Travelling waves over an arbitrary bathymetry: a local stability result

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ABSTRACT. The problem of a travelling wave in a semi-infinite channel over an arbitrary quasi-flat bathymetry is studied in the shallow-water formulation. It is shown how the streamfunction can be cast, in the vicinity of an elliptic equilibrium for the fluid flow, in the form of a nearly-integrable non-autonomous Hamiltonian with a perturbation which is *aperiodic* in time. The obtained model provides a natural example in the context of the aperiodically time-dependent Hamiltonian systems studied in Fortunati and Wiggins (2016) and the local stability of the perturbed streamlines is obtained as a consequence of this theory.
The proofs use the tools of Perturbation Theory in the real-analytic setting.

CONTENTS

1. Preliminaries

This paper deals with the problem of a propagating wave for the linear quasi-geostrophic shallow-water equation for an inviscid fluid in terms of the *streamfunction* $\psi = \psi(x, y, t)$,

$$(1) \qquad \partial_t(\Delta\psi - F\psi) + J(\psi, \delta) + \mathcal{F}\partial_x\psi = 0,$$

where $J(f_1, f_2) := \partial_x f_1 \partial_y f_2 - \partial_y f_1 \partial_x f_2$, $F, \mathcal{F} \in \mathbb{R}$ and $\delta = \delta(x, y)$, see (2).

For the derivation of equation (1) and more details about the presentation below, until (7), we refer to the comprehensive exposition of [**Ped12**].

The solutions of (1) will be studied in a *semi-infinite channel*

$$\mathcal{C} := \mathbb{R}^+ \times \mathbb{T} \ni (x, y),$$

here $\mathbb{R}^+ := (0, +\infty)$, rotating with angular velocity Ω around the vertical axis z and with a depth d, which is supposed to be "small" in a suitable scaling, i.e. $d \ll 1$. The bottom of the channel is assumed to be described by a quasi-flat bathymetry of the form

$$(2) \qquad \delta(x, y) := \mu \tilde{g}(x, y) - d,$$

in the hypothesis $0 \ll \mu \ll d$. The assumptions on $\tilde{g}$ will be discussed later.

The parameter $F = \mathcal{O}(1)$ is related to the geometric scaling of the problem, see [**Ped12**, formula 3.12.9] while $\mathcal{F} = \mathcal{O}(1)$ represents the contribution of the linear term in the well known $\beta-plane$ approximation[1].

The equation (1) is linear as the term $J(\Delta \psi - F\psi, \psi)$ has been neglected. This assumption can be thought as "appropriate" in the case of motions generated by small amplitude streamfunctions[2]. However, we remark that the linear case itself exhibits some peculiar and non-trivial difficulties due to the presence of a non-periodic bathymetry that will be described in detail later on.

Given a streamfunction ψ, the velocity field is associated via the *geostrophic equations*

$$(3) \qquad \dot{y} = \partial_x \psi, \qquad \dot{x} = -\partial_y \psi,$$

which represent the lowest order approximation in the quasi-geostrophic scaling. These exploit the intrinsic Hamiltonian structure of the problem.

The impenetrability of the channel boundaries is expressed by the conditions

$$(4) \qquad \partial_x \psi|_{y=0, 2\pi} \equiv 0, \qquad \forall x \geq 0.$$

Clearly, if $\mu = 0$ (flat bathymetry), equation (1) is simply reduced to the homogeneous equation

$$(5) \qquad \partial_t (\Delta \psi^{(0)} - F\psi^{(0)}) + \mathcal{F} \partial_x \psi^{(0)} = 0.$$

A solution for the previous equation can be chosen in the form of *travelling wave*

$$(6) \qquad \psi^{(0)}(x, y, t) = A \sin(\tilde{m}y) \cos(\kappa x + \sigma t),$$

where $A \in \mathbb{R}$ is the *amplitude*, while $(\tilde{m}, \kappa) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ and $\sigma \in \mathbb{R}$ are commonly known as *wave numbers* and *wave speed*, respectively. The parameter A will be supposed fixed once and for all.

Equation (6) is satisfied once σ has been chosen as follows (*dispersion relation*)

$$(7) \qquad \sigma = \sigma(\kappa) := \frac{\kappa \mathcal{F}}{\tilde{m}^2 + \kappa^2 + F}.$$

Note that $\sigma(-\kappa) = -\sigma(\kappa)$ and $\sigma(\kappa) \neq 0$.

Without loss of generality we shall suppose $\mathcal{F} < 0$ in such a way $\sigma < 0$ and the

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[1]More precisely, we have set $\epsilon := r_0^{-1} \tan \theta_0$ where r_0 is the Earth radius and θ_0 is the mean latitude, see [**Ped12**, formula 3.17.5] then we have chosen $\epsilon \mathcal{F} := \varepsilon$, where $\varepsilon \equiv \varepsilon_T \ll 1$, see [**Ped12**, Sec. 3.12], so that ϵ and ε have the same magnitude.

[2]In other terms, this can be seen by considering the rescaling $\psi \to \delta \psi$ then neglecting the $\mathcal{O}(\delta^2)$ contributions.

wave propagates in the direction of the positive $x-$axis.

