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Blow-up for self-interacting fractional Ginzburg-Landau equation

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Abstract. The blow-up of solutions for the Cauchy problem of fractional Ginzburg-Landau equation with non-positive nonlinearity is shown by an ODE argument. Moreover, in one dimensional case, the optimal lifespan estimate for size of initial data is obtained.

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1. Introduction

The classical complex Ginzburg-Landau (CGL) equation takes the form

$$\partial_t \psi = -(\alpha + \mathrm{i}\beta)\Delta\psi + F(\psi, \overline{\psi}), \tag{1.1}$$

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where α, β are real parameters. The standard CGL equation has a self-interaction term F of the form

$$F(\psi, \overline{\psi}) = -\sum_{j=1}^{K} (\alpha_j + \mathrm{i}\gamma_j)\psi|\psi|^{p_j - 1},$$

where α_j, β_j are real parameters. We refer to [**5**] for a review on this subject. Using the representation $\psi(t, x) = u_1(t, x) + \mathrm{i}u_2(t, x)$, where u_1, u_2 are real-valued functions, we see that the equation (1.1) can be rewritten in the form of a system of reaction diffusion equations

$$\partial_t U = A\Delta U = F(U),$$

where

$$U(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}, \quad A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}.$$

The limiting case $\alpha \to 0, \alpha_j \to 0$ leads to the nonlinear Schrödinger equation (NLS)

$$\partial_t \psi = -\mathrm{i}\beta\Delta\psi - \sum_{j=1}^{N} \mathrm{i}\gamma_j \psi|\psi|^{p_j - 1}. \tag{1.2}$$

The oscillation synchronization of phenomena modeled by Kuramoto equations (see [**4**]) lead to a system of ODE having a similar qualitative behavior

$$\partial_t \psi_k = -\mathrm{i}H_k \psi_k + F_k(\Psi, \overline{\Psi}), \quad k = 1, \cdots, N. \tag{1.3}$$

The nonlinear terms F_k in the system obey the property

$$\mathrm{Im}\ \left(F_k(\Psi, \overline{\Psi})\ \ \overline{\psi_k}\right) = 0,\ k = 1, \cdots, N.$$

This system simulates the behavior of N oscillators, so that $\Psi = (\psi_1, \cdots, \psi_N)$, with ψ_j being complex-valued functions. The nonlinearities in (1.3) are chosen so that the evolution flow associated to the Kuramoto system leaves the manifold

$$\mathcal{M} = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{N \text{ times}},$$

invariant.

The derivation of the Kuramoto system in [**4**] is based on complex Landau-Ginzburg equation (see equation (2.4.15) in [**4**])

$$\partial_t \Psi = \mathrm{i}\mathcal{H}\Psi - (\alpha + \mathrm{i}\beta)\Delta\Psi - (\alpha_1 + \mathrm{i}\beta_1)\Psi|\Psi|^2,$$

where $\Psi = (\psi_1, \cdots, \psi_N)^t$, $\mathcal{H}$ is a diagonal matrix with real entries. If β and β_1 become very large, then we have an equation very close to Schrödinger self-interacting system (1.2). As it was pointed out (p. 20, [**4**]), a chemical turbulence of a diffusion-induced type are possible only for regions intermediate between the two extreme cases, where β and β_1 are very small or very large.

Turning back to CGL equation and comparing (1.1) with Kuramoto system, we see that it is natural to take $\alpha \to 0$, $\beta_j \to 0$ so that we have the following simplified CGL equation

$$\partial_t \psi = -\mathrm{i}\beta\Delta\psi - \alpha_1 \psi|\psi|^{p-1}.$$

A similar system was discussed in [**1**] with nonlinearity typical for the Kuramoto system.

The fractional dynamics seems more adapted to synchronization models due to the considerations in [**6**], therefore we can consider the following fractional Ginzburg-Landau equations

$$\partial_t \psi = -\mathrm{i}\sqrt{-\Delta}\psi \pm \psi|\psi|^{p-1}.$$

The study of the attractive case

$$\partial_t \psi = -\mathrm{i}|D|\psi - \psi|\psi|^{p-1}, \quad |D| = \sqrt{-\Delta},$$

is initiated in [**2**], where the well-posedness is established for the cases $1 \leq n \leq 3$.

In this article, we study the repulsive case

$$\begin{cases} \partial_t u = -\mathrm{i}|D|u + |u|^{p-1}u, & t \in [0,T), \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $n \geq 1$, and $p > 1$. Our main goal is to obtain a blow-up result under the assumption that initial data are in $H^s(\mathbb{R}^n)$ with $s > n/2$, where $H^s(\mathbb{R}^n)$ is the usual Sobolev space defined by $(1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$.

