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A class of semilinear delay differential equations with nonlocal initial conditions

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Abstract. The goal of this paper is to prove an existence result for a class of semilinear delay functional differential equation subjected to nonlocal initial conditions. As applications an existence theorem referring to periodic solutions to a semilinear wave partial differential functional equation is included.

Contents

1. Introduction

Throughout X is a real Banach space, $A : D(A) \subseteq X \to X$ is a linear operator generating the C_0-semigroup of contractions $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$, $\tau \geq 0$, $f, g : [\,0, T\,] \times C([\,-\tau, 0\,]; X) \to X$ are jointly continuous functions having sublinear growth and $H : C([\,-\tau, T\,]; X) \to C([\,-\tau, 0\,]; X)$ is nonexpansive. If $u \in C([\,-\tau, T\,]; X)$ and $t \in [\,0, T\,]$, u_t is defined by $u_t(s) = u(t + s)$ for each $s \in [\,-\tau, 0\,]$.

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Let us consider the nonlinear delay differential evolution Cauchy problem

$$\left\{\begin{array}{ll} u'(t) = Au(t) + f(t,u_t) + g(t,u_t), & t \in [\,0,T\,], \\ u(t) = H(u)(t), & t \in [\,-\tau,0\,]. \end{array}\right. \tag{1.1}$$

Using some metrical combined with topological fixed-point arguments, we prove that, if f is continuous and compact, g is continuous and Lipschitz with respect to its second variable and the mapping $H : C([\,-\tau,T\,];X) \to C([\,-\tau,0\,];X)$ is nonexpansive in a certain sense to be made precise in due course, the problem (1.1) has at least one mild solution. This new theorem, inspired by those established by Vrabie [**31**] and [**33**], has applications in the study of some important classes of second order hyperbolic problems subjected to nonlocal initial conditions, applications which are not covered by the previously cited abstract results.

After the pioneering contribution of Byszewski [**14**], various results on nondelayed problems of these kind were obtained among others by: Aizicovici and Lee [**1**], Aizicovici and McKibben [**2**], Benedetti, Malaguti and Taddei [**6**], Bolojan-Nica, Infante and Precup [**8**], Byszewski and Lakshmikantham [**15**], García–Falset and Reich [**18**] and Paicu and Vrabie [**24**].

For the basic results on the classical theory of delay differential equations the interested reader is referred to Driver [**17**], Halanay [**20**] and Hale [**21**]. For the case of nonlocal problems, see Benchohra and Abbas [**5**], Burlică and Roşu [**10**], [**11**], Burlică, Necula, Roşu and Vrabie [**9**], Burlică, Roşu and Vrabie [**12**], [**13**], McKibben [**22**], Ntouyas [**23**] and Vrabie [**28**], [**29**], [**30**], [**32**], to cite only a few. Semilinear delay nonlocal problems involving measures were recently considered by Benedetti, Malaguti, Taddei and Vrabie [**7**].

The paper is divided into six sections. Section 2 collects some preliminaries referring to terminology, basic definitions and auxiliary results. Section 3 contains the statement of the main theorem, while Section 4 is concerned with an existence result for an auxiliary problem. In Section 5 we prove or main result. Here, the main and the most difficult point in the proof is Lemma 5.1. Finally, the last Section 6 includes an application to a damped semilinear delay wave equation with nonlocal initial conditions.

2. Preliminaries

We begin by introducing some basic terminology we shall use in that follows.

Definition 2.1. We say that:

(i) $f : [\,0,T\,] \times C([\,-\tau,0\,];X) \to X$ is *compact from* $C([\,-\tau,T\,];X)$ if for each bounded subset B in $C([\,-\tau,T\,];X)$, the set $\{f(t,v_t);\ v \in B,\ t \in [\,0,T\,]\}$ is relatively compact in X.

