pm2$, $b_{j1}, 2b_{j1}$ is not integer; if $l_j = \pm1, l_{j'} = \pm1$, or $l_j = \pm1, l_{j'} = \mp1$, we will prove $b_{j1} \pm b_{j'1}$ is not integer too. In fact, assuming $b_{j1} = \frac{b_1}{a_1}, b_{j'1} = \frac{b_2}{a_2}$ the fractions in lowest terms. If $\frac{b_1}{a_1} \pm \frac{b_2}{a_2}$ is a integer, from Lemma 2.2, we obtain that $a_1 = a_2$. This is contradictory to (3.10).

To sum up, (3.7) holds for all $k \in \mathbb{Z}^2$ and $1 \leq |l| \leq 2$. Thus the second part of the assumption B is satisfied.

It remains to check assumption C. It is easy to see that the Hamiltonian vector field of the perturbation $P = \bar{Q} + \bar{P} + \varepsilon K \circ \Phi$ defines a map

$$X_P : D(s,r)\times\Pi \to \mathscr{S}^2_{p-2,\mathbb{C}},$$

where $\mathscr{S}^2_{p,\mathbb{C}}$ is the phase space $\mathscr{S}^m_{p,\mathbb{C}}$ which defined in (4.3) with $m = 2, p = N + \frac{1}{2}$. We use the notation $i_\xi X_P$ for X_P evaluated at ξ, and likewise in analogous cases. For each ξ, the vector field $i_\xi X_P$, considered as a map from a subset of $\mathscr{S}^2_{p,\mathbb{C}}$ to $\mathscr{S}^2_{p-2,\mathbb{C}}$, is of the order $p - (p-2) = 2$, which strictly smaller than $d - 1 = 4$.

Moreover, it is easy to see that $i_\xi X_P$ is real analytic on $D(s,r)$ for each $\xi \in \Pi$, and $i_w X_P$ is uniformly Lipschitz on Π for each $w \in D(s,r)$. Namely, the assumption C is satisfied.

Now we consider sup norm and Lipschitz semi-norm of the perturbation P on $D(s,r)\times\Pi$, where the parameter domain

$$\Pi = \{\xi \in \mathbb{R}^2 : |\xi| \le r^{\frac{16}{11}}\}.$$

Obviously, we have

$$\|X_{\bar{Q}}\|_{r,p-2,D(s,r)\times\Pi} = O(r^2). \tag{3.12}$$

Moreover, $\bar{P}$ is at least five order of q, we get

$$\|X_{\bar{P}}\|_{r,p-2,D(s,r)\times\Pi} = O((r^{\frac{8}{11}})^5 \cdot r^{-2}) = O(r^{\frac{18}{11}}). \tag{3.13}$$

For $\varepsilon = r^{\frac{18}{11}}$, we know

$$\|X_{\varepsilon K\circ\Phi}\|_{r,p-2,D(s,r)\times\Pi} = O(r^{\frac{18}{11}}). \tag{3.14}$$

From (3.12), (3.13) and (3.14), we have

$$\|X_P\|_{r,p-2,D(s,r)\times\Pi} = O(r^{\frac{18}{11}}).$$

Since X_P is real analytic in ξ, we have

$$\|X_P\|^{\text{lip}}_{r,p-2,D(s,r)\times\Pi} = O(r^{\frac{18}{11}} \cdot r^{-\frac{16}{11}}) = O(r^{\frac{2}{11}}).$$

We choose

$$\alpha = r^{\frac{17}{11}}\gamma^{-1},$$

where γ is taken from the KAM Theorem 4.1. Set $M := M_1 + M_2$, which only depends on the index set $\mathcal{J}$. It's obvious that when r is small enough,

$$\|X_P\|_{r,p-2,D(s,r)\times\Pi} + \frac{\alpha}{M}\|X_P\|^{\text{lip}}_{r,p-2,D(s,r)\times\Pi} = O(r^{\frac{18}{11}}) = O(\varepsilon) \le \alpha\gamma,$$

which is just the smallness condition (4.5) in KAM Theorem 4.1. Therefore, the conclusion of Theorem 1.1 follows from Theorem 4.1 in Appendix.

