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On the inviscid limit of the 2D Magnetohydrodynamic system with vorticity in Yudovich-type space

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ABSTRACT. In this paper, we first prove the existence and uniqueness of solutions only with magnetic diffusion for the vorticity being Yudovich-type space, by establishing some new time weighted estimates of the magnetic field, which improves the corresponding results of [**5**] and [**16**]. Furthermore, we prove a global result on the inviscid limit of the two-dimensional Magnetohydrodynamic equations with data belonging to the Yudovich type.

CONTENTS

1. Introduction

The n-dimensional ($n \geq 2$) generalized Magnetohydrodynamic (GMHD) system can be written as

$$
(1.1)\qquad \begin{cases} u_t + u\cdot\nabla u + \nu\Lambda^{2\alpha}u = -\nabla p + b\cdot\nabla b,\ (x,t)\in\mathbb{R}^n\times\mathbb{R}^+ \\ b_t + u\cdot\nabla b + \kappa\Lambda^{2\beta}b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ (u,b)(x,0) = (u_0(x), b_0(x)), \end{cases}
$$

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where $u = u(x,t)$ denotes the velocity of the fluid, $b = b(x,t)$ stands for the magnetic field and $p = p(x,t)$ the scalar pressure. Here $\nu \geq 0$ is the viscosity of the flow and $\kappa \geq 0$ represents the resistive viscosity of the magnetic field. The parameters α and β are nonnegative constants. The operator Λ^s with $s \in \mathbb{R}$ is defined through Fourier transform

$$\Lambda = (-\Delta)^{\frac{1}{2}}, \ \widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi),$$

with

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

The nonlinear evolution equations involving the classical Laplacian describing the Wienner diffusion have been extensively studied in mathematics and physics. For the fractional Laplacian, it is remarked that it can be defined as the generator of α-stable Lévy processes, and it has found applications in many complicated engineering problems, for instance, Woyczyńskiin in [**25**] review a number of physical phenomena for which the Lévy processes and, in particular, α-stable processes can be used as a reasonable model.

There are extensive studies on global well-posedness of system (1.1). It has been proved in [**27**] that system (1.1) is globally well-posed as long as the following conditions

$$\alpha \geq \frac{1}{2} + \frac{n}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{n}{2} \tag{1.2}$$

are satisfied. When $n \geq 3$, the conditions (1.2) are sharp since the global regularity problem for the n-dimensional generalized Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nu \Lambda^{2\alpha} u = -\nabla p, \quad \nabla \cdot u = 0$$

is still challenging for $\alpha < \frac{1}{2} + \frac{n}{4}$. Indeed, the global well-posedness was proved in [**26**] if $\alpha \geq \frac{1}{2} + \frac{n}{4}$ and in [**22**] by adding a logarithmical multiplier in the dissipation $\Lambda^{1+\frac{n}{2}}$. Some regularity criteria to the 3-dimensional MHD equations in terms of the velocity field or the magnetic field are referred to [**11, 13, 24, 28**] and references therein. However, when $n = 2$, the conditions in (1.2) can be slightly weakened. Actually, Tran-Yu-Zhai in [**23**] showed that if $(u_0, b_0) \in H^s (s > 2)$, the smooth solutions to (1.1) are global in one of the following three cases:

$$\text{(i)}\ \alpha \geq \frac{1}{2},\ \beta \geq 1; \quad \text{(ii)}\ 0 \leq \alpha < \frac{1}{2},\ 2\alpha + \beta > 2; \quad \text{(iii)}\ \alpha \geq 2,\ \beta = 0. \tag{1.3}$$

In particular, they obtained the global regularity when $\alpha = 0$ and $\beta > 2$. Later, (1.3) was relaxed to the range $\alpha = 0$, $\beta > \frac{3}{2}$ in [**15**], [**29**] and [**30**]. Recently, when $\alpha = 0$, $\beta > 1$ the global well-posedness for system (1.1) was shown whenever the initial data (u_0, b_0) belongs to H^s for any $s > 2$ by Jiu and Zhao [**16**] and independently by Cao, Wu and Yuan [**5**]. Meanwhile, Fan and *etc.* [**12**] proved the global and unique regular solutions exist when $0 < \alpha < \frac{1}{2}$, $\beta = 1$. In addition, the global well-posedness on the 2-dimensional MHD equations with partial dissipations has been studied widely (see [**4, 6, 14, 18**] and references therein).

The behavior of viscous incompressible flows at the inviscid limit is a classical issue in the fluid dynamics. In bounded domain, the zero-viscosity limit of the incompressible Navier-Stokes equation with Dirichlet boundary condition is the challenging problems in fluid mechanics (see [**19**] and references therein). For the case of the whole space, it is well known that the solution of the Navier-Stokes equations converges to the one of the Euler equations. There are many researches devoting to this issue; for instance, we can refer to [**17, 10, 9, 7, 8, 2**].

In this paper, we will investigate the global well-posedness of system (1.1) with $\nu = 0$, $\kappa = 1$ and $\beta > 1$. That is,

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \ (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+ \\ b_t + u \cdot \nabla b + \Lambda^{2\beta} b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ (u,b)(x,0) = (u_0(x), b_0(x)). \end{cases} \tag{1.4}$$

In addition, we will also consider the inviscid limit of the following viscous system with $0 < \alpha \leq 1$ and $\beta > 1$

$$\begin{cases} u^\nu_t + u^\nu \cdot \nabla u^\nu + \nu \Lambda^{2\alpha} u^\nu = -\nabla p^\nu + b^\nu \cdot \nabla b^\nu, \ (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+ \\ b^\nu_t + u^\nu \cdot \nabla b^\nu + \Lambda^{2\beta} b^\nu = b^\nu \cdot \nabla u^\nu, \\ \nabla \cdot u^\nu = 0, \quad \nabla \cdot b^\nu = 0, \\ (u^\nu, b^\nu)(x,0) = (u_0(x), b_0(x)). \end{cases} \tag{1.5}$$

Our first goal here is to establish the global existence and uniqueness of solutions to system (1.4) under the low regularity assumptions on the initial data. We assume here that the initial vorticity $\omega_0 = \partial_1 u_0^2 - \partial_2 u_0^1$ is in the Yudovich class and $b_0 \in H^1$, and obtain the following result.

THEOREM 1.1. *Let $\beta > 1$. Suppose that $u_0 \in L^2$ with $\mathrm{div} u_0 = 0$ and $\omega_0 \in L^2 \cap L^\infty$, $b_0 \in H^1$ with $\mathrm{div} b_0 = 0$. Then, for any $T > 0$, system (1.4) admits a unique global solution $(u,b) \in L^\infty(0,T;H^1) \times (C([0,T];H^1) \cap L^2(0,T;H^{\beta+1}))$ satisfying*

$$\|u(t)\|^2_{H^1} + \|b(t)\|^2_{H^1} + \int_0^t \|b(s)\|^2_{H^{\beta+1}} ds \leq Ce^{Ct}, \quad \forall \ 0 < t \leq T.$$

$$\|\omega(t)\|_{L^\infty} \leq Ce^{Ct}, \quad \forall \ 0 < t \leq T.$$

Furthermore, for all $0 < t \leq T$,

$$t^s \|b(t)\|_{L^\infty} \leq Ce^{Ct}, \quad \forall \ s > 0,$$

and

$$t^{1-\epsilon} \|\nabla \nabla \times b(t)\|_{L^\infty} \leq Ce^{Ct}, \quad \text{for arbitrarily samll} \quad \epsilon > 0.$$

If we further assume $b_0 \in H^\beta$, then the solution remains the same regularity for all time, namely

$$\|b(t)\|^2_{H^\beta} + \int_0^t \|b(\tau)\|^2_{H^{2\beta}} d\tau \leq Ce^{Ct}, \quad \forall \ 0 < t \leq T.$$

Here the constants $C > 0$ only depend on the initial data.

REMARK 1.2. Theorem 1.1 extends the result in [**31**] on the 2D Euler system to the system (1.4). Compared with the papers by Jiu-Zhao [**16**] and Cao-Wu-Yuan [**5**] in which the initial data belongs to $H^s (s > 2)$, less regularity has been imposed on the initial data here; consequently, a different and simpler method has been used.

