Dynamics of PDE, Vol.15, No.2, 127-146, 2018

On the higher integrability of weak solutions to the generalized Stokes system with bounded measurable coefficients

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Communicated by Dongho Chae, received November 4, 2016.

Abstract. In this paper, we deal with the generalized Stokes and Navier-Stokes problem. The elliptic term in the equation is assumed to have form $-\operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{u}))$, where the matrix function $\boldsymbol{A}$ is uniformly positive definite, but only L^∞. Using a Meyers' type estimate we improve the integrability of gradients of local weak solutions to a generalized Stokes problem. We also show that in the case of planar motion the integrability of local weak solution to generalized Navier Stokes system can be improved. This in combination with previous result gives better properties of gradient of solutions.

Contents

1. Introduction. Statement of Main Result

Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be an open set, $0 < T < \infty$, and define $Q := \Omega \times (0,T)$. We consider the following generalized Navier-Stokes system for unknown functions

1991 *Mathematics Subject Classification.* Primary 35Q30, 34M40; Secondary 35B65, 76B03.
Key words and phrases. Navier-Stokes equations, Generalized Stokes system, higher integrability, local regularity.

$\boldsymbol{u} : Q \to \mathbb{R}^n$, $p : Q \to \mathbb{R}$

$$\boldsymbol{u}_t + \delta \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{u})) = -\nabla p + \operatorname{div} \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q, \tag{1.1}$$

where δ is either 0 or 1, $\boldsymbol{f} = \{f_{ij}\}_{i,j=1}^n$, $\boldsymbol{u} \otimes \boldsymbol{u} = \{u_i u_j\}_{i,j=1}^n$ are symmetric tensors of the second order, and we assume existence of $\lambda_1, \lambda_2 > 0$ such that $\boldsymbol{A} = \{A_{ij}^{kl}\}$ satisfies for all $i, j, k, l \in \{1, \dots, n\}$, $(\boldsymbol{x}, t) \in Q$, and $\boldsymbol{\xi} \in \boldsymbol{M}_{\text{sym}}^{n \times n}$, symmetric $n \times n$ matrices

$$A_{ij}^{kl} \in L^\infty(Q), \quad A_{ij}^{kl}(\boldsymbol{x}, t) = A_{ji}^{kl}(\boldsymbol{x}, t) = A_{kl}^{ij}(\boldsymbol{x}, t) \tag{1.2}$$

$$\lambda_1 |\boldsymbol{\xi}|^2 \le A_{ij}^{kl}(\boldsymbol{x}, t)\xi_{ij}\xi_{kl} \le \lambda_2 |\boldsymbol{\xi}|^2. \tag{1.3}$$

We study local weak solutions defined in

Definition 1.1. We say that $\boldsymbol{u} \in L^\infty_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega))^n \cap L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\Omega))^n$ is *local weak solution* to the problem (1.1) if $\operatorname{div} \boldsymbol{u} = 0$ a.e. in Q and $(1.1)_1$ holds in the weak sense, i.e.

$$\int_Q -\boldsymbol{u} \cdot \boldsymbol{\varphi}_t + (-\delta(\boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{A}\boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{f}) : \nabla \boldsymbol{\varphi} = 0$$

for all $\boldsymbol{\varphi} \in C_0^\infty(Q)^n$ with $\operatorname{div} \boldsymbol{\varphi} = 0$.

If $\boldsymbol{A}$ is a multiple of identity matrix, and $\delta = 1$, the system (1.1) is the well-known Navier-Stokes system describing a flow of the Newtonian fluid in domain Ω. The unknowns represent velocity of the fluid $\boldsymbol{u}$ and pressure in the fluid p. In this case the main elliptic term can be written in the form

$$-\operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{u})) = -\operatorname{div}(\nu \boldsymbol{D}(\boldsymbol{u})),$$

with $\nu = \nu_0 > 0$ being the viscosity of the considered fluid. Existence of the weak solutions to the Navier-Stokes system was proved in the seminal papers of Leray [**11**] and Hopf [**9**]. Lot of progress has been achieved since that time. However, in spite of the effort dedicated to the problem, the fundamental questions of regularity and uniqueness of this solution remains open. Completely different situation is in the cases when $\delta = 0$ or $n = 2$, still requiring $\boldsymbol{A} = \nu \mathbf{Id}$. In these cases the regularity is well known, see [**16**].

When developing more sophisticated models that include also other physical quantities such as temperature θ, electro-magnetic field E,... the viscosity may depend on these quantities such that $\nu = \nu(\theta, E, \dots)$. Since they might be again driven by some differential equations we, in general, cannot assume that they are smooth. Even if we know that the function ν is smooth and bounded we get that the composite function $\nu(\theta, E, \dots)$ is just L^∞. In this article we address the question of regularity of solutions to such problems.

It is well known, that the parabolic systems with bounded measurable coefficients do not have to possess maximal regularity, see for example [**4**]. However, it is possible to improve regularity of the local weak solutions at least a little. If $\delta = 0$ we show that the regularity of the local weak solutions can be improved by Meyer's trick, see [**1, 13, 15, 17**]. The main obstacle in the proof is necessary localization needed due to the local nature of the solution, and its interplay with pressure p in (1.2). It is overcome using recent results about representation of the pressure in [**19**].

In our first main result we prove that any local weak solution of the generalized Stokes problem ((1.1) with $\delta = 0$) is actually more regular. We can show better properties of it's gradient as it is written in the following theorem.

THEOREM 1.2. *There is $q > 2$ such that if $r \in [2, q]$, $\boldsymbol{u}$ is a local weak solution of* (1.1) *with $\delta = 0$ and $\boldsymbol{f} \in L^r_{\text{loc}}(Q)^{n^2}$ then*

$$\nabla \boldsymbol{u} \in L^r_{\text{loc}}(Q)^{n^2}.$$

In the case $\delta = 1$ the convective term prevents application of Theorem 1.2 to (1.1). We must first differently improve integrability of the local weak solution. This can be done if $n = 2$ by combination of Campanato technique and reverse Hölder inequality.

The result is formulated in the next theorem.

THEOREM 1.3. *If $\boldsymbol{u}$ is a local weak solution of* (1.1) *with $\delta = 1$, $n = 2$ and $\boldsymbol{f} \in L^2_{\text{loc}}(Q)^4$ then for all $r > 1$*

$$\boldsymbol{u} \in L^2_{\text{loc}}(\Omega; L^r_{loc}(0, T))^2. \tag{1.4}$$

Moreover there is $q > 2$ such that if $r \in [2, q]$, $\boldsymbol{u}$ is a local weak solution of (1.1) *with $\delta = 1$, $n = 2$ and $\boldsymbol{f} \in L^r_{\text{loc}}(\Omega; L^2_{loc}(0, T))^4$ then*

$$|\boldsymbol{u}|^2, |\nabla \boldsymbol{u}| \in L^r_{loc}(\Omega, L^2_{loc}(0, T)). \tag{1.5}$$

The main results formulated in the previous Theorems have interesting corollaries.

COROLLARY 1.4. *There is $q > 2$ such that if $r \in (2, q]$, $\boldsymbol{u}$ is a local weak solution of* (1.1) *with $\delta = 1$, $n = 2$ and $\boldsymbol{f} \in L^r_{\text{loc}}(Q)^4$ then*

$$\boldsymbol{u} \in L^{2s}_{loc}(Q), \quad \nabla \boldsymbol{u} \in L^s_{\text{loc}}(Q)^4 \quad \text{for any } s \in [2, 6 - 4/r).$$

Integrability of $\boldsymbol{u}$ is just interpolation of the results in Theorem 1.3. Integrability of $\nabla \boldsymbol{u}$ then follows from Theorem 1.2.

The results in Theorem 1.2 and 1.3 allow to improve the regularity of the weak solutions to the full Navier-Stokes-Fourier system in Q if $n = 2$

$$\begin{gathered} \operatorname{div} \boldsymbol{u} = 0, \quad \boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{\sigma} + \nabla p = \operatorname{div} f, \\ \theta_t + \boldsymbol{u} \cdot \nabla \theta + \operatorname{div} \boldsymbol{q} = \boldsymbol{\sigma} : \boldsymbol{D}(\boldsymbol{u}) + g, \end{gathered} \tag{1.6}$$

which drives unknown flow velocity $\boldsymbol{u}$, pressure p and temperature distribution θ of a heat conducting fluid in domain Ω. Note that if we know only that $\boldsymbol{u}$ is the weak solution of the problem, specially only that $\boldsymbol{u} \in L^\infty_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega))^2 \cap L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\Omega))^2$, the term on the right-hand side of the equation for temperature is only in $L^1_{\text{loc}}(Q)$. Our last result shows that if $n = 2$ and the data of the problem are sufficiently smooth, any weak solution u is actually more regular.

COROLLARY 1.5. *There is $q > 2$ such that if $r \in (2, q]$, a pair $(\boldsymbol{u}, \theta)$ is a local weak solution of* (1.6)*, see Definition 5.1, with $n = 2$, $\boldsymbol{f} \in L^r_{\text{loc}}(Q)^4$ then*

$$\boldsymbol{u} \in L^{2s}_{loc}(Q), \quad \nabla \boldsymbol{u} \in L^s_{\text{loc}}(Q)^4 \quad \text{for any } s \in [2, 6 - 4/r).$$

Here, the structure of the article is presented. In the next section we collect useful results about reconstruction of the pressure that are the base for our results. In Sections 3 and 4, Theorems 1.2 and 1.3 are proved. Section 5 is devoted to the application of our results to weak solutions of system describing flow of heat conducting fluids. In the last section we present, for readers convenience, an auxiliary result about parabolic Poincaré's inequality.

2. Existence of pressure in $\mathbb{R}^n$

As a principal obstacle in showing the result in Theorems 1.2 and 1.3 can be considered the fact that in the setting from previous section we are not able to construct to each weak solution of Stokes or Navier-Stokes problem corresponding pressure p as a function in some Lebesgue space. Instead we know that the pressure consists of a "good" part, which is indeed a function, and a "bad" part which can be represented as time derivative of some function in suitable sense, compare [**16**], [**10**], [**19**]. We state the result in the following theorems. It is based on the decomposition of the space $L^r(G)$, $r \in (1, +\infty)$, $G \subset \Omega$, into space

$$A^r(G) := \text{closure } \{\Delta\phi; \phi \in C_0^\infty(G)\} \text{ in } L^r(G)$$
$$B^r(G) := \{\varphi \in L^r(G); \Delta\varphi = 0 \text{ in } G\},$$

due to C. G. Simader.

