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Regularity criteria to the axisymmetric incompressible Magneto-hydrodynamics equations

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ABSTRACT. In this paper, we investigate the regularity criteria of axisymmetric solutions to the incompressible MHD equations, which have the form $u = u_r e_r + u_\theta e_\theta + u_z e_z$ and $b = b_\theta e_\theta$. Through establishing some *innovative* estimates, we obtain some new regularity criteria that are scaling invariant and independent of b_θ. To some extent, our work can be seen as a generation of the result by D. Chae and J. Lee [**9**] on the axisymmetric incompressible Navier-Stokes equations.

CONTENTS

1. Introduction and main results

In this paper, we are concerned with the 3D incompressible Magneto-hydrodynamics (MHD) equations

$$(1.1)\qquad \begin{cases} u_t - \Delta u + u\cdot\nabla u = -\nabla(p + \frac{1}{2}|b|^2) + b\cdot\nabla b, \\ b_t - \Delta b + u\cdot\nabla b = b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases}$$

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for $t > 0$ and $x \in \mathbb{R}^3$, where u, b represent the velocity field and the magnetic field respectively, p is a scalar pressure. In the following context, for simplicity, we denote $\pi = p + \frac{1}{2}|b|^2$. It should be noted that, if $b = 0$, (1.1) becomes the 3D incompressible Navier-Stokes equations.

It is well known that, similar to the Navier-Stokes equations, there exists globally in time smooth (strong) solution to the 2D incompressible MHD equations (see Sermange-Teman [**37**]). Also similar to the Leray-Hopf weak solutions of 3D incompressible Navier-Stokes equations, Duvaut-Lions [**13**] constructed a class of global weak solutions $(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$ which solves (1.1) in the sense of distribution. However, the uniqueness, regularity and continuous dependence on the initial data of weak solutions are still open. Naturally, given sufficient smooth and divergence free initial data, it is still unknown whether the smooth solution of 3D incompressible MHD equations exists for all time. To the best of our acknowledge, the unique strong solution $(u, b) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))$ is only local in general.

Starting with the pioneering work of Serrin [**38**], a lot of investigations are dedicated to providing sufficient conditions for the global regularity of 3D incompressible Navier-Stokes equations which states, roughly speaking, that a smooth solution u exists on the time interval $[0, T]$ as long as

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le 1, \quad \text{for} \quad q \ge 3.$$

This kind of sufficient condition is called as Serrin-type condition. Later, this result was generalized to the 3D incompressible MHD equations by He-Xin [**16**] without imposing any assumptions on the magnetic field. Most recently, there have been some progresses along this line, see [**1, 2, 7, 3, 14, 15, 18, 20, 24**] for example. In addition, some other sufficient conditions, which are in terms of one component of the velocity field only, see [**6, 26, 31, 35, 43**] and the references therein.

To solve this challenging problem for the 3D incompressible Navier-Stokes equations, many mathematicians attribute to study the case with certain symmetry assumptions which make the 3D flow close to 2D flow. A typical situation is the axisymmetric flow, that is, all velocity components (radial, angular (or swirl) and z-component) as well as the pressure are independent of the angular variable in the cylindrical coordinates. Even with this assumption, the global regularity is still widely open. However, if the swirl component of the velocity field, u_θ, is trivial, independently, Ladyzhenskaya [**27**], Ukhovskii-Yudovich [**39**] proved that the weak solutions are regular for all time (see also [**30**]). This result indicates that for the problem of global regularity, u_θ plays the crucial role.

Afterwards, in order to understand this problem better, many mathematicians are devoted to looking for suitable regularity criteria. Chae-Lee [**9**] proved that the weak solutions are smooth if there holds

$$u_r \in L^p(0, T; L^q(\Omega_\delta)), \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le 1, \quad \text{for} \quad q > 3,$$

or

$$\frac{u_r}{r} \in L^p(0, T; L^q(\Omega_\delta)), \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le 2, \quad \text{for} \quad q > \frac{3}{2},$$

where $\Omega_\delta = \{(x,y,z) \in \mathbb{R}^3 | \sqrt{x^2+y^2} < \delta\}$ is a thin cylinder with infinite height. The key point lies in that the authors established a new regularity criterion, namely,

$$w_\theta \in L^p(0,T;L^q(\Omega_\delta)) \quad \text{with} \quad \frac{2}{p}+\frac{3}{q} \le 2 \quad \text{for} \quad q > \frac{3}{2}.$$

It should be noted that the first regularity criterion doesn't work for the critical case $q = 3$. Regarding this case, in [**42**], the author obtained a sufficient condition that $\|u_r\|_{L^\infty(0,T;L^3(\Omega_{\text{loc}}))}$ is small other then bounded only. Recently, Chen-Fang-Zhang [**4**], Lei-Zhang [**29**], Wei [**40**] proved that the solution will be regular provided that $\|ru_\theta(r,z,t)\|_{L^\infty}$ is small near the symmetry axis or $\|ru_\theta(r,z,0)\|_{L^\infty}$ is small. In particular, it is shown in [**40**] that the global regularity holds true if $|ru_\theta(r,z,t)| \le |\ln r|^{-\frac{3}{2}}$ for any $0 < r \le \delta_0 \in (0, \frac{1}{2})$.

In present paper, we are dedicated to establishing some regularity criteria for a class of axisymmetric solutions to (1.1), which have the form $u = u_r e_r + u_\theta e_\theta + u_z e_z$ and $b = b_\theta e_\theta$. It is shown in this paper that the weak solutions will become smooth as long as one of following four conditions holds:

$$\text{(I)} \qquad u_r \in L^p(0,T;L^q(\Omega_\delta)), \quad \text{with} \quad \frac{2}{p}+\frac{3}{q} \le 1 \quad \text{for} \quad q > 3,$$

$$\text{(II)} \qquad \|u_r\|_{L^\infty(0,T;L^3(\Omega_\delta))} \le \frac{1}{a_0},$$

$$\text{(III)} \qquad \frac{u_r}{r} \in L^p(0,T;L^q(\Omega_\delta)), \quad \text{with} \quad \frac{2}{p}+\frac{3}{q} \le 2 \quad \text{for} \quad q > \frac{3}{2},$$

$$\text{(IV)} \qquad \|\frac{u_r}{r}\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega_\delta))} \le \frac{1}{2a_0^2},$$

where a_0 is an absolute constant in the imbedding inequality (5.1).

It should be remarked that above regularity criteria are independent of b_θ, and therefore our result can be thought as a generation of Chae-Lee's result by assuming $b_\theta \equiv 0$. Actually, in comparison with the axisymmetric incompressible Navier-Stokes equations, the presence of b_θ will bring up more stronger coupling in the nonlinearities. To overcome this difficulty, we firstly make a sensitive observation to the structure of this model. On this basis, we further make full use of its structure and then work on some delicate estimates. As a result, we establish some new estimates, which are the estimates of $\|rb_\theta\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}$ and $\|r^2 w_\theta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}$. Especially, the estimate of $\|r^2 w_\theta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}$ is also new even for the axisymmetric incompressible Navier-Stokes equations. In one word, the value of these estimates not only lies in a new discovery to this model, but also play an important role in getting the final conclusion.