We describe the difficulty in dealing with the global regularity problem and explain how our method works. Although the magnetic diffusion does provide certain regularity, it fails to produce the crucial global bounds for the vorticity equation. More precisely, energy estimates do yield for all $T>0$, $(u,b)\in L^\infty(0,T;H^1)$ and $b\in L^2(0,T;H^{\beta+1})$. However, it is not clear whether $\|\omega\|_{L^\infty(0,T;L^\infty)}$ is bounded. The lack of a global bound on ∇j ($j=\mathrm{curl}b$) in L^∞ makes it impossible to obtain a global bound for vorticity ω directly from the vorticity equation

$$\omega_t+u\cdot\nabla\omega=b\cdot\nabla j.$$

To overcome this difficulty, we add the weight $t^{1-\epsilon}$ to ∇j in order to reduce time integration of $\|\nabla j\|_{L^\infty}$. Therefore, for sufficiently large k and sufficiently small ϵ,

$$\|\omega\|_{L^\infty(0,T;L^\infty)}\le\|\omega_0\|_{L^\infty}+t^{\epsilon-\frac{1}{k\beta}}\sup_{0\le\tau\le t}\|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty}\sup_{0\le\tau\le t}\|\tau^{\frac{1}{k\beta}}b(\tau)\|_{L^\infty}, \tag{1.6}$$

where the global bound on $t^{\frac{1}{k\beta}}b(t)$ in L^∞ can be obtained (in light of the smooth effect of the magnetic equation, for details see Proposition 2.6). Since j satisfies the following fractional power dissipation equation

$$j_t+\Lambda^{2\beta}j=-u\cdot\nabla j+b\cdot\nabla\omega+Q(\nabla u,\nabla b),$$

by using the decay properties on time of the fractional power dissipation kernel, we are enable to obtain the weighted estimate of ∇j as follows:

$$\sup_{0\le t\le T}\|t^{1-\epsilon}\nabla j(t)\|_{L^\infty}\le C\Big(\|j_0\|_{L^2}+(\sup_{0\le\tau\le t\le T}\|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty})^{1-\frac{2}{p}}+\|\omega\|^{1-\frac{1}{p}}_{L^\infty(0,T;L^\infty)}\Big), \tag{1.7}$$

for some large number p ($2\le p<\infty$) and small enough number ϵ (see Proposition 2.6). Combining (1.6) with (1.7) and using the Young inequality, we get $\|\omega\|_{L^\infty(0,T;L^\infty)}<\infty$ for all $T>0$.

Our second result is a global result on the inviscid limit of system (1.5) with the same initial data as in Theorem 1.1.

THEOREM 1.3. *Let $u_0\in L^2$ be a divergence free vector field such that $\omega_0\in L^2\cap L^\infty$ and $b_0\in H^1$ with $\mathrm{div}b_0=0$. Assume that (u^ν,b^ν)(resp.(u,b)) is the solution of system (1.5) with $0<\alpha\le1$ and $\beta>1$ (resp. system (1.4) with $\beta>1$). Then, for any $T>0$, there exists a constant $C>0$ depending only on the initial data and $\nu_0>0$ depending on C and T such that for any $\nu\le\nu_0$, we have*

$$\|(u^\nu,b^\nu)-(u,b)\|_{L^2}\le(Ce^{e^{Ct}}\nu)^{e^{-e^{e^{Ct}}}},\quad\forall\ 0<t\le T.$$

The paper unfolds as follows. In Section 2, a priori estimates are established which play a vital role in proving the main results. Section 3 is devoted to the proof of Theorem 1.1 while Section 4 is devoted to the proof of Theorem 1.3. In the appendix part, for readers' convenience, we introduce briefly the Littlewood-Paley theory and inhomogeneous Besov spaces.

Throughout the paper, the $L^p(\mathbb{R}^2)$-norm of a function f is denoted by $\|f\|_{L^p}$ and the $H^s(\mathbb{R}^2)$-norm by $\|f\|_{H^s}$. In the following, we denote C a constant which may depend on $T>0$ and may be different from line to line.

2. Preliminary bounds

This section proves some useful the a priori bounds.

2.1. Basic Energy estimates.

PROPOSITION 2.1. *Let* (u, b) *be a smooth solution of system* (1.4) *with* $\beta \geq 0$. *Then, for any* $t \geq 0$, *we have*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\beta b(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{2.1}$$

PROOF. Taking the L^2-inner product of (1.4) yields

$$\frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\Big) + \|\Lambda^\beta b(t)\|_{L^2}^2 = 0,$$

where we have used the fact that $\int (b\cdot\nabla b)\cdot u\ dx + \int (b\cdot \nabla u)\cdot b\ dx = 0$. Integrating the above inequality with respect to t, one can get (2.1). □

2.2. Global H^1-bound for (u, b). The subsection provides the global H^1-bound for (u, b). Note that $\omega = \operatorname{curl} u$ and $j = \operatorname{curl} b$ satisfy the following equations

$$\begin{cases} \omega_t + u\cdot\nabla\omega = b\cdot\nabla j, \\ j_t + u\cdot\nabla j + \Lambda^{2\beta} j = b\cdot\nabla\omega + Q(\nabla u, \nabla b), \\ \omega(x,0) = \omega_0,\ j(x,0) = j_0, \end{cases} \tag{2.2}$$

where

$$Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1).$$

Then we have

PROPOSITION 2.2. *Let* (u, b) *be a smooth solution of system* (1.4) *with* $\beta \geq 1$. *Then, there exists a constant* $C = C(\|(u_0, b_0)\|_{H^1})$ *such that, for all* $t > 0$,

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta j(s)\|_{L^2}^2 ds \leq Ce^{Ct}, \tag{2.3}$$

and consequently

$$\|u(t)\|_{H^1}^2 + \|b(t)\|_{H^1}^2 + \int_0^t \|b(s)\|_{H^{\beta+1}}^2 ds \leq Ce^{Ct}. \tag{2.4}$$

Furthermore, when $\beta = 1$, *the left hand side of* (2.3) *and* (2.4) *can be bounded by an absolute constant* C *depending only on the initial data.*

PROOF OF PROPOSITION 2.2. Taking the inner product of the vorticity equation with ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 = \int (b\cdot\nabla)j\cdot\omega\ dx.$$

Similarly, taking the inner product of the current equation with j, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|j(t)\|_{L^2}^2 + \|\Lambda^\beta j(t)\|_{L^2}^2 = \int (b\cdot\nabla)\omega\cdot j\ dx + \int Q(\nabla u, \nabla b) j\ dx.$$

Note that

$$\int (b\cdot\nabla)j\cdot\omega\ dx + \int (b\cdot\nabla)\omega\cdot j\ dx = 0.$$

It follows that

$$\frac{d}{dt}\Big(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2\Big) + 2\|\Lambda^\beta j(t)\|_{L^2}^2 = 2\int Q(\nabla u, \nabla b) j\ dx.$$

By using Hölder's inequality, the Gagliardo-Nirenberg inequality and Lemma A.4, we get

$$\begin{aligned}\int Q(\nabla u,\nabla b)j\,\mathrm{d}x \leq&\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}\|j\|_{L^4}\\ \leq& C\|\omega\|_{L^2}\|j\|_{L^4}^2\\ \leq& C\|\omega\|_{L^2}\|j\|_{L^2}^{2-\frac{1}{\beta}}\|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}}.\end{aligned}$$

By Young's inequality, it is clear to verify that

$$\begin{aligned}\frac{d}{dt}\Big(\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\Big)+\|\Lambda^\beta j(t)\|_{L^2}^2 \leq& C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2\beta}{2\beta-1}}\\ \leq& C\|j\|_{L^2}^2(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2+1).\end{aligned} \tag{2.5}$$

For $\beta>1$, Sobolev embedding gives

$$\|j\|_{L^2}\leq C\|\nabla b\|_{L^2}\leq C\|b\|_{H^\beta}.$$

Plugging this inequality into (2.5) and making use of the Gronwall lemma and (2.1) yield the desired result. ☐

2.3. Global L^q-bound for ω. In this subsection, we prove a global a priori bound for ω in the space L^q for $2\leq q<\infty$. More precisely, we prove the following proposition.

PROPOSITION 2.3. *Let (u,b) be a smooth solution of system (1.4) with $\beta>1$. Then, for any*

$$2\leq q\leq\frac{2}{2-\beta},\ \textit{if } 1<\beta<2$$
$$2\leq q<\infty,\ \textit{if } \beta\geq 2,$$

there exists a constant $C=C(\|(u_0,b_0)\|_{H^1},\|\omega_0\|_{L^\infty})$ such that

$$\|\omega(t)\|_{L^q}\leq Ce^{Ct}, \tag{2.6}$$

holds for $t>0$.