We denote $\langle x \rangle = (1+|x|^2)^{1/2}$. We abbreviate $L^q(\mathbb{R}^n)$ to L^q and $\|\cdot\|_{L^q(\mathbb{R}^n)}$ to $\|\cdot\|_q$ for any q. We also denote by $\|T\|$ the operator norm of bounded operator $T : L^2 \to L^2$.

The following statements are the main results of this article.

Proposition 1.1. *Let h be a function satisfying $\frac{1}{h} \in L^\infty \cap L^2$ and*

$$\left\| \frac{1}{h}[(-\Delta)^{1/2}, h] \right\| \tag{1.5}$$

Let $u_0 \in hL^2$ satisfy

$$\|\frac{1}{h}u_0\|_2 \geq \|\frac{1}{h}[D,h]\|^{\frac{1}{p-1}} \|\frac{1}{h}\|_2. \tag{1.6}$$

If there is a solution $u \in C([0,T); hL^2)$ for (1.4), then

(1.7)

$$\|\frac{1}{h}u(t)\|_2$$

$$\geq e^{-2\|\frac{1}{h}[D,h]\|t}\Big(\|\frac{1}{h}u_0\|_2^{-p+1} + \|\frac{1}{h}[D,h]\|^{-1}\|\frac{1}{h}\|_2^{-p+1}\big\{e^{-\|\frac{1}{h}[D,h]\|(p-1)t} - 1\big\}\Big)^{-\frac{1}{p-1}}.$$

Therefore, the lifespan is estimated by

$$T \leq -\frac{2}{p-1}\|\frac{1}{h}[D,h]\|^{-1} \log\left(1 - \|\frac{1}{h}[D,h]\|\|\frac{1}{h}\|_2^{p-1}\|\frac{1}{h}u_0\|_2^{-p+1}\right). \tag{1.8}$$

We remark that for $n = 1$, we can take $h = \langle \cdot \rangle$ for Proposition 1.1. Proposition 1.1 is a blow-up result for a kind of large data of hL^2. However, in a subcritical case where $p < p_F = 3$, solutions blow up even for small L^2 initial data.

Corollary 1.2. *Let $u_0 \in L^2(\mathbb{R})\backslash\{0\}$ and $1 < p < p_F$. Then the corresponding solution in $C([0,T); L^2(\mathbb{R}))$ blows up at a finite positive time.*

Remark 1.3. If we choose $h(x) = \langle x \rangle$, the statement of our main result guarantees the blow-up of the momentum

$$Q_{-1}(t) = \int_{\mathbb{R}} \langle x \rangle^{-1} |u(t,x)|^2 dx.$$

for the solution to the fractional CGL equation

$$\partial_t u = -\mathrm{i}|D|u + |u|^{p-1}u$$

in (1.4). The blow-up mechanism is based on the differential inequality

$$Q'_{-1}(t) \geq C_0 \left(Q_{-1}(t)\right)^{(p+1)/2} - C_1 Q_{-1}(t), \quad C_0, C_1 > 0. \tag{1.9}$$

Comparing the fractional CGL equation with the classical NLS

$$\partial_t u = \mathrm{i}\Delta u + \mathrm{i}|u|^{p-1}u,$$

we see that introducing the momentum

$$Q_2(t) = \int_{\mathbb{R}^n} |x|^2 |u(t,x)|^2 dx,$$

and using a Virial identity one can show that

$$Q''_2(t) \sim E(u)(t),$$

where $E(u)(t) = \|Du(t)\|^2_{L^2} - c\|u(t)\|^{p+1}_{L^{p+1}}$ and $c > 0$ is an appropriate constant. Therefore, the blow-up mechanism for NLS is based on the estimate

$$E(u)(t) \leq -\delta, \ \delta > 0,$$

that implies differential inequality

$$Q''_2(t) \leq -\delta$$

and the last inequality can not be satisfied for the whole interval $t \in (0, \infty)$ since $Q_2(t)$ is a positive quantity.

Moreover, for large R, if u_0 is given by Rf with $f \in hL^2(\mathbb{R})$ and h satisfying (1.5), then (1.8) means $T \leq CR^{-p+1}$. In one dimensional case, this upper bound is shown to be sharp for $f \in (hL^2 \cap H^1)(\mathbb{R})$.