(ii) $g : [\,0,T\,] \times C([\,-\tau,0\,];X) \to X$ is *Lipschitz with respect to its last argument* if there exists $\ell > 0$ such that

$$\|g(t,u) - g(t,v)\| \le \ell\|u - v\|$$

for each $(t,u),(t,v) \in [\,0,T\,] \times C([\,-\tau,0\,];X)$.

Definition 2.2. By a *mild solution* of the problem (1.1) we mean a continuous function $u : [\,-\tau, T\,] \to X$ satisfying

$$u(t) = \begin{cases} H(u)(t), \ t \in [\,-\tau, 0\,] \\ S(t)H(u)(0) + \displaystyle\int_0^t S(t-s)\,[f(s,u_s) + g(s,u_s)]\,ds, \ t \in [\,0, T\,]. \end{cases}$$

Definition 2.3. Let $\omega > 0$. We say that a linear operator $A : D(A) \subseteq X \to X$ is the *infinitesimal generator of a C_0-semigroup of type* $(1, -\omega)$ if its generates a C_0-semigroup, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$, satisfying

$$\|S(t)x\| \le e^{-\omega t}\|x\|$$

for each $t \in \mathbb{R}_+$ and $x \in X$.

For easy reference, we also recall some useful results.

We begin with a compactness lemma, essentially based on Mazur Theorem, i.e. Vrabie [**27**, Theorem A.1.3, p. 292].

Lemma 2.1. *Let $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$ be a C_0-semigroup of contractions, K a compact subset in X and let $\mathcal{F}$ a family of continuous functions from $[\,a, b\,]$ to K. Then, for each $t \in [\,a, b\,]$, the set*

$$\left\{ \int_a^t S(t-s)f(s)\,ds\,;\ f \in \mathcal{F} \right\}$$

is relatively compact in X.

See Becker [**4**], or Cârjă, Necula and Vrabie [**16**, Lemma 1.5.1, p. 14].

Definition 2.4. Let X be a Banach space. We say that an operator $\mathcal{M} : X \to X$ is compact if it maps bounded subsets in X into relatively compact subsets in X.

The theorem below, due to Schaefer [**25**], is a variant of the Leray–Schauder Principle.

Theorem 2.1. (Schaefer) *Let X be a Banach space and let $\mathcal{M} : X \to X$ be a continuous, compact operator and let*

$$\mathcal{E}(\mathcal{M}) = \{x \in X; \exists \lambda \in [\,0, 1\,], \text{ such that } x = \lambda\mathcal{M}(x)\}.$$

If $\mathcal{E}(\mathcal{M})$ is bounded, then $\mathcal{M}$ has at least one fixed point.

See Schaefer [**25**] and Granas and Dugundji [**19**, Theorem 5.1, p. 123 and Theorem 5.4, p. 124].

Theorem 2.2. (Arzelà-Ascoli) *Let X be a Banach space. A subset $\mathcal{F}$ in $C([\,a, b\,]; X)$ is relatively compact if and only if:*

(*i*) *$\mathcal{F}$ is equicontinuous on $[\,a, b\,]$;*
(*ii*) *there exists a dense subset D in $[\,a, b\,]$ such that, for each $t \in D$,*

$$\mathcal{F}(t) = \{f(t);\ f \in \mathcal{F}\}$$

is relatively compact in X.

See Vrabie [**27**, Theorem A.2.1, p. 296].

3. The main abstract existence result

Definition 3.1. Let $a \in (0,T\,]$. The mapping $H : C([\,-\tau,T\,];X) \to C([\,-\tau,0\,];X)$ is called *a* $C([\,a,T\,];X)$*-contraction* if

$$\|H(u)-H(v)\|_{C([\,-\tau,0\,];X)} \le \|u-v\|_{C([\,a,T\,];X)}$$

for each $u,v \in C([\,-\tau,T\,];X)$.