In this section we assume that $G \subset \mathbb{R}^n$ is a bounded domain with $\partial G \in C^2$.

Lemma 2.1. *For all $r \in (1, +\infty)$ there holds $L^r(G) = A^r(G) \oplus B^r(G)$.*

Proof. See, for example [**19**, Corollary 2.5]. □

Now, we can formulate the first of the promised theorems

Theorem 2.2. *Let $\boldsymbol{Q} \in L^r(G\times(0,T))^{n^2}$ $(1 < r \leq 2)$ and $\boldsymbol{u} \in C_w([0,T]; L^2(G))^n$ with $\operatorname{div} \boldsymbol{u} = 0$ (in the sense of distributions). Suppose that*

$$-\int_0^T \int_G \boldsymbol{u} \cdot \boldsymbol{\varphi}_t + \int_0^T \int_G \boldsymbol{Q} : \nabla \boldsymbol{\varphi} = 0 \tag{2.1}$$

holds for all $\boldsymbol{\varphi} \in C_0^\infty(G\times(0,T))^n$ with $\operatorname{div} \boldsymbol{\varphi} = 0$. Then there exist unique functions $p_0 \in L^r(0,T; A^r(G))$ and $\tilde{p}_h \in C_w([0,T]; B^r(G))$, such that for all $t \in [0,T]$ $\int_G \tilde{p}_h = 0$ and

$$\begin{aligned} -\int_0^T \int_G \boldsymbol{u} \cdot \boldsymbol{\varphi}_t &+ \int_0^T \int_G \boldsymbol{Q} : \nabla \boldsymbol{\varphi} \\ &= \int_0^T \int_G p_0 \operatorname{div} \boldsymbol{\varphi} - \int_0^T \int_G \tilde{p}_h \operatorname{div} \boldsymbol{\varphi}_t \\ &+ \int_G u(0) \cdot \boldsymbol{\varphi}(0) \end{aligned} \tag{2.2}$$

for all $\boldsymbol{\varphi} \in C^\infty(G \times (0,T))^n$ with $\operatorname{supp}(\boldsymbol{\varphi}) \Subset G \times [0,T)$. In addition, we have the a-priori estimates

$$\|p_0\|_{L^r(G\times(0,T))} \leq c\, \|\boldsymbol{Q}\|_{L^r(G\times(0,T))}, \tag{2.3}$$

$$\|\tilde{p}_h\|_{L^\infty(0,T;L^r(G))} \leq c(\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(G))} + \|\boldsymbol{Q}\|_{L^r(G\times(0,T))}), \tag{2.4}$$

where $c > 0$ depends only on r, n, G and T.

Proof. See [**19**, Theorem 2.6]. □

We will often consider term $\boldsymbol{Q}$ in the special form

$$\boldsymbol{Q} = \sum_{i=0}^{N} \boldsymbol{Q}^i, \quad \boldsymbol{Q}^i \in L^{q_i}(0,T; L^{r_i}(G'))^{n^2}, \tag{2.5}$$

with $G' \subset G$ (we also allow $G' = G$), $\partial G' \in C^2$, and $q_i, r_i \in (1, +\infty)$, $i = 0, \ldots, N$. Under these assumption we can derive from Lemma 2.1 more information about p_0.

Lemma 2.3. *Let all assumptions of Theorem 2.2 and* (2.5) *hold. Define* $r := \min_{i=0,\ldots,N}(q_i, r_i)$*. Let* $p_0 \in L^r(0,T; A^r(G)$ *be the function constructed in Theorem 2.2. Then there are functions* $P_i \in L^{r_i}(0,T; A^{r_i}(G'))$*,* $i = 0, \ldots, N$*,* $P_h \in L^r(0,T; B^r(G'))$ *such that*

$$p_0 = \sum_{i=0}^{N} P_i + P_h \quad \text{on } G' \times (0,T), \tag{2.6}$$

$$\|P_i\|_{L^{r_i}(0,T;A^{r_i}(G'))} \le C \left\|\boldsymbol{Q}^i\right\|_{L^{r_i}(0,T;A^{r_i}(G'))} \tag{2.7}$$

$$\|P_h\|_{L^r(0,T;B^r(G'))} \le C \|p_0\|_{L^r(G'\times(0,T))}, \tag{2.8}$$

and in the sense of distributions hold the equations

$$\Delta P_i = \operatorname{div}\operatorname{div} \boldsymbol{Q}^i \quad \text{on } G' \times (0,T). \tag{2.9}$$

Proof. For a.e. $t \in (0,T)$ there is $p_0(t) \in L^r(G')$, $\boldsymbol{Q}^i(t) \in L^{r_i}(G')^{n^2}$. We define $P_i := \Delta\Phi_i$ where $\Phi_i \in W_0^{2,r_i}(\mathbb{R}^n)$ is given as the unique solution of

$$\int_{G'} \Delta\Phi_i(t)\Delta\varphi = \int_{G'} \boldsymbol{Q}^i(t) : \nabla^2\varphi \quad \forall \varphi \in C_0^\infty(G') \tag{2.10}$$

(see [**19**, Lemma 2.4]). From [**19**, Lemma 2.3] it follows

$$\|P_i(t)\|_{L^{r_i}(G')} \le C \left\|\boldsymbol{Q}^i\right\|_{L^{r_i}(G')},$$

which implies (2.7). Also (2.9) is immediately seen.

Taking in (2.2) test function $\boldsymbol{\varphi} := \nabla\psi$ with $\psi \in C_0^\infty(G' \times (0,T))$ and plugging (2.5) and (2.10) we get after integration by parts

$$0 = \int_0^T \int_{G'} (p_0 - \sum_{i=0}^{N} P_i)\Delta\psi \quad \forall \psi \in C_0^\infty(G' \times (0,T)).$$

It implies that $P_h(t) := p_0(t) - \sum_{i=0}^{N} P_i(t)$ is a harmonic function for a.e. $t \in (0,T)$. Since $P_h(t) \in B^r(G')$, $P_i(t) \in A^r(G')$, $i = 0, \ldots, N$ we have by Lemma 2.1

$$\|P_h(t)\|_{L^r(G')} \le C \|p_0(t)\|_{L^r(G')}$$

and consequently (2.8) follows.

□

Remark 2.4. We wish to remark that taking $G' = G$ in Lemma 2.3 we have $P_h = 0$.

3. Proof of Theorem 1.2

The goal is to find suitable $q > 2$. It will be fixed at the end of this section. From now we assume that $q < 2(n+2)/n$.

Steps of the Proof 1° *Pressure representation.* Let $B_R \Subset \Omega$ be a fixed ball and $I' \Subset (0,T)$ be a fixed interval. Since $\delta = 0$, $\boldsymbol{f} + \boldsymbol{A}\boldsymbol{D}(\boldsymbol{u}) \in L^2(I'; L^2(B_R))^{n^2}$, we get that $\boldsymbol{u} \in C_w(I'; L^2(B_R))^n$, and by Theorem 2.2 there exist functions

$$\tilde{p}_h \in L^\infty(I'; B^2(B_R)),$$
$$p_0 \in L^2(I'; A^2(B_R)),$$

such that the pressure corresponding to the weak solution $\boldsymbol{u}$ is in $B_R \times I'$ represented in the sense of distributions as

$$p = p_0 + \frac{\partial \tilde{p}_h}{\partial t}.$$

We wish to recall that for almost all $t \in I'$ the pressure $\tilde{p}_h(t)$ is harmonic, and thus it is smooth in B_R.

Setting $\boldsymbol{v} := \boldsymbol{u} + \nabla \tilde{p}_h$ a. e. in $B_R \times I'$ one finds that $\boldsymbol{v}$ solves in the sense of distributions in $B_R \times I'$ the equation

$$\boldsymbol{v}_t - \operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{v})) = -\nabla p_0 + \operatorname{div} \boldsymbol{g}, \tag{3.1}$$

with $\boldsymbol{g} = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{D}(\nabla \tilde{p}_h) \in L^q(I'; L^q(B_R))^{n^2}$, and $\operatorname{div} \boldsymbol{v} = 0$ a.e. in $B_R \times I'$.

2° *Localization.* Let $\phi \in C_0^\infty(B_R \times I')$ be a cut-off function. We set $\boldsymbol{W} := \phi \boldsymbol{v}$ and extend all functions from support of ϕ on $\mathbb{R}^n \times \mathbb{R}$ by zero. We take in (3.1) test functions in the form $\phi \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^{n+1})^n$. It follows that $\boldsymbol{w} \in C([0,T]; \boldsymbol{L}^2(\mathbb{R}^n)) \cap L^2(0,T; \boldsymbol{W}^{1,2}(\mathbb{R}^n))$ is a weak solution to the system

$$\operatorname{div} \boldsymbol{W} = g \quad \text{in} \quad \mathbb{R}^n \times (0,T), \tag{3.2}$$

$$\boldsymbol{W}_t - \operatorname{div} \boldsymbol{A}\boldsymbol{D}(\boldsymbol{W}) = -\nabla(\phi p_0) + \operatorname{div} \boldsymbol{G} + \boldsymbol{H} \quad \text{in} \quad \mathbb{R}^n \times (0,T), \tag{3.3}$$

$$\boldsymbol{W} = 0 \quad \text{on} \quad \mathbb{R}^n \times \{0\} \tag{3.4}$$

with functions

$$g = \nabla \phi \cdot \boldsymbol{v} \in L^2(\mathbb{R}^n \times (0,T)) \cap L^q(\mathbb{R}^n \times (0,T)),$$
$$\boldsymbol{G} = \phi \boldsymbol{g} - \boldsymbol{A}(\nabla \phi \otimes \boldsymbol{v}) \in L^2(\mathbb{R}^n \times (0,T))^{n^2} \cap L^q(\mathbb{R}^n \times (0,T))^{n^2},$$
$$\boldsymbol{H} = \boldsymbol{v} \phi_t - [\boldsymbol{A}\boldsymbol{D}(\boldsymbol{v}) - p_0 \boldsymbol{I} + \boldsymbol{g}] \nabla \phi \in L^2(\mathbb{R}^n \times (0,T))^n.$$

Regularity of g and $\boldsymbol{G}$ follows from the fact that $\boldsymbol{u} \in L^{2(n+2)/n}_{\mathrm{loc}}(Q)$ due to a-priori regularity of local weak solution, and from the properties of $\tilde{p}_h$, namely from the fact, that $\tilde{p}_h$ is harmonic in space. Note that it will be sufficient to show that $\boldsymbol{D}(\boldsymbol{W}) \in L^r(Q)^{n^2}$.