Proposition 1.4. *Let $u_0 = Rf$ with $R > 0$ sufficiently large and $f \in H^1(\mathbb{R})$. Then there exists an $H^1(\mathbb{R})$ solution for u_0 for which its lifespan is estimated by $T \geq CR^{-p+1}$ with some positive constant C.*

2. Preliminary

In this section, we recall the blow-up solutions for an ODE which gives the mechanism of blow-up for weighted L^2 norm of solutions. We also study the condition for weight functions of Corollary 1.2.

2.1. Blow-up solutions for an ODE.

Lemma 2.1. *Let $C_1, C_2 > 0$ and $q > 1$. If $f \in C^1([0,T);\mathbb{R})$ satisfies $f(0) > 0$ and*

$$f' + C_1 f = C_2 f^q \quad \text{on } [0,T) \text{ for some } T > 0,$$

then

$$f(t) = e^{-C_1 t}\left(f(0)^{-(q-1)} + C_1^{-1}C_2 e^{-C_1(q-1)t} - C_1^{-1}C_2\right)^{-\frac{1}{q-1}}.$$

Moreover, if $f(0) > C_1^{\frac{1}{q-1}} C_2^{-\frac{1}{q-1}}$, then $T < -\frac{1}{C_1(q-1)} \log(1 - C_1 C_2^{-1} f(0)^{-q+1})$.

PROOF. Let $f = e^{-C_1 t} g$. Then

$$g' = C_2 e^{-C_1(q-1)t} g^q.$$

Therefore,

$$\frac{1}{1-q}\Big(g^{1-q}(t) - g^{1-q}(0)\Big) = \frac{C_2}{C_1(1-q)}(e^{-C_1(q-1)t} - 1).$$

□

2.2. Condition for weight function.

LEMMA 2.2 (Coiffman - Meyer). *Let $p \in C^\infty(\mathbb{R}^{2n})$ satisfy the estimates*

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta}\langle \xi\rangle^{1-|\alpha|}$$

for all multi-indices α and β. Then for any Lipschitz function h,

$$\|[p(x,D),h]f\|_2 \le C\|h\|_{\mathrm{Lip}}\|f\|_2.$$

LEMMA 2.3. *Let $\phi \in C_0^\infty([0,\infty);\mathbb{R})$ satisfy*

$$\phi(\rho) = \begin{cases} 1 & \text{if} \quad 0 \le \rho \le 1, \\ 0 & \text{if} \quad \rho \ge 2. \end{cases}$$

Then

$$\left| \int_{\mathbb{R}^n} \phi(|\xi|)|\xi| e^{\mathrm{i}x\cdot\xi} d\xi \right| \le C\langle x\rangle^{-n-1}.$$

PROOF. It suffices to consider the case where $|x|$ is sufficiently large. Let $\psi \in C_0^\infty([0,\infty);\mathbb{R})$ satisfy

$$\psi(\rho) = \begin{cases} 1 & \text{if} \quad 0 \le \rho \le 1, \\ 0 & \text{if} \quad \rho \ge 2. \end{cases}$$

Let $e_1 = (1,0,\cdots,0)$. Let $\xi_1 = \xi\cdot e_1$ and $\xi' = \xi - \xi_1 e_1$. Assume $x = |x|e_1$. Then

$$\int_{\mathbb{R}^n} \phi(|\xi|)|\xi| e^{\mathrm{i}x\cdot\xi} d\xi = \int_{\mathbb{R}^n} \phi(|\xi|)|\xi| e^{\mathrm{i}|x|\xi_1} d\xi.$$

By integrating by parts k times,

$$\begin{aligned} \int_{\mathbb{R}^n} \phi(|\xi|)|\xi| e^{\mathrm{i}|x|\xi_1} d\xi &= (-\mathrm{i}|x|)^{-k} \int_{\mathbb{R}^n} \partial_1^k(\phi(|\xi|)|\xi|) e^{\mathrm{i}|x|\xi_1} d\xi \\ &= (-\mathrm{i}|x|)^{-k} \int_{\mathbb{R}^n} \partial_1^k(|\xi|)\phi(|\xi|) e^{\mathrm{i}|x|\xi_1} + R_k(\xi) d\xi, \end{aligned}$$

where $R_{n+1} \in L^1(\mathbb{R}^n)$. Here $\partial_1^k|\xi|$ is estimated by $C|\xi|^{1-k}$. Moreover,