It is customary, when dealing with travelling waves problems, see e.g. [**KW87**], to consider the following Galileian transformation

(8) $$\mathcal{G}: \qquad (y, x) = (p, q - \kappa^{-1}\sigma t).$$

The equations of motion for the streamlines given by (6) via (3) in these new variables

$$\begin{cases} \dot{p} &= -A\kappa \sin(\tilde{m}p)\sin(\kappa q) \\ \dot{q} &= \dfrac{\sigma}{\kappa} - A\tilde{m}\cos(\tilde{m}p)\cos(\kappa q) \end{cases},$$

are generated by the Hamiltonian

(9) $$\tilde{H}_0 := \frac{\sigma}{\kappa}p - A\sin(\tilde{m}p)\cos(\kappa q).$$

Under the condition

(10) $$-\sigma \leq \kappa\tilde{m}A,$$

which can be satisfied e.g. with a suitable choice of $\mathcal{F}$, it is immediate to check that the system above possesses the following equilibria in $[-\pi, \pi]^2$

(11) $$(p_e^{\pm}, q_e) = \left(\pm\arccos\left(\frac{\sigma}{\kappa\tilde{m}A}\right), 0\right), \qquad (p_h, q_h^{\pm}) = \left(0, \pm\frac{1}{\tilde{m}}\arccos\left(\frac{\sigma}{\kappa\tilde{m}A}\right)\right),$$

and that $(p_e^{\pm}, q_e)$ and $(p_h, q_h^{\pm})$ are elliptic and hyperbolic, respectively. In particular, the streamlines in the vicinity of $(p_e^{\pm}, q_e)$ are closed curves and the fluid particles will remain locally confined for all t.

The aim of this paper is to show that the above described structure is preserved, for $\mu > 0$ i.e. when the effect of the bathymetry is taken into account, in a suitable neighbourhood of (p_e^{+}, q_e) (the argument is analogous for (p_e^{-}, q_e)), at least for sufficiently small μ.

Given its remarkable geophysical application, the study of the disturbances on a fluid arising from a "rough" bottom, is a widely studied problem and it has been investigated for a long time (see e.g. [**RB73**]) in many different settings and regimes. The literature on this topic is remarkably vast and a review goes far beyond the purpose of this paper, see e.g. [**Van03**], [**Cha07**], [**CS10**], [**CLS12**] and references therein. The paper [**CLS12**], in particular, contains an extensive analysis from the Hamiltonian point of view, in which relevant cases of periodic and $O(1)$ bottoms are studied. The case of a random bottom, already investigated in [**RP83**], has been developed in [**CS10**].

More closely to the travelling wave problem, a remarkable attention has been devoted to the Rossby waves, either oceanic or atmospheric. The key interpretation of a travelling wave as an integrable and autonomous (Hamiltonian) system for the streamlines, [**WK89**], has led to the study of several classes of disturbances to be interpreted as "perturbations", see, for instance, [**Pie91**] and [**MW98**]. In this setting, phenomena such as *chaotic transport* or *chaotic mixing* have been investigated in the "perturbed" systems, either numerically or by means of a Melnikov analysis of the stable and unstable manifolds asymptotic to hyperbolic points like $(p_h, q_h^{\pm})$. In general, the above described studies stress the importance of results of stability in the presence of *aperiodically* time-dependent perturbation as "barriers" for the Lagrangian transport in fluid dynamics. See [**SW06**] for an extensive review of the subject.

Starting from the special (integrable) model given by the streamfunction of the travelling wave (6) in a neighbourhood of the point (p_e^+, q_e), see $H_0(I)$ in (13), the main step consists in the construction of a perturbation arising from a "slightly" variable bottom as described by (2). For this purpose, the bathymetry will be supposed to be flat for $x \to +\infty$ but without further assumptions on its dependence on x (e.g. periodicity). It will be shown that, at least in the vicinity of the point (p_e^+, q_e), the streamlines are described by a nearly-integrable Hamiltonian system with a non-autonomous perturbation in a suitable real-analyticity class. It might be redundant to stress that the construction of the mentioned perturbation itself, which is basically a result of existence for the solution of the "perturbed problem" (i.e. the equation (1) with $\mu > 0$), see Theorem 2.2, is the difficult part of the whole result. We anticipate that, despite the linearity of the problem, the general (non-periodic) dependence on x of the bathymetry inhibits the standard diagonalization of the (linear) perturbed operator in the Fourier space. This is a commonly used tool when dealing with the linearised operator in the context of superconvergent methods for non-linear PDEs, see e.g. [**Ber07**, Section 4.5]. The above described invertibility of the linear operator has been proven, instead, by means of perturbative methods.

Once this has been shown, it is easy to realize that, due to the general (non-periodic) dependence on x of $\tilde{g}$, the transformation $\mathcal{G}$ gives rise to an aperiodically time-dependent perturbation. The described class of Hamiltonian systems has been studied in [**FW16**] and the case at hand provides a natural example in which the mentioned results can be applied. In this way, the *perpetual stability* of the perturbed fluid flow in a neighbourhood of (p_e^+, q_e), stated in Corollary 2.3, is a direct consequence of Theorem 2.2 and of the main result of [**FW16**].

2. Analytic setting and main statement

Given $\rho \in (0, 1/2]$ we introduce the complexification of $\mathcal{C}$ by defining

$$\mathcal{C}_\rho := \mathcal{S}_\rho \times \mathbb{T}_\rho,$$

where

$$\mathcal{S}_\rho := \{x \in \mathbb{C} : \Re x > -\rho, \quad |\Im x| < \rho\}, \qquad \mathbb{T}_\rho := \{y \in \mathbb{T} : |\Im y| < \gamma\}.$$

Given a function $v : C_\rho \to \mathbb{C}$, we define, for all $\rho' < \rho$, the (x-dependent) Fourier norm

$$\|v\|_{\rho'} := \sum_{k \in \mathbb{Z}} |v_k(x)| e^{|k|\rho'},$$

where $v_k(x)$ are the Fourier coefficients $v(x, y) := \sum_{z \in \mathbb{Z}} v_k(x) e^{iky}$.
Similarly, for any function $\tilde{v} = \tilde{v}(x, y, t) : \mathcal{C}_\rho \times \mathbb{R} \to \mathbb{C}$, the norm $\|\tilde{v}\|_{\rho'}$ will be dependent on (x, t).
Let us now consider the following set $\mathcal{U} := G \times \mathbb{T} \ni (I, \varphi)$ where $G \subset \mathbb{R}$. Given $\gamma \in (0, 1/2]$, the set $\mathcal{U}$ can be complexified by considering

$$\mathcal{U}_\gamma := G_\gamma \times \mathbb{T}_\gamma, \qquad G_\gamma := \{I^* \in \mathbb{C} : |I^* - I| < \gamma, \quad \forall I \in G\},$$

and it can be endowed with the following norm

$$\|w\|_{\gamma'}^* := \sum_{k \in \mathbb{Z}} |w_k| e^{|k|\gamma'}, \qquad \forall \gamma' < \gamma,$$

where $w_k = w_k(I)$ and $\tilde{w} = \tilde{w}_k(I,t)$ if $w : \mathcal{U}_\gamma \to \mathbb{C}$ or $\tilde{w} = \tilde{w}(I,\varphi,t) : \mathcal{U}_\gamma \times \mathbb{R} \to \mathbb{C}$, respectively. Let us finally define $\mathcal{W}_\gamma := \{\eta \in \mathbb{C} : |\Im \eta| < \gamma\}$.
Throughout the paper we shall consider the following class of topographies:

