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Global regularity of logarithmically supercritical MHD system with improved logarithmic powers

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ABSTRACT. The magnetohydrodynamics system consists of a coupling of the Navier-Stokes equations and Maxwell's equation from electromagnetism. We extend the work of [**2**] on the Navier-Stokes equations to the magnetohydrodynamics system to prove its global well-posedness with logarithmically supercritical dissipation and diffusion with the logarithmic power that is improved in contrast to the previous work of [**14**]. The main difficulty is that the method in [**2**] relies heavily on the symmetry within the Navier-Stokes equation, which is lacking in the magnetohydrodynamics system due to the non-linear terms that are mixed with both velocity and magnetic fields; this difficulty may be overcome by somehow taking advantage of the symmetry within the energy formulation of the magnetohydrodynamics system appropriately.

CONTENTS

1. Introduction and Statement of Main Results

The Navier-Stokes equations (NSE) of fluid mechanics is a system of the following equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi - \nu \Delta u = 0, \tag{1.1a}$$

$$\nabla \cdot u = 0, \quad u(x,0) \triangleq u_0(x), \tag{1.1b}$$

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where $u : X \times \mathbb{R}_+ \mapsto \mathbb{R}^d, \pi : X \times \mathbb{R}_+ \mapsto \mathbb{R}$, for $d \in \mathbb{N}$ such that $d \geq 2$ and X equals either $\mathbb{R}^d$ or $\mathbb{T}^d = [0, 2\pi]^d$, represent the velocity and pressure fields respectively, u_0 denotes the initial data, and $\nu \geq 0$ the viscosity coefficient. For brevity, hereafter let us write $\frac{\partial}{\partial t} = \partial_t$ and $\int f$ for an integral over $\mathbb{R}^d$ or $\mathbb{T}^d$ when no confusion arises. Moreover, the magnetohydrodynamics (MHD) system consists of a coupling of the NSE with the Maxwell's equations from the electromagnetism as follows:

$$\partial_t u + (u \cdot \nabla) u + \nabla \pi - \nu \Delta u = (b \cdot \nabla) b, \tag{1.2a}$$

$$\partial_t b + (u \cdot \nabla) b - \eta \Delta b = (b \cdot \nabla) u, \tag{1.2b}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \quad (u, b)(x, 0) \triangleq (u_0, b_0)(x), \tag{1.2c}$$

where $b : X \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ represents the magnetic field, (u_0, b_0) denotes the initial data, and $\eta \geq 0$ the magnetic diffusivity.

Let us point out that the MHD system (1.2a)-(1.2c) at $b \equiv 0$ recovers the NSE (1.1a)-(1.1b) and thus the discussion hereafter shall formally focus on the MHD system, with immediate implications on the NSE by considering zero magnetic field. Moreover, for generality, let us consider the following fractional MHD system, the case in which $\alpha = \beta = 2$ recovers the system (1.2a)-(1.2c):

$$\partial_t u + (u \cdot \nabla) u + \nabla \pi + \nu \Lambda^\alpha u = (b \cdot \nabla) b, \tag{1.3a}$$

$$\partial_t b + (u \cdot \nabla) b + \eta \Lambda^\beta b = (b \cdot \nabla) u, \tag{1.3b}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \quad (u, b)(x, 0) \triangleq (u_0, b_0)(x), \tag{1.3c}$$

where $\Lambda^r = (-\Delta)^{\frac{r}{2}}, r \in \mathbb{R}^+$, is defined through Fourier transform with a symbol of $|\xi|^r$. For simplicity hereafter we assume $\nu = \eta = 1$; this detail may be fixed with just more care in estimates.

Let us take $L^2(X)$-inner products with (u, b) in (1.3a), (1.3b) respectively and sum to deduce the well-known uniform bound on the kinetic energy and cumulative kinetic energy dissipation and diffusion as follows:

$$\sup_{t \in [0,T]} (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)(t) + \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2, \tag{1.4}$$

where $T > 0$ is such that the solution exists over $[0, T]$. One of the most fundamental properties of the solutions to such systems of nonlinear partial differential equations that we seek to determine is the global existence of the unique solution with finite kinetic energy. It is worth mentioning here that the proof of the global regularity of the solution for the MHD system (1.2a)-(1.2c) may be arguably more difficult than that of the NSE (1.1a)-(1.1b) because once the former proof is accomplished, one may consider the initial data (u_0, b_0) with $b_0 \equiv 0$ to immediately deduce by uniqueness the global smooth solution to the NSE (1.1a)-(1.1b). In this regard, one of the most general results known in the literature states that if

$$\alpha \geq 1 + \frac{d}{2}, \quad \beta \geq 1 + \frac{d}{2}, \tag{1.5}$$

then the global existence of the unique smooth solution to the system (1.3a)-(1.3c) follows (see e.g. [**13**]). Although improving these lower bounds on the powers of the fractional dissipativity and diffusivity has presented us with an formidable challenge, based on his own previous work in [**10**], Tao in [**11**] showed that we may at least break this threshold of $1+\frac{d}{2}$ for the system (1.3a)-(1.3c) at $b \equiv 0$ logarithmically. The intuition explained by Tao concerning this new phenomenon of the global regularity of the solutions to the logarithmically supercritical equations is as follows: typically by obtaining the bound such as

$$\partial_t \|f(t)\|_H^2 \leq A(t)B(t)\|f(t)\|_H^2 \tag{1.6}$$

for some Hilbert space H and $A(t), B(t)$ both being non-negative mappings, one may deduce its bound by Gronwall's inequality if $A(t)$ is locally integrable, $B(t)$ is uniformly bounded and $f(0) \in H$. It is well-known from elementary ODE theory that one cannot possibly hope to deduce a similar result if the power of $\|f(t)\|_H$ on the right hand side of (1.6) is bigger than two, even by an arbitrarily small amount. However, although such an exponential worsening is not allowed in order to claim a global bound of f in H, a logarithmic worsening is in fact allowed; that is,

$$\partial_t \|f(t)\|_H^2 \leq A(t)B(t)\|f(t)\|_H^2 \ln(e + \|f(t)\|_H^2)$$

still leads to the desired global bound of f in H under the same conditions of $A(t)$ being locally integrable, $B(t)$ being uniformly bounded, and $f(0) \in H$. With such heuristics in mind, Tao in [**11**] proved specifically that for the following system of

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi + Du = 0, \tag{1.7a}$$

$$\nabla \cdot u = 0, \quad \int u = 0, \quad u(x,0) \triangleq u_0(x), \tag{1.7b}$$

where $x \in \mathbb{T}^d$, D is a Fourier operator with a its symbol m such that $m(k) \geq c\frac{|k|^{1+\frac{d}{2}}}{g(|k|)}$ for all sufficiently large $|k|$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a nondecreasing function such that

$$\int_1^\infty \frac{ds}{sg^2(s)} = +\infty, \tag{1.8}$$

if u_0 is smooth and compactly supported, then it admits a global smooth solution. We note that Tao actually proved in case the spatial domain was $\mathbb{R}^d$, but made a comment concerning possible generalizations to $\mathbb{T}^d$ in Remark 2.1 [**11**]. Thereafter, Wu in [**14**] extended this result to the MHD system using Besov space techniques in case $X = \mathbb{R}^d$; specifically the global well-posedness was proven for the following system

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi + D_1 u = (b \cdot \nabla)b, \tag{1.9a}$$

$$\partial_t b + (u \cdot \nabla)b + D_2 b = (b \cdot \nabla)u, \tag{1.9b}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \quad (u,b)(x,0) \triangleq (u_0, b_0)(x), \tag{1.9c}$$

in which D_1, D_2 are both Fourier operators with symbols of m_1, m_2 respectively such that $m_1(\xi) \geq \frac{|\xi|^\alpha}{g_1(\xi)}, m_2(\xi) \geq \frac{|\xi|^\beta}{g_2(\xi)}$, where α, β have the lower bounds of (1.5) and $g_1 \geq 1, g_2 \geq 1$ are both radially symmetric, nondecreasing functions that satisfy

$$\int_1^\infty \frac{ds}{s(g_1(s) + g_2(s))^2} = +\infty. \tag{1.10}$$

To be complete, the results of [**14**] also allowed more flexibility within the sum of the powers $\alpha + \beta \geq 2 + d$; however, most importantly for our discussion, the criticality of the MHD system can be best described by the lower bound on the sum of α and β, and it remains unknown if this sum could be lowered even by an arbitrarily small amount (see also [**12, 16**] for the case of zero magnetic diffusion). We also refer to [**1, 3, 4, 9, 18**] for the relevant study of logarithmically supercritical dyadic model, Euler equations, Boussinesq system, wave equation, magnetic Bénard problem respectively.

In addition, Tao in [**11**] gave a heuristic argument suggesting that the power the function g in (1.8), namely 2, may be improved to 1. Very recently, this conjecture was in fact proven in the work of [**2**] which employed a significantly different approach from those of [**11, 14**] through constructing an appropriate shell model, defining a shell solution, and proving a certain recursive inequality for the energy and dissipation of the shell solution over large modes.

Before we state our main results, we wish to describe the symmetric property of the solution to the NSE that has become indispensable in the study of the NSE for decades and is missing in the case of the MHD system. E.g. in the important work of [**8**], the authors apply a curl operator to the two-dimensional NSE, denote the vorticity by $w \triangleq \nabla \times u$, the Biot-Savart law operator by $\mathcal{K}$ so that $\mathcal{K}w = u$ and rewrite the l-th component of the Fourier decomposition of the non-linear term as

$$\begin{aligned}
((\widehat{\mathcal{K}w \cdot \nabla})w)_l =& \sum_{j,k:j+k=l} \left(\frac{ij^\perp}{|j|^2} w_j \cdot ik\right) w_k \\
=& -\frac{1}{2} \sum_{j,k:j+k=l} \left(\frac{j^\perp \cdot k}{|j|^2} w_j w_k + \frac{k^\perp \cdot j}{|k|^2} w_k w_j\right) \\
=& -\sum_{j,k:j+k=l} \frac{1}{2}(j^\perp \cdot k)\left(\frac{1}{|j|^2} - \frac{1}{|k|^2}\right) w_j w_k.
\end{aligned}$$

Such a symmetric property has played a crucial role in various other fluid equations as well, e.g. in the process of Galerkin approximation for the surface quasi-geostrophic equation in [**6**]. Here, the last equality used the obvious fact that $w_j w_k = w_k w_j$; as clear as this observation is, it also allows one to immediately realize that an analogous identity is impossible for $(b \cdot \nabla)u$ or $(u \cdot \nabla)b$ within the MHD system (1.2a)-(1.2c) because in general $u_k b_j \neq b_k u_j$, and the major obstacle in extending the approach of [**2**] to the MHD system was that this symmetric property was in fact used in many parts of the proof of [**2**] (see Remark 3.1, (3.28), (3.29), (3.43), (3.44), (3.45)). We overcome this difficulty by discovering a symmetry somehow within the energy formulation of the MHD system, despite the fact that the MHD system itself really does not have the necessary symmetry property; we believe that this symmetry within the energy formulation of the MHD system has further potentials in future works as well.