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \partial_1^n(|\xi|)\phi(|\xi|) e^{\mathrm{i}|x|\xi_1} d\xi \right| \\ &= |x|^{-1} \left| \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_1\{\partial_1^n(|\xi|)\phi(|\xi|)\}(e^{\mathrm{i}|x|\xi_1} - 1) d\xi_1 d\xi' \right| \\ &\le C \left| \int_{\mathbb{R}^n} \psi(|x||\xi|)|\xi|^{1-n}|\phi(|\xi|)| + |\partial_1\phi(|\xi|)|\} d\xi \right| \\ &+ C|x|^{-1} \left| \int_{\mathbb{R}^n} (1 - \psi(|x||\xi|))|\xi|^{-n}\{|\phi(|\xi|)| + |\partial_1\phi(|\xi|)|\} d\xi \right| \end{aligned}$$

since $|e^{\mathrm{i}|x|\xi_1} - 1| \leq |x||\xi|$. The first integral is estimated by

$$C\int_0^{2|x|^{-1}} |\phi(\rho)| + |\phi'(\rho)| d\rho \leq C\|\phi\|_{C^1(0,2)}|x|^{-1}.$$

By letting $\Psi(\rho) = \int_0^\rho |1 - \psi(\rho')| d\rho'$ and integrating by parts once again, the second integral is estimated by

$$C|x|^{-2}\int_{|x|^{-1}}^{2} \rho^{-2}\|\phi\|_{C^2(0,2)}\Psi(|x|\rho)d\rho \leq C\|\phi\|_{C^2(0,2)}|x|^{-1}.$$

This proves the lemma. □

Lemma 2.4. *Let h be a Lipschitz function on $\mathbb{R}^n$ satisfying the estimate*

$$\left\|\frac{1}{h(\cdot)}\int_{\mathbb{R}^n}\langle\cdot - y\rangle^{-n-1}h(y)f(y)dy\right\|_2 \leq C\|f\|_2$$

for any $f \in L^2$. Then $\frac{1}{h}[D, h]$ is a bounded operator from L^2 to L^2.

Proof. Let ϕ be a smooth function on $[0, \infty)$ satisfying that $\phi(\xi) = 1$ if $|\xi| \leq 1$ and $\phi(\xi) = 0$ if $|\xi| \geq 2$. Let $\phi(D)f = \mathfrak{F}^{-1}\phi\hat{f}$. We divide the proof into the following two estimate: $\|\frac{1}{h}\phi(D)(Dhf)\|_2 \leq \|f\|_2$ and $\|\frac{1}{h}[(1 - \phi(D))D, h]f\|_2 \leq \|f\|_2$.

At first,

$$\|\frac{1}{h}\phi(D)(Dhf)\|_2 \leq C\left\|\frac{1}{h(\cdot)}\int_{\mathbb{R}^n}\langle\cdot - y\rangle^{-n-1}h(y)f(y)dy\right\|_2 \leq C\|f\|_2,$$

since

$$|\mathfrak{F}^{-1}(|\cdot|\phi)| \leq C\langle x\rangle^{-n-1}.$$

Secondly, $(1 - \phi(|\xi|))|\xi|$ satisfies the condition of Lemma 2.2. So the second estimate follows from Lemma 2.2. □

Remark 2.5. $h(x) = \langle x\rangle$ satisfies the condition of Lemma 2.4. Actually h is Lipshitz and by using triangle inequality,

$$\begin{aligned}
&\left\|\langle x\rangle^{-1}\int_{\mathbb{R}^n}\langle x - y\rangle^{-n-1}\langle y\rangle f(y)dy\right\|_2 \\
&\leq \left\|\int_{\mathbb{R}^n}\langle x - y\rangle^{-n-1}f(y)dy\right\|_2 + \left\|\langle x\rangle^{-1}\int_{\mathbb{R}^n}\langle x - y\rangle^{-n}f(y)dy\right\|_2 \\
&\leq (\|\langle\cdot\rangle^{-n-1}\|_1 + \|\langle\cdot\rangle^{-1}\|_q\|\langle\cdot\rangle^{-n}\|_{q'})\|f\|_2,
\end{aligned}$$

where $n < q < \infty$.

Corollary 2.6. *Let h satisfy the condition of Lemma 2.4 and let h_R be $h_R = h(\cdot/R)$. Then*

$$\|\frac{1}{h_R}[D, h_R]\| \leq R^{-1}\|\frac{1}{h}[D, h]\|.$$

PROOF.