HYPOTHESIS 2.1. *The function $\tilde{g} : \mathcal{C} \to \mathbb{R}$ is holomorphic on $\mathcal{C}_{2\rho}$ and satisfies*

$$(12) \qquad \qquad \|\tilde{g}(x,y)\|_\rho \leq \mathcal{M}e^{-\nu|x|},$$

for some[3] $\nu > 0$.

In the above described setting, the main result states as follows

THEOREM 2.2. *Assume hypothesis 2.1 and condition (10). Then for sufficiently small[4] μ, there exists $0 < \tilde{\gamma} < \gamma$ and a set of coordinates $(I,\varphi) \in \mathcal{U}_{\tilde{\gamma}}$ for some suitable interval G, such that the solutions of (1), in a neighbourhood of (p_e^+, q_e), can be cast in the form*

$$(13) \qquad \qquad H_\mu(I,\varphi,\eta,t) := H_0(I) + \eta + H_1(I,\varphi,t;\mu),$$

where H_μ is a real-analytic function over $\mathcal{U}_{\tilde{\gamma}} \times \mathcal{W}_{\tilde{\gamma}}$ and in μ for all $t \geq 0$. We denote with η the variable canonically conjugated to t. Furthermore

$$(14) \qquad \qquad \sup_{I \in G_{\tilde{\gamma}}} \|H_1\|_{\tilde{\gamma}} \leq K e^{-\tilde{\nu}t},$$

for some $K = \mathcal{O}(\mu) > 0$ and $\tilde{\nu} := \tilde{\nu}(\nu)$ with $\tilde{\nu}(0) = 0$.

Similarly to the aperiodic dependence on t of the perturbation, the time decay (14) is another direct consequence of the transformation $\mathcal{G}$ and of (12).
The application of [**FW16**, Theorem 2.3] immediately yields, in particular[5], the following

COROLLARY 2.3. *Every solution of (13) with initial condition $I(0) \in G$ is perpetually stable.*

The paper is organized as follows: in section 3 we cast $\psi^{(0)}$ into a suitable (integrable) form, i.e. H_0, in a neighbourhood of the elliptic point by means of a real-analytic transformation of variables $\mathcal{T}$. This is done by using standard methods, the integrability of the "unperturbed" travelling wave model and its time-independence, see [**WK89**]. Subsequently, (section 5) it is shown how the disturbances given by a non-flat bottom can be treated in a perturbative setting. More precisely, the implicit equation that would be obtained from (1) by writing $\psi = \psi^{(0)} + \tilde{\psi}$, with $\tilde{\psi} = \mathcal{O}(\mu)$, is rewritten in terms of an infinite hierarchy of explicit equations by means of a classical formal expansion, see (39) and by using the *ansatz* (42) for the elements of the expansion. This is a well known approach, see e.g. [**GM04**]. Each equation of the obtained hierarchy is shown to be resolvable in a constructive way. In the final part of section 5, the convergence of the above described perturbative scheme is discussed in a suitable real-analyticity class. In this way, the constructed solution is mapped, through $\mathcal{T}$, to a function that will be the play the role of perturbation, i.e. H_1, and this will complete the construction of the Hamiltonian (13).

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[3]Note that no lower bounds on ν are imposed, consistently the threshold for the values of μ allowed in Theorem 2.2, becomes smaller and smaller with ν, see (46).

[4]See (46) for a quantitative estimate.

[5]We remark that [**FW16**, Theorem 2.3] establishes the existence of a normal form, which implies, for the class of systems considered, the perpetual stability as a consequence.

Section 5 uses some technical tools discussed separately in section 4 for reader's convenience.

3. Normal form of the unperturbed problem

LEMMA 3.1. *There exists a set of action-angle variables (I, φ) and a canonical transformations $\mathcal{T}$ casting the streamfunction $\psi^{(0)}$, in a neighbourhood of the elliptic point (p_e^+, q_e), into the form $H_0 = H_0(I)$*

PROOF. Let us recall (11). In order to study the system in a neighbourhood of the elliptic equilibrium (p_e^+, q_e), let us consider the composition of the following canonical maps

$$
\begin{aligned}
\mathcal{C}' : & \qquad (p, q) = (p' + p_e^+, q') \\
\mathcal{C}'' : & \qquad (p', q') = \left(\kappa p'', \kappa^{-1} q''\right)
\end{aligned} .
$$

Defining $H_0'' := \tilde{H}_0 \circ \mathcal{C}' \circ \mathcal{C}''$, the Hamiltonian (9) is cast in the form

$$
(15) \quad H_0'' = \sigma p'' - \frac{\sigma}{\kappa \tilde{m}} \sin(\kappa p'') \cos(q'') - \frac{\sigma}{\kappa \tilde{m}} \cos(\kappa p'') \cos(q'') \sqrt{\left(\frac{A \kappa \tilde{m}}{\sigma}\right)^2 - 1}.
$$