Before showing our main results, we would like to introduce the notations and conventions used in the remainder of this article. We denote by

$$\int \cdot \, dx = \int_{\mathbb{R}^3} \cdot \, dx,$$

and the standard Sobolev spaces by

$$L^p = L^p(\mathbb{R}^3), \; W^{k,p} = W^{k,p}(\mathbb{R}^3), \; H^k = H^{k,2}(\mathbb{R}^3)$$

for $1 \le p \le \infty$ and integer $k \ge 0$. Now, we are in the position to state the main theorems of our paper.

THEOREM 1.1. *Let (u, b) be an axisymmetric weak solution of incompressible MHD equations* (1.1) *with $b_0^r = b_0^z = 0$ and $(u_0, b_0) \in H^1(\mathbb{R}^3)$. Then it is smooth in $[0, T) \times \mathbb{R}^3$, if it holds that*

$$w_\theta \in L^p(0, T; L^q(\mathbb{R}^3)),$$

where $\frac{2}{p} + \frac{3}{q} \leq 2$ with $q > \frac{3}{2}$.

THEOREM 1.2. *If (u, b) is an axisymmetric smooth solution of incompressible MHD equations* (1.1) *with $b_0^r = b_0^z = 0$, $(u_0, b_0) \in L^2(\mathbb{R}^3)$, $r^2 w_0^\theta \in L^2(\mathbb{R}^3)$ and $(ru_0^\theta, rb_0^\theta) \in L^3(\mathbb{R}^3)$, then there holds that $r^2 w_\theta \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$.*

REMARK 1.3. The above theorem is not only a new estimate, but also plays an important role in the proof of Theorem 1.4.

THEOREM 1.4. *Assume that (u, b) is an axisymmetric weak solution of MHD equations with $b_0^r = b_0^z = 0, (u_0, b_0) \in H^1(\mathbb{R}^3), r^2 w_0^\theta \in L^2(\mathbb{R}^3), ru_0^\theta \in L^3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $(rb_0^\theta, \frac{b_0^\theta}{r}) \in L^3(\mathbb{R}^3)$. Then it is smooth in $[0, T) \times \mathbb{R}^3$, if one of the following conditions holds*

$$u_r \in L^p(0, T; L^q(\Omega_\delta)), \text{ where } \frac{2}{p} + \frac{3}{q} \leq 1 \text{ with } q > 3, \tag{1.2}$$

$$\|u_r\|_{L^\infty(0,T;L^3(\Omega_\delta))} \leq \frac{1}{a_0}, \tag{1.3}$$

$$\frac{u_r}{r} \in L^p(0, T; L^q(\Omega_\delta)), \text{ where } \frac{2}{p} + \frac{3}{q} \leq 2 \text{ with } q > \frac{3}{2}, \tag{1.4}$$

$$\|\frac{u_r}{r}\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega_\delta))} \leq \frac{1}{2a_0^2}, \tag{1.5}$$

where a_0 is an absolute constant in the imbedding inequality (5.1).

REMARK 1.5. Theorem 1.4 can be thought as a generation of Theorem 4 in [**9**] by assuming $b_\theta \equiv 0$.

REMARK 1.6. Throughout this paper, $\Omega_\delta = \{(x, y, z) \in \mathbb{R}^3 | \sqrt{x^2 + y^2} < \delta\}$ denotes a thin cylinder with radius $\delta > 0$ fixed and infinite height.

This paper is organized as follows. In Section 2, we introduce some notations and technical lemmas used for the proof of main theorems. In Section 3, we will prove Theorem 1.1. Section 4 and Section 5 are devoted to the proof of Theorem 1.2 and 1.4 respectively.

2. Preliminary

2.1. Notations.

To begin with, we would like to introduce the definition of 3D axisymmetric flow.

DEFINITION 2.1. [**Axisymmetric flow**] A vecor field $u(x,t)$ is called axisymmetric if it can be described as the form of

$$u(x,t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z \tag{2.1}$$

in the cylindrical coordinates, where $e_r = (\cos\theta, \sin\theta, 0)$, $e_\theta = (-\sin\theta, \cos\theta, 0)$, $e_z = (0,0,1)$. We call the velocity components $u_r(r,z,t)$, $u_\theta(r,z,t)$, $u_z(r,z,t)$ as radial, swirl and z-component respectively. In the following context, we will denote the components by u_r, u_θ and u_z for simplicity.

Subsequently, we get to set up the equations satisfied by the components of velocity. First of all, by recalling Definition 2.1, it is clear that the components of velocity and pressure are independent of the angular variable θ. Except that, in the cylindrical coordinates, the gradient operator and laplacian operator has the expression $\nabla = e_r\partial_r + \frac{1}{r}e_\theta\partial_\theta + e_z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$ respectively. Therefore, for the axisymmetric solutions with the form $u = u_r e_r + u_\theta e_\theta + u_z e_z$ and $b = b_\theta e_\theta$, we can rewrite (1.1) as

$$\begin{cases} \partial_t u_r - (\tilde{\Delta} - \dfrac{1}{r^2})u_r + \tilde{u}\cdot\tilde{\nabla}u_r = -\partial_r\pi + \dfrac{u_\theta^2}{r} - \dfrac{b_\theta^2}{r}, \\ \partial_t u_\theta - (\tilde{\Delta} - \dfrac{1}{r^2})u_\theta + \tilde{u}\cdot\tilde{\nabla}u_\theta = -\dfrac{u_\theta u_r}{r}, \\ \partial_t u_z - \tilde{\Delta}u_z + \tilde{u}\cdot\tilde{\nabla}u_z = -\partial_z\pi, \\ \partial_t b_\theta - (\tilde{\Delta} - \dfrac{1}{r^2})b_\theta + \tilde{u}\cdot\tilde{\nabla}b_\theta = \dfrac{u_r b_\theta}{r}, \\ \partial_r u_r + \dfrac{u_r}{r} + \partial_z u_z = 0, \end{cases} \tag{2.2}$$

where $\tilde{u} = u_r e_r + u_z e_z$, $\tilde{\nabla} = e_r\partial_r + e_z\partial_z$ and $\tilde{\Delta} = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$.

REMARK 2.2. It should be noted that for any function $f(r,z,t)$ which is independent of the angular variable θ (such as u_r, u_θ, u_z), it clearly holds that $\|\nabla f\|_{L^2(\mathbb{R}^3)} = \|\tilde{\nabla} f\|_{L^2(\mathbb{R}^3)}$.