3° *Homogenization of the divergence equation*
We make use of the homogenous Sobolev space $\dot{W}^{1,r}(\mathbb{R}^n)$ which contains all functions $f \in W^{1,1}_{loc}(\mathbb{R}^n)$ such that $\nabla f \in L^r(\mathbb{R}^n)$ together with the mean property $f_{B_1} = 0$. This space is a Banach space with respect to the norm $\|\nabla f\|_r$. According to Lemma A.2 (Korn's inequality) we may equip the space of vector valued functions

$\dot{W}^{1,r}(\mathbb{R}^n)^n$ with the equivalent norm

$$\|\boldsymbol{u}\|_{\dot{W}^{1,r}} = \|\boldsymbol{D}(\boldsymbol{u})\|_r, \quad \boldsymbol{u} \in \dot{W}^{1,r}(\mathbb{R}^n)^n.$$

Now we use the bounded operator $\boldsymbol{B} : L^r(\mathbb{R}^n) \to \dot{W}^{1,r}(\mathbb{R}^n)^n$ introduced in Theorem A.3 and Remark A.4. It fulfils for $f \in L^r(\mathbb{R}^n)$ that

$$\operatorname{div} \boldsymbol{B}(f) = f, \quad \int_{\mathbb{R}^n} \boldsymbol{B}(g) \cdot \boldsymbol{\varphi} dx = 0 \quad \forall \varphi \in C^\infty_{\mathrm{c,div}}(\mathbb{R}^n)^n.$$

We now define

$$\boldsymbol{w} := \boldsymbol{W} - \boldsymbol{W}_{B_1} - \boldsymbol{B}(g), \quad \tilde{\boldsymbol{G}} = \boldsymbol{G} + \boldsymbol{A}\boldsymbol{D}(\boldsymbol{B}(g)).$$

We easily see that $\boldsymbol{w} \in L^2(0,T;\dot{W}^{1,2}(\mathbb{R}^n)^n)$ is a weak solution to

$$\operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T), \tag{3.5}$$

$$\boldsymbol{w}_t - \operatorname{div} \boldsymbol{A}\boldsymbol{D}(\boldsymbol{w}) = -\nabla\pi + \operatorname{div} \tilde{\boldsymbol{G}} + \boldsymbol{H} \quad \text{in} \quad \mathbb{R}^n \times (0,T), \tag{3.6}$$

$$\boldsymbol{w}(0) \in \boldsymbol{V}_{grad}, \tag{3.7}$$

where $\boldsymbol{V}_{grad}$ denotes the space of all $\boldsymbol{v} \in L^1_{loc}(\mathbb{R}^n)^n$ such that $\int_{\mathbb{R}^n} \boldsymbol{v} \cdot \boldsymbol{\varphi} dx = 0$ for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{c},\sigma}(\mathbb{R}^n)^n$ (cf. Theorem A.3). A vector function $\boldsymbol{w}$ is called a weak solution to (3.5)–(3.7) if $\operatorname{div} \boldsymbol{w} = 0$ a. e. in $\mathbb{R}^n \times (0,T)$ and the following integral identity is fulfilled for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{c}}(\mathbb{R}^n \times [0,T))^n$ with $\operatorname{div} \boldsymbol{\varphi} = 0$

$$\int_Q -\boldsymbol{w} \cdot \boldsymbol{\varphi}_t + \boldsymbol{A}\boldsymbol{D}(\boldsymbol{w}) : \boldsymbol{D}(\boldsymbol{\varphi}) dxdt = \int_Q \tilde{\boldsymbol{G}} : \boldsymbol{D}(\boldsymbol{\varphi}) + \boldsymbol{H} \cdot \boldsymbol{\varphi} dxdt. \tag{3.8}$$

Since $\boldsymbol{D}(\boldsymbol{B}(g)) \in L^r(\mathbb{R}^n)^{n^2} \cap L^2(\mathbb{R}^n)^{n^2}$ it will be sufficient to show that $\boldsymbol{D}(\boldsymbol{w}) \in L^r(Q)^{n^2}$. To verify this, we first show that the weak solution to (3.5)–(3.7) is unique (for fixed $\tilde{\boldsymbol{G}}$ and $\boldsymbol{H}$), and second, arguing as in [**5**], we use Meyer's trick to obtain a solution to (3.5)–(3.7) fulfilling the required higher integrability.

4° *Uniqueness*

Let $\boldsymbol{w} \in L^2(0,T;\dot{\boldsymbol{W}}^{1,2}(\mathbb{R}^n))$ be a weak solution to (3.5)–(3.7) with $\tilde{\boldsymbol{G}} = \boldsymbol{0}, \boldsymbol{H} = \boldsymbol{0}$. By using the standard theory of the Stokes equation, we get a weak solution $\boldsymbol{W} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^n)) \cap L^2(0,T;\boldsymbol{W}^{1,2}(\mathbb{R}^n))$ of the problem

$$\operatorname{div} \boldsymbol{W} = 0 \quad \text{in} \quad Q, \tag{3.9}$$

$$\boldsymbol{W}_t - \operatorname{div} \boldsymbol{D}(\boldsymbol{W}) = -\nabla\Pi + \operatorname{div}(\boldsymbol{D}(\boldsymbol{w}) - \boldsymbol{A}\boldsymbol{D}(\boldsymbol{w})) \quad \text{in} \quad Q, \tag{3.10}$$

$$\boldsymbol{W} = \boldsymbol{0} \quad \text{on} \quad \mathbb{R}^n \times \{0\} \tag{3.11}$$

(e. g. see in [**16**]).

Then $\boldsymbol{U} = \boldsymbol{w} - (\boldsymbol{W} - \boldsymbol{W}_{B_1}) \in L^2(0,T;\dot{\boldsymbol{W}}^{1,2}(\mathbb{R}^n))$ satisfies the Stokes equation with right-hand side $\boldsymbol{0}$ and $\boldsymbol{U}(0) \in \boldsymbol{V}_{grad}$. Accordingly, $\omega_{ij} = \partial_i U^j - \partial_j U^i$ $(i,j = 1,\dots,n)$ is caloric with $\omega_{ij}(0) = 0$. Thus, $\omega_{ij} = 0$. Recalling that $\operatorname{div} \boldsymbol{U} = 0$, we find $\Delta \boldsymbol{U} = \boldsymbol{0}$, and thus $\Delta\nabla\boldsymbol{U} = \boldsymbol{0}$. Owing to $\nabla\boldsymbol{U} \in L^2(Q;\mathbb{R}^{n^2})$ we see that $\boldsymbol{U}$ is constant. On the other hand, observing $\boldsymbol{U}_{B_1} = 0$ it follows that $\boldsymbol{U} \equiv \boldsymbol{0}$. Hence, $\boldsymbol{w} = \boldsymbol{W} - \boldsymbol{W}_{B_1}$. Consequently, $\boldsymbol{W}$ solves (3.5), (3.6) with right-hand side $\tilde{\boldsymbol{G}} = \boldsymbol{0}$ and $\boldsymbol{H} = \boldsymbol{0}$ satisfying the initial condition $\boldsymbol{W}(0) = \boldsymbol{0}$ in $\mathbb{R}^n$. From the energy equality for $\boldsymbol{W}$ we get $\boldsymbol{W} = \boldsymbol{w} + \boldsymbol{W}_{B_1} \equiv \boldsymbol{0}$, and thus $\boldsymbol{w} = -\boldsymbol{W}_{B_1}$. Recalling that $\boldsymbol{w}_{B_1} = 0$ we conclude that $\boldsymbol{w} \equiv \boldsymbol{0}$.

5° *Higher integrability via Meyer's trick.*

Note that our assumption $q \le 2(n+2)/n$ guarantees that $L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^n))) \cap L^2(0,T;\boldsymbol{W}^{1,2}(\mathbb{R}^n)) \hookrightarrow \boldsymbol{L}^q(\mathbb{R}^n \times (0,T))$.

We rewrite (3.6) as follows,

$$\boldsymbol{w}_t - \lambda_2 \operatorname{div} \boldsymbol{D}(\boldsymbol{w}) = -\nabla\pi - \operatorname{div}((\lambda_2 \boldsymbol{I} - \boldsymbol{A})\boldsymbol{D}(\boldsymbol{w})) - \operatorname{div} \tilde{\boldsymbol{G}} + \boldsymbol{H} \quad \text{in} \quad Q. \tag{3.12}$$

Our aim is to construct a weak solution to (3.5)–(3.7) by applying Banach's fixed point theorem while solving (3.5), (3.12), (3.7) with a given function $\tilde{\boldsymbol{w}}$ in place of $\boldsymbol{w}$ on the right-hand side of (3.12).