PROOF. Multiplying the vorticity equation by $|\omega|^{q-2}\omega$ with $2\leq q<\infty$ and integrating the resulting equation over $\mathbb{R}^2$, we have

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^q}^q=\int b\cdot\nabla j|\omega|^{q-2}\omega\,dx\leq\|b\|_{L^\infty}\|\nabla j\|_{L^q}\|\omega\|_{L^q}^{q-1}.$$

It follows that

$$\|\omega(t)\|_{L^q}\leq\|\omega_0\|_{L^2\cap L^\infty}+\int_0^t\|b(s)\|_{L^\infty}\|\nabla j(s)\|_{L^q}\,ds. \tag{2.7}$$

Recall the Sobolev inequalities

$$\|b\|_{L^\infty}\leq C\|b\|_{H^\beta},\quad \beta>1,$$

and

$$\|\nabla j\|_{L^q}\leq C\|\nabla j\|_{H^{\beta-1}}\leq C\|j\|_{H^\beta},$$

where

$$2\leq q\leq\frac{2}{2-\beta},\ \text{if } 1<\beta<2$$
$$2\leq q<\infty,\ \text{if } \beta\geq 2.$$

By plugging these two Sobolev inequalities into (2.7) and using Young's inequality and the estimates in Proposition 2.2, we get the desired estimate (2.6). The proof of the proposition is finished. □

REMARK 2.4. Since the following Sobolev inequality

$$\|u\|_{L^\infty} \leq C\,\|u\|_{L^2}^{\frac{q-2}{2(q-1)}}\,\|\nabla u\|_{L^q}^{\frac{q}{2(q-1)}} \leq C\,\|u\|_{L^2}^{\frac{q-2}{2(q-1)}}\,\|\omega\|_{L^q}^{\frac{q}{2(q-1)}}$$

holds for $q > 2$, it follows from (2.1) and (2.6) that

$$\|u\|_{L^\infty(0,t;L^\infty)} \leq Ce^{Ct}. \tag{2.8}$$

2.4. Yudovich-type regularity through decay estimates of $\|\nabla j\|_{L^\infty}$. The goal of this subsection is to establish the Yudovich type regularity for the vorticity ω, namely, the estimate of $\|\omega\|_{L^\infty(0,t;L^\infty)}$. Due to the lack of a global bound on ∇j in L^∞, the global bound for vorticity ω does not follow directly from the vorticity equation

$$\omega_t + u\cdot\nabla\omega = b\cdot\nabla j.$$

To overcome this difficulty, we add the weight $t^{1-\epsilon}$ to ∇j in order to reduce time integration of $\|\nabla j\|_{L^\infty}$. More details will be provided in the following Proposition 2.7.

Let us first recall some decay estimates of solutions to the homogeneous linear fractional power dissipative equations.

LEMMA 2.5 ([**21**]). *There exists a constant $C > 0$ such that for every solution v of the Cauchy problem*

$$\begin{cases} v_t + \Lambda^{2\alpha} v = 0, & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ v(x,0) = v_0(x), \end{cases} \tag{2.9}$$

the following estimates

$$\|v\|_{L^p(\mathbb{R}^n)} = \|S_\alpha(t)v_0\|_{L^p(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}\|v_0\|_{L^r(\mathbb{R}^n)},$$

$$\|\Lambda^\nu v\|_{L^p(\mathbb{R}^n)} = \|\Lambda^\nu S_\alpha(t)v_0\|_{L^p(\mathbb{R}^n)} \leq Ct^{-\frac{\nu}{2\alpha}-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}\|v_0\|_{L^r(\mathbb{R}^n)},$$

hold true, for any $1 \leq r \leq p \leq \infty$, $\alpha > 0$, $\nu > 0$. Here $S_\alpha(t) = e^{-t(-\Delta)^\alpha}$.

We also need the following weighted estimate of b in $L^\infty(0,T;L^\infty)$.

PROPOSITION 2.6. *Let (u,b) be a smooth solution of system (1.4) with $\beta > 1$. Then, there exists a constant $C = C(\|(u_0,b_0)\|_{H^1})$ such that, for all $s > 0$*

$$t^s\|b(t)\|_{L^\infty} \leq Ce^{Ct}, \quad \forall\ t > 0.$$

PROOF. By Duhamel's principle, we write the second equation in system (1.4) as

$$b(x,t) = S_\beta(t)b_0 + \int_0^t S_\beta(t-\tau)(-u\cdot\nabla b + b\cdot\nabla u)(\tau)\ d\tau.$$

Using Lemma 2.5 gives

$$t^s\|S_\beta(t)b_0\|_{L^\infty} \leq Ct^{s-\frac{1}{p\beta}}\,\|b_0\|_{L^p} \leq Ct^{s-\frac{1}{p\beta}}\,\|b_0\|_{H^1}\,.$$

Since $s>0$, choosing sufficiently large p ensures $s-\frac{1}{p\beta}>0$. By Lemma 2.5 and Hölder's inequality, for some $q\geq 1$,

$$\begin{aligned}\left\|\int_0^t S_\beta(t-\tau)(-u\cdot\nabla b)(\tau)\,d\tau\right\|_{L^\infty}&\leq\int_0^t\|\nabla\cdot S_\beta(t-\tau)(ub)(\tau)\|_{L^\infty}\,d\tau\\&\leq\int_0^t(t-\tau)^{-\frac{1}{2\beta}-\frac{1}{\beta q}}\|u(\tau)\|_{L^{2q}}\|b(\tau)\|_{L^{2q}}\,d\tau\\&\leq Ct^{1-\frac{1}{2\beta}-\frac{1}{\beta q}}\|u\|_{L^\infty(0,T;H^1)}\|b\|_{L^\infty(0,T;H^1)}.\end{aligned}$$

Similarly, for some $1\leq q<\frac{1}{2-\beta}$,

$$\begin{aligned}\left\|\int_0^t S_\beta(t-\tau)(b\cdot\nabla u)(\tau)\,d\tau\right\|_{L^\infty}&\leq\int_0^t(t-\tau)^{-\frac{1}{\beta q}}\|\nabla u(\tau)\|_{L^{2q}}\|b(\tau)\|_{L^{2q}}\,d\tau\\&\leq Ct^{1-\frac{1}{\beta q}}\|\omega\|_{L^\infty(0,T;L^{2q})}\|b\|_{L^\infty(0,T;H^1)}.\end{aligned}$$

Collecting these estimates combined with (2.4) and (2.6) yields, for all $t>0$

$$t^s\|b(t)\|_{L^\infty}\leq Ce^{Ct},\quad s>0.$$

□

Now, we are ready to establish the following estimates which is a key to complete the proof of Theorem 1.1.

PROPOSITION 2.7. *Let (u,b) be a smooth solution of system (1.4) with $\beta>1$. Then, for arbitrarily small $\epsilon>0$, there exists a constant $C=C(\|(u_0,b_0)\|_{H^1},\|\omega_0\|_{L^\infty})$ such that*

$$\sup_{t>0}\|t^{1-\epsilon}\nabla j(t)\|_{L^\infty}+\|\omega\|_{L^\infty(0,T;L^\infty)}\leq Ce^{Ct}. \tag{2.10}$$

PROOF. By Duhamel's principle, the solution to the equation of j (2.2) can be written as the integral form:

$$j(x,t)=S_\beta(t)j_0+\int_0^t S_\beta(t-\tau)\big(-u\cdot\nabla j+b\cdot\nabla\omega+Q(\nabla u,\nabla b)\big)(\tau)\,d\tau. \tag{2.11}$$

Therefore, for any $t>0$ and arbitrily small ϵ, we have

$$\|t^{1-\epsilon}\nabla j(t)\|_{L^\infty}\leq t^{1-\epsilon}(J_0+J_1+J_2+J_3), \tag{2.12}$$

where

$$\begin{aligned}J_0&=\|\nabla S_\beta(t)j_0\|_{L^\infty},\\J_1&=\int_0^t\|\nabla S_\beta(t-\tau)(u\cdot\nabla)j(\tau)\|_{L^\infty}\,d\tau,\\J_2&=\int_0^t\|\nabla S_\beta(t-\tau)(b\cdot\nabla)\omega(\tau)\|_{L^\infty}\,d\tau,\\J_3&=\int_0^t\|\nabla S_\beta(t-\tau)Q(\nabla u,\nabla b)(\tau)\|_{L^\infty}\,d\tau.\end{aligned}$$

As for J_0, using Lemma 2.5 yields

$$t^{1-\epsilon}J_0\leq Ct^{1-\frac{1}{\beta}-\epsilon}\|j_0\|_{L^2}, \tag{2.13}$$

To bound J_1, by Lemma 2.5, we have for any $2 < p < \infty$,

$$J_1 \le C \int_0^t (t-\tau)^{-\frac{1}{2\beta}-\frac{1}{p\beta}} \|(u\cdot\nabla)j(\tau)\|_{L^p}\ d\tau.$$