4. Appendix:The KAM Theorem

Consider a small perturbation $H = N + P$ of an infinite dimensional Hamiltonian in the parameter dependent normal form

$$N = \sum_{1\le j\le m} \omega_j(\xi) y_j + \sum_{j\in\mathbb{N}_*} \Omega_j z_j \bar{z}_j \tag{4.1}$$

on a phase space

$$\mathscr{S}^m_p = \mathbb{T}^m \times \mathbb{R}^m \times \ell^2_p \times \ell^2_p \ni (x,y,z,\bar{z})$$

with symplectic structure

$$\sum_{1\le j\le m} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{j\ge1} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j,$$

where

$$\ell^2_p = \{z \in \ell^2(\mathbb{N},\mathbb{R}) : \|z\|^2 = \sum_{j\ge1} |z_j|^2 j^{2p} < \infty\},$$

where $p \geq 0$. The tangential frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ and normal frequencies $\Omega = (\Omega_1, \Omega_2, \cdots)$ are real analytic in the space coordinates and Lipschitz in the parameters. The Hamiltonian N depends on parameters

$$\xi \in \Pi \subset \mathbb{R}^m,$$

where Π is a compact Cantor set of $\mathbb{R}^m$ of positive Lebesgue measure. Moreover, for each $\xi \in \Pi$, its Hamiltonian vector field

$$X_P = ((\sigma_{j_b} P_{y_b})_{1\leq b\leq m}, -(\sigma_{j_b} P_{x_b})_{1\leq b\leq m}, \mathrm{i}(\sigma_j P_{\bar{z}_j})_{j\in\mathbb{Z}_*}, -\mathrm{i}(\sigma_j P_{z_j})_{j\in\mathbb{Z}_*})^T$$

defines near $T_0 := \mathbb{T}^m \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ a real analytic map

$$X_P : \mathscr{S}_p^m \to \mathscr{S}_q^m,$$

where

$$p - d = \tilde{d}.$$

To give the KAM theorem we need to introduce some domains and norms. For $s, r > 0$, we introduce the complex T_0-neighborhoods

$$D(s,r) = \{|\Im x| < s\} \times \{|y| < r^2\} \times \{\|z\|_p + \|\bar{z}\|_p < r\} \tag{4.2}$$

$$\subset \mathbb{C}^m \times \mathbb{C}^m \times \ell^2_{p,\mathbb{C}} \times \ell^2_{p,\mathbb{C}} = \mathscr{S}^m_{p,\mathbb{C}}, \tag{4.3}$$

and weighted norm for $W = (X, Y, Z, \bar{Z}) \in \mathscr{S}^m_{q,\mathbb{C}}$,

$$\|W\|_{r,q} = |X| + \frac{|Y|}{r^2} + \frac{\|Z\|_q}{r} + \frac{\|\bar{Z}\|_q}{r},$$

where $|\cdot|$ denotes the sup-norm for complex vectors. Furthermore, for a map $W : U \times \Pi \to \mathscr{S}^m_{q,\mathbb{C}}$, such as the Hamiltonian vector field X_P, we define the norms

$$\|W\|^{\sup}_{r,q;U\times\Pi} = \sup_{(w,\xi)\in U\times\Pi} \|W(w,\xi)\|_{r,q},$$

$$\|W\|^{\mathrm{lip}}_{r,q;U\times\Pi} = \sup_{\xi,\zeta\in\Pi,\xi\neq\zeta} \frac{\|\Delta_{\xi\zeta} W\|^{\sup}_{r,q;U}}{|\xi - \zeta|},$$

where $\Delta_{\xi\zeta} W = i_\xi W - i_\zeta W$, and

$$\|i_\xi W\|^{\sup}_{r,q;U} = \sup_{w\in U} \|W(w,\xi)\|_{r,q}.$$

In a completely analogous manner, the Lipschitz semi-norm of the frequencies ω is defined as

$$|\omega|^{\mathrm{lip}}_{\Pi} = \sup_{\xi,\zeta\in\Pi,\xi\neq\zeta} \frac{\|\Delta_{\xi\zeta}\omega\|}{|\xi - \zeta|},$$

and the Lipschitz semi-norm of $\tilde{\Omega} : \Pi \to \ell^\infty_{-\delta}$ is defined as

$$|\tilde{\Omega}|^{\mathrm{lip}}_{-\delta,\Pi} = \sup_{\xi,\zeta\in\Pi,\xi\neq\zeta} \frac{\|\Delta_{\xi\zeta}\tilde{\Omega}\|_{-\delta}}{|\xi - \zeta|}$$

for any real number δ. Note that $|\tilde{\Omega}|^{\mathrm{lip}}_{-\delta,\Pi} = |\Omega|^{\mathrm{lip}}_{-\delta,\Pi}$, since $\bar{\Omega} = \Omega - \tilde{\Omega}$ is independent of ξ.