Then we finally consider the following transfomation

$$
\mathcal{C}''' : \quad (p'', q'') = ((\kappa \tilde{m})^{-1} P, Q).
$$

The equations of motion in the new variables are still canonical. In fact, by setting $\lambda := \sqrt{\sigma^{-2} (A \kappa \tilde{m})^2 - 1} \in \mathbb{R}^+$ by (10), their Hamiltonian reads as

$$
(16) \qquad\qquad H_0'''(P, Q) = \sigma[P - (\sin P + \lambda \cos P) \cos Q].
$$

Note that the elliptic equilibrium is now $(P^*, Q^*) = (0, 0)$. A Taylor expansion of (16) centred at the origin yields

$$
(17) \qquad\qquad H_0'''(P, Q) = \frac{\sigma \lambda}{2}(P^2 + Q^2) + \mathcal{O}_3(P, Q).
$$

It is well known, see e.g. [IAKN14], that there exists a canonical transformation $\mathcal{B}$ casting the Hamiltonian (17) into the following *Birkhoff normal form*

$$
(18) \qquad\qquad H_0^{\mathcal{B}}(P, Q) = \frac{\sigma \lambda}{2}(P^2 + Q^2) + \sum_{k \geq 1} \alpha_k (P^2 + Q^2)^{2k},
$$

for bounded $\{\alpha_k\} \in \mathbb{R}$. The transformation is analytic in some complex neighbourhood of the origin, i.e. in a set $D_R := \{(P, Q) \in \mathbb{C}^2 : |P|, |Q| < R\}$ with some positive $R = \mathcal{O}(1)$, see e.g. [Gio12].

The last step consists in the use of the well known transformation

$$
(19) \qquad\qquad \mathcal{P} : \qquad (P, Q) = \left(\sqrt{\frac{2I}{-\sigma \lambda}} \cos \varphi, \sqrt{\frac{2I}{-\sigma \lambda}} \sin \varphi\right).
$$

Hence, the required change of variable is

$$
(20) \qquad\qquad \mathcal{T} : (x, y) \to (I, \varphi), \qquad \mathcal{T} := \mathcal{G} \circ \mathcal{C}' \circ \mathcal{C}'' \circ \mathcal{C}''' \circ \mathcal{B} \circ \mathcal{P},
$$

where $\mathcal{G}$ has been defined in (8). Note that the interval G mentioned in the statement of Theorem 2.2 is given by the values $\Re(P^2 + Q^2)/2$ as from (19) and condition $|P|, |Q| < R$.

Given the real-analyticity of the maps of the composition (20), the map $\mathcal{T}$ will be real-analytic too. This implies that the same property holds for H_0, being the image through $\mathcal{T}$ of $\psi^{(0)}$, which is clearly real-analytic.

4. Intermezzo: some technical tools

4.1. Estimates on a second order ODE. Given a function $f : \mathcal{S}_{\hat{\rho}} \to \mathbb{C}$ we denote with $\check{f}(x)$ and $\check{\check{f}}(x)$ the derivatives $f'(x)$ and $f''(x)$, respectively.

PROPOSITION 4.1. *Let $R : \mathcal{S}_{\hat{\rho}} \to \mathbb{C}$ such that $|R(x)| \leq M \exp(-\nu|x|)$ with $\nu, \hat{\rho} \in (0, 1]$ and consider the following differential equation*

$$(21) \qquad \check{\check{B}}(x) - 2i\alpha \check{B}(x) - \beta_m^2 B(x) = R(x),$$

where $\alpha > 0$ and $\{\beta_m^2\}_{m \in \mathbb{Z}} \subset \mathbb{R}^+$. Setting

$$(22) \qquad \delta_m := \sqrt{\beta_m^2 - \alpha^2} \in \mathbb{C},$$

let us suppose that $\delta_m \neq 0$ at least for some m, then define $\delta_- := \inf_{m \in \mathbb{Z}}\{|\delta_m| : \beta_m^2 \neq \alpha^2\}$ and $\delta_+ := \sup_{m \in \mathbb{Z}}\{|\delta_m| : \beta_m^2 < \alpha^2\}$ or $\delta_+ := 0$ if $\beta_m^2 \geq \alpha^2$ for all m. Then the following bounds hold

$$(23) \qquad |B(x)| \leq \frac{\mathfrak{G}M}{\nu^3(1 + |\delta_m|)} e^{-\frac{\nu|x|}{2}}, \qquad |\check{B}(x)| \leq \mathfrak{G}M e^{-\frac{\nu|x|}{2}}, \qquad \forall x \in \mathcal{S}_{\hat{\rho}},$$

where $\mathfrak{G} := 32(2 + \alpha)e^{(\alpha + 2\nu + |\delta_+|)}\delta_-^{-1}\nu^{-3}$.

REMARK 4.2. The key feature of this statement consists in the possibility to provide an explicit estimate for $|\check{B}(x)|$. This avoids the use of a Cauchy bound and the consequent domain restriction, playing a key role in the convergence argument of Section 5.
The possibility to consider a sequence $\{\beta_m\}$ instead of $\beta \in \mathbb{R}$ enable us to provide the uniform estimates (23), being δ_- a lower bound for $\delta_m > 0$.

The proof uses some ideas from [**FW14**, Proposition 2].

PROOF. We shall replace β_m, δ_m with β, δ for simplicity throughout the proof. The cases $\beta > \alpha$, $\beta = \alpha$ and $\beta < \alpha$ will be examined separately. The general solution of equation (21) is
$$(24)$$
$$B(x) = \begin{cases} e^{i\alpha x}(K_1 + xK_2) + \displaystyle\int_0^x (x - y)e^{i\alpha(x-y)}R(y)dy & \text{if } \beta = \alpha \\ e^{i\alpha x}(K_1 e^{\delta x} + K_2 e^{-\delta x}) + \dfrac{1}{\delta}\displaystyle\int_0^x e^{i\alpha(x-y)}\sinh(\delta(x-y))R(y)dy & \text{otherwise} \end{cases}$$