The main result of the paper is:

Theorem 3.1. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of type* $(1,-\omega)$, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$. *Let* $f : [\,0,T\,] \times C([\,-\tau,0\,];X) \to X$ *be a continuous and compact from* $C([\,-\tau,T\,];X)$ *mapping – see Definition* 2.1 *– for which there exist* $k \in \mathbb{R}_+$ *and* $m \in \mathbb{R}_+$ *such that*

$$\|f(t,v)\| \le k\|v\|_{C([\,-\tau,0\,];X)} + m \tag{3.2}$$

for each $(t,v) \in [\,0,T\,] \times C([\,-\tau,0\,];X)$*, let* $g : [\,0,T\,] \times C([\,-\tau,0\,];X) \to X$ *be continuous on* $[\,0,T\,]\times C([\,-\tau,0\,];X)$ *and Lipschitz with respect to its last argument whose Lipschitz constant* ℓ *satisfies*

$$k+\ell<\omega, \tag{3.3}$$

and let $H : C([\,-\tau,T\,];X) \to C([\,-\tau,0\,];X)$ *be a* $C([\,a,T\,];X)$*-contraction – see Definition* 3.1. *Then the nonlocal problem* (1.1) *has at least one mild solution.*

It should be noticed that condition (3.3) is exactly the stability hypothesis in Poincaré-Liapunov Theorem. See Vrabie [**34**, Theorem 3.5.1, p. 169]. This shows that the dominant term in the problem (1.1) is A. One may easily see that, the case $A \equiv 0$ is ruled out by the fact that the C_0-semigroup generated by A is of type $(1,-\omega)$, where $\omega > 0$. It would be of great interest to know if the result remains still valid for $\omega = 0$. This is certainly case if assume a stronger condition on the history function H, i.e., if $H(u)=\psi$ for each $u \in C([\,-\tau,T\,;X)$, where $\psi \in C([\,-\tau,T\,;X)$ is fixed. For details on this simpler but useful in applications case, see Vrabie [**33**].

4. An auxiliary lemma

The goal of this section is to prove an auxiliary local existence result – interesting by itself – referring to the Cauchy problem

$$\begin{cases} u' = Au(t) + g(t,u_t) + h(t), & t \in [\,0,T\,] \\ u(t) = H(u)(t), & t \in [\,-\tau,0\,]. \end{cases} \tag{4.4}$$

Lemma 4.1. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of type* $(1,-\omega)$, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$. *Let* $g : [\,0,T\,] \times C([\,-\tau,0\,];X) \to X$ *be continuous on* $[\,0,T\,] \times C([\,-\tau,0\,];X)$ *and Lipschitz with respect to its last argument whose Lipschitz constant* $\ell > 0$ *satisfies* (3.3)*, and let* $H : C([\,-\tau,T\,];X) \to C([\,-\tau,0\,];X)$ *be a* $C([\,a,T\,];X)$*-contraction – see Definition* 3.1. *Then, for each* $h \in L^1(0,T\,;X)$*, the problem* (4.4) *has a unique mild solution defined on* $[\,0,T\,]$.

Before passing to the proof of Lemma 4.1, we state the following simple and useful

Remark 4.1. If $H : C([\,-\tau, T\,]; X) \to C([\,-\tau, 0\,]; X)$ is a $C([\,a, T\,]; X)$-contraction, then its values depend only on the values of the argument function on $[\,a, T\,]$. Indeed, if $u, v \in C([\,-\tau, T\,]; X)$ are such that $u(t) = v(t)$ for each $t \in [\,a, T\,]$, then $H(u) = H(v)$.

This explains why, in that follows, whenever $H : C([\,-\tau, T\,]; X) \to C([\,-\tau, 0\,]; X)$ is a $C([\,a, T\,]; X)$-contraction, we simply consider that H is defined merely on $C([\,a, T\,]; X)$.

We can now pass to the proof of Lemma 4.1.