Let $r \in [2,q]$. Set $X_r = L^2(0,T;\dot{\boldsymbol{W}}^{1,2}(\mathbb{R}^n)) \cap L^r(0,T;\dot{\boldsymbol{W}}^{1,r}(\mathbb{R}^n))$ equipped with the following norm

$$\|\boldsymbol{w}\|_{X_r} := \|\boldsymbol{D}(\boldsymbol{w})\|_2 + \|\boldsymbol{D}(\boldsymbol{w})\|_r, \quad \boldsymbol{w} \in X_r.$$

Note that due to Korn's inequality $\|\cdot\|_{X_r}$ defines a norm (see Lemma A.2 and inequality (A.1)). Furthermore, X_r is a Banach space. We define the operator $\mathbb{T}_r : X_r \to X_r$, by setting

$$\mathbb{T}_r(\tilde{\boldsymbol{w}}) := \boldsymbol{W} - \boldsymbol{W}_{B_1},$$

where $\boldsymbol{W} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^n)) \cap L^2(0,T;\boldsymbol{W}^{1,2}(\mathbb{R}^n))$ stands for the unique weak solution to

$$\operatorname{div} \boldsymbol{W} = 0 \quad \text{in} \quad Q, \tag{3.13}$$

$$\begin{aligned} &\boldsymbol{W}_t - \lambda_2 \operatorname{div} \boldsymbol{D}(\boldsymbol{W}) \\ &\qquad = -\nabla\pi - \operatorname{div}((\lambda_2 \boldsymbol{I} - \boldsymbol{A})\boldsymbol{D}(\tilde{\boldsymbol{w}})) - \operatorname{div} \tilde{\boldsymbol{G}} + \boldsymbol{H} \quad \text{in} \quad Q, \end{aligned} \tag{3.14}$$

$$\boldsymbol{W} = \boldsymbol{0} \quad \text{on} \quad \mathbb{R}^n \times \{0\}. \tag{3.15}$$

We claim that $\mathbb{T}_r(\tilde{\boldsymbol{w}}) \in X_r$. In fact, this immediately follows from the maximal regularity of the Stokes equation in $\mathbb{R}^n$, what is equivalent to the maximal regularity of the heat equation (cf. [**8**]), taking into accout the Helmholtz projection on the whole space.

Next, our aim is to show that there exists $q \in (2, 2(n+2)/2)$, such that for any $r \in [2,q]$, $\mathbb{T}_r$ is strictly contractive, and hence has a unique fixed point. Let $\tilde{\boldsymbol{w}}_1, \tilde{\boldsymbol{w}}_2 \in X_r$. We define $\tilde{\boldsymbol{w}} = \tilde{\boldsymbol{w}}_1 - \tilde{\boldsymbol{w}}_2$ and set $\boldsymbol{w}_1 = \mathbb{T}_r(\tilde{\boldsymbol{w}}_1)$ and $\boldsymbol{w}_2 = \mathbb{T}_r(\tilde{\boldsymbol{w}}_2)$. According to the definition of $\mathbb{T}_r$ we have $\boldsymbol{w} := \boldsymbol{w}_1 - \boldsymbol{w}_2 = \boldsymbol{W} - \boldsymbol{W}_{B_1}$, where $\boldsymbol{W} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^n)) \cap L^2(0,T;\boldsymbol{W}^{1,2}(\mathbb{R}^n))$ is the unique weak solution to the following Stokes system

$$\operatorname{div} \boldsymbol{W} = 0 \quad \text{in} \quad Q, \tag{3.16}$$

$$\lambda_2^{-1}\boldsymbol{W}_t - \operatorname{div} \boldsymbol{D}(\boldsymbol{W}) = -\nabla\pi - \operatorname{div}((\boldsymbol{I} - \lambda_2^{-1}\boldsymbol{A})\boldsymbol{D}(\tilde{\boldsymbol{w}})) \quad \text{in} \quad Q, \tag{3.17}$$

$$\boldsymbol{W} = \boldsymbol{0} \quad \text{on} \quad \mathbb{R}^n \times \{0\}. \tag{3.18}$$

By testing (3.17) with $\boldsymbol{W}$, along with (1.3) it is readily seen that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |\boldsymbol{D}(\boldsymbol{w})|^2 dxdt &\le \int_0^T \int_{\mathbb{R}^n} (\boldsymbol{I} - \lambda_2^{-1}\boldsymbol{A})\boldsymbol{D}(\tilde{\boldsymbol{w}}) : \boldsymbol{D}(\boldsymbol{w}) dxdt \\ &\le (1 - \lambda_1\lambda_2^{-1}) \int_0^T \int_{\mathbb{R}^n} |\boldsymbol{D}(\tilde{\boldsymbol{w}})|\,|\boldsymbol{D}(\boldsymbol{w})| dxdt. \end{aligned} \tag{3.19}$$

Here we have used the fact that $\boldsymbol{I} - \lambda_2^{-1}\boldsymbol{A}$ is positive. With the help of Cauchy-Schwarz's inequality from (3.19) we infer

$$\|\boldsymbol{D}(\boldsymbol{w})\|_2 \le (1 - \lambda_1\lambda_2^{-1})\|\boldsymbol{D}(\tilde{\boldsymbol{w}})\|_2. \tag{3.20}$$

On the other hand, by means of the maximal regularity of the Stokes equation, once more appealing to (1.3), for each r, $2 \le r$ there exists a constant $C_r > 0$, such that

$$\|\boldsymbol{D}(\boldsymbol{w})\|_r \le C_r(1-\lambda_1\lambda_2^{-1})\|\boldsymbol{D}(\tilde{\boldsymbol{w}})\|_r. \tag{3.21}$$

Thus, we are in a position to apply Riesz-Thorin's interpolation theorem, to $\mathbb{T}_r$ with $\tilde{\boldsymbol{G}} = \boldsymbol{0}$ and $\boldsymbol{H} = \boldsymbol{0}$. Accordingly, by the aid of (3.20) and (3.21) with $r = 2(n+2)/n$ we find

$$C_r \le C_2^{1-\theta_r} C_*^{\theta_r}, \quad \text{where} \quad \frac{1-\theta_r}{2} + \frac{\theta_r n}{2(n+2)} = \frac{1}{r} \quad \text{and } C_* = C_{2(n+2)/n}$$

for shortness. We calculate $\theta_r = (1/r - 1/2)/(n/(2(n+2)) - 1/2)$. As $C_2 \le 1$ we get

$$\|\boldsymbol{D}(\boldsymbol{w})\|_r \le C_*^{\theta_r}(1-\lambda_1\lambda_2^{-2})\|\boldsymbol{D}(\tilde{\boldsymbol{w}})\|_r. \tag{3.22}$$

Since $(1-\lambda_1\lambda_2^{-1}) < 1$ and $\theta_r \to 0$ as $r \to 2^+$ we can choose $q \in (2, 2(n+2)/n)$, such that

$$C_*^{\theta_r}(1-\lambda_1\lambda_2^{-1}) < 1 \quad \text{for all } r \in [2,q]. \tag{3.23}$$

We also may assume that $C_* \ge 1$. Hence, (3.20) and (3.21) lead to

$$\begin{aligned} \|\boldsymbol{w}\|_{X_r} &= \|\boldsymbol{D}(\boldsymbol{w})\|_2 + \|\boldsymbol{D}(\boldsymbol{w})\|_r \\ &\le C_*^{\theta_r}(1-\lambda_1\lambda_2^{-1})(\|\boldsymbol{D}(\tilde{\boldsymbol{w}})\|_2 + \|\boldsymbol{D}(\tilde{\boldsymbol{w}})\|_r) \\ &= C_*^{\theta_r}(1-\lambda_1\lambda_2^{-1})\|\tilde{\boldsymbol{w}}\|_{X_r}. \end{aligned} \tag{3.24}$$

Thanks to (3.23) the mapping $\mathbb{T}_r$ is strictly contractive for any $r \in [2,q]$. Thus, by Banach's fixed point theorem $\mathbb{T}_r$ has a unique fixed point $\boldsymbol{w} \in X_r$, which in turn appears to be the desired weak solution to (3.5)–(3.7). This completes the proof of the Theorem 1.1.

4. Proof of Theorem 1.3

1° *Pressure representation.* Fix an open square $G \Subset \Omega$ and $I' \Subset (0,T)$ an open interval. Let Ω', Ω'' be two open sets such that $G \Subset \Omega' \Subset \Omega'' \Subset \Omega$. Using the orthogonal decomposition

$$L^2(\Omega'') = A^2(\Omega'') \oplus B^2(\Omega'')$$

there exists $p_0 \in L^2(I'; L^2(\Omega''))$ such that

$$-\Delta p_0 = \operatorname{div}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{A}\boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{f}) \quad \text{in} \quad \Omega''\times I'.$$

Then in the sense of distributions there holds $p = p_0 + \frac{\partial \tilde{p}_h}{\partial t}$ in $\Omega'' \times I'$, where $\tilde{p}_h \in L^\infty(I'; B^2(\Omega''))$ (cf. Theorem 2.2 and Lemma 2.3). Accordingly, the function $\boldsymbol{v} := \boldsymbol{u} + \nabla\tilde{p}_h$ solves in the sense of distributions in $\Omega' \times I'$ the equation

$$\boldsymbol{v}_t + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v}) - \operatorname{div}(\boldsymbol{A}(x,t)\boldsymbol{D}(\boldsymbol{v})) = -\nabla p_0 + \operatorname{div}\boldsymbol{g}$$

with $\boldsymbol{g} = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{D}(\nabla\tilde{p}_h) + \boldsymbol{u}\otimes\nabla\tilde{p}_h \in L^2(I'; L^2(\Omega'))^4$.

2° *Localization in time.* Let $\rho \in C^\infty(\mathbb{R})$ with $\rho \equiv 0$ in $\mathbb{R}\setminus I'$. Setting $\boldsymbol{w} := \rho\boldsymbol{v}$, $\sigma := \rho p_0$, and extending all functions to $\Omega' \times (-\infty, 0)$ by 0, it holds in the sense of distributions in $\Omega' \times (-\infty, T)$

$$\boldsymbol{w}_t + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{w}) - \operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{w})) = -\nabla\sigma - \operatorname{div}\boldsymbol{G}, \tag{4.1}$$

where

$$\boldsymbol{G} := \rho \boldsymbol{g} + \rho' \boldsymbol{w} \in L^2(-\infty, T; L^2(\Omega'))^4.$$

Note that from (4.1) it follows that $\boldsymbol{w}_t \in L^2(-\infty, T; (W^{1,2}(\Omega'))^*)$, and it is allowed to test (4.1) with functions $\boldsymbol{\varphi} \in L^2(-\infty, T; W_0^{1,2}(\Omega'))^2$.