Case $\beta > \alpha$. Hence $\delta \in \mathbb{R}^+$. Denote $x =: \xi + i\eta \in \mathcal{S}_{\hat{\rho}}$. By using their path-independence, we write the integral appearing in (24) as

$$\int_0^x e^{-i\alpha y}e^{\pm \delta y}R(y)dy = \int_0^\xi e^{-i\alpha\xi'}e^{\pm\delta\xi'}R(\xi')d\xi' \\ + ie^{-i\alpha\xi}e^{\pm\delta\xi}\int_0^\eta e^{\alpha\eta'}e^{\pm i\delta\eta'}R(\xi + i\eta')d\eta'.$$

Now we choose

$$(25) \qquad K_1 := -\frac{1}{2\delta}\int_0^{+\infty} e^{-(i\alpha+\delta)y}R(y)dy, \qquad K_2 = 0.$$

Note that $|K_1| < +\infty$. In this way, by setting

$$
\begin{aligned}
(26) \qquad \mathcal{I}_1 &:= e^{(i\alpha+\delta)x}\left(K_1 + \frac{1}{2\delta}\int_0^x e^{-(i\alpha+\delta)y}R(y)dy\right) \\
\mathcal{I}_2 &:= -\frac{e^{(i\alpha-\delta)x}}{2\delta}\int_0^x e^{-(i\alpha-\delta)y}R(y)dy
\end{aligned}
$$

we have $B(x) = \mathcal{I}_1 + \mathcal{I}_2$ where, in particular

$$
(27) \qquad \mathcal{I}_1 = \frac{e^{(i\alpha+\delta)\xi}e^{(i\delta-\alpha)\eta}}{2\delta}\left[ie^{-(i\alpha+\delta)\xi}\int_0^\eta e^{(\alpha-i\delta)\eta'}R(\xi+i\eta')d\eta'\right.
$$

$$
\left. - \int_\xi^{+\infty} e^{-(i\alpha+\delta)\xi'}R(\xi')d\xi'\right]
$$

Hence, by hypothesis

$$
(28) \qquad |\mathcal{I}_1| \leq \frac{Me^{\delta\xi-\alpha\eta}}{2\delta}\left(e^{-\delta\xi}\int_0^\eta e^{\alpha\eta'-\nu\xi}d\eta' + \int_\xi^{+\infty}e^{-(\delta+\nu)\xi'}d\xi'\right)
$$

$$
\leq \frac{Me^{(\alpha+\nu)\hat\rho}}{2\delta}\left(|\eta|e^{\alpha|\eta|-\nu\xi} + \frac{e^{-\nu\xi}}{\delta+\nu}\right)
$$

$$
\leq Me^{(\alpha+\nu)\hat\rho}(2\delta)^{-1}\left[e + (\delta+\nu)^{-1}\right]e^{-\nu\xi}
$$

$$
\leq 2Me^{(\alpha+\nu)\hat\rho}(\nu\delta)^{-1}e^{-\nu\xi}.
$$

On the other hand,

$$
(29) \qquad |\mathcal{I}_2| \leq \frac{Me^{-(\alpha\eta+\delta\xi)}}{2\delta}\left(e^{\delta\xi}e^{\nu(\rho-\xi)}\int_0^\eta d\eta' + \int_0^\xi e^{(\delta-\nu)\xi'}d\xi'\right)
$$

$$
\leq \frac{Me^{(\alpha+\nu)\hat\rho}}{2\delta}\left(e^{1-\nu\xi} + \frac{e^{-\delta\xi}-e^{-\nu\xi}}{\nu-\delta}\right)
$$

$$
\leq \frac{Me^{(\alpha+\nu)\hat\rho}}{2\delta}\left(e + \xi e^{-\nu\xi/2}\right)e^{-\nu\xi/2}
$$

$$
\leq 2Me^{(\alpha+\nu)\hat\rho}(\nu\delta)^{-1}e^{-\nu\xi/2}.
$$

By using the elementary inequality $e^{-\nu\xi} \leq e^{-\nu|x|}e^{\nu\hat\rho}$ valid for all $x \in \mathcal{S}_{\hat\rho}$,

$$
(30) \qquad |B(x)| \leq |\mathcal{I}_1| + |\mathcal{I}_2| \leq 4Me^{(\alpha+2\nu)\hat\rho}(\nu\delta)^{-1}e^{-\nu|x|/2}.
$$

As for the first derivative of $B(x)$, we have

$$
\check{B}(x) = i\alpha B(x) + \delta(\mathcal{I}_1 - \mathcal{I}_2) + \delta^{-1}R(x),
$$

hence, by (30), (28), (29) and by hypothesis, respectively

$$
(31) \qquad |\check{B}(x)| \leq M(4\alpha(\nu\delta)^{-1}e^{(\alpha+2\nu)\hat\rho} + 4\nu^{-1}e^{(\alpha+\nu)\hat\rho} + \delta^{-1})e^{-\nu|x|}
$$

$$
\leq 4Me^{(\alpha+2\nu)\hat\rho}(2+\alpha)\nu^{-1}e^{-\nu|x|/2}.
$$

Case $\beta < \alpha$. In this case $\delta = i\gamma$ with $\gamma > 0$. We shall choose K_1 and $\mathcal{I}_1$ as in (25) and (26), respectively, and

$$
\mathcal{I}_2 := e^{i(\alpha-\gamma)x}\left(K_2 - \frac{1}{2\delta}\int_0^x e^{-i(\alpha-\delta)y}R(y)dy\right) \; ; \; K_2 := \frac{1}{2\delta}\int_0^{+\infty}e^{-i(\alpha+\gamma)y}R(y)dy.
$$