Then, by Hölder's inequality and the Sobolev interpolation inequality, we can get

$$\begin{aligned} J_1 \le & C \int_0^t (t-\tau)^{-\frac{1}{2\beta}-\frac{1}{p\beta}} \|u(\tau)\|_{L^\infty} \|\nabla j(\tau)\|_{L^p}\ d\tau \\ \le & C \int_0^t (t-\tau)^{-\frac{1}{2\beta}-\frac{1}{p\beta}} \|u(\tau)\|_{L^\infty} \|\nabla j(\tau)\|_{L^2}^{\frac{2}{p}} \|\nabla j(\tau)\|_{L^\infty}^{1-\frac{2}{p}}\ d\tau \\ \le & C\|u\|_{L^\infty(0,t;L^\infty)} (\sup_{0\le\tau\le t} \|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty})^{1-\frac{2}{p}} \\ & \int_0^t (t-\tau)^{-\frac{1}{2\beta}-\frac{1}{p\beta}} \tau^{(1-\epsilon)(-1+\frac{2}{p})} \|\nabla j(\tau)\|_{L^2}^{\frac{2}{p}}\ d\tau \\ \le & C\|u\|_{L^\infty(0,t;L^\infty)} (\sup_{0\le\tau\le t} \|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty})^{1-\frac{2}{p}} \\ & \times \|\nabla j\|_{L^2(0,t;L^2)}^{\frac{2}{p}} \Big(\int_0^t (t-\tau)^{(-\frac{1}{2\beta}-\frac{1}{p\beta})\frac{p}{p-1}} \tau^{-(1-\epsilon)\frac{p-2}{p-1}}\ d\tau\Big)^{1-\frac{1}{p}}. \end{aligned}$$

By choosing $p > \frac{2\beta+2}{2\beta-1}$, the Beta function $\mathbf{B}(1+(-\frac{1}{2\beta}-\frac{1}{p\beta})\frac{p}{p-1}, 1-(1-\epsilon)\frac{p-2}{p-1})$ is bounded. As a result,

$$\begin{aligned} & \Big(\int_0^t (t-\tau)^{(-\frac{1}{2\beta}-\frac{1}{p\beta})\frac{p}{p-1}} \tau^{-(1-\epsilon)\frac{p-2}{p-1}}\ d\tau\Big)^{1-\frac{1}{p}} \\ = & t^{-\frac{1}{2\beta}-\frac{1}{p\beta}+\frac{1}{p}+\epsilon\frac{p-2}{p}} \Big(\int_0^1 (1-\tau)^{(-\frac{1}{2\beta}-\frac{1}{p\beta})\frac{p}{p-1}} \tau^{-(1-\epsilon)\frac{p-2}{p-1}}\ d\tau\Big)^{1-\frac{1}{p}} \\ = & t^{-\frac{1}{2\beta}-\frac{1}{p\beta}+\frac{1}{p}+\epsilon\frac{p-2}{p}} \mathbf{B}(1+(-\frac{1}{2\beta}-\frac{1}{p\beta})\frac{p}{p-1}, 1-(1-\epsilon)\frac{p-2}{p-1}) \\ \le & Ct^{-\frac{1}{2\beta}-\frac{1}{p\beta}+\frac{1}{p}+\epsilon\frac{p-2}{p}}, \end{aligned}$$

where the constant C doesn't depend on t. Hence, we deduce that

$$\begin{aligned} & t^{1-\epsilon}J_1 \\ \le & Ct^{-\frac{1}{2\beta}-\frac{1}{p\beta}+\frac{1}{p}+1-\epsilon\frac{2}{p}} (\sup_{0\le\tau\le t} \|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty})^{1-\frac{2}{p}} \|u\|_{L^\infty(0,t;L^\infty)} \|\nabla j\|_{L^2(0,t;L^2)}^{\frac{2}{p}}. \end{aligned} \tag{2.14}$$

For any $2 < p < \infty$, by using Lemma 2.5 combined with Hölder's inequality and the Sobolev interpolation inequality again, we can estimate J_2 as follows

$$\begin{aligned} J_2 \le & \int_0^t \|\nabla\cdot\nabla S_\beta(t-\tau)(b\omega)(\tau)\|_{L^\infty}\ d\tau \\ \le & C\int_0^t (t-\tau)^{-\frac{1}{\beta}-\frac{1}{p\beta}} \|b(\tau)\|_{L^{2p}} \|\omega(\tau)\|_{L^{2p}}\ d\tau \\ \le & C\int_0^t (t-\tau)^{-\frac{1}{\beta}-\frac{1}{p\beta}} \|b(\tau)\|_{L^{2p}} \|\omega(\tau)\|_{L^2}^{\frac{1}{p}} \|\omega(\tau)\|_{L^\infty}^{1-\frac{1}{p}}\ d\tau \\ \le & C\|b\|_{L^\infty(0,t;H^1)} \|\omega\|_{L^\infty(0,t;L^\infty)}^{1-\frac{1}{p}} \|\omega\|_{L^\infty(0,t;L^2)}^{\frac{1}{p}} \int_0^t (t-\tau)^{-\frac{1}{\beta}-\frac{1}{p\beta}}\ d\tau, \end{aligned}$$

where the divergence-free condition of b has been used here. Choosing $p > \frac{1}{\beta-1}$ gives to

$$t^{1-\epsilon} J_2 \leq C t^{-\frac{1}{\beta}-\frac{1}{p\beta}+2-\epsilon} \|b\|_{L^\infty(0,t;H^1)} \|\omega\|_{L^\infty(0,t;L^\infty)}^{1-\frac{1}{p}} \|\omega\|_{L^\infty(0,t;L^2)}^{\frac{1}{p}}. \tag{2.15}$$

To deal with J_3, for any $2 < p < \infty$, we first use Lemma 2.5 and the Hölder inequality to obtain

$$J_3 \leq C \int_0^t (t-\tau)^{-\frac{1}{2\beta}-\frac{1}{p\beta}} \|\omega(\tau)\|_{L^{2p}} \|\nabla b(\tau)\|_{L^{2p}}\, d\tau.$$

Then, similar to the argument as leading to (2.14) or (2.15), for $p > \frac{2}{\beta-1}$, we get

$$t^{1-\epsilon} J_3 \leq C t^{-\frac{1}{2\beta}-\frac{1}{p\beta}+\frac{3}{2}-\epsilon} \|\nabla b\|_{L^2(0,t;L^{2p})} \|\omega\|_{L^\infty(0,t;L^\infty)}^{1-\frac{1}{p}} \|\omega\|_{L^\infty(0,t;L^2)}^{\frac{1}{p}}. \tag{2.16}$$

By the a priori estimates in the previous subsections, we can deduce that $\|u\|_{L^\infty(0,t;L^\infty)}$, $\|b\|_{L^\infty(0,t;H^1)}$, $\|\nabla j\|_{L^2(0,t;L^2)}$, $\|\nabla b\|_{L^2(0,t;L^{2p})}$ are bounded. Therefore, plugging these estimates (2.13)-(2.16) into (2.12) and then taking the supremum with respect to t on the resulting inequality yield, for some $p > \max\{\frac{2\beta+2}{2\beta-1}, \frac{2}{\beta-1}, 2\}$,

$$\sup_{t>0} \|t^{1-\epsilon}\nabla j(t)\|_{L^\infty} \leq C e^{Ct}\Big(\|j_0\|_{L^2} + (\sup_{0\leq\tau\leq t} \|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty})^{1-\frac{1}{p}} + \|\omega\|_{L^\infty(0,T;L^\infty)}^{1-\frac{1}{p}}\Big). \tag{2.17}$$

Now, let us consider the vorticity equation in (2.2). By maximum principle and the Hölder inequality, we have for sufficiently large k and sufficiently small ϵ,

$$\begin{aligned}\|\omega(t)\|_{L^\infty} \leq& \|\omega_0\|_{L^\infty} + \int_0^t \|(b\cdot\nabla) j(\tau)\|_{L^\infty}\, ds \\ \leq& \|\omega_0\|_{L^\infty} + t^{\epsilon-\frac{1}{k\beta}} \sup_{0\leq\tau\leq t} \|\tau^{1-\epsilon}\nabla j(\tau)\|_{L^\infty} \sup_{0\leq\tau\leq t} \|\tau^{\frac{1}{k\beta}} b(\tau)\|_{L^\infty}.\end{aligned}$$

Combining this inequality with (2.17) and then using the Young inequality yield (2.10). Thus, we complete the proof. □

3. Proof of Theorem 1.1

This section proves Theorem 1.1. We will divide the proof into two parts. The first part deals with the existence of solution to system (1.4) and the second one deals with the uniqueness.