3° *Proof of* (1.4). We reformulate standard energy estimates in space time setting to the time space setting, i.e. we obtain estimates where we first integrate over time variable t and then over x. Let $\eta \in C_0^\infty(\Omega' \times (0,T))$, $\eta = 1$ on G. We set

$$\boldsymbol{\varphi}(x,t) = -\!\!\!\!\!\!\int_{t-h}^{t} \eta^2(x,s)(\boldsymbol{w}(x,s+h) - \boldsymbol{w}(x,s))ds.$$

The function $\boldsymbol{\varphi}$ is well-defined for all $t \in \mathbb{R}$ if $0 < h < h_0 := \operatorname{dist}(\operatorname{spt}\eta, \{0,T\})$ and $\boldsymbol{\varphi} \in L^2(-\infty, T; W^{1,2}(\Omega'))^2$, it may be used as test function in (4.1). It holds

$$-\int_{\mathbb{R}} \int_{\mathbb{R}^2} \boldsymbol{w}(x,t) \boldsymbol{\varphi}_t(x,t) dx dt = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \eta^2 \frac{|\boldsymbol{w}(x,t+h) - \boldsymbol{w}(x,t)|^2}{h} dt dx.$$

Since $\boldsymbol{u} \otimes \boldsymbol{w}, \boldsymbol{AD}(\boldsymbol{w}), \boldsymbol{G} \in L^2(0,T; L^2(\Omega'))^4$, $\sigma \in L^2(0,T; L^2(\Omega'))$ we easily estimate all other terms in the weak formulation of (4.1) and get

$$\exists C > 0, \forall h \in (0, h_0) : \int_{\mathbb{R}^2} \int_{\mathbb{R}} \eta^2 \frac{|\boldsymbol{w}(x,t+h) - \boldsymbol{w}(x,t)|^2}{h} dt dx \le C.$$

Now we divide this inequality by $h^{-\alpha}$ with $\alpha < 1$, integrate over $h \in (0, h_0)$ and after some simple manipulation we arrive at the estimate for $\boldsymbol{z} := \boldsymbol{w}\eta$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}} |\boldsymbol{z}(x,t)|^2 dt dx + \int_0^{h_0} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{|\boldsymbol{z}(x,t+h) - \boldsymbol{z}(x,t)|^2}{h^{1+\alpha}} dt dx dh \le C. \tag{4.2}$$

Note that (4.2) means $\boldsymbol{z} \in W^{\alpha/2,2}(0,T; L^2(\mathbb{R}^2))^2$ for all $\alpha < 1$. From (4.2) it follows that for a.e. $x \in G$

$$\int_{\mathbb{R}} |\boldsymbol{z}(x,t)|^2 dt + \int_0^{h_0} \int_{\mathbb{R}} \frac{|\boldsymbol{z}(x,t+h) - \boldsymbol{z}(x,t)|^2}{h^{1+\alpha}} dt dh < +\infty$$

holds. Embedding theorem for Sobolev-Slobodeckii spaces gives for $q = 2/(1-\alpha)$

$$\left(\int_{\mathbb{R}} |\boldsymbol{z}(x,t)|^q dt \right)^{\frac{2}{q}} \le C \left(\int_0^{h_0} \int_{\mathbb{R}} \frac{|\boldsymbol{z}(x,t+h) - \boldsymbol{z}(x,t)|^2}{h^{1+\alpha}} ds dh + \int_{\mathbb{R}} |\boldsymbol{z}(x,t)|^2 dt \right).$$

Integrating this over $x \in G$, and using the estimate (4.2), we get for all $q > 1$ that $\boldsymbol{w} \in L^2(G; L^q_{\text{loc}}(0,T))^2$, i.e., (1.4) since G was arbitrary.

4° *Proof of* (1.5). It appears that the space-time regularity of $\boldsymbol{w}$ can be improved in space. First it is observed that

$$\exists C > 0 : \|\boldsymbol{u}\|_{L^4(\Omega' \times (0,T))} + \|\nabla \boldsymbol{w}\|_{L^2(\Omega' \times (0,T))} < C.$$

Next, we set $R_0 = \operatorname{dist}(G, \partial\Omega')/2$, and fix $(x_0, t_0) \in G \times (0,T)$ and $R \in (0, R_0)$. Clearly, we have $Q_{2R} = B_{2R}(x_0) \times (t_0 - 4R^2, t_0) \subset \Omega' \times (-\infty, T)$. Let $\psi \in C^\infty(Q_{2R})$ denote a cut-off function appropriate to $Q_{2R} = B_{2R}(x_0) \times (t_0 - 4R^2, t_0)$ satisfying $\psi = 1$ on $Q_{3R/2}$, $\psi = 0$ in $\mathbb{R}^2 \setminus B_{2R} \times (-\infty, t_0 - 4R^2)$ and $|\nabla\psi| \le C/R$, $|\psi_t| \le C/R^2$.

For a. e. $t \in (t_0 - 4R^2, t_0)$ testing (4.1) by $\boldsymbol{\varphi} = (\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}})\psi^3 \chi_{(0,t)}$ yields

$$\begin{aligned}
&\int_{B_{2R}} \psi^3(t)|\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}}|^2 dx + 2\int_0^t \int_{B_{2R}} \psi^3 \boldsymbol{A}\boldsymbol{D}(\boldsymbol{w}) : \boldsymbol{D}(\boldsymbol{w}) dxds \\
&\quad = -6 \int_0^t \int_{B_{2R}} \psi^2 \boldsymbol{A}\boldsymbol{D}(\boldsymbol{w}) : (\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}) \otimes \nabla\psi dxds \\
&\qquad + 3 \int_0^t \int_{B_{2R}} \psi^2 |\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}|^2 \boldsymbol{u} \cdot \nabla\psi dxds \\
&\qquad + 2 \int_0^t \int_{B_{2R}} \psi^3 \boldsymbol{G} : \nabla\boldsymbol{w} + 3\psi^2 \boldsymbol{G} : (\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}) \otimes \nabla\psi dxds \\
&\qquad + 6 \int_0^t \int_{B_{2R}} \psi^2 \sigma(\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}) \cdot \nabla\psi dxds \\
&\qquad + 3 \int_0^t \int_{B_{2R}} \psi^2 |\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}|^2 \psi_t dxds.
\end{aligned}$$

For the sake of notational simplification in what follows we set $I_R = I_R(t_0) = (t_0 - R^2, t_0)$. By means of Korn's inequality observing

$$\int_{Q_{2R}} \psi^3 |\nabla\boldsymbol{w}|^2 dxdt \le c \int_{Q_{2R}} \psi^3 |\boldsymbol{D}(\boldsymbol{w})|^2 dxdt + \frac{c}{R^2} \int_{Q_{2R}} |\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}|^2 dxdt,$$

and by standard estimates employing Hölder's and Young's inequalities, and (1.3) one gets for $\epsilon > 0$

$$\begin{aligned}
&\|\psi^{3/2}(\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}})\|^2_{L^\infty(I_{2R};L^2(B_{2R}))} + \int_{Q_{2R}} \psi^3 |\nabla\boldsymbol{w}|^2 dxdt \\
&\quad \le \frac{c}{R^2}(1 + \varepsilon^{-1}) \int_{Q_{2R}} |\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}|^2 dxdt \\
&\qquad + \frac{c}{R} \Big(\int_{Q_{2R}} |\psi(\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}})|^{8/3} dxdt \Big)^{3/4} \\
&\qquad + c \int_{Q_{2R}} |\boldsymbol{G}|^2 dxdt + c\varepsilon \int_{Q_{2R}} |\sigma|^2 dxdt.
\end{aligned} \tag{4.3}$$

We estimate in turn all terms appearing on the right-hand side of (4.3). We begin our discussion by estimating the second term on the right-hand side of (4.3). First we make use of Hölder's inequality to obtain for almost all $t \in (t_0 - 4R^2, t_0)$

$$\|\psi(t)(\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}})\|^{8/3}_{8/3} \le \|\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}}\|^{4/3}_{L^2(B_{2R})} \|\psi(t)^2(\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}})\|^{4/3}_4. \tag{4.4}$$

On the other hand, by virtue of an inequality due to Galiardo-Nirenberg noting that

$$\forall\, z \in W^{1,2}(\mathbb{R}^2): \ \|z\|^2_4 \le \|\nabla z\|^{1/2}_2 \|z\|^{1/2}_2, \tag{4.5}$$

we see that

$$\begin{aligned}
&\|\psi(t)^2(\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}})\|^{4/3}_4 \\
&\le cR^{-2/3} \|\psi(t)(\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}})\|^{4/3}_{L^2(B_{2R})} + c\|\psi(t)^2(\boldsymbol{w}(t) - \boldsymbol{w}_{Q_{2R}})\|^{2/3}_2 \|\psi(t)^2 \nabla\boldsymbol{w}(t)\|^{2/3}_2.
\end{aligned}$$

Inserting this inequality into the right-hand side of (4.4), we find

$$\begin{aligned}
&\|\psi(t)(\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}})\|_{8/3}^{8/3}\\
&\quad\le cR^{-2/3}\|\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}}\|_{L^2(B_{2R})}^{4/3}\|\psi(t)(\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}})\|_2^{4/3}\\
&\qquad+c\|\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}}\|_{L^2(B_{2R})}^{4/3}\|\psi(t)^2(\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}})\|_2^{2/3}\|\psi^2\nabla\boldsymbol{w}(t)\|_2^{2/3}\\
&\quad\le cR^{-2/3}\|\psi(t)^{3/2}(\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}})\|_2^{2/3}\|\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}}\|_{L^2(B_{2R})}^{2}\\
&\qquad+c\|\psi(t)^{3/2}(\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}})\|_2^{2/3}\|\boldsymbol{w}(t)-\boldsymbol{w}_{Q_{2R}}\|_{L^2(B_{2R})}^{4/3}\|\psi(t)^{3/2}\nabla\boldsymbol{w}(t)\|_2^{2/3}.
\end{aligned}$$