Note that $|K_2| < \infty$ by hypothesis on $R(x)$. The procedure is similar to (28) and (29) simply by replacing δ with $i\gamma$. The only difference lies in the term $\exp(\gamma\eta)$ which can be bounded by $\exp(|\delta_+|)$. This yields

$$
(32) \qquad
\begin{aligned}
|\mathcal{I}_1| &\leq 2Me^{(\alpha+\nu)\hat{\rho}}(\gamma\nu)^{-1}e^{-\nu\xi/2} \\
|\mathcal{I}_2| &\leq 2Me^{[(\alpha+\nu)\hat{\rho}+|\delta_+|]}(\gamma\nu)^{-1}e^{-\nu\xi/2}
\end{aligned} \; ,
$$

then

$$
(33) \qquad
\begin{aligned}
|B(x)| &\leq 4Me^{(\alpha+2\nu)\hat{\rho}}(\nu\gamma)^{-1}e^{-\nu|x|/2} \\
|\check{B}(x)| &\leq 4Me^{[(\alpha+\nu)\hat{\rho}+|\delta_+|]}\nu^{-1}e^{-\nu|x|/2}
\end{aligned} \; .
$$

Case $\beta = \alpha$. This case ($\delta = 0$) can be treated in a similar way. We only mention that one can choose

$$
K_1 = \int_0^{+\infty} \xi' e^{-i\alpha\xi'} R(\xi')d\xi', \qquad K_2 = -\int_0^{+\infty} e^{-i\alpha\xi'} R(\xi')d\xi',
$$

obtaining

$$
(34) \qquad |B(x)|, |\check{B}(x)| \leq 16M(1+\alpha)\nu^{-3}e^{(\alpha+2\nu)\hat{\rho}}e^{-\nu|x|/2}.
$$

The required bounds follow immediately by collecting those obtained in (30), (31), (33) and (34) and using the fact that $|\delta_m|^{-1} \leq 2[\delta_-(1+|\delta_m|)]^{-1}$. $\qquad\square$

4.2. A bound on certain brackets of two sequences. Let us now consider two sequences $\mathbf{f} := \{f_n(x)\}_{n\in\mathbb{Z}}$ and $\mathbf{g} := \{g_n(x)\}_{n\in\mathbb{Z}}$, then define (formally), for all $m \in \mathbb{Z}$,

$$
(35) \qquad [\mathbf{f}, \mathbf{g}]_m := \sum_{l\in\mathbb{Z}} l(f_l\check{g}_{m-l} - g_l\check{f}_{m-l}),
$$

which is a sequence of functions indexed by m.

PROPOSITION 4.3. *Let $\{f_l(x)\}$ and $\{g_l(x)\}$ be two sequences, with $f_l(x), g_l(x) : \mathcal{S}_{\hat{\rho}} \to \mathbb{C}$ for all $l \in \mathbb{Z}$, such that, for all $\hat{\rho} < \rho$ and all $l \in \mathbb{Z} \setminus \{0\}$,*

$$
(36) \qquad |f_l(x)| \leq \frac{K(x)}{|l|}e^{-|l|\hat{\rho}}, \qquad |g_l(x)| \leq L(x)e^{-|l|\rho}
$$

where $K(x)$ and $L(x)$ are bounded functions. Then, denoted $\delta := \rho - \hat{\rho} > 0$, one has

$$
(37) \qquad |[\mathbf{f}, \mathbf{g}]_m| \leq 8K(x)L(x)\delta^{-3}e^{-|m|\hat{\rho}}.
$$

PROOF. Along the lines of [**Gio03**, Lemma 4.1], by a Cauchy estimate

$$
|\check{g}_l(x)| \leq \delta^{-1}|g_l(x)|, \quad \forall x \in \mathcal{S}_{\hat{\rho}},
$$

hence

$$
\begin{aligned}
|[\mathbf{f}, \mathbf{g}]_m| &\leq K(x)L(x)\delta^{-1}\sum_{l\neq 0}\left[e^{-|l|\hat{\rho}}e^{-|m-l|\rho} + |l|e^{-|l|\rho}e^{-|m-l|\hat{\rho}}\right] \\
&= K(x)L(x)\delta^{-1}\sum_{l\neq 0}\left[e^{-|m-l|\delta} + |l|e^{-|l|\delta}\right]e^{-(|l|+|m-l|)\hat{\rho}} \\
&\leq 4K(x)L(x)\delta^{-1}\left[\delta^{-1} + \delta^{-2}\right]e^{-|m|\hat{\rho}},
\end{aligned}
$$

where we have used the elementary inequality $\sum_{l\geq 1} l^a \exp(-l\delta) \leq 2\delta^{-(1+a)}$, for all $\delta \in (0, 1/2]$ and $a = 0, 1$. $\qquad\square$

REMARK 4.4. We anticipate here that the presence of the term $|l|^{-1}$ in the first of (36) will play a crucial role in the proof of Lemma 5.2. Without its contribution it would have not been possible to "preserve" the term $\exp(-|m|\hat{\rho})$ in (37). As it is well known in the real-analytic functions context, see e.g. [**Gio03**], the absence of this term would have implied a domain restriction, namely, the $\exp(-|m|\hat{\rho})$ above, would have been replaced by a term of the form $Cd^{-1}\exp(-(1-d)|m|\hat{\rho})$, for some $d > 0$ (restriction) and some constant $C > 0$.

As a standard procedure in perturbation theory one chooses a sequence of d's e.g. $d_j \sim j^{-2}$ in such a way the final domain $\Pi_{j\geq 1}(1-d_j)\rho$ is non-trivial. Unfortunately, the accumulation of these d_j^{-1} would invalidate the convergence argument of Lemma 5.2 as can be easily seen from (50) and (51). This kind of obstruction to Cauchy's majorants method is a well known phenomenon and it cannot be overcome unless the above mentioned "artificial" *small divisors* are controlled with some specialized argument, see for instance [**Gio12**].