Similarly, in the cylindrical coordinates, the vorticity $w = \nabla\times u$ and $j = \nabla\times b$ can be expressed as

$$w = w_r e_r + w_\theta e_\theta + w_z e_z, \tag{2.3}$$

and

$$j = j_r e_r + j_z e_z, \tag{2.4}$$

where $w_r = -\partial_z u_\theta$, $w_\theta = \partial_z u_r - \partial_r u_z$, $w_z = \frac{1}{r}\partial_r(ru_\theta)$ and $j_r = -\partial_z b_\theta$, $j_z = \frac{1}{r}\partial_r(rb_\theta)$. As a result, through some basic calculations, we can reduce the equations satisfied by the components of vorticity, that is

$$(2.5)\qquad \begin{cases} \partial_t w_r - (\tilde{\Delta} - \frac{1}{r^2})w_r + \tilde{u}\cdot\tilde{\nabla}w_r = (w_r\partial_r + w_z\partial_z)u_r, \\ \partial_t w_\theta - (\tilde{\Delta} - \frac{1}{r^2})w_\theta + \tilde{u}\cdot\tilde{\nabla}w_\theta = \frac{u_r w_\theta}{r} + \frac{\partial_z(u_\theta^2 - b_\theta^2)}{r}, \\ \partial_t w_z - \tilde{\Delta}w_z + \tilde{u}\cdot\tilde{\nabla}w_z = (w_r\partial_r + w_z\partial_z)u_z, \\ \partial_t j_r - (\tilde{\Delta} - \frac{1}{r^2})j_r + \tilde{u}\cdot\tilde{\nabla}j_r = (\partial_z u_r\partial_r b_\theta + \partial_z u_z\partial_z b_\theta) - \frac{1}{r}\partial_z(u_r b_\theta), \\ \partial_t j_z - \tilde{\Delta}j_z + \tilde{u}\cdot\tilde{\nabla}j_z = \tilde{u}\cdot\tilde{\nabla}(\frac{b_\theta}{r}) + \frac{1}{r}\partial_r(u_r b_\theta) - \frac{1}{r}\partial_r(r\tilde{u})\cdot\tilde{\nabla}b_\theta. \end{cases}$$

2.2. Some a priori estimates.

In this subsection, we will first introduce some useful estimates in this paper. To begin with, we would like to recall the well-known relation between the velocity and vorticity by [**32**], namely

$$\nabla u(x) = \mathcal{P}w(x) + Cw(x),$$

where $\mathcal{P}$ is a singular integral operator of Calderon-Zygmund type. Thanks to Calderon-Zygmund inequality [**23**], the following lemma holds true.

Lemma 2.3. [**8**] *Let $u \in W^{1,p}(\mathbb{R}^3)$ be a velocity field with its divergence free and vorticity w, then the inequality*

$$(2.6)\qquad \|\nabla u\|_{L^p(\mathbb{R}^3)} \le C_p\|w\|_{L^p(\mathbb{R}^3)}$$

holds for any $p \in (1,\infty)$, where the constant C_p depends only on p.

As a special fluid, the 3D axisymmetric flow also have one particular property, which are shown as follows.

Lemma 2.4. [**9**] *Suppose that $u = u(r,z) \in W^{1,p}(\mathbb{R}^3)$ is an axisymmetric field with zero divergence, then there holds*

$$(2.7)\qquad \|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)} \le C_p\|w_\theta\|_{L^p(\mathbb{R}^3)},\ \ \forall p \in (1,\infty),$$

where the constant C_p is an absolute value depending only on p.

To the end, it is necessary to introduce some closed estimates for the single component of velocity and magnetic fields. The acquirement of these estimates is due to the special structure of model (2.2), which play the fundamental role in the proof of main theorems.

Lemma 2.5. *If (u,b) solves the system (1.1) with $u = u_r e_r + u_\theta e_\theta + u_z e_z$ and $b = b_\theta e_\theta$, then the inequality*

$$(2.8)\qquad \|ru_\theta\|_{L^p(\mathbb{R}^3)} \le C_1$$

holds for any $p \in [2,\infty]$, where $C_1 = C(\|ru_0^\theta\|_{L^p(\mathbb{R}^3)})$.

Proof. According to $(2.2)^2$ and some basic calculations, it is clear that the quantity ru_θ solves the equation

$$(2.9)\qquad \partial_t(ru_\theta) + \tilde{u}\cdot\tilde{\nabla}(ru_\theta) - (\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2)(ru_\theta) + \frac{2}{r}\partial_r(ru_\theta) = 0.$$

Then by multiplying $|ru_\theta|^{p-2}(ru_\theta)$ on both sides of (2.9) and then integrating on $\mathbb{R}^3$, it follows that

$$\frac{1}{p}\frac{d}{dt}\|ru_\theta\|_{L^p}^p - \int |ru_\theta|^{p-2}ru_\theta(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2)ru_\theta dx \tag{2.10}$$
$$= -\frac{1}{p}\int (u_r\partial_r + u_z\partial_z)|ru_\theta|^p dx - \int \frac{2}{r}\partial_r(ru_\theta)|ru_\theta|^{p-2}(ru_\theta)dx.$$

Next, thanks integrating by part and the fact that $dx = 2\pi r dr dz$, we conclude

$$-\int |ru_\theta|^{p-2}ru_\theta(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2)ru_\theta dx = \frac{4(p-1)}{p^2}\|\nabla|ru_\theta|^{\frac{p}{2}}\|_{L^2}^2,$$

$$-\frac{1}{p}\int (u_r\partial_r + u_z\partial_z)|\Omega|^p dx = 0,$$

and

$$-\int \frac{2}{r}\partial_r(ru_\theta)|ru_\theta|^{p-2}(ru_\theta)dx = -\frac{4\pi}{p}\int_{-\infty}^{\infty}\int_0^{\infty}\partial_r|ru_\theta|^p drdz = 0.$$

which together with (2.10) yields, after applying Gronwall's inequality, that for all $p \in (2, \infty)$,

$$\|ru_\theta\|_{L^p}^p + \frac{4(p-1)}{p}\int_0^T \|\nabla|ru_\theta|^{\frac{p}{2}}\|_{L^2}^2 dt \le \|ru_0^\theta\|_{L^p}^p. \tag{2.11}$$

Regarding for the case $p = \infty$, through dropping up the second term in the left hand of (2.11) and letting $p \to \infty$, we then derive

$$\|ru_\theta\|_{L^\infty} \le \|ru_0^\theta\|_{L^\infty}.$$

Thus, we finish all the proof. □

Lemma 2.6. *Assume that (u, b) with the form of $u = u_r e_r + u_\theta e_\theta + u_z e_z$ and $b = b_\theta e_\theta$ is a smooth solution of the system (1.1), then there holds*

$$\|\frac{b_\theta}{r}\|_{L^p(\mathbb{R}^3)} \le C_2, \quad \forall p \in [2, \infty], \tag{2.12}$$

where $C_2 = C(\|\frac{b_0^\theta}{r}\|_{L^p(\mathbb{R}^3)})$.