Proof. Clearly, for each $v \in C([\,-\tau, T\,]; X)$ the problem

$$\begin{cases} u' = Au(t) + h(t), & t \in [\,0, T\,] \\ u(t) = H(v)(t), & t \in [\,-\tau, 0\,] \end{cases} \tag{4.5}$$

has a unique solution $u : [\,-\tau, T\,] \to X$. Accordingly, we can define the operator $Q : C([\,a, T\,]; X) \to C([\,a, T\,]; X)$ by

$$Q(v) := u_{|[\,a,T\,]},$$

where u is the unique mild solution of the problem (4.5) on $[\,-\tau, T\,]$. See also Remark 4.1. Let us observe that

$$\|Q(v)(t) - Q(w)(t)\| \le e^{-\omega a}\|v - w\|_{C([\,a,T\,];X)}$$

for each $t \in [\,a, T\,]$ and consequently Q is a strict contraction. By the Banach Fixed Point Theorem, Q has a unique fixed point which is a mild solution of the problem

$$\begin{cases} u' = Au(t) + h(t), & t \in [\,0, T\,] \\ u(t) = H(u)(t), & t \in [\,-\tau, 0\,]. \end{cases} \tag{4.6}$$

Next, let us consider the problem

$$\begin{cases} u' = Au(t) + g(t, v_t) + h(t), & t \in [\,0, T\,] \\ u(t) = H(u)(t), & t \in [\,-\tau, 0\,], \end{cases} \tag{4.7}$$

where $v \in C([\,-\tau, T\,]; X)$. Let $P : C([\,-\tau, T\,]; X) \to C([\,-\tau, T\,]; X)$ be defined by

$$P(v) := u,$$

where u is the unique mild solution of the problem (4.7) which is nothing but (4.6) with $h(t)$ replaced by $g(t, v_t) + h(t)$. We have

$$\|P(v)(t) - P(w)(t)\| \le e^{-\omega t}\|P(v) - P(w)\|_{C([\,a,T\,];X)} + \frac{\ell}{\omega}\left(1 - e^{-\omega t}\right)\|v - w\|_{C([\,-\tau,T\,];X)}$$

for each $t \in (0, T\,]$.

Since

$$\|P(v) - P(w)\|_{C([\,-\tau,0\,];X)} \le \|P(v) - P(w)\|_{C([\,a,T\,];X)} \le \|P(v) - P(w)\|_{C([\,0,T\,];X)},$$

we get

$$\begin{aligned} \|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} &= \max\{\|P(v) - P(w)\|_{C([\,-\tau,0\,];X)}, \|P(v) - P(w)\|_{C([\,0,T\,];X)}\} \\ &= \|P(v) - P(w)\|_{C([\,0,T\,];X)}. \end{aligned}$$

Therefore

$$\|P(v)(t) - P(w)(t)\| \le e^{-\omega t}\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} + \frac{\ell}{\omega}\left(1 - e^{-\omega t}\right)\|v - w\|_{C([\,-\tau,T\,];X)}$$

for each $t \in (0, T\,]$.

Let $\theta \in [\,-\tau, T\,]$ be such that

$$\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} = \|P(v)(\theta) - P(w)(\theta)\|.$$

We distinguish between two cases.

Case 1 If $\theta > 0$, we have

$$\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} = \|P(v) - P(w)\|_{C([\,0,T\,];X)}.$$

Taking $t = \theta$ in the above inequality yields

$$\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} \leq \frac{\ell}{\omega}\|v - w\|_{C([\,-\tau,T\,];X)}.$$

Case 2 If $\theta \leq 0$, from the nonlocal initial condition in (4.7), recalling that H is a $C([\,a,T\,];X)$-contraction – see Definition 3.1, we deduce

$$\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} = \|P(v)(\theta) - P(w)(\theta)\|$$

$$= \|H(P(v))(\theta) - H(P(w))(\theta)\| \leq \|P(v) - P(w)\|_{C([\,a,T\,];X)} \leq \|P(v) - P(w)\|_{C([\,-\tau,T\,];X)}.$$