5. The perturbative setting

5.1. Formal scheme. Let us consider equation (1) in which the bathymetry has been chosen of the form (2) and write

$$(38) \qquad \psi(x,y,t) = \psi^{(0)}(x,y,t) + \tilde{\psi}(x,y,t),$$

then expanding[6], formally

$$(39) \qquad \tilde{\psi}(x,y,t) = \sum_{j=1}^{\infty} \lambda^j \psi^{(j)}(x,y,t),$$

where λ is an auxiliary parameter which will be thought to be equal to one (*bookkeeping parameter*, see e.g. [**Eft12**]). On the other hand, let $g := \mu\tilde{g}$, then we get

$$J(\psi, \lambda g - d) = \sum_{j=1}^{\infty} \lambda^j \left[\partial_x \psi^{(j-1)} \partial_y g - \partial_y \psi^{(j-1)} \partial_x g \right] = \sum_{j=1}^{\infty} \lambda^j J(g, \psi^{(j-1)}).$$

In this setting, equation (1) is equivalent to the following hierarchy of recursive equations (obtained by balancing the powers of λ)

$$(40) \qquad \partial_t(\Delta\psi^{(j)} - F\psi^{(j)}) + \mathcal{F}\partial_x \psi^{(j)} = J(g, \psi^{(j-1)}),$$

for all $j \geq 1$, where the order zero is given by (5).

The following statement shows the possibility to solve, at least formally, the equations of the hierarchy up to an arbitrarily high order in a constructive way.

PROPOSITION 5.1. *For all $j \geq 1$, it is possible to determine a sequence of functions $\{b_{m,n}^{(j)}(x)\}$ satisfying*

$$(41\text{a}) \qquad b_{-m,n}^{(j)}(x) = -b_{m,n}^{(j)}(x)$$

$$(41\text{b}) \qquad b_{\mp m,n}^{(j)}(x) = \bar{b}_{\pm m,-n}^{(j)}(x)$$

---

[6]A similar expansion of the streamfunction, despite with respect to a different parameter, was already considered in [**RB73**].

in particular $b_{0,n}^{(j)}(x) \equiv 0$ ($\bar{z}$ denotes the complex-conjugate of $z \in \mathbb{C}$) and such that the elements

$$\psi^{(j)}(x,y,t) = \sum_{(m,n) \in \mathcal{J}} b_{m,n}^{(j)}(x) e^{i(my+\sigma(n)t)}, \tag{42}$$

where $\mathcal{J} := \{(m,n) : m \in \mathbb{Z} \setminus \{0\}, n = \pm\kappa\}$, are real solutions of the hierarchy (40) for all $(x,y) \in \mathcal{C}$ and all $t \in \mathbb{R}$, under the boundary conditions (4).

PROOF. Let us preliminarily observe that, from the standard theory of Fourier series, the conditions (41a) and (41b) hold if, and only if, $\psi^{(j)}(x,y,t)$ is odd in y and real, respectively. In particular, (41a) implies that the conditions (4) are satisfied. Note that (6) can be written in the form (42) with

$$b_{m,n}^{(0)}(x) = \begin{cases} A(4i)^{-1} \operatorname{sign}(m) \exp(inx) & \text{if} \quad (m,n) = (\pm\tilde{m}, \pm\kappa) \\ 0 & \text{otherwise} \end{cases}. \tag{43}$$

Hence (41a) holds, being (41b) obvious.

Then we can suppose that (42) satisfies the equations (40) with $b_{m,n}^{(j)}(x)$ recursively determined and satisfying (41a) and (41b) up to a level $j-1$ and proceed by induction. Let us write

$$g(x,y) = \sum_{l \in \mathbb{Z} \setminus \{0\}} g_l(x) e^{ily},$$

where, by hypothesis, $g_0(x) \equiv 0$ and

$$g_{-l}(x) = -g_l(x), \quad g_{-l}(x) = \bar{g}_l(x). \tag{44}$$

Hence,

$$\begin{aligned}
J(g, \psi^{(j-1)}) &= i \sum_{n=\pm\kappa} e^{i\sigma(n)t} \left[\left(\sum_l \check{g}_l e^{ily} \right) \left(\sum_m m b_{m,n}^{(j-1)} e^{imy} \right) \right. \\
&\quad \left. - \left(\sum_l l g_l e^{ily} \right) \left(\sum_m \check{b}_{m,n}^{(j-1)} e^{imy} \right) \right] \\
&= i \sum_{n=\pm\kappa} e^{i\sigma(n)t} \sum_{m \neq 0} \left[\sum_{k'} k' \left(b_{k',n}^{(j-1)} \check{g}_{m-k'} - g_{k'} \check{b}_{m-k',n}^{(j-1)} \right) \right] \\
&= i \sum_{(m,n) \in \mathcal{J}} [\mathbf{b}_{\cdot,n}^{(j-1)}, \mathbf{g}]_m e^{i(my+\sigma(n)t)},
\end{aligned}$$

where the formula of the product à la Cauchy, the definition (35) and the fact that $[\mathbf{b}_{\cdot,n}^{(j-1)}, \mathbf{g}]_0 = 0$ due to (41a) and (44) have been used.

On the other hand, by substituting $\psi^{(j)}$ as in (42) in the l.h.s. of (40), we get that the latter is satisfied if the following linear differential equation

$$\sigma(n) \check{\check{b}}_{m,n}^{(j)} - i\mathcal{F}\check{b}_{m,n}^{(j)} - \sigma(n)(F + m^2) b_{m,n}^{(j)} = [\mathbf{b}_{\cdot,n}^{(j-1)}, \mathbf{g}]_m, \tag{45}$$

holds true for all $(m,n) \in \mathcal{J}$. Note that $[\mathbf{b}_{\cdot,n}^{(j-1)}, \mathbf{g}]_m$ is a function of x, known by hypothesis. By dividing both sides of (45) by $\sigma(n) < 0$, we have that equation (45) is of the form (21). Hence $b_{m,n}^{(j)}(x)$ can be uniquely determined by (24). In addition, $b_{m,n}^{(j)}(x)$ satisfies the relations (41a) and (41b) if $[\mathbf{b}_{\cdot,n}^{(j-1)}, \mathbf{g}]_m$ does. The latter property is a straightforward check under the hypothesis (44).