THEOREM 1.1. *Consider for* $d \in \mathbb{N}, d \geq 2$, *the system (1.9a)-(1.9c) in* $\mathbb{T}^d$ *under the additional condition that* $\int_{\mathbb{T}^d} u = \int_{\mathbb{T}^d} b = 0$. *Suppose that the Fourier operators* D_1, D_2 *of (1.9a), (1.9b) have Fourier symbols of* m_1, m_2 *respectively such that*

$$m_1(k) \geq \frac{|k|^\alpha}{g_1(|k|)}, \quad m_2(k) \geq \frac{|k|^\beta}{g_2(|k|)}, \tag{1.11}$$

where the parameters α, β *have the lower bounds described in (1.5),* $g_1 : [0, \infty) \mapsto [1, \infty), g_2 : [0, \infty) \mapsto [1, \infty)$ *are both nondecreasing functions such that* $x^{-\alpha} g_1(x)$,

$x^{-\beta}g_2(x)$ are nonincreasing and satisfy

$$\int_1^\infty \frac{ds}{s(g_1(s)+g_2(s))} = +\infty. \tag{1.12}$$

Then, given smooth and periodic initial data, the system (1.9a)-(1.9c) has a unique smooth solution for all time $t > 0$.

REMARK 1.2.

(1) Comparing (1.10) and (1.12), we see that Theorem 1.1 improves the previous results of [**13, 14**], and as the MHD system at $b \equiv 0$ recovers the NSE, Theorem 1.1 also extends the work of [**2**]. Further flexibility between the parameters α, β may be possible as in the case of [**14**]; we choose not to pursue this direction due to complex interactions of the Fourier modes within the non-linear terms in the proof of Theorem 1.1, while we do point out that the necessary changes must be made prior to the equations (3.63), (3.64), (3.65), (3.66). For generalizations to the case of $\mathbb{R}^d$, we refer to Remark 2.9 [**2**].

(2) We mention one relevant open problem. In [**17**], the d-dimensional Boussinesq system with zero thermal diffusion and fractional dissipation with the equivalent strength of the fractional NSE, specifically $\alpha \geq 1 + \frac{d}{2}$, was proven to be globally well-posed. It is not clear to the author if that result may be logarithmically improved by any of the methods in [**2, 11, 12, 14, 16**].

2. Preliminaries

Let us assume the most difficult case where (1.5) and (1.11) are held with equalities instead of inequalities: $\alpha = \beta = 1 + \frac{d}{2}$, $m_1(k) = \frac{|k|^\alpha}{g_1(|k|)}$, $m_2(k) = \frac{|k|^\beta}{g_2(|k|)}$, which motivates us to denote for convenience $\lambda \triangleq 2^{1+\frac{d}{2}}$. We denote by $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$, by

$$H^r(\mathbb{T}^d) \triangleq \left\{ f = (f_k)_{k\in\mathbb{Z}^d} : \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^r |f_k|^2 < \infty \right\},$$

the usual Sobolev space with order $r \in \mathbb{R}^+$, by V^m the set of functions in $H^m(\mathbb{T}^d)$ those are mean zero and divergence-free, and by V^m_{weak} the space V^m equipped with weak topology.

We first consider the system (1.9a)-(1.9c) under the mean zero condition of $\int_{\mathbb{T}^d} u = \int_{\mathbb{T}^d} b = 0$ on the Fourier side and state an equivalent definition of its solution:

DEFINITION 2.1. A solution to the system (1.9a)-(1.9c) under the condition that $\int_{\mathbb{T}^d} u = \int_{\mathbb{T}^d} b = 0$ is a family $(u_k, b_k)_{k\in\mathbb{Z}^d}$ where each $(u_k, b_k) = (\hat{u}, \hat{b})_k$ is a differentiable mapping from $[0,\infty)$ to $\mathbb{C}^d$ and satisfy

$$\partial_t u_k = -\frac{|k|^\alpha}{g_1(|k|)} u_k - i \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \langle u_h, k\rangle P_k u_{k-h} + i \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \langle b_h, k\rangle P_k b_{k-h}, \tag{2.1a}$$

$$\partial_t b_k = -\frac{|k|^\beta}{g_2(|k|)} b_k - i \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \langle u_h, k\rangle b_{k-h} + i \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \langle b_h, k\rangle u_{k-h}, \tag{2.1b}$$

$$\langle u_k, k\rangle = \langle b_k, k\rangle = 0, \quad u_{-k} = \overline{u}_k,\ b_{-k} = \overline{b}_k, \quad u_0 = b_0 = 0, \tag{2.1c}$$

where $P_k(w) \triangleq w - \frac{\langle w,k\rangle k}{|k|^2}$.

Following [**2**] (we also refer to [**5**] for Littlewood-Paley theory on $\mathbb{T}^d$), we let $\Phi : [0,\infty) \mapsto [0,1]$ be a smooth mapping such that $\Phi|_{[0,1]} \equiv 1, \Phi|_{[2,\infty)} \equiv 0$ and $\Phi'|_{[1,2]} < 0$. For $x \geq 0$, we define $\psi(x) \triangleq \Phi(x) - \Phi(2x)$ so that ψ is smooth, $\text{supp}(\psi) \subset \left(\frac{1}{2}, 2\right)$ and

$$\sum_{n=0}^{\infty} \psi\left(\frac{x}{2^n}\right) = 1 - \Phi(2x) \equiv 1\ \forall\ x \geq 1. \tag{2.2}$$

We furthermore define $\psi_n : \mathbb{R}^d \mapsto [0,1]$ for all $n \in \mathbb{N}_0$ by $\psi_n(x) \triangleq \psi(2^{-n}|x|)$ so that

$$\sum_{n\in\mathbb{N}_0} \psi_n(x) \equiv 1\ \forall\ x \in \mathbb{Z}^d \setminus \{0\} \tag{2.3}$$

by (2.2) and $\text{supp}(\psi_n) \subset \{x \in \mathbb{R}^d : 2^{n-1} < |x| < 2^{n+1}\}$. As in the classical Littlewood-Paley theory, we define

$$P_n^u(x) \triangleq \sum_{k\in\mathbb{Z}^d} \psi_n(k) u_k e^{i\langle k,x\rangle}, \quad P_n^b(x) \triangleq \sum_{k\in\mathbb{Z}^d} \psi_n(k) b_k e^{i\langle k,x\rangle}, \tag{2.4}$$

so that we may write $u(x) = \sum_{n\in\mathbb{N}_0} P_n^u(x), b(x) = \sum_{n\in\mathbb{N}_0} P_n^b(x)$. However, due to the lack of orthogonality,

$$\sum_{n\in\mathbb{Z}} \|P_n^u\|_{L^2}^2 \neq \sum_{k\in\mathbb{Z}^d} |u_k|^2 = \|u\|_{L^2}^2, \quad \sum_{n\in\mathbb{Z}} \|P_n^b\|_{L^2}^2 \neq \sum_{k\in\mathbb{Z}^d} |b_k|^2 = \|b\|_{L^2}^2. \tag{2.5}$$

This motivates us to use the square-averaged Littlewood-Paley decomposition as in [**2**]:

$$X_n^u(t) \triangleq \left(\sum_{k\in\mathbb{Z}^d} \psi_n(k) |u_k(t)|^2\right)^{\frac{1}{2}}, \quad X_n^b(t) \triangleq \left(\sum_{k\in\mathbb{Z}^d} \psi_n(k) |b_k(t)|^2\right)^{\frac{1}{2}} \tag{2.6}$$

so that due to (2.6), Fubini's theorem, (2.3) and (2.5),

$$\sum_{n\in\mathbb{N}_0} |X_n^u|^2 = \|u\|_{L^2}^2, \qquad \sum_{n\in\mathbb{N}_0} |X_n^b|^2 = \|b\|_{L^2}^2. \tag{2.7}$$

Definition 2.2. We call $X^u \triangleq (X_n^u(t))_{n\in\mathbb{N}_0, t\geq 0}, X^b \triangleq (X_n^b(t))_{n\in\mathbb{N}_0, t\geq 0}$, the shell approximations of u, b respectively if they satisfy (2.6).

The following local well-posednedss result is an MHD version of Theorem A.1 [**2**] on the NSE (actually on the Leray-alpha model from [**15**], the special case of which is the NSE); its proof follows the approach of using mollifiers as in [**7**]. It is well-known that this method of proving local well-posednesss using mollifiers may be immediately extended to the MHD system (1.9a)-(1.9c) considering its solution as the $2d$-dimensional vector field (u, b) and thus we only state it and refer to Appendix [**2**] for its proof:

LEMMA 2.3. *Let* $m \geq 2 + \frac{d}{2}$, $(u_0, b_0) \in V^m \times V^m$. *Then there exists* $T > 0$ *such that the system (1.9a)-(1.9c) admits a unique solution* (u, b) *on* $[0, T]$ *and*

$$u \in L^\infty([0,T]; V^m) \cap Lip([0,T]; V_{m-\alpha}) \cap C([0,T]; V^m_{weak}), \int_0^T \|D_1^{\frac{1}{2}} u\|^2_{H^m} d\tau < \infty,$$

$$b \in L^\infty([0,T]; V^m) \cap Lip([0,T]; V_{m-\beta}) \cap C([0,T]; V^m_{weak}), \int_0^T \|D_2^{\frac{1}{2}} b\|^2_{H^m} d\tau < \infty.$$

Moreover, both u, b *are right-continuous, with values in* V^m *for the strong topology. Finally, if* T^* *is the maximal time of existence of the solution and* $T^* < \infty$, *then*

$$\limsup_{t \nearrow T^*} \|(u,b)(t)\|_{H^m} = +\infty. \tag{2.8}$$

REMARK 2.4. If $f \in H^\gamma(\mathbb{T}^d)$ ($f = u$ or b) and $X^f \triangleq (X^f_n(t))_{n \in \mathbb{N}_0, t \geq 0}$ is its shell approximation, then by (2.6), Fubini's theorem, it follows that

$$\sum_{n \in \mathbb{N}_0} 2^{2\gamma n} |X^f_n|^2 \approx \|f\|^2_{H^\gamma}. \tag{2.9}$$

Moreover, it may be immediately verified using (2.9) that $f \in C^\infty(\mathbb{T}^d)$ if and only if $\sup_{n \in \mathbb{N}_0} 2^{\delta n} X^f_n < \infty$ for all $\delta > 0$.

3. Proof of Theorem 1.1

By the blow-up criterion (2.8) of Lemma 2.3, in order to prove Theorem 1.1, it suffices to prove that $\limsup_{t \nearrow T^*} \|(u,b)\|_{H^m} < \infty$; by (2.9) we obtain the following reduction:

PROPOSITION 3.1. *Under the hypothesis of Theorem 1.1, if* (u, b) *is a solution to the system (1.9a)-(1.9c) in* $[0, T^*)$, X^u, X^b *are shell approximations of* u, b *respectively, and*

$$\sup_{t \in [0,T^*)} \left(\sum_{n \in \mathbb{N}_0} 2^{2mn} |X^u_n(t)|^2 + \sum_{n \in \mathbb{N}_0} 2^{2mn} |X^b_n(t)|^2 \right) < \infty, \tag{3.1}$$

then $T^* = +\infty$.