Now, integration over (t_0-4R^2,t_0) yields

$$\begin{aligned}
&\frac{c}{R}\bigg(\int_{Q_{2R}}|\psi(\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}})|^{8/3}dxdt\bigg)^{3/4}\\
&\qquad\le\frac{c}{R^{3/2}}\|\psi^{3/2}(\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}})\|_{L^\infty(I_{2R};L^2(B_{2R}))}^{1/2}\bigg(\int_{Q_{2R}}|\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}}|^2dxdt\bigg)^{3/4}\\
&\qquad\quad+\frac{c}{R}\|\psi^{3/2}(\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}})\|_{L^\infty(I_{2R};L^2(B_{2R}))}^{1/2}\times\\
&\qquad\quad\times\bigg(\int_{Q_{2R}}|\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}}|^2dxdt\bigg)^{1/2}\bigg(\int_{Q_{2R}}\psi^3|\nabla\boldsymbol{w}|^2dxdt\bigg)^{1/4}.
\end{aligned}$$

Finally, by the aid of Young's inequality we obtain

$$\begin{aligned}
&\frac{c}{R}\bigg(\int_{Q_{2R}}|\psi(\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}})|^{8/3}dxdt\bigg)^{3/4}\\
&\qquad\le\frac{c}{R^2}\int_{Q_{2R}}|\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}}|^2dxdt+\frac12\|\psi^{3/2}(\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}})\|_{L^\infty(I_{2R};L^2(B_{2R}))}^{2}\\
&\qquad\quad+\frac12\int_{Q_{2R}}\psi^3|\nabla\boldsymbol{w}|^2dxdt.
\end{aligned}$$

Replacing the second integral of the right-hand side of (4.3) be the above estimate, we arrive at

$$\begin{aligned}
&\|\psi^{3/2}(\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}})\|_{L^\infty(I_{2R};L^2(B_{2R}))}^{2}+\int_{Q_{2R}}\psi^3|\nabla\boldsymbol{w}|^2dxdt\\
&\qquad\le\frac{c}{R^2}(1+\varepsilon^{-1})\int_{Q_{2R}}|\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}}|^2dxdt\\
&\qquad\quad+c\int_{Q_{2R}}|\boldsymbol{G}|^2dxdt+c\varepsilon\int_{Q_{2R}}|\sigma|^2dxdt.
\end{aligned}\tag{4.6}$$

In order to estimate the first term on the right-hand side of (4.6) we make use of the Poincaré-type inequality (B.2) for $n=2$ and $E=\frac43$ (see also Remark B.2). Accordingly, there holds

$$\begin{aligned}
\int_{Q_{2R}}|\boldsymbol{w}-\boldsymbol{w}_{Q_{2R}}|^2dxdt&\le cR^{4/3}\int_{I_{2R}}\bigg(\int_{B_{2R}}(|\nabla\boldsymbol{w}|+|\sigma|)^{3/2}dx\bigg)^{4/3}dt\\
&\quad+cR^2\int_{Q_{2R}}|\boldsymbol{G}|^2dxdt.
\end{aligned}\tag{4.7}$$

It remains to estimate the last integral on the right-hand side of (4.6) involving the pressure σ. For this purpose, we decompose the pressure σ on $Q_{3R/2}$ according to Lemma 2.3 into three parts, i.e. $\sigma = P_0 + P_1 + P_h$

$$\begin{aligned} \Delta P_0 &= -\operatorname{div}\operatorname{div}(\boldsymbol{u} \otimes (\boldsymbol{w} - \boldsymbol{w}_{B_{3R/2}})), \\ \Delta P_1 &= \operatorname{div}\operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{G}), \\ \Delta P_h &= 0. \end{aligned}$$

Here

$$P_1, P_2 \in L^2(0,T; A^2(B_{3R/2})), \quad P_h \in L^2(0,T; B^2(B_{3R/2})).$$

Lemma 2.3 together with (4.5) and Poincaré's inequality gives the estimates

$$\begin{aligned} \|P_0\|^2_{L^2(Q_{3R/2})} &\le \|\boldsymbol{u} \otimes (\boldsymbol{w} - \boldsymbol{w}_{B_{3R/2}})\|^2_{L^2(Q_{3R/2})} \\ &\le \|\boldsymbol{u}\|^2_{L^4(Q_{2R})} \|\boldsymbol{w} - \boldsymbol{w}_{B_{3R/2}}\|^2_{L^4(Q_{3R/2})}. \\ &\le c\|\boldsymbol{u}\|^2_{L^4(Q_{2R})} \|\psi^{3/2}(\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}})\|_{L^\infty(I_{2R}; L^2(B_{2R}))} \times \\ &\qquad \times \left(\int_{Q_{2R}} \psi^3 |\nabla \boldsymbol{w}|^2 dxdt \right)^{1/2}. \end{aligned}$$

Lemma 2.3 also implies

$$\|P_1\|^2_{L^2(Q_R)} \le c \int_{Q_{2R}} (\psi^3 |\nabla \boldsymbol{w}|^2 + |\boldsymbol{G}|^2) dxdt$$

and

$$\|P_h\|^2_{L^2(I_R; L^{3/2}(B_{3R/2}))} \le c\|\sigma\|^2_{L^2(I_R; L^{3/2}(B_{3R/2}))}, \tag{4.8}$$

which yields with help of the mean value property of harmonic functions

$$\begin{aligned} \|P_h\|^2_{L^2(Q_R)} &\le cR^{-2/3} \int_{I_R} \left(\int_{B_{3R/2}} |P_h|^{3/2} dx \right)^{4/3} dt \\ &\le cR^{-2/3} \int_{I_R} \left(\int_{B_{3R/2}} |\sigma|^{3/2} dx \right)^{4/3} dt. \end{aligned} \tag{4.9}$$

Thus,

$$\begin{aligned} \int_{Q_R} |\sigma|^2 dxdt \le c \int_{Q_{2R}} & \psi^3 |\nabla \boldsymbol{w}|^2 + |\boldsymbol{G}|^2 dxdt \\ & + cR^{-2/3} \int_{I_R} \left(\int_{B_{2R}} |\sigma|^{3/2} dx \right)^{4/3} dt \\ & + \frac{1}{2} \left\| \psi^{3/2}(\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}}) \right\|^2_{L^\infty(I_{2R}; L^2(B_{2R}))}. \end{aligned} \tag{4.10}$$

Furthermore, again making use of (4.5) it follows

$$\begin{aligned} \int_{Q_R} |\boldsymbol{w}|^4 dxdt \le c \Bigg\{ & R^{-2/3} \int_{I_R} \left(\int_{B_R} |\boldsymbol{w}|^3 dx \right)^{4/3} dt \\ & + \|\nabla \boldsymbol{w}\|^2_{L^2(Q_{2R})} \|\psi^{3/2}(\boldsymbol{w} - \boldsymbol{w}_{Q_{2R}})\|^2_{L^\infty(I_{2R}, L^2(B_{2R}))} \Bigg\}. \end{aligned} \tag{4.11}$$

Combining (4.6), (4.7), (4.10) and (4.11), we are lead to the estimate

$$
\begin{aligned}
&\int_{Q_R} (|\nabla \boldsymbol{w}| + |\sigma| + |\boldsymbol{w}|^2)^2 dxdt \\
&\le cR^{-2/3} \int_{I_{2R}(t_0)} \bigg(\int_{B_{2R}} (|\nabla \boldsymbol{w}| + |\sigma| + |\boldsymbol{w}|^2)^{3/2} dx \bigg)^{4/3} dt \qquad (4.12)\\
&\quad + c\varepsilon \int_{Q_{2R}} (|\nabla \boldsymbol{w}| + |\sigma| + |\boldsymbol{w}|^2)^2 dxdt + c\int_{Q_{2R}} |\boldsymbol{G}|^2 dxdt,
\end{aligned}
$$

where the constants $c > 0$ are independent of t_0, x_0 and R.

REMARK 4.1. Note that the first term on the right-hand side of (4.12) may be replaced by

$$
cR^{2-4/E} \int_{I_{2R}} \bigg(\int_{B_{2R}} (|\nabla \boldsymbol{w}| + |\sigma| + |\boldsymbol{w}|^2)^E dx \bigg)^{2/E} dt
$$

if we use in (4.7) parabolic Poincaré inequality (B.2) for $n = 2$, and in (4.9) Hölder's inequality with $E \in (1, 2)$.

Next, define $N := \Big[\dfrac{T}{R^2}\Big]$, and set

$$
\tau_m := T - mR^2, \quad m = 0, 1, \ldots, N.
$$

By this definition we have chosen a partition $T = \tau_0 > \tau_1 > \ldots > \tau_N \ge 0$ of the interval $[0, T]$ with $0 \le \tau_N \le R^2$ and $\tau_{j-1} - \tau_j = R^2$ for $j = 1, \ldots, N$. Replacing in (4.12) t_0 by τ_m and summing up these inequalities from $m = 0$ to $m = N$, and recalling that $\boldsymbol{w} = 0$ for $t < 0$, it shows that

$$
\begin{aligned}
\int_0^T \int_{B_R} h^2 dxdt \le cR^{-2/3} &\int_0^T \bigg(\int_{B_{2R}} h^{3/2} dx \bigg)^{4/3} dt \\
&+ c\varepsilon \int_0^T \int_{B_{2R}} h^2 dxdt + c\int_0^T \int_{B_{2R}} |\boldsymbol{G}|^2 dxdt,
\end{aligned}
$$

where

$$
h(x, t) := |\nabla \boldsymbol{w}(x, t)| + |\sigma(x, t)| + |\boldsymbol{w}(x, t)|^2, \quad (x, t) \in \Omega' \times (0, T).
$$

Now, using Fubini's theorem along with Minkowski's inequality [**18**, Appendix], we find

$$
\begin{aligned}
\int_{B_R} \int_0^T h^2 dtdx \le cR^{-2/3} &\bigg(\int_{B_{2R}} \bigg(\int_0^T h^2 dt \bigg)^{3/4} dx \bigg)^{4/3} \\
&+ c\varepsilon \int_{B_{2R}} \int_0^T h^2 dtdx + c\int_{B_{2R}} \int_0^T |\boldsymbol{G}|^2 dtdx.
\end{aligned}
$$