This completes the proof and the formal resolvability of the hierarchy (40). $\qquad\square$

5.2. Convergence. In this section, the convergence of the formal scheme built in the previous section is addressed, and it can be stated as in the following

LEMMA 5.2. *Set* $\beta_m^2 := F + m^2$ *and let* m^* *be such that* $\delta_{m^*} = \delta_-$ *as defined in Proposition 4.1. If* μ *satisfies*

$$(46) \qquad \mathcal{L}(\mu) := \frac{32\mu\mathcal{M}\mathfrak{G}(|m^*| + 1)}{\sigma\rho^2} \leq \frac{1}{2},$$

then $\tilde{\psi}$, *as defined in (39), is a real-analytic function on* $\mathcal{C}_{\rho/4}$, *satisfying*

$$(47) \qquad ||\tilde{\psi}(x, y, t)||_{\rho/4} \leq C(\mu)e^{-\nu\frac{|x|}{2}}.$$

where $C(\mu)$ *is a* $O(\mu)$ *constant.*

Before proceeding with the proof, let us observe that the statement above implies Theorem 2.2. The term $H_0(I)$ is given by Lemma 3.1. On the other hand, by setting

$$H_1(x, y, t) := \tilde{\psi} \circ \mathcal{T},$$

where $\mathcal{T}$ has been defined in (20) we clearly have H_1 is $\mathcal{O}(\mu)$ by (47). Furthermore, as $\tilde{\psi}$ is real-analytic on $\mathcal{C}_{\rho/4}$ and $\mathcal{T}$ is a real-analytic map then H_1 will be real-analytic as well, more precisely in $\mathcal{U}_{\tilde{\gamma}}$ for some $\tilde{\gamma} > 0$. Finally, it is clear that the transformation $\mathcal{G}$ itself, produces a linear growth in time of $\Re x$. Hence, the function $\tilde{\psi}$ which varies with a general dependence in x, see (42), will be (at least in general) *aperiodic* in time and, by (47) and (8), it will decay in time as in (14). □

PROOF. Following the classical approach of the Cauchy majorants, the strategy consists in showing, by induction, that there exists a suitable infinitesimal sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that

$$(48) \qquad |m||b_{m,n}^{(j)}(x)|, |\check{b}_{m,n}^{(j)}(x)| \leq \begin{cases} \epsilon_0 e^{-|m|\frac{\rho}{2}} & \text{if } j = 0 \\ \epsilon_j e^{-|m|\frac{\rho}{2}} e^{-\nu\frac{|x|}{2}} & \text{otherwise} \end{cases}.$$

First of all we note that, only a finite number of $b_{m,n}^{(0)}$ are different from zero, i.e., $b_{m,n}^{(0)}(x) \equiv 0$ for all $|m| \geq \tilde{M}$. Furthermore, by (43)

$$\tilde{\epsilon} := \max_{\substack{|m|\leq\tilde{M} \\ n=\pm\kappa}} \{\sup_{x\in\mathcal{S}_\rho} |b_{m,n}^{(0)}(x)|, \sup_{x\in\mathcal{S}_\rho} |\check{b}_{m,n}^{(0)}(x)|\} = \frac{\kappa A}{4}.$$

By setting $\epsilon_0 := \tilde{M}\tilde{\epsilon}$ the induction basis ($j = 0$) is clearly true.
Let us suppose the statement to be true for j and proceed by induction. By (12) we have

$$(49) \qquad |g_l(x)| \leq \epsilon e^{-|l|\rho} e^{-\nu|x|}, \qquad \forall l \in \mathbb{Z} \setminus \{0\},$$

hence we can set $K(x) := \epsilon_j$ (we disregard here the term $\exp(-\nu|x|/2)$) and $L(x) := \epsilon\exp(-\nu|x|)$ and we use Proposition 4.3 with $\hat{\rho} = \rho/2$. The latter yields, in particular,

$$|[\mathbf{b}_{\cdot,n}^{(j)}, \mathbf{g}]_m| \leq 32\epsilon_j\epsilon\rho^{-1} e^{-|m|\frac{\rho}{2}} e^{-\nu|x|}.$$

We are now able to give estimate on the solutions of equation (45). In fact, by setting $\alpha \leftarrow \mathcal{F}/(2\sigma)$ and β_m as in the statement, the equation (45) is of the form (21) with $R := \sigma^{-1}[\mathbf{b}_{\cdot,n}^{(j)}, \mathbf{g}]_m$.
Hence, by Proposition 4.1 (with $\hat{\rho} \leftarrow \rho/2$) we get

$$|m||b_{m,n}^{(j+1)}(x)|, |\check{b}_{m,n}^{(j+1)}(x)| \leq 32\mathfrak{G}(|m^*| + 1)\sigma^{-1}\rho^{-2}\epsilon\epsilon_j e^{-|m|\frac{\rho}{2}} e^{-\nu|x|/2},$$

where we have used the bound $(1 + |\delta_m|)^{-1} \le (|m^*| + 1)/|m|$. Hence recalling that $\epsilon = \mu\mathcal{M}$ and by setting

$$(50) \qquad \epsilon_{j+1} := \mathcal{L}(\mu)\epsilon_j,$$

the inductive step is complete.

Furthermore, (50) implies that $\epsilon_j := \mathcal{L}^j(\mu)\epsilon_0$, then by (48) and condition (46),

$$
\begin{aligned}
(51) \qquad \left\|\tilde{\psi}\right\|_{\rho/4} &\le \sum_{j \ge 1} \sum_{(m,n) \in \mathcal{J}} |b_{m,n}^{(j)}(x)| e^{|m|\frac{\rho}{4}} \\
&\le 2e^{-\nu|x|/2} \sum_{j \ge 1} \epsilon_j \sum_{m \in \mathbb{Z}\backslash\{0\}} e^{-|m|\frac{\rho}{4}} \\
&\le 4\epsilon_0 \mathcal{L}(\mu) e^{-\frac{\rho}{4}} \left[(1 - e^{-\frac{\rho}{4}})(1 - \mathcal{L}(\mu))\right]^{-1} e^{-\nu|x|/2},
\end{aligned}
$$

which implies (47). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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