DEFINITION 3.2. We define

$$I \triangleq \left\{ (l, m, n) \in \mathbb{N}_0^3 : \max\{l, m, n\} - \max\left\{ \{l, m, n\} \setminus \{\max\{l, m, n\}\} \right\} \leq 2 \right\} \tag{3.2}$$

and let $X(t) \triangleq (X^u(t), X^b(t)) = (X^u_n(t), X^b_n(t))_{n \in \mathbb{N}_0}$ where $X^u_n : [0, \infty) \mapsto \mathbb{R}, X^b_n : [0, \infty) \mapsto \mathbb{R}$. We say $X = (X^u, X^b)$ is a shell solution of the system (1.9a)-(1.9c) if there exists $(\chi^u, \chi^b) \triangleq (\chi_n^u, \chi_n^b)_{n\in\mathbb{N}_0}$ and $\phi \triangleq (\phi^1_{(l,m,n)}, \phi^2_{(l,m,n)}, \phi^3_{(l,m,n)})_{(l,m,n)\in I}$ such that

$$\begin{aligned}
&\partial_t(|X_n^u|^2 + |X_n^b|^2)(t) \\
&= -\chi_n^u|X_n^u|^2(t) - \chi_n^b|X_n^b|^2(t) \\
&\quad - \sum_{l,m\in\mathbb{N}_0,(l,m,n)\in I} \phi^1_{(l,m,n)}(t)X_l^u(t)X_m^u(t)X_n^u(t) \\
&\qquad + \phi^3_{(l,m,n)}(t)X_l^u(t)X_m^b(t)X_n^b(t) \\
&\qquad - \phi^2_{(l,m,n)}(t)[X_l^b(t)X_m^b(t)X_n^u(t) + X_l^b(t)X_m^u(t)X_n^b(t)]
\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}_0, t > 0$, where the sum is absolutely convergent,

$$\phi^i_{(l,m,n)}(t) = -\phi^i_{(l,n,m)}(t) \quad \forall\, (l,m,n) \in I, t \geq 0, i = 1,2,3, \tag{3.4}$$

and there exists $c_1, c_2, c_3 > 0$ such that

$$\chi_n^u(t) \geq c_1 \frac{2^{\alpha n}}{g_1(2^{n+1})}, \quad \chi_n^b(t) \geq c_2 \frac{2^{\beta n}}{g_2(2^{n+1})}, \tag{3.5}$$

$$|\phi^i_{(l,m,n)}(t)| \leq c_3 2^{(1+\frac{d}{2})\min\{l,m,n\}} \quad \forall\, (l,m,n) \in I, t \geq 0, i = 1,2,3. \tag{3.6}$$

Furthermore, a shell solution X is said to satisfy the energy inequality over a time interval $[0,T]$ if $\sum_{n\in\mathbb{N}_0} X_n^2(0) = \sum_{n\in\mathbb{N}_0} |X_n^u|^2(0) + |X_n^b|^2(0)$ is finite and for all $t \in [0,T]$,

$$\sum_{n\in\mathbb{N}_0} |X_n|^2(t) + \int_0^t \chi_n^u|X_n^u|^2(\tau)d\tau + \int_0^t \chi_n^b|X_n^b|^2(\tau)d\tau \leq \sum_{n\in\mathbb{N}_0} X_n^2(0). \tag{3.7}$$

Following [**2**], we define $(F_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ the tail and energy bound of X respectively:

Definition 3.3. For a shell solution of the system (1.9a)-(1.9c) $X = (X^u, X^b) = (X_n^u, X_n^b)_{n\in\mathbb{N}_0}$, we define $(F_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ by $F_n(t) \triangleq F_n^u(t) + F_n^b(t), d_n(t) \triangleq (|d_n^u(t)|^2 + |d_n^b(t)|^2)^{\frac{1}{2}}$ where

$$F_n^u(t) \triangleq \sum_{k\geq n}|X_k^u|^2(t),\ F_n^b(t) \triangleq \sum_{k\geq n}|X_k^b|^2(t), \tag{3.8a}$$

$$d_n^u(t) \triangleq \left(F_n^u(t) + \sum_{h\geq n}\int_0^t \chi_h^u|X_h^u|^2(\tau)d\tau\right)^{\frac{1}{2}}, \tag{3.8b}$$

$$d_n^b(t) \triangleq \left(F_n^b(t) + \sum_{h\geq n}\int_0^t \chi_h^b|X_h^b|^2(\tau)d\tau\right)^{\frac{1}{2}}. \tag{3.8c}$$

In addition, we define

$$\overline{d}_n \triangleq \max_{s\in[0,t]} d_n^u(s) + d_n^b(s), \quad \overline{d}_n^u \triangleq \max_{s\in[0,t]} d_n^u(s), \quad \overline{d}_n^b \triangleq \max_{s\in[0,t]} d_n^b(s). \tag{3.9}$$

Proposition 3.4. *If (u,b) is a solution to the system (1.9a)-(1.9c), and X^u, X^b are shell approximations of u, b respectively, then $X = (X^u, X^b)$ is a shell solution to the system (1.9a)-(1.9c).*

Before we prove Proposition 3.4, we need several lemmas. Firstly, the following concerns one of the properties of a shell solution, specifically (3.5).

LEMMA 3.5. *Let X^u, X^b be the shell approximations of u, b respectively.*

(1) *If*

$$\chi_n^u(t) \triangleq \begin{cases} \frac{2}{|X_n^u|^2(t)} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k)\frac{|k|^\alpha}{g_1(|k|)}|u_k(t)|^2 & \text{if } X_n^u(t) \neq 0, \\ \frac{2^{\alpha n-\alpha+1}}{g_1(2^{n+1})} & \text{if } X_n^u(t) = 0, \end{cases} \tag{3.10}$$

then

$$\chi_n^u(t) \geq \frac{2^{\alpha n-\alpha+1}}{g_1(2^{n+1})} \quad \forall\, n \in \mathbb{N}_0, t \geq 0. \tag{3.11}$$

(2) *If*

$$\chi_n^b(t) \triangleq \begin{cases} \frac{2}{|X_n^b|^2(t)} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k)\frac{|k|^\beta}{g_2(|k|)}|b_k(t)|^2 & \text{if } X_n^b(t) \neq 0, \\ \frac{2^{\beta n-\beta+1}}{g_2(2^{n+1})} & \text{if } X_n^b(t) = 0, \end{cases} \tag{3.12}$$

then

$$\chi_n^b(t) \geq \frac{2^{\beta n-\beta+1}}{g_2(2^{n+1})} \quad \forall\, n \in \mathbb{N}_0, t \geq 0. \tag{3.13}$$

PROOF OF LEMMA 3.5. The proof of Lemma 3.5 is similar to that of Proposition 2.10 [**2**]; we sketch it here for completeness. As $\text{supp}(\psi_n) \subset \{x \in \mathbb{R}^d : 2^{n-1} < |x| < 2^{n+1}\}$, and $g_i, i = 1, 2$, are both nondecreasing, we deduce

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_n(k)|k|^\alpha}{g_1(|k|)}|u_k(t)|^2 \geq & \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k)\frac{2^{(n-1)\alpha}}{g_1(2^{n+1})}|u_k(t)|^2 \\ =& \frac{2^{(n-1)\alpha}}{g_1(2^{n+1})}|X_n^u|^2(t) \end{aligned} \tag{3.14}$$

due to (2.6). Thus, in case $X_n^u(t) \neq 0$, (3.10) and (3.14) immediately lead to (3.11); the case in which $X_n^u(t) = 0$ is clear. The inequality (3.13) may be proven by a completely analogous procedure; this completes the proof of Lemma 3.5. ☐

The following in particular concerns more properties of the shell solution, specifically (3.4), (3.6).

LEMMA 3.6. *Let X^u and X^b be shell approximations of u and b respectively, I be defined as in (3.2) and*

(3.15a)

$$\phi^1_{(l,m,n)}(t) \triangleq \begin{cases} \frac{2}{X_l^u(t)X_m^u(t)X_n^u(t)} \sum_{h,k\in\mathbb{Z}^d, h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k) \\ \qquad \times Im\{\langle u_h(t), k\rangle\langle u_{k-h}(t), u_k(t)\rangle\} & \text{if } X_l^u(t)X_m^u(t)X_n^u(t) \neq 0, \\ 0 & \text{if } X_l^u(t)X_m^u(t)X_n^u(t) = 0, \end{cases}$$

(3.16a)

$$\phi^2_{(l,m,n)}(t) \triangleq \begin{cases} \frac{2}{X_l^b(t)X_m^b(t)X_n^u(t)+X_l^b(t)X_m^u(t)X_n^b(t)} \sum_{h,k\in\mathbb{Z}^d, h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k) \\ \quad \times[Im\{\langle b_h(t),k\rangle\langle b_{k-h}(t),u_k(t)\rangle\} + Im\{\langle b_h(t),k\rangle\langle u_{k-h}(t),b_k(t)\rangle\}] \\ \qquad \text{if } X_l^b(t)X_m^b(t)X_n^u(t)+X_l^b(t)X_m^u(t)X_n^b(t)\neq 0, \\ 0 \qquad \text{if } X_l^b(t)X_m^b(t)X_n^u(t)+X_l^b(t)X_m^u(t)X_n^b(t)= 0, \end{cases}$$

(3.17a)

$$\phi^3_{(l,m,n)}(t) \triangleq \begin{cases} \frac{2}{X_l^u(t)X_m^b(t)X_n^b(t)} \sum_{h,k\in\mathbb{Z}^d, h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k) \\ \quad \times Im\{\langle u_h(t),k\rangle\langle b_{k-h}(t),b_k(t)\rangle\} \qquad \text{if } X_l^u(t)X_m^b(t)X_n^b(t)\neq 0, \\ 0 \qquad \text{if } X_l^u(t)X_m^b(t)X_n^b(t)= 0, \end{cases}$$

$\forall\ (l,m,n)\in\mathbb{N}_0^3, t\geq 0$. *Then*

(1) $\phi^i_{(l,m,n)}(t)=0\ \forall (l,m,n)\ \notin I, t\geq 0, i=1,2,3,$

(2) $\phi^i_{(l,m,n)}(t) = -\phi^i_{(l,n,m)}(t)$ *and consequently* $\phi^i_{(l,m,m)}(t)=0\ \forall\ (l,m,n)\in\mathbb{N}_0^3, t\geq 0, i=1,2,3,$

(3) *there exists* $c_3=c_3(d,L)>0$, *where* L *is the Lipschitz constant of* $\sqrt{\psi}$, *such that*

$$|\phi^i_{(l,m,n)}(t)|\leq c_3 2^{(1+\frac{d}{2})\min\{l,m,n\}} \qquad \forall\ (l,m,n)\in I, t\geq 0, i=1,2,3. \tag{3.18}$$

Remark 3.7. We emphasize that the more natural choice of defining

(3.19a)

$$\phi^{2*}_{(l,m,n)}(t) \triangleq \begin{cases} \frac{2}{X_l^b(t)X_m^b(t)X_n^u(t)} \sum_{h,k\in\mathbb{Z}^d, h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k) \\ \quad \times Im\{\langle b_h(t),k\rangle\langle b_{k-h}(t),u_k(t)\rangle\} \qquad \text{if } X_l^b(t)X_m^b(t)X_n^u(t)\neq 0, \\ 0 \qquad \text{if } X_l^b(t)X_m^b(t)X_n^u(t)= 0, \end{cases}$$

(3.19b)

$$\phi^{4*}_{(l,m,n)}(t) \triangleq \begin{cases} \frac{2}{X_l^b(t)X_m^u(t)X_n^b(t)} \sum_{h,k\in\mathbb{Z}^d, h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k) \\ \quad \times Im\{\langle b_h(t),k\rangle\langle u_{k-h}(t),b_k(t)\rangle\} \qquad \text{if } X_l^b(t)X_m^u(t)X_n^b(t)\neq 0, \\ 0 \qquad \text{if } X_l^b(t)X_m^u(t)X_n^b(t)= 0, \end{cases}$$

instead of $\phi^2_{(l,m,n)}(t)$ in (3.16a) will not work for the proof of Lemma 3.6 due to the lack of symmetry; we will point out this issue within the proof.