Proof. Through defining $\Omega = \frac{b_\theta}{r}$, recalling $(2.2)^4$ and some basic calculations, it follows that Ω satisfies the equation

$$\partial_t\Omega + \tilde{u}\cdot\tilde{\nabla}\Omega - (\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\Omega = 0. \tag{2.13}$$

Subsequently, by multiplying $|\Omega|^{p-2}\Omega$ on both sides of (2.13) and then integrating on $\mathbb{R}^3$, one has

$$\frac{1}{p}\frac{d}{dt}\|\Omega\|_{L^p}^p - \int |\Omega|^{p-2}\Omega(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\Omega dx \tag{2.14}$$
$$= -\frac{1}{p}\int (u_r\partial_r + u_z\partial_z)|\Omega|^p dx.$$

Then, by integrating by parts, the incompressible condition $(2.2)^5$ and the fact $dx = 2\pi r dr dz$, it follows that

$$-\frac{1}{p}\int (u_r\partial_r + u_z\partial_z)|\Omega|^p dx = 0$$

and

$$-\int |\Omega|^{p-2}\Omega(\partial_r^2+\frac{3}{r}\partial_r+\partial_z^2)\Omega dx=\frac{4(p-1)}{p^2}\|\nabla|\Omega|^{\frac{p}{2}}\|_{L^2}^2+\frac{4\pi}{p^2}\int|\Omega(t,0,z)|^p dz\geq 0,$$

which yields, after employing (2.14) and Gronwall's inequality, that

$$\|\Omega(t)\|_{L^p}\leq\|\Omega_0\|_{L^p},\quad \forall\, p\in(2,\infty). \tag{2.15}$$

As for the case $p=\infty$, it suffices to let $p\to\infty$ in (2.14), and then one can get

$$\|\Omega(t)\|_{L^\infty}\leq\|\Omega_0\|_{L^\infty}. \tag{2.16}$$

In the end, by combining the estimates (2.15) and (2.16), we can conclude the conclusion, that is,

$$\|\frac{b_\theta}{r}\|_{L^p}\leq\|\frac{b_0}{r}\|_{L^p},\quad \forall\, p\in[2,\infty].$$

□

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, it is *essential* to introduce the following Serrin-type criterion for the MHD system (1.1).

LEMMA 3.1. [**16**] *Suppose* $(u_0,\, b_0)\in H^1(\mathbb{R}^3)$*, then any Leray-Hopf weak solution of incompressible MHD equations* (1.1) *is smooth if there holds that*

$$u\in L^p(0,T;L^q(\mathbb{R}^3)), \tag{3.1}$$

where $\frac{2}{p}+\frac{3}{q}\leq 1$ *with* $q>3$.

The following is standard energy estimates, which holds for any weak solutions of system (1.1).

PROPOSITION 3.1. [**Energy estimates**] If (u,b) solves the system (1.1), then the following energy estimates

$$\|u\|_{L^2(\mathbb{R}^3)}^2+\|b\|_{L^2(\mathbb{R}^3)}^2+\int_0^T(\|\nabla u\|_{L^2(\mathbb{R}^3)}^2+\|\nabla b\|_{L^2(\mathbb{R}^3)}^2)dt\leq\ C \tag{3.2}$$

hold, where the constant C depends only on $\|u_0\|_{L^2(\mathbb{R}^3)}$ and $\|b_0\|_{L^2(\mathbb{R}^3)}$.

Thanks to the regularity criterion in Lemma 3.1 and energy estimates (3.2), to prove Theorem 1.1, it suffices to verify (3.1).

Proof of Theorem 1.1. In this part, our strategy is to verify the estimate of $L^\infty(0,T;H^1(\mathbb{R}^3))$ for velocity field. Therefore, according to Lemma 2.3, it is sufficient to do the estimate of $L^\infty(0,T;L^2(\mathbb{R}^3))$ for its vorticity w. Recalling (2.3) and (2.4), the vorticity w and j have the forms of $w=\nabla\times u=w_re_r+w_\theta e_\theta+w_ze_z$ and $j=\nabla\times b=j_re_r+j_ze_z$ respectively. Then, by multiplying each equation in (2.5) with w_r,w_θ,w_z,j_r,j_z respectively and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2+\|j\|_{L^2}^2)+\|\nabla w\|_{L^2}^2+\|\nabla j\|_{L^2}^2\\
=\ &\int(w_r\partial_r+w_z\partial_z)u_rw_rdx+\int\frac{w_\theta^2u_r}{r}dx+\int\frac{w_\theta}{r}\partial_z(u_\theta^2-b_\theta^2)dx,\\
&+\int(w_r\partial_r+w_z\partial_z)u_zw_zdx+\int Ej_rdx+\int Fj_zdx\\
=\ &I+II+III+IV+V+VI,
\end{aligned}$$

where $E = (\partial_z u_r \partial_r b_\theta + \partial_z u_z \partial_z b_\theta) - \frac{1}{r}\partial_z(u_r b_\theta)$ and $F = \tilde{u} \cdot \tilde{\nabla}(\frac{b_\theta}{r}) + \frac{1}{r}\partial_r(u_r b_\theta) - \frac{1}{r}\partial_r(r\tilde{u}) \cdot \tilde{\nabla} b_\theta$.

To estimate above terms, we can apply Hölder inequality, Gagliardo-Nirenberg inequality, Lemma 2.4 and Young inequality to derive

$$\begin{aligned} I + IV &= \int w_r \partial_r u_r w_r dx + \int w_z \partial_z u_r w_r dx \\ &\quad + \int w_r \partial_r u_r w_z dx + \int w_z \partial_z u_r w_z dx \\ &\leq \|w_\theta\|_{L^q} \|w\|^2_{L^{\frac{2q}{q-1}}} \\ &\leq \|w_\theta\|_{L^q} \|\nabla w\|^{\frac{3}{q}}_{L^2} \|w\|^{\frac{2q-3}{q}}_{L^2} \\ &\leq \frac{1}{10}\|\nabla w\|^2_{L^2} + C\|w_\theta\|^{\frac{2q}{2q-3}}_{L^q} \|w\|^2_{L^2}, \end{aligned}$$

$$\begin{aligned} II &= \int \frac{w_\theta^2 u_r}{r} dx \\ &\leq \|w_\theta\|_{L^q} \|w\|^2_{L^{\frac{2q}{q-1}}} \\ &\leq \|w_\theta\|_{L^q} \|\nabla w\|^{\frac{3}{q}}_{L^2} \|w\|^{\frac{2q-3}{q}}_{L^2} \\ &\leq \frac{1}{10}\|\nabla w\|^2_{L^2} + C\|w_\theta\|^{\frac{2q}{2q-3}}_{L^q} \|w\|^2_{L^2}, \end{aligned}$$