3.1. Existence. Firstly, by smoothing out the initial data (u_0, b_0), we can get a sequence of smooth initial data $(u_0^n, b_0^n)_{n\in\mathbb{N}} = J_n(u_0, b_0)$, where J_n, $n = 1, 2, 3...$ is the spectral cut-off defined by

$$\widehat{J_n f}(\xi) = 1_{[0,n]}(|\xi|)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^2.$$

By Bernstein's inequality in Lemma A.1, we obtain that those smooth initial data are bounded in the spaces given in Theorem 1.1 and belong to all the Sobolev spaces H^s. Now let us consider the following system:

$$\begin{cases} \partial_t u^n + u^n\cdot\nabla u^n = -\nabla p^n + b^n\cdot\nabla b^n, \\ \partial_t b^n + u^n\cdot\nabla b^n + \Lambda^{2\beta} b^n = b^n\cdot\nabla u^n, \\ \nabla\cdot u^n = \nabla\cdot b^n = 0, \\ (u^n, b^n)|_{t=0} = J_n(u_0, b_0). \end{cases} \tag{3.1}$$

According to the result of Jiu and Zhao [**16**], for any $T > 0$, we can find a sequence of smooth global solutions $\{(u^n, b^n)\}_{n\in\mathbb{N}} \in \big(L^\infty(0,T;H^s)\big)^2$, $\{b^n\}_{n\in\mathbb{N}} \in L^2(0,T;H^{s+\beta})$, $s > \max\{2,\beta\}$ satisfying, $\forall\ 0 < t \le T$,

$$\|(u^n,b^n)\|_{H^s}^2 + \int_0^t \|b^n(s)\|_{H^{s+\beta}}^2 ds \le C(\|J_n(u_0,b_0)\|_{H^s}) = C(n^{s-1}, \|(u_0,b_0)\|_{H^1}).$$

By mimicking computations in Section 2, we conclude that

$$\begin{aligned}
&\{u^n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^\infty(0,T;L^2),\\
&\{\omega^n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^\infty(0,T;L^2\cap L^\infty),\\
&\{b^n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^\infty(0,T;H^1)\cap L^2(0,T;H^{\beta+1}).
\end{aligned}$$

Secondly, in order to show that $\{(u^n,b^n)\}_{n\in\mathbb{N}}$ converges (up to extraction), a uniform boundness information on $\{(\partial_t u^n, \partial_t b^n)\}_{n\in\mathbb{N}}$ is needed. In fact, applying the Leray projector $\mathcal{P}$ on both sides of the first equation in (3.1), we can get

$$\partial_t u^n = \mathcal{P}(-u^n\cdot\nabla u^n + b^n\cdot\nabla b^n).$$

The bounds of $\{u^n\}_{n\in\mathbb{N}}$ imply that $\{\partial_t u^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H^{-1})$. As for the magnetic field,

$$\partial_t b^n = -\Lambda^{2\beta} b^n - u^n\cdot\nabla b^n + b^n\cdot\nabla u^n.$$

The previous bounds of $\{b^n\}_{n\in\mathbb{N}}$ imply that $\{\partial_t b^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^{1-\beta})$. Since $L^2 \hookrightarrow H^{-1}$ and $H^1 \hookrightarrow H^{1-\beta}$ are locally compact, the classical Aubin-Lions argument ensures that, up to extraction, $\{(u^n,b^n)\}_{n\in\mathbb{N}}$ strongly converges to some (u,b) in $L^\infty(0,T;H^{-1}_{loc}) \times L^2(0,T;H^{1-\beta}_{loc})$. This strong convergence together with the uniform bounds of $\{(u^n,b^n)\}_{n\in\mathbb{N}}$ stated in the first step enables us to pass to limit in the system (3.1) in the sense of distributions. Moreover, we can conclude that the solution (u,b) satisfies

$$u \in L^\infty(0,T;L^2), \quad \omega \in L^\infty(0,T;L^2\cap L^\infty),$$
$$b \in L^\infty(0,T;H^1)\cap L^2(0,T;H^{\beta+1}).$$

Finally, we prove the time continuity of b. For any $T \ge t_2 > t_1 \ge 0$, by the fact that H^1 is equivalent to Besov space $B^1_{2,2}$ (see Definition A.1 in Appendix), we have

(3.2)
$$\begin{aligned}
&\|b(t_2)-b(t_1)\|_{H^1}\\
=&\Big(\sum_{q\ge -1} 2^{2q}\left\|\Delta_q\big(b(t_2)-b(t_1)\big)\right\|_{L^2}^2\Big)^{\frac12}\\
\le&\Big(\sum_{-1\le q\le N} 2^{2q}\left\|\Delta_q\big(b(t_2)-b(t_1)\big)\right\|_{L^2}^2\Big)^{\frac12} + \Big(\sum_{q\ge N} 2^{2q}\left\|\Delta_q\big(b(t_2)-b(t_1)\big)\right\|_{L^2}^2\Big)^{\frac12},
\end{aligned}$$

where the integer N is to be determined and Δ_q is the frequency localization operator (see the definition in Appendix). If we assume that $b \in \widetilde{L}^\infty(0,T;H^1)$ (see Definition A.2 in Appendix), then for any $\epsilon > 0$, we can choose $N = N(\epsilon)$ satisfying

$$\Big(\sum_{q\ge N(\epsilon)} 2^{2q}\|\Delta_q b\|_{L^\infty(0,T;L^2)}^2\Big)^{\frac12} \le \frac{\epsilon}{2}.$$

which ensures that the high-frequency term of (3.2) can be bounded by $\frac{\epsilon}{2}$. In fact, applying Δ_q on the magnetic equation yields

$$\Delta_q b_t + \Delta_q \Lambda^{2\beta} b = \Delta_q(b \cdot \nabla u) - \Delta_q(u \cdot \nabla b).$$

By Duhamel's principle, we have

$$\Delta_q b(x,t) = S_\beta(t)\Delta_q b_0 + \int_0^t S_\beta(t-\tau)\big(\Delta_q(b\cdot\nabla u) - \Delta_q(u\cdot\nabla b)\big)(\tau)\, d\tau.$$

Therefore, using Lemma 2.5 and divergence free condition of u and b yield

$$2^q\|\Delta_q b\|_{L^\infty(0,t;L^2)} \le 2^q\|\Delta_q b_0\|_{L^2} + 2\int_0^t (t-\tau)^{-\frac{1}{2\beta}} 2^q \|\Delta_q(bu)(\tau)\|_{L^2}\, d\tau.$$

Taking the l^2-norm and using the definition of Besov space, we get

$$\|b\|_{\widetilde{L}^\infty(0,t;H^1)} \le \|b_0\|_{H^1} + 2\int_0^t (t-\tau)^{-\frac{1}{2\beta}} \|(bu)(\tau)\|_{H^1}\, d\tau.$$

Since $\|bu\|_{H^1} \le \|u\|_{H^1}\|b\|_{L^\infty} + \|b\|_{H^1}\|u\|_{L^\infty}$, we have

$$\begin{aligned}\int_0^t (t-\tau)^{-\frac{1}{2\beta}}\|bu\|_{H^1}\, d\tau \le& \|u\|_{L^\infty(0,t;H^1)}\|b\|_{L^2(0,t;L^\infty)}\Big(\int_0^t (t-\tau)^{-\frac{1}{\beta}}\, d\tau\Big)^{\frac12} \\ &+ \|b\|_{L^\infty(0,t;H^1)}\|u\|_{L^\infty(0,t;L^\infty)}\int_0^t (t-\tau)^{-\frac{1}{2\beta}}\, d\tau \\ \le& Ce^{Ct},\end{aligned}$$

where the Sobolev embedding and the previous estimates in Section 2 have been used. For the low-frequency term,

$$\begin{aligned}&\Big(\sum_{-1\le q\le N(\epsilon)} 2^{2q}\left\|\Delta_q\big(b(t_2)-b(t_1)\big)\right\|_{L^2}^2\Big)^{\frac12} \\ &= \Big(\sum_{-1\le q\le N(\epsilon)} 2^{2q}\left\|\int_{t_1}^{t_2}\partial_t\Delta_q b(t)\, dt\right\|_{L^2}^2\Big)^{\frac12} \\ &\le \Big(\sum_{-1\le q\le N(\epsilon)} 2^{2q\beta}\Big(\int_{t_1}^{t_2} 2^{q(1-\beta)}\left\|\partial_t\Delta_q b(t)\right\|_{L^2}\, dt\Big)^2\Big)^{\frac12} \\ &\le 2^{\beta N}\|\partial_t b\|_{L^2(0,T;H^{1-\beta})}(t_2-t_1)^{\frac12},\end{aligned}$$

where Hölder's inequality has been used in the last inequality. Consequently, combining these two parts with (3.2) yield

$$\limsup_{t_2-t_1\to 0}\|b(t_2)-b(t_1)\|_{H^1} \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the continuity (in time) of b is obtained.