In order to prove Lemma 3.6, we need to establish several lemmas first. The following result is a slight generalization of Lemma 2.12 [**2**] and will be necessary for the case of the MHD system subsequently:

Lemma 3.8. *Let $(u_k)_{k\in\mathbb{Z}^d}, (b_k)_{k\in\mathbb{Z}^d}$ and $(v_k)_{k\in\mathbb{Z}^d}$ be complex fields over $\mathbb{Z}^d$ such that for all $k \in \mathbb{Z}^d$, $\langle k, f_k\rangle = 0, \overline{f}_k = f_{-k}\ \forall\ f \in \{u, b, v\}$. Then for all $h \in \mathbb{Z}^d$,*

$$\begin{aligned}(3.20)\qquad &\sum_{k\in\mathbb{Z}^d} \psi_m(k-h)\psi_n(k) Im\{\langle u_h, k\rangle\langle b_{k-h}, v_k\rangle\}\\ &= -\sum_{k\in\mathbb{Z}^d} \psi_m(k)\psi_n(k-h) Im\{\langle u_h, k\rangle\langle v_{k-h}, b_k\rangle\}.\end{aligned}$$

Proof of Lemma 3.8. We see that

$$\psi_m(k-h) = \psi_m(h-k) = \psi_m(k'), \quad \psi_n(k) = \psi_n(-k) = \psi_n(k'-h) \tag{3.21}$$

if $k' = h - k$ because ψ_m defined at (2.3) is radial. Moreover,

$$\langle u_h, k\rangle = \langle u_h, h - k'\rangle = -\langle u_h, k'\rangle, \tag{3.22}$$

again with $k' = h - k$ due to divergence-free properties. Finally,

$$\langle b_{k-h}, v_k\rangle = \langle b_{-k'}, v_{h-k'}\rangle = \langle \overline{b}_{k'}, \overline{v}_{k'-h}\rangle = \langle v_{k'-h}, b_{k'}\rangle. \tag{3.23}$$

Applying (3.21), (3.22), (3.23) immediately deduces the desired property (3.20). □

The following is also a slight generalization of of Lemma 2.13 [**2**] which will be necessary for the case of the MHD system subsequently:

Lemma 3.9. *Let X^u, X^b, X^v be shell approximations of u, b, v respectively. Then for all $e, f, g \in \mathbb{N}_0, t \geq 0$,*

$$\sum_{h\in\mathbb{Z}^d} \psi_e(h)|u_h| \sum_{k\in\mathbb{Z}^d} \sqrt{\psi_f(k)\psi_g(k-h)}|b_k||v_{k-h}| \leq 2^{\frac{d(e+3)}{2}} X_e^u X_f^b X_g^v. \tag{3.24}$$

Proof of Lemma 3.9. We compute for all $h \in \mathbb{Z}^d$,

$$\sum_{k\in\mathbb{Z}^d} \sqrt{\psi_f(k)\psi_g(k-h)}|b_k||v_{k-h}| \leq X_f^b X_g^v \tag{3.25}$$

by Hölder's inequality, and (2.6). Now if $S_e \triangleq \mathbb{Z}^d \cap \text{supp}(\psi_e)$, then the cardinality of S_e may be bounded by $|S_e| \leq (2^{e+2}+1)^d \leq 2^{(e+3)d}$ because $\psi_e(x) = \psi(2^{-e}|x|)$ and $\text{supp}(\psi) \subset \left(\frac{1}{2}, 2\right)$. Hence,

$$\sum_{h\in\mathbb{Z}^d} \psi_e(h)|u_h| \leq \left(|S_e| \sum_{h\in S_e} |\psi_e(h)||u_h|^2\right)^{\frac{1}{2}} \leq (2^{(e+3)d})^{\frac{1}{2}} X_e^u(t) \tag{3.26}$$

by Hölder' inequality, that $|\psi_e| \leq 1$, and (2.6). Therefore, we conclude (3.24) from (3.25) and (3.26). □

We are now ready to prove Lemma 3.6:

PROOF OF LEMMA 3.6. We prove part (2) first: we may compute from (3.15a), (3.17a), Lemma 3.8,

$$\begin{aligned}\phi^1_{(l,m,n)}(t) =&\frac{2}{X^u_l(t)X^u_m(t)X^u_n(t)}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle u_h,k\rangle\langle u_{k-h},u_k\rangle\}\\ =&\frac{-2}{X^u_lX^u_mX^u_n}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k)\psi_n(k-h)Im\{\langle u_h,k\rangle\langle u_{k-h},u_k\rangle\}\\ =&-\phi^1_{(l,n,m)}(t),\end{aligned}\tag{3.27a}$$

$$\begin{aligned}\phi^3_{(l,m,n)}(t) =&\frac{2}{X^u_l(t)X^b_m(t)X^b_n(t)}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle u_h,k\rangle\langle b_{k-h},b_k\rangle\}\\ =&\frac{-2}{X^u_lX^b_mX^b_n}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k)\psi_n(k-h)Im\{\langle u_h,k\rangle\langle b_{k-h},b_k\rangle\}\\ =&-\phi^3_{(l,n,m)}(t),\end{aligned}\tag{3.27b}$$

in cases $X^u_l(t)X^u_m(t)X^u_n(t)\neq 0, X^u_l(t)X^b_m(t)X^b_n(t)\neq 0$ respectively, and therefore $\phi^1_{(l,m,m)}=0,\phi^3_{(l,m,m)}=0$. From such computations, we clearly see that the choice of $\phi^{2*}_{(l,m,n)},\phi^{4*}_{(l,m,n)}$ from (3.19a) and (3.19b) will not work; in fact

$$\begin{aligned}&\phi^{2*}_{(l,m,n)}(t)\\ =&\frac{2}{X^b_l(t)X^b_m(t)X^u_n(t)}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle\}\\ =&\frac{-2}{X^b_lX^b_mX^u_n}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k)\psi_n(k-h)Im\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}\\ =&-\phi^{4*}_{(l,n,m)}(t)\neq-\phi^{2*}_{(l,n,m)}(t),\end{aligned}\tag{3.28}$$

$$\begin{aligned}&\phi^{4*}_{(l,m,n)}(t)\\ =&\frac{2}{X^b_l(t)X^u_m(t)X^b_n(t)}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}\\ =&\frac{-2}{X^b_lX^u_mX^b_n}\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k)\psi_n(k-h)Im\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle\}\\ =&-\phi^{2*}_{(l,n,m)}(t)\neq-\phi^{4*}_{(l,n,m)}(t)\end{aligned}\tag{3.29}$$

by (3.19a), (3.19b) and Lemma 3.8. Remarkably by combining the two in the form of (3.16a), we actually regain symmetry which allows us to compute as follows: in case $X^b_l(t)X^b_m(t)X^u_n(t)+X^b_l(t)X^u_m(t)X^b_n(t)\neq 0$,

$$\begin{aligned}&\phi^2_{(l,m,n)}(t)\\ =&\frac{2}{X^b_l(t)X^b_m(t)X^u_n(t)+X^b_l(t)X^u_m(t)X^b_n(t)}\\ &\times[\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle\}\\ &+\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}]\end{aligned}\tag{3.30}$$

$$
\begin{aligned}
&= \frac{-2}{X_l^b X_m^b X_n^u + X_l^b X_m^u X_n^b} \\
&\quad \times [\sum_{h,k \in \mathbb{Z}^d, h \neq 0} \psi_l(h)\psi_m(k)\psi_n(k-h) Im\{\langle b_h, k\rangle \langle u_{k-h}, b_k\rangle\} \\
&\quad + \sum_{h,k \in \mathbb{Z}^d, h \neq 0} \psi_l(h)\psi_m(k)\psi_n(k-h) Im\{\langle b_h, k\rangle \langle b_{k-h}, u_k\rangle\}]
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
&= \frac{-2}{X_l^b X_n^b X_m^u + X_l^b X_n^u X_m^b} \\
&\quad \times [\sum_{h,k \in \mathbb{Z}^d, h \neq 0} \psi_l(h)\psi_n(k-h)\psi_m(k) Im\{\langle b_h, k\rangle \langle b_{k-h}, u_k\rangle\} \\
&\quad + \sum_{h,k \in \mathbb{Z}^d, h \neq 0} \psi_l(h)\psi_n(k-h)\psi_m(k) Im\{\langle b_h, k\rangle \langle u_{k-h}, b_k\rangle\}] = -\phi^2_{(l,n,m)}(t)
\end{aligned} \tag{3.32}
$$

by (3.16a) and Lemma 3.8.

Next, we can prove part (1) following the exact same argument in the proof of Proposition 2.11 (1) [**2**]; we only sketch it for completeness. If without loss of generality, we have $\max\{|h|, |k-h|, |k|\} = |k|$, $\max\{|h|, |k-h|\} = |h|$, then $|k| \leq 2|h|$. On the other hand, if $\phi^i_{(l,m,n)} \neq 0$ so that in particular $\psi_l(h) \neq 0$ and $\psi_n(k) \neq 0$, then $2^{n-1} < 2^{l+2}$; therefore, to have $\phi^i_{(l,m,n)} \neq 0$, necessarily $n \leq l+2$ so that $(l, m, n) \in I$ due to (3.2).

Finally, we prove part (3): for $(l, m, n) \in I$ such that without loss of generality $m < n$ we consider two cases, namely $n - m > 2$ and $n - m \in \{1, 2\}$.