$$\begin{aligned} III &= \int \frac{u_\theta}{r} \partial_z u_\theta w_\theta dx - \int \frac{b_\theta}{r} \partial_z u_\theta w_\theta dx \\ &= -\int (w_z - \partial_r u_\theta) w_r w_\theta dx + \int (j_z - \partial_r b_\theta) j_r w_\theta dx \\ &= \int \partial_r u_\theta w_r w_\theta dx - \int w_z w_r w_\theta dx + \int j_z j_r w_\theta dx - \int \partial_r b_\theta j_r w_\theta dx \\ &\leq \|w_\theta\|_{L^q} (\|w\|^2_{L^{\frac{2q}{q-1}}} + \|j\|^2_{L^{\frac{2q}{q-1}}}) \\ &\leq \frac{1}{10}(\|\nabla w\|^2_{L^2} + \|\nabla j\|^2_{L^2}) + C\|w_\theta\|^{\frac{2q}{2q-3}}_{L^q} (\|w\|^2_{L^2} + \|j\|^2_{L^2}), \end{aligned}$$

and

$$\begin{aligned} V &= \int [(\partial_z u_r \partial_r b_\theta + \partial_z u_z \partial_z b_\theta) - \frac{1}{r}\partial_z(u_r b_\theta)] j_r dx \\ &= \int (\partial_z u_r \partial_r b_\theta + \partial_z u_z \partial_z b_\theta - \frac{u_r}{r}\partial_z b_\theta) j_r dx - \int \frac{b_\theta}{r} \partial_z u_r j_r dx \\ &\leq \|w_\theta\|_{L^q} (\|w\|^2_{L^{\frac{2q}{q-1}}} + \|j\|^2_{L^{\frac{2q}{q-1}}}) \\ &\leq \frac{1}{10}(\|\nabla w\|^2_{L^2} + \|\nabla j\|^2_{L^2}) + C\|w_\theta\|^{\frac{2q}{2q-3}}_{L^p} (\|w\|^2_{L^2} + \|j\|^2_{L^2}). \end{aligned}$$

Regarding the last term, initially, we rewrite F as

$$\begin{aligned} F &= \frac{1}{r}\partial_r(u_r b_\theta) - \frac{1}{r}\partial_r(r\tilde{u})\cdot\tilde{\nabla} b_\theta + \tilde{u}\cdot\tilde{\nabla}(\frac{b_\theta}{r}) \\ &= \frac{1}{r}\partial_r u_r b_\theta + \frac{1}{r}\partial_r b_\theta u_r - \frac{1}{r}\partial_r b_\theta u_r - \partial_r u_r \partial_r b_\theta \\ &\quad -\frac{1}{r}u_z\partial_z b_\theta - \partial_r u_z \partial_z b_\theta + u_r\partial_r(\frac{b_\theta}{r}) + u_z\partial_z(\frac{b_\theta}{r}) \\ &= \frac{1}{r}\partial_r u_r b_\theta + \frac{1}{r}\partial_r b_\theta u_r - \partial_r u_r\partial_r b_\theta - \partial_r u_z \partial_z b_\theta \\ &\quad -\frac{1}{r^2}u_r b_\theta, \end{aligned}$$

and then by employing Hölder inequality, Gagliardo-Nirenberg inequality, Lemma 2.4 and Young inequality again, it follows that

$$\begin{aligned} VI &= \int(\frac{1}{r}\partial_r u_r b_\theta + \frac{1}{r}\partial_r b_\theta u_r) j_z dx - \int \frac{u_r}{r}\frac{b_\theta}{r} j_z dx \\ &\quad -\int(\partial_r u_r \partial_r b_\theta + \partial_r u_z \partial_z b_\theta) j_z dx \\ &\leq \|w_\theta\|_{L^q}(\|w\|^2_{L^{\frac{2q}{q-1}}} + \|j\|^2_{L^{\frac{2q}{q-1}}}) \\ &\leq \frac{1}{10}(\|\nabla w\|^2_{L^2} + \|\nabla j\|^2_{L^2}) + C\|w_\theta\|^{\frac{2q}{2q-3}}_{L^q}(\|w\|^2_{L^2} + \|j\|^2_{L^2}). \end{aligned}$$

Thus, summing up all above estimates and applying the Gronwall's inequality yield

$$\begin{aligned} &\|w\|^2_{L^2} + \|j\|^2_{L^2} + \int_0^T(\|\nabla w\|^2_{L^2} + \|\nabla j\|^2_{L^2})dt \\ \leq\ & C(\|w_0\|^2_{L^2} + \|j_0\|_{L^2})e^{\int_0^T \|w_\theta\|^{\frac{2q}{2q-3}}_{L^q} dt}, \end{aligned}$$

which together with Lemma 2.3 reduces that

$$\|\nabla u\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C,$$

i.e.,

$$u \in L^\infty(0,T;L^6(\mathbb{R}^3)).$$

Finally, with the help of Lemma 3.1, we can finish the proof of this theorem. □

4. Proof of Theorem 1.2

Before proving Theorem 1.2, we would like to introduce the following estimate which plays a fundamental role in the proof of Theorem 1.2. Moreover, to the best of our acknowledge, this is a *new* estimate.

LEMMA 4.1. *If (u,b) is a smooth solution of system (1.1) with the form of $u = u_r e_r + u_\theta e_\theta + u_z e_z$ and $b = b_\theta e_\theta$, then it holds that*

$$\|rb_\theta\|^3_{L^3(\mathbb{R}^3)} + \int_0^T \|\nabla|rb_\theta|^{\frac{3}{2}}\|^2_{L^2(\mathbb{R}^3)}dt \leq\ C_3, \tag{4.1}$$

where the constant C_3 depends only on $T, \|u_0\|_{L^2(\mathbb{R}^3)}, \|b_0^\theta\|_{L^2(\mathbb{R}^3)}$ and $\|rb_0^\theta\|_{L^3(\mathbb{R}^3)}$.

Proof. According to $(2.2)^4$, it follows that the quantity rb_θ satisfies the following equation

$$\partial_t(rb_\theta)+\tilde{u}\cdot\tilde{\nabla}(rb_\theta)-\tilde{\Delta}(rb_\theta)+\frac{2}{r}\partial_r(rb_\theta)=2u_rb_\theta. \tag{4.2}$$

Then, by multiplying (4.2) with $rb_\theta|rb_\theta|$, integrating on $\mathbb{R}^3$ and energy estimates (3.2), it follows that

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\|rb_\theta\|_{L^3}^3+\frac{8}{9}\|\nabla|rb_\theta|^{\frac{3}{2}}\|_{L^2}^2\\
=\;& 2\int u_rb_\theta rb_\theta|rb_\theta|dx\\
\le\;& C\|u_r\|_{L^2}\|b_\theta\|_{L^6}\|rb_\theta\|_{L^6}^2\\
\le\;& C\|\nabla b_\theta\|_{L^2}\|rb_\theta\|_{L^3}^{\frac{1}{2}}\|rb_\theta\|_{L^9}^{\frac{3}{2}}\\
\le\;& C\|\nabla b_\theta\|_{L^2}\|rb_\theta\|_{L^3}^{\frac{1}{2}}\|\nabla|rb_\theta|^{\frac{3}{2}}\|_{L^2}\\
\le\;& \frac{1}{2}\|\nabla|rb_\theta|^{\frac{3}{2}}\|_{L^2}^2+4C\|\nabla b_\theta\|_{L^2}^2(1+\|rb_\theta\|_{L^3}^3),
\end{aligned}$$

which yields, after applying Gronwall's inequality and energy estimates (3.2), that

$$\|rb_\theta\|_{L^3}^3+\int_0^T\|\nabla|rb_\theta|^{\frac{3}{2}}\|_{L^2}^2dt\le C\|rb_0^\theta\|_{L^3}^3.$$