3.2. Uniqueness. In this subsection, we prove the uniqueness of the solution presented in Theorem 1.1.

Let (u_1, b_1, p_1) and (u_2, b_2, p_2) be solutions of (1.4) with the same initial data. Denote $\delta u = u_2 - u_1$, $\delta b = b_2 - b_1$ and $\delta p = p_2 - p_1$. It is easy to verify that $(\delta u, \delta b, \delta p)$ satisfies the following system:

$$\begin{cases}\partial_t\delta u + u_2\cdot\nabla\delta u = -\nabla\delta p - \delta u\cdot\nabla u_1 + b_2\cdot\nabla\delta b + \delta b\cdot\nabla b_1, \\ \partial_t\delta b + u_2\cdot\nabla\delta b + \Lambda^{2\beta}\delta b = -\delta u\cdot\nabla b_1 + b_2\cdot\nabla\delta u + \delta b\cdot\nabla u_1.\end{cases} \tag{3.3}$$

A standard energy method shows that, for any $2 \le p < \infty$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2\Big)\\
\le&\|\nabla u_1\|_{L^p}\big(\|\delta u\|^2_{L^{\frac{2p}{p-1}}} + \|\delta b\|^2_{\frac{2p}{p-1}}\big) + 2\|\nabla b_1\|_{L^\infty}\|\delta u\|_{L^2}\|\delta b\|_{L^2}\\
\le&\|\nabla u_1\|_{L^p}\big(\|\delta u\|_{L^2}^{2-\frac{2}{p}}\|\delta u\|_{L^\infty}^{\frac{2}{p}} + \|\delta b\|_{L^2}^{2-\frac{2}{p}}\|\delta b\|_{L^\infty}^{\frac{2}{p}}\big) + 2\|\nabla b_1\|_{L^\infty}\|\delta u\|_{L^2}\|\delta b\|_{L^2},
\end{aligned}
$$

where the interpolation inequality has been used in the last inequality. For any $\epsilon > 0$, let

$$X_\epsilon(t) = \big(\|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2 + \epsilon^2\big)^{\frac{1}{2}},$$

we can get

$$\frac{d}{dt}X_\epsilon(t) \le \|\nabla u_1\|_{L^p}X_\epsilon^{1-\frac{2}{p}}(t)\big(\|\delta u\|_{L^\infty}^{\frac{2}{p}} + \|\delta b\|_{L^\infty}^{\frac{2}{p}}\big) + \|\nabla b_1\|_{L^\infty}X_\epsilon(t). \tag{3.4}$$

By the regularity of the magnetic fields, it infers that $\|\nabla b_1(t)\|_{L^\infty}$ is integrable. Therefore, by setting

$$Y_\epsilon(t) = X_\epsilon(t)e^{-\int_0^t \|\nabla b_1(s)\|_{L^\infty}\ ds},$$

we can get

$$
\begin{aligned}
\frac{d}{dt}Y_\epsilon^{\frac{2}{p}}(t) =&\frac{2}{p}Y_\epsilon^{-1+\frac{2}{p}}(t)\frac{d}{dt}Y_\epsilon(t)\\
\le&2\frac{\|\nabla u_1\|_{L^p}}{p}\big(\|\delta u\|_{L^\infty}^{\frac{2}{p}} + \|\delta b\|_{L^\infty}^{\frac{2}{p}}\big)e^{-\frac{2}{p}\int_0^t \|\nabla b_1(s)\|_{L^\infty}\ ds}\\
\le&2\frac{\|\nabla u_1\|_{L^p}}{p}\big(\|\delta u\|_{L^\infty}^{\frac{2}{p}} + \|\delta b\|_{L^\infty}^{\frac{2}{p}}\big).
\end{aligned}
$$

Integrating in time on the last inequality from 0 to t gives to

$$Y_\epsilon(t) \le \Big(\epsilon^{\frac{2}{p}} + 2\int_0^t \frac{\|\nabla u_1\|_{L^p}}{p}\big(\|\delta u\|_{L^\infty}^{\frac{2}{p}} + \|\delta b\|_{L^\infty}^{\frac{2}{p}}\big)\ ds\Big)^{\frac{p}{2}}.$$

Letting ϵ tend to 0, we can deduce that, for all $t \ge 0$,

$$
\begin{aligned}
&\|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2\\
\le&\Big(2\int_0^t \frac{\|\nabla u_1\|_{L^p}}{p}\big(\|\delta u\|_{L^\infty}^{\frac{2}{p}} + \|\delta b\|_{L^\infty}^{\frac{2}{p}}\big)\ ds\Big)^p e^{2\int_0^t \|\nabla b_1(s)\|_{L^\infty}\ ds}\\
\le&C2^{p-1}\Big(\|\delta u\|^2_{L_t^\infty(L^\infty)}\big(\int_0^t \frac{\|\nabla u_1\|_{L^p}}{p}\ ds\big)^p + \|\delta b\|^2_{L_t^2(L^\infty)}\big(\int_0^t \big(\frac{\|\nabla u_1\|_{L^p}}{p}\big)^{\frac{p}{p-1}}\ ds\big)^{p-1}\Big),
\end{aligned}
\tag{3.5}
$$

where in the last inequality we used the Hölder inequality and the fact that

$$(a+b)^p \le 2^{p-1}(a^p + b^p),\ \text{for any}\ p \ge 1.$$

By virtue of Lemma A.4, for any $p \ge 2$,

$$\frac{\|\nabla u_1\|_{L^p}}{p} \le C\|\omega_1\|_{L^2\cap L^\infty} \le Ce^{ct},$$

with C only depending on the $\|(u_0, b_0)\|_{H^1}$, $\|\omega_0\|_{L^\infty}$. Therefore, plugging this inequality into (3.5), one can find a time $T_0 > 0$ such that $\int_0^t \|w_1\|_{L^2\cap L^\infty}ds < \frac{1}{4}$ and $\int_0^t \|w_1\|^{\frac{p}{p-1}}_{L^2\cap L^\infty}ds < \frac{1}{4}$ hold for $0 \le t \le T_0$. Then letting p tend to ∞ in (3.5), one can obtain that $(\delta u, \delta b) \equiv 0$ on $[0, T_0]$. Noting that δu and δb are continuous in time with values in L^2, we fianlly conclude that $(\delta u, \delta b) \equiv 0$ on $[0, T]$ by means of a standard connectivity argument.

3.3. Higher regularity. This subsection contributes to proving the remaining part of Theorem 1.1, it suffices to give a global bound for the current j in the space-time space $L^\infty(0, T; H^{\beta-1})$.

Applying $\Lambda^{\beta-1}$ on both sides of the second equation of (2.2) and then taking the L^2-norm on the resulting equation, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\beta-1}j(t)\|_{L^2}^2 + \|\Lambda^{2\beta-1}j(t)\|_{L^2}^2 = I_1 + I_2 + I_3, \tag{3.6}$$

where I_1, I_2 and I_3 are given by

$$\begin{aligned} I_1 &= -\int \Lambda^{\beta-1}(u\cdot\nabla j)\cdot\Lambda^{\beta-1}j\ dx, \\ I_2 &= \int \Lambda^{\beta-1}(b\cdot\nabla\omega)\cdot\Lambda^{\beta-1}j\ dx, \\ I_3 &= \int \Lambda^{\beta-1}Q(\nabla u, \nabla b)\Lambda^{\beta-1}j\ dx. \end{aligned}$$

By the Plancherel theorem, we write

$$I_1 = \int \Lambda^{\beta-1}(uj)\cdot\Lambda^{\beta-1}\nabla j\ dx = \int (uj)\cdot\Lambda^{2\beta-2}\nabla j\ dx.$$

By Hölder's inequality and Young's inequality, we have

$$\begin{aligned} I_1 \leq& \|u\|_{L^2}\|j\|_{L^\infty}\|\Lambda^{2\beta-1}j\|_{L^2} \\ \leq& \frac{1}{4}\|\Lambda^{2\beta-1}j\|_{L^2}^2 + C\|u\|_{L^2}^2\|j\|_{H^\beta}^2, \end{aligned} \tag{3.7}$$

where the following Sobolev embedding is used,

$$H^\beta \hookrightarrow L^\infty,\ \beta > 1.$$

To deal with I_2, similar to the argument as to (3.7), we infer that

$$I_2 \leq \frac{1}{4}\|\Lambda^{2\beta-1}j\|_{L^2}^2 + C\|\omega\|_{L^2}^2\|b\|_{H^\beta}^2. \tag{3.8}$$

To bound I_3, we use the Plancherel theorem and Hölder's inequality to get

$$I_3 \leq \|\omega\|_{L^2}\|\nabla b\|_{L^\infty}\|\Lambda^{2\beta-2}j\|_{L^2}. \tag{3.9}$$

Inserting the Sobolev inequality

$$\|\Lambda^{2\beta-2}j\|_{L^2} \leq \|\Lambda^{\beta-1}j\|_{L^2}^{\frac{1}{\beta}}\|\Lambda^{2\beta-1}j\|_{L^2}^{1-\frac{1}{\beta}}$$

with $\beta > 1$ into (3.9) and then using the Young inequality, we obtain

$$\begin{aligned} I_3 \leq& \frac{1}{4}\|\Lambda^{2\beta-1}j\|_{L^2}^2 + C\big(\|\Lambda^{\beta-1}j\|_{L^2}^2 + \|\omega\|_{L^2}^2\|\nabla b\|_{L^\infty}^2\big) \\ \leq& \frac{1}{4}\|\Lambda^{2\beta-1}j\|_{L^2}^2 + C\big(\|\Lambda^{\beta-1}j\|_{L^2}^2 + \|\omega\|_{L^2}^2\|j\|_{H^\beta}^2\big). \end{aligned} \tag{3.10}$$

In the last inequality, we have used the Sobolev embedding $H^\beta \hookrightarrow L^\infty$ for $\beta > 1$. Plugging (3.7), (3.8), (3.10) into (3.6) and using the Gronwall lemma together with estimate (2.4) yield

$$\|\Lambda^{\beta-1}j\|_{L^2}^2 + \int_0^t \|\Lambda^{2\beta-1}j(s)\|_{L^2}^2 ds \leq Ce^{Ct}.$$

This inequality combined with (2.3) yields, for $0 \le t \le T$

$$\|j\|^2_{L^\infty(0,T;H^{\beta-1})} + \int_0^t \|j(s)\|^2_{H^{2\beta-1}}\, ds \le Ce^{Ct},$$

where the fact that $\|f\|_{H^s}$ is equivalent to $\|f\|_{L^2} + \|\Lambda^s f\|_{L^2}$ is used.