In the above inequality taking $\varepsilon := \frac{1}{2c}$, we deduce

$$
⨍_{B_R} U^2 dx \le c_1 \bigg(⨍_{B_{2R}} U^{3/2} dx \bigg)^{4/3} + \frac{1}{2} ⨍_{B_{2R}} U^2 dx + c_2 ⨍_{B_{2R}} F^2 dx,
$$

for all $x_0 \in G$ and $R \in (0, R_0)$, where

$$
U(x) = \bigg(\int_0^T h^2(x, t) dt \bigg)^{\frac{1}{2}}, \quad F(x) = \bigg(\int_0^T |\boldsymbol{G}(x, t)|^2 dt \bigg)^{\frac{1}{2}}, \quad x \in \Omega.
$$

Here the constants $c_1, c_2 > 0$ are independent of the choice of x_0 and $R \in (0, R_0)$. Thus, by virtue of the higher integrability result of Giaquinta-Modica (cf. [**7**, Theorem V.1.2]) based on Gehring's Lemma we get $q > 2$ depending only on c_1, such that if $\boldsymbol{F} \in L^r(G)$, i.e. $\boldsymbol{G} \in L^r(G, L^2(0,T))$, for a $r \in [2, q]$, then $U \in L^r_{loc}(G)$, i.e., $h \in L^r_{loc}(G, L^2(0,T))$. Here we want to emphasize that q indeed depends only on c_1 as it is seen from [**7**, Lemma V.1.2 and (V.1.6)], and it is not necessary to assume that $F \in L^{q_0}(G)$ for some $q_0 > 2$, compare [**7**, assumption on f below Theorem V.1.1]. Consequently, we have

$$|\boldsymbol{w}|^2, |\nabla \boldsymbol{w}| \in L^r_{loc}(G; L^2(0,T)). \tag{4.13}$$

Since the set G was arbitrary the statement (1.5) of Theorem 1.3 follows from the definition of $\boldsymbol{w}$.

5. Application: Heat conducting fluid

In this section we apply the results proved in previous section to system describing flow of Newtonian heat conducting fluid. Fluid in the domain Ω is described by unknown velocity $\boldsymbol{u} : Q \to \mathbb{R}^2$, pressure $p : Q \to \mathbb{R}$ and temperature $\theta : Q \to \mathbb{R}$. From conservation laws we have in Q the following system for these quantities

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0, \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div} \boldsymbol{\sigma} &= \operatorname{div} \boldsymbol{f}, \\ \theta_t + \boldsymbol{u} \cdot \nabla\theta + \operatorname{div} \boldsymbol{q} &= \boldsymbol{\sigma} : \boldsymbol{D}(\boldsymbol{u}) + g. \end{aligned} \tag{5.1}$$

We assume that the extra stress tensor and the heat flux have the form

$$\sigma = \mu(\theta)\boldsymbol{D}(\boldsymbol{u}) \text{ and } \boldsymbol{q} = -\kappa(\theta)\nabla\theta$$

and

$$\exists C_1, C_2 > 0, \forall \theta \in \mathbb{R} : \mu(\theta), \kappa(\theta) \in (C_1, C_2).$$

Existence of weak solution to such problem was shown in [**3**], [**2**] under suitable boundary and initial conditions and in more general setting. Here we are interested only in local weak solutions and so we do not need to introduce boundary and initial conditions.

Definition 5.1. We say that pair $(\boldsymbol{u}, \theta) : Q \to \mathbb{R}^2 \times \mathbb{R}$ is local weak solution to (5.1) if $\boldsymbol{u} \in L^\infty_{\mathrm{loc}}(0,T; L^2_{\mathrm{loc}}(\Omega))^2 \cap L^2_{\mathrm{loc}}(0,T; W^{1,2}_{\mathrm{loc}}(\Omega))^2$, with $\operatorname{div} \boldsymbol{u} = 0$ a.e. in Q, $\theta \in L^\infty_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(\Omega))$, and for all solenoidal vector functions $\boldsymbol{\varphi} \in C^\infty_0(Q)^2$ ($\operatorname{div} \boldsymbol{\varphi} = 0$) and scalar functions $\psi \in C^\infty_0(Q)$ holds

$$\begin{aligned} \int_Q -\boldsymbol{u} \cdot \boldsymbol{\varphi}_t + (\mu(\theta)\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{u} \otimes \boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{\varphi}) &= \int_Q f : \boldsymbol{D}(\boldsymbol{\varphi}), \\ \int_Q -\theta\psi_t - \theta\boldsymbol{u} \cdot \nabla\psi + \kappa(\theta)\nabla\theta \cdot \nabla\psi &= \int_Q \mu(\theta)|\boldsymbol{D}(\boldsymbol{u})|^2\psi + g\psi. \end{aligned} \tag{5.2}$$

Proof of Theorem 1.5. Local weak solution satisfies assumptions of Theorems 1.3 and Corollary 1.4 on $\Omega \times I$ where $I \Subset (0,T)$. This concludes the proof. □

Acknowledgements

Research of Petr Kaplický was supported by the grant GACR 201/09/0917, and partially also by the research project MSM 0021620839. Research of Jörg Wolf was supported by the Nečas Center for Mathematical Modeling, project LC06052, financed by MSMT of the Czech Republic.

Appendix A. Homogenous Sobolev spaces

By $\dot{W}^{1,q}(\mathbb{R}^n)$ we denote the space of all $f \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$, such that $\partial_i f \in L^q(\mathbb{R}^n)$ for all $i = 1, \ldots, n$, and $f_{B_1} = 0$. Then, compare variant of Poincaré's inequality in [**12**, Theorem 1.51 and Lemma 1.50],

$$\int_{B_R} |f|^q dx \le cR^q \|\nabla f\|_q^q \quad \forall\, f \in \dot{W}^{1,q}(\mathbb{R}^n), \quad \forall\, 1 \le R < +\infty. \tag{A.1}$$

The space $\dot{W}^{1,q}(\mathbb{R}^n)$ becomes a normed space, by setting

$$\|f\|_{\dot{W}^{1,q}} := \|\nabla f\|_q, \quad f \in \dot{W}^{1,q}(\mathbb{R}^n).$$

In fact, $\dot{W}^{1,q}(\mathbb{R}^n)$ is a Banach space, which can be easily proved by the aid of (A.1). Indeed, if $\{f_k\}$ is a Cauchy sequence in $\dot{W}^{1,q}(\mathbb{R}^n)$, from (A.1) it follows that $\{f_k|_{B_R}\}$ is a Cauchy sequence in $L^q(B_R)$ for all $0 < R < +\infty$. This yields a unique function $f \in L^q_{\rm loc}(\mathbb{R}^n)$ with $f_{B_1} = 0$ and

$$f_k \to f \quad \text{in} \quad L^q(B_R) \quad \text{as} \quad k \to +\infty, \quad \forall\, 0 < R < +\infty.$$

On the other hand, $\{\nabla f_k\}$ is a Cauchy sequence in $L^q(\mathbb{R}^n)$ which shows that $\nabla f \in L^q(\mathbb{R}^n)$. Thus, $f \in \dot{W}^{1,q}(\mathbb{R}^n)$ and $f_k \to f$ in $\dot{W}^{1,q}(\mathbb{R}^n)$ as $k \to +\infty$.

We have the following density result

Lemma A.1. *For every $f \in \dot{W}^{1,q}(\mathbb{R}^n)$ there exists $\varphi_k \in C^\infty_{\rm c}(\mathbb{R}^n)$ ($k \in \mathbb{N}$), such that*

$$\varphi_k - (\varphi_k)_{B_1} \to f \quad \textit{in} \quad \dot{W}^{1,q}(\mathbb{R}^n) \quad \textit{as} \quad k \to +\infty.$$

Proof. Let $\phi \in C^\infty_{\rm c}(B_2)$ with $\phi \equiv 1$ on B_1. Define $\phi_k(x) = \phi(x/k)$. We set

$$g_k = \phi_k(f - f_{B_{2k}\setminus B_k}) - \Big(\phi_k(f - f_{B_{2k}\setminus B_k})\Big)_{B_1}.$$

Then, $\nabla(g_k - f) = \nabla(g_k - (f - f_{B_{2k}\setminus B_k})) = (\phi_k - 1)\nabla f + \nabla\phi_k(f - f_{B_{2k}\setminus B_k})$. Thus, by employing the Poincaré inequality we estimate

$$\begin{aligned}\|\nabla(g_k - f)\|_q^q &\le 2^{q-1}\bigg(\int_{\mathbb{R}^n} |(1-\phi_k)\nabla f|^q dx + \int_{B_{2k}\setminus B_k} |\nabla\phi_k(f - f_{B_{2k}\setminus B_k})|^q dx\bigg)\\ &\le c\int_{\mathbb{R}^n\setminus B_k} |\nabla f|^q dx.\end{aligned}$$

Since the right-hand side converges to zero as $k \to +\infty$ we get

$$g_k \to f \quad \text{in} \quad \dot{W}^{1,q}(\mathbb{R}^n) \quad \text{as} \quad k \to +\infty.$$

Now, the assertion follows by using a standard mollification argument. □

With the help of Lemma A.1 we are able to show the following Korn's inequality

Lemma A.2. *Let $1 < q < +\infty$. There holds*

$$\int_{\mathbb{R}^n} |\nabla \boldsymbol{u}|^q \le C_{Korn} \int_{\mathbb{R}^n} |\boldsymbol{D}(\boldsymbol{u})|^q \quad \forall\, \boldsymbol{u} \in \dot{\boldsymbol{W}}^{1,q}(\mathbb{R}^n). \tag{A.2}$$

Proof. Let $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{\rm c}(\mathbb{R}^n)$. As $-\Delta\boldsymbol{\varphi} = -\operatorname{div}(2\boldsymbol{D}(\boldsymbol{\varphi}) - \boldsymbol{I}\operatorname{div}\boldsymbol{\varphi})$ we may use the Calderón-Zygmund inequality to get

$$\int_{\mathbb{R}^n} |\nabla\boldsymbol{\varphi}|^q \le C_{Korn} \int_{\mathbb{R}^n} |\boldsymbol{D}(\boldsymbol{\varphi})|^q.$$

Thus, (A.2) immediately follows by the aid of Lemma A.1. □

By $\boldsymbol{V}_{grad}$ we denote the space of all $\boldsymbol{w} \in \boldsymbol{L}^1_{\rm loc}$ such that

$$\int_{\mathbb{R}^n} \boldsymbol{w}\cdot\boldsymbol{\varphi} dx = 0 \quad \forall\,\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{{\rm c},\sigma}(\mathbb{R}^n).$$

Next we shall solve the divergence equation $\operatorname{div}\boldsymbol{u} = g$ in the homogeneous Sobolev space $\dot{\boldsymbol{W}}^{1,q}(\mathbb{R}^n)$ by the following

Theorem A.3. *For every $1 < q < +\infty$ there exists a unique bounded linear operator $\boldsymbol{B}_q : L^q(\mathbb{R}^n) \to \dot{\boldsymbol{W}}^{1,q}(\mathbb{R}^n)$, such that*

(i) $\operatorname{div}\boldsymbol{B}_q(f) = f$ *for all* $f \in L^q(\mathbb{R}^n)$;
(ii) $\boldsymbol{B}_q(f) \in \boldsymbol{V}_{grad}$ *for all* $f \in L^q(\mathbb{R}^n)$.