□

Proof of Theorem 1.2: First of all, by multiplying $(2.5)^2$ with r^4w_θ and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|r^2w_\theta\|_{L^2}^2+\int(\tilde{u}\cdot\tilde{\nabla})w_\theta(r^4w_\theta)dx\\
=\;&\int(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r)w_\theta(r^4w_\theta)dx+\int r^3w_\theta\partial_z{u_\theta}^2dx-\int r^3w_\theta\partial_z{b_\theta}^2dx\\
&+\int r^3u_r{w_\theta}^2dx-\int r^2{w_\theta}^2dx.
\end{aligned} \tag{4.3}$$

Then, by integrating by part and some basic calculations, we can further deduce the equality

$$\int(\tilde{u}\cdot\tilde{\nabla})w_\theta(r^4w_\theta)dx=\int\tilde{u}\cdot\tilde{\nabla}(r^2w_\theta)(r^2w_\theta)dx-2\int r^3u_r{w_\theta}^2dx,$$

and

$$\begin{aligned}
&\int(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r)w_\theta(r^4w_\theta)dx\\
=\;&\int(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r)(r^2w_\theta)(r^2w_\theta)dx-4\int rw_\theta\partial_r(r^2w_\theta)dx+4\int r^2w_\theta^2dx.
\end{aligned}$$

Subsequently, by applying Lemma 2.5, (4.1), Hölder inequality and Gagliardo-Nirenberg inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|r^2w_\theta\|_{L^2}^2+\int|\nabla(r^2w_\theta)|^2dx\\
=\;&\int(rb_\theta{}^2-ru_\theta{}^2)\partial_z(r^2w_\theta)dx+3\int r^3u_rw_\theta{}^2dx-4\int rw_\theta\cdot\partial_r(r^2w_\theta)dx\\
&+3\int r^2w_\theta^2dx\\
\le\;&\|rb_\theta\|_{L^3}\|b_\theta\|_{L^6}\|\partial_z(r^2w_\theta)\|_{L^2}+\|ru_\theta\|_{L^3}\|u_\theta\|_{L^6}\|\partial_z(r^2w_\theta)\|_{L^2}\\
&+\|u_r\|_{L^2}\left(\int(r^2w_\theta)^3w_\theta dx\right)^{\frac{1}{2}}+\|\partial_r(r^2w_\theta)\|_{L^2}\left(\int(r^2w_\theta)w_\theta dx\right)^{\frac{1}{2}}\\
&+C\|w_\theta\|_{L^2}\|r^2w_\theta\|_{L^2}\\
\le\;&C(\|\nabla b_\theta\|_{L^2}+\|\nabla u_\theta\|_{L^2})\|\partial_z(r^2w_\theta)\|_{L^2}+C\|r^2w_\theta\|_{L^6}^{\frac{3}{2}}\|w_\theta\|_{L^2}^{\frac{1}{2}}\\
&+\|\partial_r(r^2w_\theta)\|_{L^2}\|r^2w_\theta\|_{L^2}^{\frac{1}{2}}\|w_\theta\|_{L^2}^{\frac{1}{2}}+C\|w_\theta\|_{L^2}\|r^2w_\theta\|_{L^2}\\
\le\;&\frac{1}{2}\|\nabla(r^2w_\theta)\|_{L^2}^2+C\|r^2w_\theta\|_{L^2}^2+C(\|u\|_{H^1}^2+\|b\|_{H^1}^2),
\end{aligned}$$

which further implies, after employing Gronwall's inequality and energy estimates (3.2), that

$$\|r^2w_\theta\|_{L^2}^2+\int_0^T\|\nabla(r^2w_\theta)\|_{L^2}^2dt\le\; C(T,\|rw_0^\theta\|_{L^2},\|ru_0^\theta\|_{L^3},C_3).$$

Thus, we finish all the proof. □

5. Proof of Theorem 1.4

To begin with, we would like to introduce a smooth cut-off function $0\le\phi(r)\le 1$ with $\delta>0$ fixed such that

$$\phi(r)=\begin{cases}1, & r\le\frac{\delta}{2},\\ 0, & r>\delta,\end{cases}$$

and the imbedding constant a_0 of well-known Sobolev inequality

$$\|f\|_{L^6(\mathbb{R}^3)}\le a_0\|\nabla f\|_{L^2(\mathbb{R}^3)}. \tag{5.1}$$

Then, it is necessary to list the following Lemma which is useful in the proof of Theorem 1.4.

Lemma 5.1. *Suppose that (u,b) solves the system (1.1) with $u=u_re_r+u_\theta e_\theta+u_ze_z$ and $b=b_\theta e_\theta$, if one of the conditions (1.2)-(1.5) holds, then*

$$\|u_\theta\phi\|_{L^4(\mathbb{R}^3)}^4+\int_0^T\|\nabla|u_\theta\phi|^2\|_{L^2(\mathbb{R}^3)}^2dt\le\; C_4,$$

where C_4 depends only on T, $\|u_0^\theta\|_{L^4(\mathbb{R}^3)}$, $\|ru_0^\theta\|_{L^3(\mathbb{R}^3)}$.

Proof. By multiplying $(2.2)^1$ with $u_\theta\phi^2|u_\theta\phi|^2$, integrating on $\mathbb{R}^3$ and integrating by part, we have

$$\begin{aligned} & \frac{1}{4}\frac{d}{dt}\|u_\theta\phi\|_{L^4}^4+\frac{3}{4}\|\nabla|u_\theta\phi|^2\|_{L^2}^2+\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2 \\ = & \int u_r\partial_r\phi u_\theta^4\phi^3dx+3\int\phi^2(\partial_r\phi)^2u_\theta^4dx \\ & +2\int u_\theta^3\partial_r u_\theta\phi^3\partial_r\phi dx-\int\frac{u_r}{r}u_\theta^4\phi^4dx \\ = & \int u_r\partial_r\phi u_\theta^4\phi^3dx+\int\phi^2(\partial_r\phi)^2u_\theta^4dx \\ & +\int\partial_r(u_\theta\phi)^2u_\theta^2\phi\partial_r\phi dx-\int\frac{u_r}{r}u_\theta^4\phi^4dx \\ = & I+II+III+IV. \end{aligned}$$