Thus

$$\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} = \|P(v) - P(w)\|_{C([\,a,T\,];X)}.$$

Accordingly, there exists $\theta_0 \in [\,a, T\,]$ such that

$$\|P(v) - P(w)\|_{C([\,-\tau,T\,];X)} = \|P(v)(\theta_0) - P(w)(\theta_0)\|.$$

Therefore, we can always assume that we are in **Case 1**. But this implies that P is a strict contraction of constant $\dfrac{\ell}{\omega}$. See (3.3). Consequently, P has a unique fixed point which is a mild solution of the problem (4.4). □

5. Proof of the main result

We shall prove this theorem with the help of the two lemmas below which ore nontrivial replicas of some previous results in Vrabie [**33**] which, in turn, generalize some results referring to the non-delayed case, established in Vrabie [**27**, Lemma 10.2.2, p. 233 and Lemma 10.3.4, p. 234]. It should be noticed however that the proofs of both lemmas below are different from the proof of their just mentioned counterparts.

Lemma 5.1. *Let $\omega > 0$ and $\tau > 0$, and let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of type $(1, -\omega)$, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$. Let $g : [\,0,T\,] \times C([\,-\tau,0\,];X) \to X$ be continuous on $[\,0,T\,] \times C([\,-\tau,0\,];X)$ and Lipschitz with respect to its last argument with Lipschitz constant $\ell > 0$, and let $H : C([\,0,T\,];X) \to C([\,-\tau,0\,];X)$ be a $C([\,a,T\,];X)$-contraction – see Definition 3.1. Then, the mapping $h \mapsto \mathcal{S}(h)$, i.e., the unique solution of the problem (4.4) corresponding to h, is continuous from $L^1(0,T;X)$ to $C([\,-\tau,0\,];X)$ in the following weak sense: if $h \in L^1(0,T;X)$ and $(h_m)_{m\in\mathbb{N}}$ is a sequence in $L^1(0,T;X)$ such that*

$$\lim_m \mathcal{H}_m(t) = \mathcal{H}(t)$$

uniformly for $[\,0,T\,]$, where

$$\mathcal{H}_m(t) = \int_0^t S(t-s)h_m(s)\,ds \quad \textit{and} \quad \mathcal{H}(t) = \int_0^t S(t-s)h(s)\,ds,$$

then

$$\lim_m \mathcal{S}(h_m) = \mathcal{S}(h)$$

uniformly for $t \in [-\tau, T]$.

Proof. Let $h \in L^1(0,T;X)$ and let $(h_m)_{m\in\mathbb{N}}$ be any sequence in $L^1(0,T;X)$ such that

$$\lim_m \mathcal{H}_m(t) = \mathcal{H}(t)$$

uniformly for $[0,T]$. Let us denote by

$$t_m := \inf\{t \in [0,T];\ \|[\mathcal{S}(h_m)](t) - [\mathcal{S}(h)](t)\| = \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)}\}.$$

At this point, let us remark that there are only two complementary possibilities:

(I) $(t_m)_{m\in\mathbb{N}}$ is convergent and
(II) $(t_m)_{m\in\mathbb{N}}$ is not convergent.

We analyze these two possibilities separately.

(I) First, let us assume that $(t_m)_{m\in\mathbb{N}}$ is convergent to some $t \in [0,T]$, i.e., that

$$\lim_m t_m = t.$$

Also here, we distinguish between two cases.