Firstly, let us consider the case $n - m > 2$. The assumption implies $m = \min\{l, m, n\}$ and $|l - n| \leq 2$ by definition of I. Now we first work on $\phi^1_{(l,m,n)}$: from the summation within (3.15a), we see that every non-zero term must be such that $\psi_l(h)\psi_m(k-h)\psi_n(k) \neq 0$ so that considering their support, we must have $|k-h| \leq |k|$. With this in mind, we write $\langle u_h, k\rangle = \langle u_h, k-h\rangle$ due to divergence-free condition and compute from (3.15a) in case $X_l^u(t)X_m^u(t)X_n^u(t) \neq 0$ as follows:

$$
\begin{aligned}
&|\phi^1_{(l,m,n)}(t)| \\
=&| \frac{2}{X_l^u(t)X_m^u(t)X_n^u(t)} \\
&\times \sum_{h,k \in \mathbb{Z}^d, h \neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k) Im\{\langle u_h, k-h\rangle \langle u_{k-h}, u_k\rangle\}| \\
\leq& \frac{2}{X_l^u X_m^u X_n^u} \sum_{h,k' \in \mathbb{Z}^d, h \neq 0} \psi_l(h)\psi_m(k')\psi_n(k'+h)|u_h||k'||u_{k'}||u_{k'+h}| \\
\leq& \frac{2}{X_l^u X_m^u X_n^u} \sum_{h,k' \in \mathbb{Z}^d} \psi_l(h)\psi_m(k')\psi_n(k'+h)|u_h|2^{m+1}|u_{k'}||u_{k'+h}| \\
\leq& \frac{2^{m+2}}{X_l^u X_m^u X_n^u} \sum_{k' \in \mathbb{Z}^d} \psi_m(k')|u_{k'}| \sum_{h \in \mathbb{Z}^d} \sqrt{\psi_l(h)\psi_n(k'+h)}|u_h||u_{k'+h}| \\
\leq& 2^{\min\{l,m,n\}(1+\frac{d}{2})} c_3,
\end{aligned} \tag{3.33}
$$

for $c_3 = 2^{2+(\frac{d3}{2})}$ where we defined $k' \triangleq k - h$, and used that $\text{supp}(\psi_m) = \{x \in \mathbb{R}^d : 2^{m-1} < |x| < 2^{m+1}\}$, that $\psi_l, \psi_n \in [0, 1]$, Lemma 3.9, that $m = \min\{l, m, n\}$.

Similarly we work on $\phi^3_{(l,m,n)}$ to deduce from (3.17a) in case $X^u_l(t)X^b_m(t)X^b_n(t) \neq 0$,

$$\begin{aligned}
&|\phi^3_{(l,m,n)}(t)| \\
=&|\frac{2}{X^u_l(t)X^b_m(t)X^b_n(t)} \\
&\times \sum_{h,k\in\mathbb{Z}^d,h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle u_h, k-h\rangle\langle b_{k-h}, b_k\rangle\}| \\
\leq&\frac{2}{X^u_l X^b_m X^b_n} \sum_{h,k'\in\mathbb{Z}^d,h\neq 0} \psi_l(h)\psi_m(k')\psi_n(k'+h)|u_h||k'||b_{k'}||b_{k'+h}| \\
\leq&\frac{2}{X^u_l X^b_m X^b_n} \sum_{h,k'\in\mathbb{Z}^d} \psi_l(h)\psi_m(k')\psi_n(k'+h)|u_h|2^{m+1}|b_{k'}||b_{k'+h}| \\
\leq&\frac{2^{m+2}}{X^u_l X^b_m X^b_n} \sum_{k'\in\mathbb{Z}^d} \psi_m(k')|b_{k'}| \sum_{h\in\mathbb{Z}^d} \sqrt{\psi_l(h)\psi_n(k'+h)}|u_h||b_{k'+h}| \\
\leq&2^{\min\{l,m,n\}(1+\frac{d}{2})}c_3,
\end{aligned} \tag{3.34}$$

for $c_3 = 2^{2+(\frac{d3}{2})}$ by that $\langle u_h, k\rangle = \langle u_h, k-h\rangle$, that $k' \triangleq k-h$, that $\text{supp}(\psi_m) = \{x \in \mathbb{R}^d : 2^{m-1} < |x| < 2^{m+1}\}$, that $\psi_l, \psi_n \in [0,1]$, that $m = \min\{l,m,n\}$.

Finally, from (3.16a) in case $X^b_l(t)X^b_m(t)X^u_n(t) + X^b_l X^u_m(t)X^b_n(t) \neq 0$,

$$\begin{aligned}
&|\phi^2_{(l,m,n)}(t)| \\
=&|\frac{2}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \\
&\times [\sum_{h,k\in\mathbb{Z}^d,h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle b_h, k-h\rangle\langle b_{k-h}, u_k\rangle\} \\
&\quad + \sum_{h,k\in\mathbb{Z}^d,h\neq 0} \psi_l(h)\psi_m(k-h)\psi_n(k)Im\{\langle b_h, k-h\rangle\langle u_{k-h}, b_k\rangle\}]| \\
\leq&\frac{2}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \\
&\times \sum_{h,k'\in\mathbb{Z}^d} \psi_l(h)\psi_m(k')\psi_n(k'+h)|b_h||k'|[|b_{k'}||u_{k'+h}| + |u_{k'}||b_{k'+h}|] \\
\leq&\frac{2^{m+2}}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \\
&\times [\sum_{k'\in\mathbb{Z}^d} \psi_m(k')|b_{k'}| \sum_{h\in\mathbb{Z}^d} \sqrt{\psi_l(h)\psi_n(k'+h)}|b_h||u_{k'+h}| \\
&\quad + \sum_{k'\in\mathbb{Z}^d} \psi_m(k')|u_{k'}| \sum_{h\in\mathbb{Z}^d} \sqrt{\psi_l(h)\psi_n(k'+h)}|b_h||b_{k'+h}|] \\
\leq&2^{\min\{l,m,n\}(1+\frac{d}{2})}c_3
\end{aligned} \tag{3.35}$$

for $c_3 = 2^{2+(\frac{d3}{2})}$, by that $\langle b_h, k\rangle = \langle b_h, k-h\rangle$, that $\psi_l, \psi_n \in [0,1]$, Lemma 3.9, and that $m = \min\{l,m,n\}$.

Secondly, let us consider the case $n - m \in \{1,2\}$; i.e. $n = m+1$ or $n = m+2$ and hence $n > m$. By hypothesis, $(l,m,n) \in I$. If $l = \max\{l,m,n\}$, then $l - n \leq 2$.

If $n = \max\{l, m, n\}$, then $l \leq n$. Either way, $l \leq n+2$ and it can be checked that $\min\{l, m, n\} \geq l-4$. Now we first work on $\phi^1_{(l,m,n)}$: in case $X^u_l(t)X^u_m(t)X^u_n(t) \neq 0$,

$$\begin{aligned}
&|\phi^1_{(l,m,n)}(t)| \\
\leq&\frac{2}{X^u_l X^u_m X^u_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h) \\
&\quad \times |\sum_{k\in\mathbb{Z}^d:2^{n-1}<|k|<2^{n+1}} \psi_m(k-h)\psi_n(k) Im\{\langle u_h, k\rangle\langle u_{k-h}, u_k\rangle\}| \\
=&\frac{1}{X^u_l X^u_m X^u_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h) \\
&\quad \times |\sum_{k\in\mathbb{Z}^d:2^{n-1}<|k|<2^{n+1}} [\psi_m(k-h)\psi_n(k) - \psi_m(k)\psi_n(k-h)] \\
&\qquad \times Im\{\langle u_h, k\rangle\langle u_{k-h}, u_k\rangle\}| \\
\leq&\frac{2^{n+1}}{X^u_l X^u_m X^u_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h) \\
&\quad \times |u_h| \sum_{k\in\mathbb{Z}^d} |\psi_m(k-h)\psi_n(k) - \psi_m(k)\psi_n(k-h)||u_{k-h}||u_k|
\end{aligned} \tag{3.36}$$

by (3.15a), Lemma 3.8, and that $\text{supp}(\psi_n) \subset \{x \in \mathbb{R}^d : 2^{n-1} < |x| < 2^{n+1}\}$. Now we observe that

$$\begin{aligned}
&|\sqrt{\psi_m(k-h)\psi_n(k)} - \sqrt{\psi_m(k)\psi_n(k-h)}| \\
\leq&\sqrt{\psi_n(k)}|\sqrt{\psi(2^{-m}|k-h|)} - \sqrt{\psi(2^{-m}|k|)}| \\
&+ \sqrt{\psi_m(k)}|\sqrt{\psi(2^{-n}|k|)} - \sqrt{\psi(2^{-n}|k-h|)}| \\
\leq&\sqrt{\psi_n(k)}L2^{-m}\,||k-h|-|k|| + \sqrt{\psi_m(k)}L2^{-n}\,||k|-|k-h|| \leq \frac{L|h|}{2^{n-3}}
\end{aligned} \tag{3.37}$$

where L is the Lipschitz coefficient of $\sqrt{\psi}$, by that $\psi_n, \psi_m \in [0,1]$, and that $m \geq n-2$. Moreover,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \left(\sqrt{\psi_m(k-h)\psi_n(k)} + \sqrt{\psi_m(k)\psi_n(k-h)}\right)|u_{k-h}||u_k| \\
=&\sum_{k\in\mathbb{Z}^d} \sqrt{\psi_m(k-h)\psi_n(k)}|u_{k-h}||u_k| + \sqrt{\psi_m(k-h)\psi_n(k)}|u_k||u_{k-h}| \\
=&2\sum_{k\in\mathbb{Z}^d} \sqrt{\psi_m(k-h)\psi_n(k)}|u_k||u_{k-h}|.
\end{aligned} \tag{3.38}$$

Therefore, we estimate continuing from (3.36) as

$$\begin{aligned}
&|\phi^1_{(l,m,n)}(t)| \\
\leq&\frac{2^{n+2}}{X^u_l X^u_m X^u_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h)|u_h| \left(\frac{L|h|}{2^{n-3}}\right) \sum_{k\in\mathbb{Z}^d} \sqrt{\psi_m(k-h)\psi_n(k)}|u_{k-h}||u_k| \\
\leq&\frac{2^{n+2}}{X^u_l X^u_m X^u_n} \frac{L2^{l+1}}{2^{n-3}} [2^{\frac{d(l+3)}{2}} X^u_l X^u_m X^u_n] \leq 2^{(\frac{d}{2}+1)\min\{l,m,n\}} c_3
\end{aligned} \tag{3.39}$$

for $c_3 = L2^{10+\frac{7d}{2}}$ by (3.37), (3.38), that $|h| \leq 2^{l+1}$ due to supp(ψ_l), Lemma 3.9 and that $l-4 \leq \min\{l,m,n\}$.

Next, we may work on $\phi^3_{(l,m,n)}$ similarly: in case $X^u_l(t)X^b_m(t)X^b_n(t) \neq 0$,

$$\begin{aligned}
&|\phi^3_{(l,m,n)}(t)|\\
\leq&\frac{2^{n+1}}{X^u_l X^b_m X^b_n}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}\psi_l(h)|u_h|\\
&\qquad\sum_{k\in\mathbb{Z}^d}|\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)||b_{k-h}||b_k|\\
\leq&\frac{2^{n+1}}{X^u_l X^b_m X^b_n}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}\psi_l(h)|u_h|\frac{L|h|}{2^{n-3}}\\
&\qquad\times\sum_{k\in\mathbb{Z}^d}|\sqrt{\psi_m(k-h)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k-h)}||b_{k-h}||b_k|
\end{aligned}\tag{3.40}$$

by (3.17a), Lemma 3.8, supp(ψ_n) and (3.37). Moreover, we may rewrite similarly to (3.38),

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\left(\sqrt{\psi_m(k-h)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k-h)}\right)|b_{k-h}||b_k|\\
=&\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}|b_{k-h}||b_k|+\sqrt{\psi_m(k-h)\psi_n(k)}|b_k||b_{k-h}|\\
=&2\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}|b_{k-h}||b_k|.
\end{aligned}\tag{3.41}$$

Applying (3.41) to (3.40) leads to

$$\begin{aligned}
&|\phi^3_{(l,m,n)}(t)|\\
\leq&\frac{L2^{l+6}}{X^u_l X^b_m X^b_n}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}\psi_l(h)|u_h|\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}|b_{k-h}||b_k|\\
\leq&L2^{(\frac{d}{2}+1)(l-4)}2^{10+\frac{7d}{2}}\leq 2^{(\frac{d}{2}+1)\min\{l,m,n\}}c_3
\end{aligned}\tag{3.42}$$

for $c_3 = L2^{10+\frac{d7}{2}}$ by $|h| \leq 2^{l+1}$ considering supp(ψ_l), Lemma 3.9, and that $l-4 \leq \min\{l,m,n\}$.