To estimate the first term, by noticing $\mathrm{Spt}(\phi\partial_r\phi)\subset\Omega_\delta\backslash\Omega_{\frac{\delta}{2}}$, applying Hölder inequality, Sobolev imbedding inequality, Lemma 2.5 and Lemma 4.1, there holds

$$\begin{aligned} I \le\ & C(\delta)\||u_\theta\phi|^2\|_{L^6}\|u_r\|_{L^4}\|ru_\theta\|_{L^3}\|u_\theta\phi\|_{L^4} \\ \le\ & C\|\nabla|u_\theta\phi|^2\|_{L^2}\|u\|_{H^1}\|u_\theta\phi\|_{L^4} \\ \le\ & \frac{1}{16}\|\nabla|u_\theta\phi|^2\|_{L^2}^2+C\|u\|_{H^1}^2(1+\|u_\theta\phi\|_{L^4}^4). \end{aligned}$$

Then by applying Gagliardo-Nirenberg inequality, Hölder inequality and Sobolev imbedding inequality, it yields

$$\begin{aligned} & II+III \\ \le\ & \||u_\theta\phi|^2\|_{L^3}\|u_\theta\|_{L^3}^2+\|\nabla|u_\theta\phi|^2\|_{L^2}\|u_\theta\phi\|_{L^4}\|u_\theta\|_{L^4} \\ \le\ & \||u_\theta\phi|^2\|_{L^2}^{\frac{1}{2}}\|\nabla|u_\theta\phi|^2\|_{L^2}^{\frac{1}{2}}\|u_\theta\|_{L^2}\|\nabla u_\theta\|_{L^2} \\ & +\|\nabla|u_\theta\phi|^2\|_{L^2}\|u_\theta\phi\|_{L^4}\|u_\theta\|_{L^4} \\ \le\ & \|u_\theta\phi\|_{L^4}\|\nabla|u_\theta\phi|^2\|_{L^2}^{\frac{1}{2}}\|u_\theta\|_{L^2}\|\nabla u_\theta\|_{L^2} \\ & +\|\nabla|u_\theta\phi|^2\|_{L^2}\|u_\theta\phi\|_{L^4}\|u_\theta\|_{L^4} \\ \le\ & \frac{1}{16}\|\nabla|u_\theta\phi|^2\|_{L^2}^2+C\|u\|_{H^1}^2(1+\|u_\theta\phi\|_{L^4}^4). \end{aligned}$$

Regarding the last term, we will deal with it by the following way. If assumption (1.2) holds, one has

$$\begin{aligned} IV \le\ & \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2+C\int u_\theta^4\phi^4u_r^2dx \\ \le\ & \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2+C\|u_r\|_{L^q(\Omega_\delta)}^2\|u_\theta^2\phi^2\|_{L^{\frac{2q}{q-2}}}^2 \\ \le\ & \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2+C\|u_r\|_{L^q(\Omega_\delta)}^2\|u_\theta^2\phi^2\|_{L^2}^{2(1-\frac{3}{q})}\|\nabla|u_\theta\phi|^2\|_{L^2}^{\frac{6}{q}} \\ \le\ & \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2+\frac{1}{4}\|\nabla|u_\theta\phi|^2\|_{L^2}^2+C\|u_r\|_{L^p(\Omega_\delta)}^{\frac{2q}{q-3}}\|u_\theta\phi\|_{L^4}^4. \end{aligned}$$

If assumption (1.3) holds, it follows that

$$\begin{aligned}
IV &\le \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2 + \frac{1}{2}\int u_\theta^4\phi^4 u_r^2 dx\\
&\le \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2 + \frac{1}{2}\|u_r\|_{L^3(\Omega_\delta)}^2\|u_\theta^2\phi^2\|_{L^6}^2\\
&\le \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2 + \frac{1}{2}a_0^2\|u_r\|_{L^3(\Omega_\delta)}^2\|\nabla|u_\theta\phi|^2\|_{L^2}^2\\
&\le \frac{1}{2}\|\frac{u_\theta^2\phi^2}{r}\|_{L^2}^2 + \frac{1}{2}\|\nabla|u_\theta\phi|^2\|_{L^2}^2.
\end{aligned}$$

If assumption (1.4) holds, we have

$$\begin{aligned}
IV &\le \|\frac{u_r}{r}\|_{L^q(\Omega_\delta)}\|u_\theta^2\phi^2\|_{L^{\frac{2q}{q-1}}}^2\\
&\le \|\frac{u_r}{r}\|_{L^q(\Omega_\delta)}\|u_\theta^2\phi^2\|_{L^2}^{2(1-\frac{3}{2q})}\|\nabla|u_\theta\phi|^2\|_{L^2}^{\frac{3}{q}}\\
&\le \frac{1}{4}\|\nabla|u_\theta\phi|^2\|_{L^2}^2 + C\|\frac{u_r}{r}\|_{L^p(\Omega_\delta)}^{\frac{2q}{2q-3}}\|u_\theta\phi\|_{L^4}^4.
\end{aligned}$$

If assumption (1.5) holds, it yields that

$$\begin{aligned}
IV &\le \|\frac{u_r}{r}\|_{L^{\frac{3}{2}}(\Omega_\delta)}\|u_\theta^2\phi^2\|_{L^6}^2\\
&\le {a_0}^2\|\frac{u_r}{r}\|_{L^{\frac{3}{2}}(\Omega_\delta)}\|\nabla|u_\theta\phi|^2\|_{L^2}^2\\
&\le \frac{1}{2}\|\nabla|u_\theta\phi|^2\|_{L^2}^2.
\end{aligned}$$

At last, by summing up all above estimates, employing the assumptions (1.2)-(1.5), energy estimates (3.2) and Gronwall's inequality, we can conclude that

$$\|u_\theta\phi\|_{L^4}^4 + \int_0^T \|\nabla|u_\theta\phi|^2\|_{L^2}^2 dt \le \; C(T, \|u_0\|_{L^2}, \|b_0\|_{L^2}, \|u_0^\theta\|_{L^4}, \|ru_0^\theta\|_{L^3}).$$

□

Proof of Theorem 1.4: According to Theorem 1.1, to prove Theorem 1.4, it suffices to verify

$$w_\theta \in L^\infty(0,T;L^2(\mathbb{R}^3)). \tag{5.2}$$

Keep this in mind, by multiplying $(2.5)^2$ with $w_\theta\phi^2$ and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_\theta\phi\|_{L^2}^2 + \|\nabla(w_\theta\phi)\|_{L^2}^2 + \|\frac{w_\theta\phi}{r}\|_{L^2}^2\\
=& \int u_r\partial_r\phi w_\theta^2\phi dx + \int w_\theta^2(\partial_r\phi)^2 dx\\
&+\int \frac{w_\theta\phi^2}{r}\partial_z(u_\theta^2 - b_\theta^2)dx + \int \frac{w_\theta^2\phi^2 u_r}{r}dx\\
=& I + II + III + IV.
\end{aligned}$$