$$\begin{aligned}\frac{1}{h_R(x)}[D,h_R]f(x) &= \frac{1}{h(\frac{x}{R})}\int_{\mathbb{R}^n} e^{\mathrm{i}(x-y)\cdot\xi}|\xi|\{h(\frac{y}{R})-h(\frac{x}{R})\}f(y)d\xi dy\\ &= R^n\frac{1}{h(\frac{x}{R})}\int_{\mathbb{R}^n} e^{\mathrm{i}(\frac{x}{R}-y)\cdot R\xi}|\xi|\{h(y)-h(\frac{x}{R})\}f(Ry)d\xi dy\\ &= R^{-1}\frac{1}{h(\frac{x}{R})}\int_{\mathbb{R}^n} e^{\mathrm{i}(\frac{x}{R}-y)\cdot\xi}|\xi|\{h(y)-h(\frac{x}{R})\}f(Ry)d\xi dy\\ &= R^{-1}\frac{1}{h(\frac{x}{R})}[D,h]f_{R^{-1}}(\frac{x}{R}).\end{aligned}$$

This implies

$$\|\frac{1}{h_R}[D,h_R]f\|_2 = R^{-1+n/2}\|\frac{1}{h}[D,h]f_{R^{-1}}\|_2 \le R^{-1}\|\frac{1}{h}[D,h]\|\|f\|_2.$$

□

3. Proof

3.1. Proof of Proposition 1.1. Let $u(t,x)=h(x)v(t,x)$. Then

$$\mathrm{i}\partial_t v + Dv + \frac{1}{h}[D,h]v = \mathrm{i}h^{p-1}|v|^{p-1}v. \tag{3.1}$$

Multiplying both hand sides of (3.1) by $\overline{v}$, integrating over $\mathbb{R}^n$, and taking the imaginary part of the resulting integrals, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 &= \int_{\mathbb{R}^n} h(x)^{p-1}|v(t,x)|^{p+1}dx - \mathrm{Im}\int_{\mathbb{R}^n}\overline{v(t,x)}\frac{1}{h(x)}[D,h]v(t,x)dx\\ &\ge \int_{\mathbb{R}^n} h(x)^{p-1}|v(t,x)|^{p+1}dx - \|\frac{1}{h}[D,h]\|\|v(t)\|_2^2\\ &\ge \|\frac{1}{h}\|_2^{-p+1}\|v(t)\|_2^{p+1} - \|\frac{1}{h}[D,h]\|\|v(t)\|_2^2,\end{aligned}$$

where we used the following estimate:

$$\|v(t)\|_2 \le \|\frac{1}{h^{\frac{p-1}{p+1}}}\|_{\frac{2(p+1)}{p-1}}\|h^{\frac{p-1}{p+1}}v(t)\|_{p+1}.$$

Then (1.7) follows from Lemma 2.1 with $q=(p+1)/2$.

3.2. Proof of Corollary 1.2. Let $h_R(x)=\langle x/R\rangle$ with $R>0$. h_R satisfies (1.5) and $1/h_R\in(L^\infty\cap L^2)(\mathbb{R})$, and $\frac{1}{h_R}u_0\to u_0$ in L^2 as $R\to\infty$. Moreover, $\|\frac{1}{h_R}[D,h_R]\|\sim R^{-1}$, and $\|\frac{1}{h_R}\|_2\sim R^{1/2}$. Therefore

$$\text{RHS (1.6)} \sim R^{\frac{1}{2}-\frac{1}{p-1}} \to 0$$

as $R\to\infty$ if $p<3$. It means that for any $u_0\in L^2(\mathbb{R})\backslash\{0\}$, there exists R_0 such that (1.6) is satisfied with $h(x)=\langle x/R_0\rangle$.

3.3. Proof of Proposition 1.4. The local well-posedness in $H^1(\mathbb{R})$ is easily obtained by the Sobolev embedding and standard contraction argument. By multiplying (1.4) by $\overline{u}$ and $(-\Delta)\overline{u}$, integrating over $\mathbb{R}$, we obtain

$$\frac{d}{dt}\|u(t)\|_2^2 = \|u(t)\|_{p+1}^{p+1} \le C\|u(t)\|_{H^1(\mathbb{R})}^{p+1},$$
$$\frac{d}{dt}\|\nabla u(t)\|_2^2 = \mathrm{Re}\int_{\mathbb{R}} \nabla(|u(t,x)|^{p-1}u(t,x)) \cdot \overline{\nabla u(t,x)}dx \le C\|u(t)\|_{H^1(\mathbb{R})}^{p+1},$$

where $\|f\|_{H^1(\mathbb{R})}^2 = \|f\|_2^2 + \|\nabla f\|_2^2$. By solving the following ordinary differential equality:

$$\frac{d}{dt}U(t) = CU(t)^{\frac{p+1}{2}},$$

we get

$$\|u(t)\|_{H^1(\mathbb{R})} \le \left(\|u_0\|_{H^1(\mathbb{R})}^{-(p-1)} - \frac{C(p-1)}{2}t\right)^{-\frac{1}{p-1}}.$$

This proves the Proposition 1.4.

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