Next, with the choice of $\phi^{2*}_{(l,m,n)}(t)$ in (3.19a) or $\phi^{4*}_{(l,m,n)}(t)$ in (3.19b) at Remark 3.1, it is clear that an analogous inequality cannot be obtained because although we may proceed as e.g. in the case of $\phi^{2*}_{(l,m,n)}$,

$$
\begin{aligned}
|\phi^{2*}_{(l,m,n)}(t)| \leq & \frac{1}{X_l^b X_m^b X_n^u} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h) \\
& | \sum_{k\in\mathbb{Z}^d:2^{n-1}<|k|<2^{n+1}} \psi_m(k-h)\psi_n(k) Im\{\langle b_h, k\rangle\langle b_{k-h}, u_k\rangle\} \\
& \qquad + \psi_m(k-h)\psi_n(k) Im\{\langle b_h, k\rangle\langle b_{k-h}, u_k\rangle\}| \\
= & \frac{1}{X_l^b X_m^b X_n^u} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h) \\
& | \sum_{k\in\mathbb{Z}^d:2^{n-1}<|k|<2^{n+1}} \psi_m(k-h)\psi_n(k) Im\{\langle b_h, k\rangle\langle b_{k-h}, u_k\rangle\} \\
& \qquad - \psi_m(k)\psi_n(k-h) Im\{\langle b_h, k\rangle\langle u_{k-h}, b_k\rangle\}|
\end{aligned}
\tag{3.43}
$$

in case $X_l^b(t)X_m^b(t)X_n^u(t) \neq 0$ by Lemma 3.8, it is clear at this point that we cannot possibly write this in the form of (3.36) due to in particular the difference of $\langle b_{k-h}, u_k\rangle$ and $\langle u_{k-h}, b_k\rangle$; i.e. the lack of symmetry. Even if we choose to continue and rewrite the right hand side of (3.43) as

$$
\begin{aligned}
& \frac{1}{X_l^b X_m^b X_n^u} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \psi_l(h) | \sum_{k\in\mathbb{Z}^d:2^{n-1}<|k|<2^{n+1}} \\
& \quad \times [\psi_m(k-h)\psi_n(k) - \psi_m(k)\psi_n(k-h)] Im\{\langle b_h, k\rangle\langle b_{k-h}, u_k\rangle\} \\
& \quad + \psi_m(k)\psi_n(k-h) Im\{[\langle b_h, k\rangle\langle b_{k-h}, u_k\rangle - \langle b_h, k\rangle\langle u_{k-h}, b_k\rangle]\}|,
\end{aligned}
\tag{3.44}
$$

we will not be able to deduce an analogous equality to (3.38) precisely due to the lack of symmetry, namely $|b_{k-h}|, |u_k|$:

$$
\begin{aligned}
& \sum_{k\in\mathbb{Z}^d} \left(\sqrt{\psi_m(k-h)\psi_n(k)} + \sqrt{\psi_m(k)\psi_n(k-h)}\right) |b_{k-h}||u_k| \\
= & \sum_{k\in\mathbb{Z}^d} \sqrt{\psi_m(k-h)\psi_n(k)}|b_{k-h}||u_k| + \sqrt{\psi_m(k-h)\psi_n(k)}|b_k||u_{k-h}| \\
\neq & 2\sum_{k\in\mathbb{Z}^d} \sqrt{\psi_m(k-h)\psi_n(k)}|b_{k-h}||u_k|.
\end{aligned}
\tag{3.45}
$$

However, with our choice of $\phi^2_{(l,m,n)}$ in (3.16a), the very much needed symmetry is regained as follows: in case $X^b_l(t)X^b_m(t)X^u_n(t) + X^b_l(t)X^u_m(t)X^b_n(t) \neq 0$,

$$
\begin{aligned}
&|\phi^2_{(l,m,n)}(t)| \\
\leq & \frac{2}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \psi_l(h) \\
& \times |\sum_{k \in \mathbb{Z}^d : 2^{n-1} < |k| < 2^{n+1}} \psi_m(k-h)\psi_n(k) \\
& \qquad \times [Im\{\langle b_h, k\rangle \langle b_{k-h}, u_k\rangle\} + Im\{\langle b_h, k\rangle \langle u_{k-h}, b_k\rangle\}]| \\
= & \frac{1}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \psi_l(h) \\
& \times |\sum_{k \in \mathbb{Z}^d : 2^{n-1} < |k| < 2^{n+1}} \psi_m(k-h)\psi_n(k) Im\{\langle b_h, k\rangle \langle b_{k-h}, u_k\rangle\} \\
& - \sum_{k \in \mathbb{Z}^d : 2^{n-1} < |k| < 2^{n+1}} \psi_m(k)\psi_n(k-h) Im\{\langle b_h, k\rangle \langle u_{k-h}, b_k\rangle\} \\
& + \sum_{k \in \mathbb{Z}^d : 2^{n-1} < |k| < 2^{n+1}} \psi_m(k-h)\psi_n(k) Im\{\langle b_h, k\rangle \langle u_{k-h}, b_k\rangle\} \\
& - \sum_{k \in \mathbb{Z}^d : 2^{n-1} < |k| < 2^{n+1}} \psi_m(k)\psi_n(k-h) Im\{\langle b_h, k\rangle \langle b_{k-h}, u_k\rangle\}|
\end{aligned}
\tag{3.46}
$$

by (3.16a), Lemma 3.8, and now we see that we can make the appropriate arrangement to continue this estimate by

$$
\begin{aligned}
&|\phi^2_{(l,m,n)}(t)| \\
\leq & \frac{2^{n+1}}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \psi_l(h)|b_h| \\
& \qquad \times \sum_{k \in \mathbb{Z}^d} |\psi_m(k-h)\psi_n(k) - \psi_m(k)\psi_n(k-h)||b_{k-h}||u_k| \\
& \qquad + |\psi_m(k-h)\psi_n(k) - \psi_m(k)\psi_n(k-h)||u_{k-h}||b_k| \\
\leq & \frac{2^{n+1}}{X^b_l X^b_m X^u_n + X^b_l X^u_m X^b_n} \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \psi_l(h)|b_h| \sum_{k \in \mathbb{Z}^d} \frac{L|h|}{2^{n-3}} \\
& \times [\sqrt{\psi_m(k-h)\psi_n(k)} + \sqrt{\psi_m(k)\psi_n(k-h)}][|b_{k-h}||u_k| + |u_{k-h}||b_k|]
\end{aligned}
\tag{3.47}
$$

due to (3.37). Now despite that with $\phi^{2*}_{(l,m,n)}$ in (3.19a), we could not find the necessary symmetry as we saw in (3.45), with our choice of $\phi^2_{(l,m,n)}$ in (3.16a), we will see that the necessary symmetry exists and allows us to deduce the following identity:

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\left(\sqrt{\psi_m(k-h)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k-h)}\right)[|b_{k-h}||u_k|+|u_{k-h}||b_k|]\\
&=\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}[|b_{k-h}||u_k|+|u_{k-h}||b_k|]\\
&\quad+\sqrt{\psi_m(k-h)\psi_n(k)}[|b_k||u_{k-h}|+|u_k||b_{k-h}|]\\
&=2\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}[|b_k||u_{k-h}|+|u_k||b_{k-h}|].
\end{aligned}\tag{3.48}
$$

Applying (3.48) in (3.47), we deduce

$$
\begin{aligned}
|\phi^2_{(l,m,n)}(t)|\leq&\frac{2^{n+2}}{X_l^bX_m^bX_n^u+X_l^bX_m^uX_n^b}\left(\frac{L2^{l+1}}{2^{n-3}}\right)\sum_{h\in\mathbb{Z}^d\setminus\{0\}}\psi_l(h)|b_h|\\
&\times\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}[|b_k||u_{k-h}|+|u_k||b_{k-h}|]\\
\leq& L2^{(\frac{d}{2}+1)(l-4)}2^{10+\frac{7d}{2}}\leq 2^{(\frac{d}{2}+1)\min\{l,m,n\}}c_3.
\end{aligned}\tag{3.49}
$$

for $c_3=L2^{10+\frac{7d}{2}}$ by that $|h|\leq 2^{l+1}$ considering $\text{supp}(\psi_l)$, Lemma 3.9 and that $l-4\leq\min\{l,m,n\}$. This completes the proof of Lemma 3.6. □

We are now ready to prove Proposition 3.4.

Proof of Proposition 3.4. We compute

$$
\begin{aligned}
\frac{1}{2}\partial_t X_n^2=&-\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\alpha}{g_1(|k|)}|u_k|^2-\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\beta}{g_2(|k|)}|b_k|^2\\
&-\sum_{h,k\in\mathbb{Z}^d,h\neq0}\psi_n(k)[\text{Im}\{\langle u_h,k\rangle\langle u_{k-h},u_k\rangle\}-\text{Im}\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle\}\\
&\qquad+\text{Im}\{\langle u_h,k\rangle\langle b_{k-h},b_k\rangle\}-\text{Im}\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}]
\end{aligned}\tag{3.50}
$$

by (2.6), (2.1a), (2.1b). We consider χ_n^u,χ_n^b as defined in Lemma 3.5 so that

$$
\begin{aligned}
&-\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\alpha}{g_1(|k|)}|u_k|^2-\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\beta}{g_2(|k|)}|b_k|^2\\
&\qquad=-\frac{1}{2}\chi_n^u|X_n^u|^2-\frac{1}{2}\chi_n^b|X_n^b|^2.
\end{aligned}\tag{3.51}
$$