Suppose the normal form N described above satisfies the following assumptions:
Assumption A: Frequency Asymptotics. There exist two real numbers $d > 1$ and $\delta < d - 1$ such that the following holds. First, the frequencies Ω_n are real valued functions of ξ of the form

$$\Omega_n(\xi) = \bar{\Omega}_n + \tilde{\Omega}_n(\xi),$$

where $\bar{\Omega}_n$ is independent of ξ and of the form $\tilde{\Omega}_n = cn^d + \cdots$, where the dots stand for an expansion in lower order terms in n. Second, the functions

$$\xi \mapsto \frac{\tilde{\Omega}_n(\xi)}{n^\delta},\ n \geq 1$$

are uniformly Lipschitz on Π, or equivalently, the map

$$\tilde{\Omega} : \Pi \to \ell^\infty_{-\delta},\ \xi \mapsto \tilde{\Omega}(\xi) = (\tilde{\Omega}_n(\xi))_{n\geq 1}$$

is Lipschitz on Π.

Assumption B: Nondegeneracy. The map $\xi \to \omega(\xi)$ between Π and its image is a homeomorphism which is Lipschitz continuous in both directions. Moreover, for every $k \in \mathbb{Z}^m$ and $l \in \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$ (here $|l| = \sum_{j\geq 1} |l_j|$), the resonance set

$$\Re_{kl} = \{\xi \in \Pi : \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\} \tag{4.4}$$

has Lebesgue measure zero.

Assumption C: Regularity. There is a neighbourhood U of T_0 in $\mathscr{S}^m_{p,\mathbb{C}}$ such that P is defined on $U \times \Pi$, and its Hamiltonian vector field defines a map

$$X_P : U \times \Pi \to \mathscr{S}^m_{q,\mathbb{C}},$$

where q satisfies

$$p - q < d - 1.$$

Moreover, $i_\xi X_P$ is real analytic on U for each $\xi \in \Pi$, and $i_w X_P$ is uniformly Lipschitz on Π for each $w \in U$.

We introduce one more constant. By assumption A and B,

$$|\omega|^{\text{lip}}_\Pi + |\Omega|^{\text{lip}}_\Pi \leq M < \infty.$$

Finally observe that if X_P satisfies assumption C, then it does so with the T_0-neighbourhoods $D(s, r)$ for all $s > 0, r > 0$ sufficiently small.

Under the above conditions, we have the following KAM theorem.

Theorem 4.1. *Suppose N is a family of Hamiltonians of the form (4.1) defined on a phase space $\mathscr{S}^m_p$ and depending on parameters in Π so that assumption A and B are satisfied. Then there exists a positive constant γ depending only on m, d, δ, the frequencies ω and Ω and the real number $s > 0$ such that for every perturbed Hamiltonian $H = N + P$ that satisfies assumption C and the smallness condition*

$$\epsilon = \|X_P\|^{\sup}_{r,q,D(s,r)\times\Pi} + \frac{\alpha}{M}\|X_P\|^{\text{lip}}_{r,q,D(s,r)\times\Pi} \leq \alpha\gamma \tag{4.5}$$

for some $r > 0$ and $0 < \alpha < 1$, the following holds. There exist

(i) a Cantor set $\Pi_\alpha \subset \Pi$ with meas$(\Pi \setminus \Pi_\alpha) \to 0$ $(\alpha \to 0)$,

(ii) a Lipschitz family of real analytic torus embeddings $\Phi : \mathbb{T}^m \times \Pi_\alpha \to \mathscr{S}^m_p$,

(iii) a Lipschitz map $\phi : \Pi_\alpha \to \mathbb{R}^m$,

such that for each $\xi \in \Pi_\alpha$, the map Φ restricted to $\mathbb{T}^m \times \{\xi\}$ is a real analytic embedding of a rotational frequencies $\phi(\xi)$ for the perturbed Hamiltonian H at ξ. In other words,

$$t \mapsto \Phi(\theta + t\phi(\xi), \xi), t \in \mathbb{R}$$

is a real analytic, quasi-periodic solution for the Hamiltonian $i_\xi H$ for every $\theta \in \mathbb{T}^m$ and $\xi \in \Pi_\alpha$.

Moreover, each embedding is real analytic on $D(s/2) = \{|\Im x| < s/2\}$, *and*

$$\|\Phi - \Phi_0\|^{\sup}_{r,p,D(s/2)\times\Pi_\alpha} + \frac{\alpha}{M}\|\Phi - \Phi_0\|^{\mathrm{lip}}_{r,p,D(s/2)\times\Pi_\alpha} \le \frac{c\epsilon}{\alpha},$$

$$|\phi - \omega|^{\sup}_{\Pi_\alpha} + \frac{\alpha}{M}|\phi - \omega|^{\mathrm{lip}}_{\Pi_\alpha} \le c\epsilon,$$

where

$$\Phi_0 : \mathbb{T}^m \times \Pi \to T_0,\ (x,\xi) \mapsto (x,0,0,0)$$

is the trivial embedding for each ξ, *and* c *is a positive constant which depends on the same parameters as* γ.

Proof. The proof can be found in [**11**]. □

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