4. Inviscid limit for an initial vorticity in L^∞

This section contributes to the proof of Theorem 1.3. First, we recall the following Osgood lemma which is a slight generalization of Lemma 3.4 in [**1**] where the function c is a constant.

LEMMA 4.1. *Let ρ be a measurable function from $[t_0, T]$ to $[0, a]$, γ a locally integrable function from $[t_0, T]$ to $\mathbb{R}^+$, and μ a continuous and nondecreasing function from $[0, a]$ to $\mathbb{R}^+$. Assume that, for some nonnegative nondecreasing continuous c, the function ρ satisfies*

$$\rho(t) \le c(t) + \int_{t_0}^t \gamma(\tau)\mu(\rho(\tau))\, d\tau.$$

Then if $c(t)$ is positive, then for a.e. $t \in [t_0, T]$,

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(c(t)) \le \int_{t_0}^t \gamma(\tau)\, d\tau, \tag{4.1}$$

with $\mathcal{M}(x) = \int_x^a \frac{1}{\mu(r)}\, dr$.

PROOF. Arguing by density, it suffices to consider the case where the functions γ and ρ are continuous. Now, consider the following continuous function:

$$R_c(t) := c(t) + \int_{t_0}^t \gamma(\tau)\mu(\rho(\tau))\, d\tau.$$

Because μ is nondecreasing, we have

$$\begin{aligned} R_c'(t) &= c'(t) + \gamma(t)\mu(\rho(t)) \\ &\le c'(t) + \gamma(t)\mu(R_c(t)). \end{aligned}$$

Hence, we get

$$\begin{aligned} -\frac{d}{dt}\mathcal{M}(R_c(t)) &= \frac{1}{\mu(R_c(t))} R_c'(t) \\ &\le \frac{c'(t)}{\mu(R_c(t))} + \gamma(t). \end{aligned} \tag{4.2}$$

Since $R_c(t) \ge c(t)$ and μ is nondecreasing,

$$\frac{c'(t)}{\mu(R_c(t))} \le \frac{c'(t)}{\mu(c(t))} = -\frac{d}{dt}\mathcal{M}(c(t)).$$

Plugging this inequality into (4.2) and integrating, we thus get (4.1). □

PROOF OF THEOREM 1.3. It is easy to obtain that system (1.5) with the same initial data as Theorem 1.1 has a unique solution (u^ν, b^ν) satisfying

$$\|u^\nu(t)\|^2_{L^2} + \|b^\nu(t)\|^2_{L^2} + 2\nu \int_0^t \|\Lambda^\alpha u^\nu(s)\|^2_{L^2} ds + 2\int_0^t \|\Lambda^\beta b^\nu(s)\|^2_{L^2} ds \le \|u_0\|^2_{L^2} + \|b_0\|^2_{L^2},$$

the vorticity $\omega^\nu = \partial_1 u_2^\nu - \partial_2 u_1^\nu$ satisfies the following estimates

$$\|\omega^\nu\|_{L^\infty(0,T;L^p)} \le Ce^{Ct}, \quad 2 \le p \le \infty,$$

and the current $j^\nu = \partial_1 b_2^\nu - \partial_2 b_1^\nu$ satisfies

$$\|j^\nu(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta j^\nu(s)\|_{L^2}^2 ds \le Ce^{Ct},$$

where the constant C is uniformly bounded on ν.

Let $\overline{u} = u^\nu - u$, $\overline{b} = b^\nu - b$ and $\overline{p} = p^\nu - p$. One denotes also $\overline{\omega} = \omega^\nu - \omega$. The vector field $\overline{u}$ and $\overline{b}$ satisfy

$$\begin{cases} \overline{u}_t + u^\nu \cdot \nabla \overline{u} + \nu \Lambda^{2\alpha} \overline{u} + \nabla \overline{p} = -\nu \Lambda^{2\alpha} u - \overline{u} \cdot \nabla u + b^\nu \cdot \nabla \overline{b} + \overline{b} \cdot \nabla b, \\ \overline{b}_t + u^\nu \cdot \nabla \overline{b} + \Lambda^{2\beta} \overline{b} = -\overline{u} \cdot \nabla b + b^\nu \cdot \nabla \overline{u} + \overline{b} \cdot \nabla u, \\ \nabla \cdot \overline{u} = \nabla \cdot \overline{b} = 0, \\ \overline{u}(x,0) = \overline{b}(x,0) = 0. \end{cases} \tag{4.3}$$

The energy estimate gives that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\overline{u}\|_{L^2}^2 + \|\overline{b}\|_{L^2}^2) + \nu \|\Lambda^\alpha \overline{u}\|_{L^2}^2 + \|\Lambda^\beta \overline{b}\|_{L^2}^2 \\ &= -\nu \int \overline{u} \Lambda^{2\alpha} u \, dx - \int \overline{u} \cdot \nabla u \overline{u} \, dx - \int \overline{u} \cdot \nabla b \overline{b} \, dx + \int \overline{b} \cdot \nabla u \overline{b} \, dx + \int \overline{b} \cdot \nabla b \overline{u} \, dx. \end{aligned} \tag{4.4}$$

Now we estimate the right hand terms one by one. By the Hölder and the Young inequalities, we get

$$\begin{aligned} -\nu \int \overline{u} \Lambda^{2\alpha} u \, dx = &- \nu \int \Lambda^\alpha \overline{u} \Lambda^\alpha u \, dx \\ \le &\nu \|\Lambda^\alpha \overline{u}\|_{L^2} \|\Lambda^\alpha u\|_{L^2} \\ \le &\frac{1}{2}\nu \|\Lambda^\alpha \overline{u}\|_{L^2}^2 + \frac{1}{2}\nu \|\Lambda^\alpha u\|_{L^2}^2 . \end{aligned}$$

Since $\omega \in L^\infty(0,T;L^\infty)$, for any $T > 0$, we have for any $p \ge 2$,

$$\|\nabla u\|_{L^p} \le Cp \|\omega\|_{L^p} \le Cp(\|\omega\|_{L^2} + \|\omega\|_{L^\infty}).$$

Therefore,

$$\sup_{p \ge 2} \frac{\|\nabla u\|_{L^p}}{p} \le Ce^{Ct}.$$

This together with Hölder's inequality yields

$$\begin{aligned} -\int \overline{u} \cdot \nabla u \overline{u} \, dx \le &\|\nabla u\|_{L^p} \|\overline{u}\|_{L^{\frac{2p}{p-1}}}^2 \\ \le &Cpe^{Ct} \|\overline{u}\|_{L^2}^{2-\frac{2}{p}} \|\overline{u}\|_{L^\infty}^{\frac{2}{p}} . \end{aligned}$$

In addition, by the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} M := \|\overline{u}\|_{L^\infty} &\le C(\|\overline{\omega}\|_{L^4} + \|\overline{u}\|_{L^2}) \\ &\le C(\|\omega^\nu\|_{L^4} + \|\omega\|_{L^4} + \|u^\nu\|_{L^2} + \|u\|_{L^2}) \le Ce^{Ct}. \end{aligned}$$

Hence, for $2 \le p < \infty$

$$-\int \overline{u} \cdot \nabla u \overline{u} \, dx \le Cpe^{Ct} \|\overline{u}\|_{L^2}^{2-\frac{2}{p}} M^{\frac{2}{p}} \le Cpe^{Ct} \|\overline{u}\|_{L^2}^{2-\frac{2}{p}} .$$

Integrating by parts and using the Hölder inequality, we get

$$\begin{aligned}\int \overline{b}\cdot\nabla u\overline{b}\,dx &= -\int \overline{b}\cdot\nabla\overline{b}u\,dx\\ &\le \|\overline{b}\|_{L^2}\|\nabla\overline{b}\|_{L^2}\|u\|_{L^\infty}\\ &\le \|\overline{b}\|_{L^2}\|\overline{b}\|_{H^\beta}(\|\omega\|_{L^3}+\|u\|_{L^2})\\ &\le\frac{1}{2}\|\Lambda^\beta\overline{b}\|_{L^2}^2+C\|\overline{b}\|_{L^2}^2(\|\omega\|_{L^3}^2+\|u\|_{L^2}^2+1).\end{aligned}$$