For the first two terms, noticing that $\mathrm{Spt}(\partial_r\phi)\subset\Omega_\delta\backslash\Omega_{\frac{\delta}{2}}$, applying Gagliardo-Nirenberg inequality and Theorem 1.2, it follows that

$$\begin{aligned}
& I+II \\
\leq\ & C(\delta)\|w_\theta\phi\|_{L^6}\|r^2w_\theta\|_{L^2}\|u_r\|_{L^3}+C(\delta)\|w_\theta\|_{L^2}^2 \\
\leq\ & C\|\nabla(w_\theta\phi)\|_{L^2}\|u_r\|_{L^3}+C\|w_\theta\|_{L^2}^2 \\
\leq\ & \frac{1}{8}\|\nabla(w_\theta\phi)\|_{L^2}+C\|u\|_{H^1}^2.
\end{aligned}$$

Then, by employing integrating by part and Lemma 2.6, it is clear that

$$\begin{aligned}
III\ =\ & \int\frac{w_\theta\phi^2}{r}\partial_z u_\theta^2dx+\int\partial_z(w_\theta\phi)\frac{b_\theta^2\phi}{r}dx \\
\leq\ & C\|\frac{w_\theta\phi}{r}\|_{L^2}\|\partial_z{u_\theta}^2\phi\|_{L^2}+C\|\partial_z(w_\theta\phi)\|_{L^2}\|\frac{b_\theta}{r}\|_{L^3}\|b_\theta\|_{L^6} \\
\leq\ & C\|\frac{w_\theta\phi}{r}\|_{L^2}\|\partial_z{u_\theta}^2\phi\|_{L^2}+C\|\partial_z(w_\theta\phi)\|_{L^2}\|\frac{b_\theta}{r}\|_{L^3}\|\nabla b_\theta\|_{L^2} \\
\leq\ & C\|\frac{w_\theta\phi}{r}\|_{L^2}\left[\|\partial_z{u_\theta}^2\|_{L^2(\Omega_{\frac{\delta}{2}})}+C(\delta)\|\partial_z(ru_\theta)^2\|_{L^2(\Omega_\delta\backslash\Omega_{\frac{\delta}{2}})}\right] \\
& +C\|\partial_z(w_\theta\phi)\|_{L^2}\|\frac{b_\theta}{r}\|_{L^3}\|\nabla b_\theta\|_{L^2} \\
\leq\ & \frac{1}{8}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{8}\|\partial_z(w_\theta\phi)\|_{L^2}^2+C\|\partial_z{u_\theta}^2\|_{L^2(\Omega_{\frac{\delta}{2}})}^2 \\
& +C(\delta)\|\partial_z(ru_\theta)^2\|_{L^2(\Omega_\delta\backslash\Omega_{\frac{\delta}{2}})}^2+C\|\nabla b_\theta\|_{L^2}^2 \\
\leq\ & \frac{1}{8}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{8}\|\partial_z(w_\theta\phi)\|_{L^2}^2+C\|\partial_z{u_\theta}^2\|_{L^2(\Omega_{\frac{\delta}{2}})}^2 \\
& +C\|\partial_z(ru_\theta)^2\|_{L^2}^2+C\|\nabla b_\theta\|_{L^2}^2.
\end{aligned}$$

As for the last term, we will deal with it by the similar way as before. Suppose assumption (1.2) holds, one has

$$\begin{aligned}
IV\ \leq\ & \frac{1}{4}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\|u_rw_\theta\phi\|_{L^2}^2 \\
\leq\ & \frac{1}{4}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+C\|u_r\|_{L^q(\Omega_\delta)}^2\|w_\theta\phi\|_{L^{\frac{2q}{q-2}}}^2 \\
\leq\ & \frac{1}{4}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+C\|u_r\|_{L^q(\Omega_\delta)}^2\|w_\theta\phi\|_{L^2}^{2(1-\frac{3}{q})}\|\nabla(w_\theta\phi)\|_{L^2}^{\frac{6}{q}} \\
\leq\ & \frac{1}{4}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{4}\|\nabla(w_\theta\phi)\|_{L^2}^2+C\|u_r\|_{L^q(\Omega_\delta)}^{\frac{2q}{q-3}}\|w_\theta\phi\|_{L^2}^2.
\end{aligned}$$

Suppose assumption (1.3) holds, it leads to

$$\begin{aligned}
IV\ \leq\ & \frac{1}{2}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{2}\|u_rw_\theta\phi\|_{L^2}^2 \\
\leq\ & \frac{1}{2}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{2}\|u_r\|_{L^3(\Omega_\delta)}^2\|w_\theta\phi\|_{L^6}^2 \\
\leq\ & \frac{1}{2}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{2}a_0^2\|u_r\|_{L^3(\Omega_\delta)}^2\|\nabla(w_\theta\phi)\|_{L^2}^2 \\
\leq\ & \frac{1}{2}\|\frac{w_\theta\phi}{r}\|_{L^2}^2+\frac{1}{2}\|\nabla(w_\theta\phi)\|_{L^2}^2.
\end{aligned}$$

Suppose assumption (1.4) holds, we have

$$\begin{aligned} IV &\leq C\|\frac{u_r}{r}\|_{L^q(\Omega_\delta)}\|w_\theta\phi\|^2_{L^{\frac{2q}{q-1}}} \\ &\leq C\|\frac{u_r}{r}\|_{L^q(\Omega_\delta)}\|w_\theta\phi\|^{2(1-\frac{3}{2q})}_{L^2}\|\nabla(w_\theta\phi)\|^{\frac{3}{q}}_{L^2} \\ &\leq \frac{1}{4}\|\nabla(w_\theta\phi)\|^2_{L^2}+C\|\frac{u_r}{r}\|^{\frac{2q}{2q-3}}_{L^q(\Omega_\delta)}\|w_\theta\phi\|^2_{L^2}. \end{aligned}$$

Suppose assumption (1.5), it yields

$$\begin{aligned} IV &\leq \|\frac{u_r}{r}\|_{L^{\frac{3}{2}}(\Omega_\delta)}\|w_\theta\phi\|^2_{L^6} \\ &\leq {a_0}^2\|\frac{u_r}{r}\|_{L^{\frac{3}{2}}(\Omega_\delta)}\|\nabla(w_\theta\phi)\|^2_{L^2} \\ &\leq \frac{1}{2}\|\nabla(w_\theta\phi)\|^2_{L^2}. \end{aligned}$$

In the end, by summing up all above estimates, making use of Lemma 5.1, (2.11), assumptions (1.2)-(1.5) and Gronwall's inequality, it follows that

$$\|w_\theta\|^2_{L^2(\Omega_{\frac{\delta}{2}})} \leq C.$$

Thus, with the help of Theorem 1.2, it is clear to deduce that

$$\begin{aligned} \|w_\theta\|^2_{L^2} &= C\int_{-\infty}^{\infty}\int_0^{\infty}|w_\theta|^2 rdrdz \\ &\leq C\int_{-\infty}^{\infty}\int_0^{\frac{\delta}{2}}|w_\theta|^2 rdrdz + C\int_{-\infty}^{\infty}\int_{\frac{\delta}{2}}^{\infty}|r^2 w_\theta|^2 rdrdz \\ &\leq C, \end{aligned}$$

which implies the conclusion. □

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