Proof. For $\varphi \in C^\infty_{\rm c}(\mathbb{R}^n)$ we define

$$\boldsymbol{B}_q(\varphi) := \nabla N * \varphi - (\nabla N * \varphi)_{B_1}, \quad i = 1,\dots,n,$$

where N stands for the Newton potential. By the well-known Calderón-Zygmund inequality we find

$$\|\nabla\boldsymbol{B}_q(\varphi)\|_q \le C_{CZ}\|\varphi\|_q.$$

Thus, $\boldsymbol{B}_q(\varphi) \in \dot{\boldsymbol{W}}^{1,q}(\mathbb{R}^n)$ fulfilling $\operatorname{div}\boldsymbol{B}_q(\varphi) = \varphi$. It is also readily seen that $\boldsymbol{B}_q(\varphi) \in \boldsymbol{V}_{grad}$.

Since $C^\infty_{\rm c}(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$, $\boldsymbol{B}_q$ can be extended to a bounded linear operator from $L^q(\mathbb{R}^n)$ into $\dot{\boldsymbol{W}}^{1,q}(\mathbb{R}^n)$ satisfying (i). To see (ii), for $f \in L^q(\mathbb{R}^n)$ we may choose a sequence $\{f_k\}$ in $C^\infty_{\rm c}(\mathbb{R}^n)$ such that $f_k \to f$ in $L^q(\mathbb{R}^n)$ as $k \to +\infty$. Employing (A.1) (see appendix below), we obtain $\boldsymbol{B}_q(f_k)|_{B_R} \to \boldsymbol{B}_q(f)|_{B_R}$ in $\boldsymbol{L}^q(B_R)$ for all $0 < R < +\infty$. This gives for any $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{{\rm c},\sigma}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \boldsymbol{B}_q(f)\cdot\boldsymbol{\varphi} dx = \lim_{k\to 0}\int_{\mathbb{R}^n} \boldsymbol{B}_q(f_k)\cdot\boldsymbol{\varphi} dx = 0. \tag{A.3}$$

Therefore, $\boldsymbol{B}_q(f) \in \boldsymbol{V}_{grad}$.

Uniqueness. Let $\boldsymbol{v} \in \dot{\boldsymbol{W}}^{1,q}(\mathbb{R}^n) \cap \boldsymbol{V}_{grad}$ such that $\operatorname{div}\boldsymbol{v} = f$. Set $\boldsymbol{w} = \boldsymbol{B}_q(f) - \boldsymbol{v}$. According to $\boldsymbol{w} \in \boldsymbol{V}_{grad}$ we get a function $\phi \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$ such that $\boldsymbol{w} = \nabla\phi$ a. e. in $\mathbb{R}^n$ (cf. Galdi [**6**, Lemma III.1.1, pp. 144]). Having $\operatorname{div}\boldsymbol{w} = 0$, it follows that $\Delta\phi = 0$. In particular, $\partial_i\partial_j\phi$ is harmonic for all $i, j = 1,\dots,n$. As $\partial_i\partial_j\phi \in L^q(\mathbb{R}^n)$ this function must vanish. Therefore, $\boldsymbol{w} = \nabla\phi$ is constant. Thanks to $\boldsymbol{w}_{B_1} = 0$ we get $\boldsymbol{w} \equiv \boldsymbol{0}$. □

Remark A.4. By the definition of $\boldsymbol{B}_q$ we see that

$$\boldsymbol{B}_q(f) = \boldsymbol{B}_r(f) \quad \forall\, f \in L^q \cap L^r(\mathbb{R}^n).$$

Thus, we may neglect the subscript q and write shortly $\boldsymbol{B}(f)$.

Appendix B. Parabolic Poincaré inequality

In this section we prove Poincaré-type inequality for solutions to the equation

$$\boldsymbol{w}_t + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{w}) - \operatorname{div}(\boldsymbol{A}\boldsymbol{D}(\boldsymbol{w})) = -\nabla\sigma - \operatorname{div}\boldsymbol{G} \quad \text{in} \quad \Omega'\times(0,T), \tag{B.1}$$

which is related to (4.1). Here $\Omega'\subset\mathbb{R}^n$, $n\ge 2$, denotes an open set.

Lemma B.1. *Let $\boldsymbol{G}\in L^2(\Omega'\times(0,T))^{n^2}$, $\boldsymbol{u}\in L^2(\Omega'\times(0,T))^n$ with $\nabla\cdot\boldsymbol{u}=0$ in the sense of distributions, and $\boldsymbol{w}\in L^\infty(0,T;L^2(\Omega'))^n\cap L^2(0,T;W^{1,2}(\Omega'))^n$, $\sigma\in L^2(\Omega'\times(0,T))$ satisfying* (B.1) *in the sense of distributions. Then for every $E\in((2n)/(n+2),2]$ there is a constant $c>0$ such that for all $Q_R=B_R(x_0)\times(t_0-R^2,t_0)\subset\Omega'\times(0,T)$ it holds*

$$\int_{Q_R}|\boldsymbol{w}-\boldsymbol{w}_{Q_R}|^2 \le c(1+\|u\|_2^2)R^{n+2-\frac{2n}{E}}\int_{t_0-R^2}^{t_0}\left(\int_{B_R}(|\nabla\boldsymbol{w}|+|\sigma|)^E dx\right)^{2/E} dt + cR^2\int_{Q_R}|\boldsymbol{G}|^2. \tag{B.2}$$

Proof. Setting $\boldsymbol{H}=-\boldsymbol{A}\boldsymbol{D}(\boldsymbol{w})+\boldsymbol{G}+\boldsymbol{u}\otimes(\boldsymbol{w}-\boldsymbol{w}_{B_R})+\boldsymbol{I}\sigma$, the equation (B.1) becomes

$$\boldsymbol{w}_t = -\operatorname{div}\boldsymbol{H} \quad \text{in} \quad \Omega'\times(0,T). \tag{B.3}$$

Let $Q_R=B_R(x_0)\times(t_0-R^2,t_0)\subset\Omega'\times(0,T)$ be arbitrarily chosen. We proceed as in [**14**, Section 4]. First we find a cut-off function $\eta\in C_0^\infty(B_R)$ satisfying $0\le\eta\ge\le 1$, $\eta=1$ on $B_{R/2}$, and $\nabla\eta\le cR^{-1}$. Define,

$$\tilde{\boldsymbol{w}}_R(t)=\frac{1}{\int_{B_R}\eta(x)dx}\int_{B_R}\boldsymbol{w}(x,t)\eta(x)dx, \quad t\in(0,T).$$

Following [**14**, (4.4)], using Sobolev-Poincaré's inequality, we find for each $E\in(2n/(n+2),2)$ a constant $c>0$ such that

$$\int_{Q_R}|\boldsymbol{w}-\boldsymbol{w}_{Q_R}|^2 \le cR^{n+2-\frac{2n}{E}}\int_{t_0-R^2}^{t_0}\|\nabla\boldsymbol{w}(t)\|_E^2\,dt + cR^{n-2}\int_{t_0-R^2}^{t_0}\int_{t_0-R^2}^{t_0}|\tilde{\boldsymbol{w}}_R(t)-\tilde{\boldsymbol{w}}_R(s)|^2 dtds. \tag{B.4}$$

It remains to estimate the last integral in (B.4). For this the equation (B.3) is used. Again repeating steps from [**14**, Theorem 4.1] together with Hölder's inequality and

Sobolev-Poincaré's inequality, we obtain

$$
\begin{aligned}
&|\tilde{\boldsymbol{w}}(t)-\tilde{\boldsymbol{w}}(s)|^2 \le cR^{-2n-2}\left(\int_{Q_R}|\boldsymbol{H}|\right)^2\\
&\le cR^{-\frac{2n}{E}}\int_{t_0-R}^{t_0}\|\boldsymbol{H}(t)\|_E^2 dt\\
&\le cR^{-\frac{2n}{E}}\int_{t_0-R}^{t_0}\|\boldsymbol{D}(\boldsymbol{w}(t))\|_E^2+\|\sigma(t)\|_E^2 dt\\
&\qquad + cR^{-n}\int_{t_0-R}^{t_0}\|\boldsymbol{G}(t))\|_2^2 dt + cR^{-2n-2}\|u\|_2^2\int_{t_0-R}^{t_0}\|\boldsymbol{w}(t)-\boldsymbol{w}_{B_R}(t)\|_2^2 dt\\
&\le cR^{-\frac{2n}{E}}\int_{t_0-R}^{t_0}\|\boldsymbol{D}(\boldsymbol{w}(t))\|_E^2+\|\sigma(t)\|_E^2 dt\\
&\qquad + cR^{-n}\int_{t_0-R}^{t_0}\|\boldsymbol{G}(t))\|_2^2 dt + cR^{-\frac{2n}{E}}\|u\|_2^2\int_{t_0-R^2}^{t_0}\|\nabla\boldsymbol{w}(t)\|_E^2\, dt.
\end{aligned}
$$

Inserting this estimate into the last integral on the right-hand side of (B.4), we complete the proof. □

Remark B.2. If $n=2$ we require $E\in(1,2]$. In particularly, $E=3/2$ satisfies the assumptions.

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