Similarly, by the Sobolev embedding, we have

$$\begin{aligned}-\int \overline{u}\cdot\nabla b\overline{b}\,dx+\int \overline{b}\cdot\nabla b\overline{u}\,dx &\le \|\nabla b\|_{L^\infty}\|\overline{u}\|_{L^2}\|\overline{b}\|_{L^2}\\ &\le\frac{1}{2}\|\nabla b\|_{H^\beta}(\|\overline{u}\|_{L^2}^2+\|\overline{b}\|_{L^2}^2).\end{aligned}$$

Plugging these estimates into (4.4) gives

$$\begin{aligned}&\frac{d}{dt}(\|\overline{u}\|_{L^2}^2+\|\overline{b}\|_{L^2}^2)\\ &\le Cpe^{Ct}\|\overline{u}\|_{L^2}^{2-\frac{2}{p}}+C\nu\|\Lambda^\alpha u\|_{L^2}^2+C(\|b\|_{H^{\beta+1}}+e^{Ct})(\|\overline{u}\|_{L^2}^2+\|\overline{b}\|_{L^2}^2).\end{aligned}\tag{4.5}$$

Take $g_\nu(t)=\|\overline{u}\|_{L^2}^2+\|\overline{b}\|_{L^2}^2+\delta$, $0<\delta\le\nu$ and define T_ν the maximal time

$$T_\nu=\max\Big\{t\le T:\sup_{\tau\in[0,t]}g_\nu(\tau)\le\frac{1}{e^2}\Big\}.$$

For all $t\in(0,T_\nu)$, one choose $p=\ln\left(\frac{1}{g_\nu(t)}\right)$ in (4.5) to get

$$\frac{d}{dt}g_\nu(t)\le Ce^{Ct}g_\nu^{1+\frac{1}{\ln g_\nu(t)}}(t)\ln\Big(\frac{1}{g_\nu(t)}\Big)+C\nu\|\Lambda^\alpha u\|_{L^2}^2+C(\|b\|_{H^{\beta+1}}+e^{Ct})g_\nu(t).$$

The Gronwall's lemma yields

$$\begin{aligned}g_\nu(t)&\le Ce^{C\int_0^t(e^{C\tau}+\|b\|_{H^{\beta+1}})d\tau}\Big(\delta+\nu\int_0^t\|\Lambda^\alpha u\|_{L^2}^2\,d\tau\\ &\quad+\int_0^t e^{C\tau}\ln\big(\frac{1}{g_\nu(\tau)}\big)g_\nu^{1+\frac{1}{\ln g_\nu(\tau)}}(\tau)d\tau\Big)\\ &\le Ce^{e^{Ct}}\Big(\delta+\nu+\int_0^t e^{C\tau}\ln\big(\frac{1}{g_\nu(\tau)}\big)g_\nu^{1+\frac{1}{\ln g_\nu(\tau)}}(\tau)d\tau\Big).\end{aligned}$$

By the definition of δ, we have

$$\begin{aligned}g_\nu(t)&\le Ce^{e^{Ct}}\Big(2\nu+\int_0^t e^{C\tau}\ln\big(\frac{1}{g_\nu(\tau)}\big)g_\nu^{1+\frac{1}{\ln g_\nu(\tau)}}(\tau)d\tau\Big)\\ &\le Ce^{e^{Ct}}\Big(\nu+\int_0^t e^{C\tau}\ln\big(\frac{1}{g_\nu(\tau)}\big)g_\nu(\tau)d\tau\Big).\end{aligned}$$

Assuming $Ce^{e^{CT}}\nu<1$ and applying Osgood lemma 4.1 with $\mu(r)=r\ln\frac{1}{r}$, we get for all $t\in(0,T^\nu)$,

$$-\ln(-\ln g_\nu(t))+\ln(-\ln Ce^{e^{Ct}}\nu)\le Ce^{e^{Ct}}.$$

As a result,

$$g_\nu(t)\le(Ce^{e^{Ct}}\nu)^{e^{-e^{e^{Ct}}}},\quad\forall 0\le t<T^\nu.$$

If we choose ν which is satisfied by

$$Ce^{e^{CT}}\nu \le \frac{3}{4e^2},$$

then there exists $\nu_0 > 0$ such that for $\nu \le \nu_0$

$$g_\nu(t) \le \frac{3}{4e^2} < \frac{1}{e^2}.$$

Using the standard continuity argument, for $\nu \le \nu_0$, we have $T_\nu = T$ and consequently

$$g_\nu(t) \le (Ce^{e^{Ct}}\nu)^{e^{-e^{e^{Ct}}}}, \quad \forall 0 \le t \le T.$$

□

Appendix A. Littlewood-Paley theory and Besov spaces

In this section, we recall some basic facts about Besov spaces, Bernstein's inequalities and the Littlewood-Paley decomposition. For more details, it is referred to [**1, 20**] and references therein.

Let (χ, φ) be a couple of smooth functions with values in $[0,1]$ such that χ is supported in the ball $\{\xi \in \mathbb{R}^n \big| |\xi| \le \frac{4}{3}\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^n \big| \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ and

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1 \quad \text{for each } \xi \in \mathbb{R}^n.$$

For every $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the dyadic blocks as

$$\Delta_{-1}u = \chi(D)u \quad \text{and} \quad \Delta_j u := \varphi(2^{-j}D)u \quad \text{for each } j \in \mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u := \chi(2^{-j}D)u.$$

It may be easily checked that

$$u = \sum_{j\ge -1} \Delta_j u$$

holds in $\mathcal{S}'(\mathbb{R}^n)$. In addition, for two tempered distributions u and v, we also recall the notion of paraproducts

$$T_u v = \sum_j S_{j-1}u\Delta_j v, \qquad R(u,v) = \sum_{|i-j|\le 2} \Delta_i u \Delta_j v$$

and Bony's decomposition

$$uv = T_u v + T_v u + R(u,v).$$

Definition A.1. For $s \in \mathbb{R}$, $(p,q) \in [1,+\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, we set

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} := \Big(\sum_{j\ge -1} 2^{jsq} \|\Delta_j u\|^q_{L^p(\mathbb{R}^n)} \Big)^{\frac{1}{q}} \quad \text{if} \quad q < +\infty$$

and

$$\|u\|_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{j\ge -1} 2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}.$$

Then we define inhomogeneous Besov spaces as

$$B^s_{p,q}(\mathbb{R}^n) := \big\{ u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B^s_{p,q}(\mathbb{R}^n)} < +\infty \big\}.$$

DEFINITION A.2. For $s \in \mathbb{R}$, $1 \le p, q, \sigma \le \infty$, $I = [0, T]$, the inhomogeneous space-time Besov spaces are defined as

$$\widetilde{L}^q(I; B^s_{p,\sigma}(\mathbb{R}^n)) =$$
$$\{u \in \mathbb{D}'(I, \mathcal{S}'(\mathbb{R}^n)) : \|u\|_{\widetilde{L}^q(I; B^s_{p,\sigma}(\mathbb{R}^n))} = \left\| 2^{js} \|\Delta_j u\|_{L^q(I; L^p(\mathbb{R}^n))} \right\|_{l^\sigma} < \infty\}.$$

It should be remarked that the usual Sobolev spaces H^s coincide with Besov spaces $B^s_{2,2}$. The following lemma is the well-known Bernstein inequality.

LEMMA A.3 (**Bernstein's inequality**). *Let $\mathcal{B}$ be a ball of $\mathbb{R}^n$, and $\mathcal{C}$ be a ring of $\mathbb{R}^n$. There exists a positive constant C such that for all integer $k \ge 0$, all $1 \le a \le b \le \infty$ and $u \in L^a(\mathbb{R}^n)$, the following estimates are satisfied:*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b(\mathbb{R}^n)} \le C^{k+1} \lambda^{k+n(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a(\mathbb{R}^n)}, \ supp\hat{u} \subset \lambda\mathcal{B},$$

$$C^{-(k+1)} \lambda^k \|u\|_{L^a(\mathbb{R}^n)} \le \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a(\mathbb{R}^n)} \le C^{k+1} \lambda^k \|u\|_{L^a(\mathbb{R}^n)}, \ supp\hat{u} \subset \lambda\mathcal{C}.$$

At the end of this section, we recall a lemma which are used frequently in establishing the a prior estimates, especially in proving the uniqueness of solutions and the zero-viscosity limit problem.

LEMMA A.4 ([**3**]). *For any $p \in (1, \infty)$, there holds*

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \le C \frac{p^2}{p-1} \|\omega\|_{L^p(\mathbb{R}^n)}, \tag{A.1}$$

where C does not depend on p.

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