Moreover, we may write $\sum_{l\in\mathbb{N}_0}\psi_l(h)=1,\ \sum_{m\in\mathbb{N}_0}\psi_m(k-h)=1\ \forall\ h,k\in\mathbb{Z}^d, 0\neq h\neq k$ so that

$$
\begin{aligned}
&-\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_n(k)[\mathrm{Im}\{\langle u_h,k\rangle\langle u_{k-h},u_k\rangle\}-\mathrm{Im}\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle\}\\
&\qquad\qquad+\mathrm{Im}\{\langle u_h,k\rangle\langle b_{k-h},b_k\rangle\}-\mathrm{Im}\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}]\\
&=-\sum_{l,m\in\mathbb{N}_0}\sum_{h,k\in\mathbb{Z}^d}\psi_l(h)\psi_m(k-h)\psi_n(k)\mathrm{Im}\{\langle u_h,k\rangle\langle u_{k-h},u_k\rangle\}\\
&\quad+\sum_{l,m\in\mathbb{N}_0}\sum_{h,k\in\mathbb{Z}^d}\psi_l(h)\psi_m(k-h)\psi_n(k)\\
&\qquad\times[\mathrm{Im}\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle+\mathrm{Im}\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}]\\
&\quad-\sum_{l,m\in\mathbb{N}_0}\sum_{h,k\in\mathbb{Z}^d}\psi_l(h)\psi_m(k-h)\psi_n(k)\mathrm{Im}\{\langle u_h,k\rangle\langle b_{k-h},b_k\rangle\}
\end{aligned}
\tag{3.52}
$$

by Fubini's theorem. By definitions of $\phi^1_{(l,m,n)},\phi^2_{(l,m,n)},\phi^3_{(l,m,n)}$ from Lemma 3.6, we now deduce

$$
\begin{aligned}
&-\sum_{h,k\in\mathbb{Z}^d,h\neq 0}\psi_n(k)[\mathrm{Im}\{\langle u_h,k\rangle\langle u_{k-h},u_k\rangle\}-\mathrm{Im}\{\langle b_h,k\rangle\langle b_{k-h},u_k\rangle\}\\
&\qquad\qquad+\mathrm{Im}\{\langle u_h,k\rangle\langle b_{k-h},b_k\rangle\}-\mathrm{Im}\{\langle b_h,k\rangle\langle u_{k-h},b_k\rangle\}]\\
&=-\frac{1}{2}\sum_{l,m\in\mathbb{N}_0}\phi^1_{(l,m,n)}X^u_lX^u_mX^u_n-\phi^2_{(l,m,n)}[X^b_lX^b_mX^u_n+X^b_lX^u_mX^b_n]\\
&\qquad+\phi^3_{(l,m,n)}X^u_lX^b_mX^b_n.
\end{aligned}
\tag{3.53}
$$

By Lemma 3.6 (1), $\phi^i_{(l,m,n)}(t)=0\ \forall\ t\geq 0, i=1,2,3$ if $(l,m,n)\notin I$; thus, applying (3.51), (3.53) in (3.50) and multiplying by 2 gives (3.3). Moreover, by Lemma 3.6 and Lemma 3.5, (3.4), (3.5), (3.6) are all satisfied. This implies that X is a shell solution of the system (1.9a)-(1.9c) and completes the proof of Proposition 3.4. □

Having proved Proposition 3.4, our next task is to obtain a recursive inequality for the shell solution. Before we do so in Proposition 3.11, we need to prove the following lemma which is a slight generalization of Lemma 3.4 [**2**], that we will need subsequently in the case of the MHD system.

LEMMA 3.10. *Let X^u and X^b be shell approximations of u and b respectively and $\phi^i_{(l,m,n)}, i=1,2,3$, be defined by (3.15a), (3.16a), (3.17a) respectively. Then*

$$
\begin{aligned}
&-\sum_{(l,m,h)\in I,h\leq n-1}\phi^1_{(l,m,h)}X^u_lX^u_mX^u_h+\phi^3_{(l,m,h)}X^u_lX^b_mX^b_h\\
&\qquad\qquad-\phi^2_{(l,m,h)}[X^b_lX^b_mX^u_h+X^b_lX^u_mX^b_h]\\
&=\sum_{(l,m,h)\in I,m\leq n-1<h}\phi^1_{(l,m,h)}X^u_lX^u_mX^u_h+\phi^3_{(l,m,h)}X^u_lX^b_mX^b_h\\
&\qquad\qquad-\phi^2_{(l,m,h)}[X^b_lX^b_mX^u_h+X^b_lX^u_mX^b_h].
\end{aligned}
\tag{3.54}
$$

Proof of Lemma 3.10. If $h \leq n-1$, then $\min\{l,m,h\} \leq h \leq n-1$. Now we write

$$
\begin{aligned}
&- \sum_{(l,m,h)\in I, h\leq n-1} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h] \\
&= - \sum_{(l,m,h)\in I, h\leq n-1, m<h} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h] \\
&\quad - \sum_{(l,m,h)\in I, h\leq n-1, m>h} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h]
\end{aligned} \tag{3.55}
$$

where we used that $\phi^i_{(l,m,n)} = 0 \; \forall \; i = 1,2,3$ by Lemma 3.6 (2). Now we rewrite

$$
\begin{aligned}
&\sum_{(l,m,h)\in I, h\leq n-1, m>h} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h] \\
&= - \sum_{(l,m,h)\in I, h\leq n-1, m>h} \phi^1_{(l,h,m)} X^u_l X^u_m X^u_h + \phi^3_{(l,h,m)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,h,m)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h] \\
&= - \sum_{(l,m',h')\in I, m'\leq n-1, m'<h'} \phi^1_{(l,m',h')} X^u_l X^u_{h'} X^u_{m'} + \phi^3_{(l,m',h')} X^u_l X^b_{h'} X^b_{m'} \\
&\qquad - \phi^2_{(l,m',h')} [X^b_l X^b_{h'} X^u_{m'} + X^b_l X^u_{h'} X^b_{m'}]
\end{aligned} \tag{3.56}
$$

by Lemma 3.6 (2), that $(l,h',m') \in I$ if and only if $(l,m',h') \in I$. Substituting (3.56) in (3.55) gives (3.54):

$$
\begin{aligned}
&- \sum_{(l,m,h)\in I, h\leq n-1} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h] \\
&= - \sum_{(l,m,h)\in I, h\leq n-1, m<h} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h] \\
&\quad + \sum_{(l,m',h')\in I, m'\leq n-1, m'<h'} \phi^1_{(l,m',h')} X^u_l X^u_{h'} X^u_{m'} + \phi^3_{(l,m',h')} X^u_l X^b_{h'} X^b_{m'} \\
&\qquad - \phi^2_{(l,m',h')} [X^b_l X^b_{h'} X^u_{m'} + X^b_l X^u_{h'} X^b_{m'}] \\
&= \sum_{(l,m,h)\in I, m\leq n-1<h} \phi^1_{(l,m,h)} X^u_l X^u_m X^u_h + \phi^3_{(l,m,h)} X^u_l X^b_m X^b_h \\
&\qquad - \phi^2_{(l,m,h)} [X^b_l X^b_m X^u_h + X^b_l X^u_m X^b_h],
\end{aligned}
$$

where we emphasize again that the symmetry of $\phi^2_{(l,m,n)}(t)$ was utilized here. □

PROPOSITION 3.11. *Let $X = (X^u, X^b)$ be a shell solution of the system (1.9a)-(1.9c) that satisfies the energy inequality over $[0,t]$, and $(d_n)_{n\in\mathbb{N}_0}$ be its sequence of energy bounds. Then there exists $c_4 > 0$, that does not depend on t, such that*

$$\begin{aligned} d_n^2(t) \leq & F_n(0) + c_4 \sum_{l=0}^{n-1} \frac{\overline{d}_l}{2^{-(1+\frac{d}{2})l}} \\ & \times \sum_{m\geq n-2} \left(\frac{g_1(2^{m+1})}{2^{\alpha m}} + \frac{g_2(2^{m+1})}{2^{\beta m}} \right) (|d_m|^2 - |d_{m+1}|^2)(t). \end{aligned} \tag{3.57}$$

PROOF OF PROPOSITION 3.11. We fix $n \in \mathbb{N}_0$ and compute

$$\begin{aligned} \partial_t \sum_{h=0}^{n-1} X_h^2 = & - \sum_{h=0}^{n-1} \chi_h^u |X_h^u|^2 + \chi_h^b |X_h^b|^2 \\ & + \sum_{(l,m,h)\in I, m\leq n-1<h} \phi^1_{(l,m,h)} X_l^u X_m^u X_h^u + \phi^3_{(l,m,h)} X_l^u X_m^b X_h^b \\ & - \phi^2_{(l,m,h)} [X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] \end{aligned} \tag{3.58}$$

by (3.3), Lemma 3.10 so that integrating over $[0,t]$ gives

$$\begin{aligned} \sum_{h=0}^{n-1} X_h^2(t) - \sum_{h=0}^{n-1} X_h^2(0) = & - \int_0^t \sum_{h=0}^{n-1} \chi_h^u |X_h^u|^2 + \chi_h^b |X_h^b|^2 ds \\ & + \int_0^t \sum_{(l,m,h)\in I, m<n\leq h} \phi^1_{(l,m,h)} X_l^u X_m^u X_h^u + \phi^3_{(l,m,h)} X_l^u X_m^b X_h^b \\ & - \phi^2_{(l,m,h)} [X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] ds. \end{aligned} \tag{3.59}$$

This implies

$$\begin{aligned} d_n^2(t) \leq & F_n(0) - \int_0^t \sum_{(l,m,h)\in I, m<n\leq h} \phi^1_{(l,m,h)} X_l^u X_m^u X_h^u + \phi^3_{(l,m,h)} X_l^u X_m^b X_h^b \\ & - \phi^2_{(l,m,h)} [X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] ds \end{aligned} \tag{3.60}$$

by Definition 3.3, (3.8a), (3.7), (3.59). By (3.6), this implies

$$\begin{aligned} d_n^2(t) \leq & F_n(0) + c_3 \int_0^t \sum_{(l,m,h)\in I, m<n\leq h} 2^{(1+\frac{d}{2})\min\{l,m\}} \\ & \times [X_l^u X_m^u X_h^u + X_l^u X_m^b X_h^b + X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] ds. \end{aligned} \tag{3.61}$$

We estimate the sum as follows:

$$
\begin{aligned}
&\sum_{(l,m,h)\in I, m<n\le h} 2^{(1+\frac{d}{2})\min\{l,m\}}[X_l^u X_m^u X_h^u + X_l^u X_m^b X_h^b + X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] \\
=&\sum_{(l,m,h)\in I, m<n\le h, l<m} 2^{(1+\frac{d}{2})l}[X_l^u X_m^u X_h^u + X_l^u X_m^b X_h^b + X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] \\
&+\sum_{(l,m,h)\in I, m<n\le h, m\le l} 2^{(1+\frac{d}{2})m}[X_l^u X_m^u X_h^u + X_l^u X_m^b X_h^b + X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] \\
=&\sum_{(l,m,h)\in I, m<n\le h, l<m} 2^{(1+\frac{d}{2})l}[X_l^u X_m^u X_h^u + X_l^u X_m^b X_h^b + X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] \\
&+\sum_{(l,m,h)\in I, l<n\le h, l\le m} 2^{(1+\frac{d}{2})l}[X_m^u X_l^u X_h^u + X_m^u X_l^b X_h^b + X_m^b X_l^b X_h^u + X_m^b X_l^u X_h^b] \\
\le& 2\sum_{(l,m,h)\in I, l<n\le h, l\le m} 2^{(1+\frac{d}{2})l}[X_l^u X_m^u X_h^u + X_l^u X_m^b X_h^b + X_l^b X_m^b X_h^u + X_l^b X_m^u X_h^b] \\
\le& 2\sum_{l=0}^{n-1} 2^{(1+\frac{d}{2})l}\overline{d}_l^u \sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^u X_h^u| + 2\sum_{l=0}^{n-1} 2^{(1+\frac{d}{2})l}\overline{d}_l^u \sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^b X_h^b| \\
&+ 2\sum_{l=0}^{n-1} 2^{(1+\frac{d}{2})l}\overline{d}_l^b \sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^b X_h^u| + 2\sum_{l=0}^{n-1} 2^{(1+\frac{d}{2})l}\overline{d}_l^b \sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^u X_h^b|
\end{aligned} \tag{3.62}
$$

by (3.9), that $l < h$ and $l \le m$ so that $|h-m| \le 2$ due to the definition of I in (3.2). We may compute by Young's inequality and (3.5),

$$
\begin{aligned}
2\sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^u X_h^u| \le& \sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_h^u|^2 + |X_m^u|^2 \\
\le& 10c_1^{-1}\sum_{m\ge n-2}\frac{g_1(2^{m+1})}{2^{\alpha m}}\chi_m^u|X_m^u|^2.
\end{aligned} \tag{3.63}
$$

Identical procedures lead to

$$
2\sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^b X_h^b| \le 10c_2^{-1}\sum_{m\ge n-2}\frac{g_2(2^{m+1})}{2^{\beta m}}\chi_m^b|X_m^b|^2, \tag{3.64}
$$

$$
\begin{aligned}
&2\sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^b X_h^u| \\
\le& 5c_1^{-1}\sum_{m\ge n-2}\frac{g_1(2^{m+1})}{2^{\alpha m}}\chi_m^u|X_m^u|^2 + 5c_2^{-1}\sum_{m\ge n-2}\frac{g_2(2^{m+1})}{2^{\beta m}}\chi_m^b|X_m^b|^2,
\end{aligned} \tag{3.65}
$$

$$
\begin{aligned}
&2\sum_{h\ge n}\sum_{m=h-2}^{h+2}|X_m^u X_h^b| \\
\le& 5c_2^{-1}\sum_{m\ge n-2}\frac{g_2(2^{m+1})}{2^{\beta m}}\chi_m^b|X_m^b|^2 + 5c_1^{-1}\sum_{m\ge n-2}\frac{g_1(2^{m+1})}{2^{\alpha m}}\chi_m^u|X_m^u|^2.
\end{aligned} \tag{3.66}
$$

Thus, applying (3.62), (3.63), (3.64), (3.65), (3.66) in (3.61) we have

$$\begin{aligned} d_n^2(t) \leq & F_n(0) + c_3 \sum_{l=0}^{n-1} 2^{(1+\frac{d}{2})l} [\overline{d}_l^u \int_0^t 10c_1^{-1} \sum_{m\geq n-2} \frac{g_1(2^{m+1})}{2^{\alpha m}} \chi_m^u |X_m^u|^2 ds \\ & + \overline{d}_l^u \int_0^t 10c_2^{-1} \sum_{m\geq n-2} \frac{g_2(2^{m+1})}{2^{\beta m}} \chi_m^b |X_m^b|^2 ds \\ & + \overline{d}_l^b \int_0^t 10c_1^{-1} \sum_{m\geq n-2} \frac{g_1(2^{m+1})}{2^{\alpha m}} \chi_m^u |X_m^u|^2 ds \\ & + \overline{d}_l^b \int_0^t 10c_2^{-1} \sum_{m\geq n-2} \frac{g_2(2^{m+1})}{2^{\beta m}} \chi_m^b |X_m^b|^2 ds]. \end{aligned} \tag{3.67}$$

We further estimate

$$\int_0^t \chi_m^u |X_m^u|^2 ds \leq |d_m^u|^2(t) - |d_{m+1}^u|^2(t), \tag{3.68a}$$

$$\int_0^t \chi_m^b |X_m^b|^2 ds \leq |d_m^b|^2(t) - |d_{m+1}^b|^2(t), \tag{3.68b}$$

because F_m^u, F_m^b defined by (3.8a) are both nonincreasing in m. Applying (3.68a), (3.68b) in (3.67) gives

$$\begin{aligned} d_n^2(t) \leq & F_n(0) + 10 \left(\frac{c_3}{\min\{c_1, c_2\}} \right) \\ & \times \sum_{l=0}^{n-1} 2^{(1+\frac{d}{2})l} [\overline{d}_l^u \sum_{m\geq n-2} \frac{g_1(2^{m+1})}{2^{\alpha m}} (|d_m^u|^2(t) - |d_{m+1}^u|^2(t)) \\ & + \overline{d}_l^u \sum_{m\geq n-2} \frac{g_2(2^{m+1})}{2^{\beta m}} (|d_m^b|^2(t) - |d_{m+1}^b|^2(t)) \\ & + \overline{d}_l^b \sum_{m\geq n-2} \frac{g_1(2^{m+1})}{2^{\alpha m}} (|d_m^u|^2(t) - |d_{m+1}^u|^2(t)) \\ & + \overline{d}_l^b \sum_{m\geq n-2} \frac{g_2(2^{m+1})}{2^{\beta m}} (|d_m^b|^2(t) - |d_{m+1}^b|^2(t))] \\ \leq & F_n(0) + c_4 \sum_{l=0}^{n-1} \frac{\overline{d}_l}{2^{-(1+\frac{d}{2})l}} \sum_{m\geq n-2} [\frac{g_1(2^{m+1})}{2^{\alpha m}} (|d_m^u|^2(t) - |d_{m+1}^u|^2(t)) \\ & + \frac{g_2(2^{m+1})}{2^{\beta m}} (|d_m^b|^2(t) - |d_{m+1}^b|^2(t))] \end{aligned} \tag{3.69}$$

for $c_4 = 20 \left(\frac{c_3}{\min\{c_1, c_2\}} \right)$. Finally,

$$\begin{aligned} & \frac{g_1(2^{m+1})}{2^{\alpha m}} (|d_m^u|^2 - |d_{m+1}^u|^2)(t) + \frac{g_2(2^{m+1})}{2^{\beta m}} (|d_m^b|^2 - |d_{m+1}^b|^2)(t) \\ \leq & \left(\frac{g_1(2^{m+1})}{2^{\alpha m}} + \frac{g_2(2^{m+1})}{2^{\beta m}} \right) (|d_m|^2 - |d_{m+1}|^2)(t) \end{aligned} \tag{3.70}$$

where we used that d_m^u, d_m^b defined in (3.8b), (3.8c) are both nonincreasing in m. Applying (3.70) in (3.69) finally gives (3.57). $\square$

Having completed our work thus far, the rest of the proof of Theorem 1.1 follows from Section 4 [**2**], and thus instead of repeat its proof verbatim, we now only explain furthermore. The final step of the proof of Theorem 1.1 is the following proposition:

PROPOSITION 3.12. *Let $X = (X^u, X^b)$ be a shell solution of the system (1.9a)-(1.9c) that satisfies the energy inequality on $[0, T^*)$. If $\sup_{n\in\mathbb{N}_0} 2^{mn}|X_n(0)| < \infty \; \forall\, m \geq 1$, then*

$$\sup_{t\in[0,T^*)} \sup_{n\in\mathbb{N}_0} 2^{mn}|X_n(t)| < \infty \quad \forall\, m \geq 1. \tag{3.71}$$

Let us first make the following observation.

LEMMA 3.13. *A sequence $(b_n)_{n\geq -1}$ defined by*

$$b_n \triangleq \frac{1}{g_1(2^{n+1}) + g_2(2^{n+1})} \tag{3.72}$$

has the following properties:

(1) $(b_n)_{n\geq -1}$ *is nonincreasing,*
(2) $(\lambda^n b_n)_{n\geq -1}$ *is nondecreasing, where* $\lambda = 2^{1+\frac{d}{2}} = 2^\alpha = 2^\beta$,
(3) $\sum_{n=1}^\infty b_n = +\infty$.

PROOF. The fact that $(b_n)_{n\geq -1}$ is nonincreasing follows from the hypothesis that g_1, g_2 are both nondecreasing. Moreover, the fact that $(\lambda^n b_n)_{n\geq -1}$ is nondecreasing may be seen considering that $x^{-\alpha} g_1(x), x^{-\beta} g_2(x)$ are both nonincreasing due to hypothesis. Finally, in order to prove that $\displaystyle\sum_{n=1}^\infty b_n = +\infty$, one may compute that

$$\begin{aligned}
\infty = &\int_1^\infty \frac{ds}{s(g_1(s) + g_2(s))} \\
\leq &\frac{1}{g_1(2^0) + g_2(2^0)} \int_{2^0}^{2^1} \frac{ds}{s} + \frac{1}{g_1(2^1) + g_2(2^1)} \int_{2^1}^{2^2} \frac{ds}{s} + \dots \\
= &\ln(2) \sum_{n=-1}^\infty \frac{1}{g_1(2^{n+1}) + g_2(2^{n+1})} = \ln(2) \sum_{n=-1}^\infty b_n
\end{aligned}$$

where we used (1.12), that g_1, g_2 are both nondecreasing, and (3.72). $\square$

By (3.57) and (3.72), we obtain

$$d_n^2(t) \leq F_n(0) + c_4 \lambda^2 \sum_{l=0}^{n-1} \frac{\overline{d}_l}{\lambda^{n-l}} \sum_{m\geq n-2} \frac{1}{\lambda^{m-(n-2)}} \left(\frac{1}{b_m}\right) (|d_m|^2 - |d_{m+1}|^2)(t).$$

We denote by

$$Q_n \triangleq \sum_{l=0}^{n-1} \frac{\overline{d}_l}{\lambda^{n-l}}, \quad R_n(t) \triangleq \sum_{j=n}^\infty \frac{|d_j|^2(t) - |d_{j+1}|^2(t)}{\lambda^{j-n} b_j} \tag{3.73}$$

so that we obtain

$$d_n^2(t) \leq F_n(0) + c_4 Q_n R_{n-2}(t) \tag{3.74}$$

where we relabeled $c_4\lambda^2$ by c_4. The definition of Q_n, R_n in (3.73) and the inequality (3.74) are identical to those in [**2**] (see pg. 2021, in particular the equation (4-1) [**2**]). The properties of $(b_n)_{n\geq -1}$ in Lemma 3.13 is identical to those described on pg. 2021 [**2**]. Using (3.73), (3.74), Lemma 3.13, the work of Section 4 [**2**] which only relies on the definitions of Q_n, R_n and properties of $(b_n)_{n\geq -1}$ (see its Lemma 4.2, Lemma 4.3, Lemma 4.4), only with slight appropriate modifications, we are able to obtain the following result:

LEMMA 3.14. *(Lemma 4.5 [**2**]) For every $t > 0, M > 0$, there exists $c_M > 0$ such that*

$$\overline{d}_n^2 \leq c_M \lambda^{-Mn}, \quad Q_n \leq c_M \lambda^{-n}. \tag{3.75}$$

As an immediate consequence of Lemma 3.14, we may now prove Proposition 3.9.

PROOF OF PROPOSITION 3.12. Due to Lemma 3.14, we know in particular that for all $M > 0$, there exists $c_M > 0$ such that for any $t \in [0, T^*)$, $|d_n^u(t)|^2 + |d_n^b(t)|^2 \leq c_M \lambda^{-Mn}$. Therefore,

$$|X_n^u(t)|^2 + |X_n^b(t)|^2 \leq c_M \lambda^{-Mn}$$

by (3.8b), (3.8c), (3.8a). Now for a fixed $m \geq 1$, we may take $M > \frac{2m}{\alpha}$ so that

$$2^{2mn}(|X_n^u|^2 + |X_n^b|^2)(t) \leq c_M.$$

Taking a square root, supremum over $n \in \mathbb{N}_0$ and $t \in [0, T^*)$ finally gives (3.71). □

Proposition 3.12 and Proposition 3.4 imply (3.1) which concludes the proof of Theorem 1.1.

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