Case 1. If $t > 0$, then, there exist $t^* > 0$ and $m_0 \in \mathbb{N}$ such that, for each $m \in \mathbb{N}$, $m \geq m_0$, we have

$$0 < t^* \leq t_m. \tag{5.8}$$

Hence

$$\begin{aligned}
\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)} &= \|[\mathcal{S}(h_m)](t_m) - [\mathcal{S}(h)](t_m)\| \\
\leq e^{-\omega t_m}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} &+ \left\|\int_0^{t_m} S(t_m - s)(h_m)(s) - h(s))\,ds\right\| \\
+ \int_0^{t_m} \|S(t_m - s)\| &\|g(s, (\mathcal{S}(h_m))_s) - g(s, (\mathcal{S}(h))_s)\|\,ds \\
&\leq e^{-\omega t_m}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)} \\
+\frac{\ell}{\omega}\left(1 - e^{-\omega t_m}\right)\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} &+ \|\mathcal{H}_m - \mathcal{H}\|_{C([0,T];X)}.
\end{aligned}$$

Observing that

$$\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} = \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)},$$

and that, by (5.8), $e^{-\omega t_m} < e^{-\omega t^*}$, we deduce

$$\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)} \leq \frac{\omega}{(\omega - \ell)(1 - e^{-\omega t^*})}\|\mathcal{H}_m - \mathcal{H}\|_{C([0,T];X)}$$

for each $m \in \mathbb{N}$. But this shows that

$$\lim_m \mathcal{S}(h_m) = \mathcal{S}(h).$$

Case 2. If $t = 0$, the above inequality rewrites

$$\begin{aligned}
\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)} &\leq e^{-\omega t_m}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} \\
+\frac{\ell}{\omega}\left(1 - e^{-\omega t_m}\right)\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} &+ \|\mathcal{H}_m - \mathcal{H}\|_{C([0,T];X)}.
\end{aligned}$$

Passing to the lim sup both sides, we get

$$\begin{aligned}
\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)} &\leq \limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} \\
&\leq \limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)}.
\end{aligned}$$

Consequently,

$$\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)} = \limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)}. \tag{5.9}$$

Since

$$\|\mathcal{S}(h_m)-\mathcal{S}(h)\|_{C([-\tau,T];X)} = \max\{\|\mathcal{S}(h_m)-\mathcal{S}(h)\|_{C([-\tau,0];X)}, \|\mathcal{S}(h_m)-\mathcal{S}(h)\|_{C([0,T];X)}\},$$

we deduce that

$$\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} = \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([0,T];X)}.$$

Then, from (5.9), we get

$$\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} = \limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)}. \tag{5.10}$$

On the other hand, for each $m \in \mathbb{N}$, there exists $s_m \in [a, T]$ such that

$$\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} = \|\mathcal{S}(h_m)(s_m) - \mathcal{S}(h)(s_m)\|$$

$$\leq e^{-\omega s_m}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} + \left\|\int_0^{s_m} S(s_m - s)(h_m)(s) - h(s))\, ds\right\|$$

$$+\int_0^{s_m} \|S(s_m - s)\|\|g(s, (\mathcal{S}(h_m))_s) - g(s, (\mathcal{S}(h))_s)\|\, ds$$

$$\leq e^{-\omega s_m}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)}$$

$$+\frac{\ell}{\omega}\left(1 - e^{-\omega s_m}\right)\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} + \|\mathcal{H}_m - \mathcal{H}\|_{C([0,T];X)}.$$

Hence,

$$\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} \leq e^{-\omega s_m}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)}$$

$$+\frac{\ell}{\omega}\left(1 - e^{-\omega s_m}\right)\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} + \|\mathcal{H}_m - \mathcal{H}\|_{C([0,T];X)}.$$

Consequently

$$\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)}$$

$$\leq \frac{\ell}{\omega}\|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} + \frac{1}{(1 - e^{-\omega s_m})}\|\mathcal{H}_m - \mathcal{H}\|_{C([0,T];X)}.$$

Passing to the lim sup both sides in the last inequality, recalling that $0 < a \leq s_m$ for each $m \in \mathbb{N}$, and taking into account of (5.9) and (5.10), we get

$$\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} \leq \frac{\ell}{\omega}\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)}.$$

By (3.3), we have $\dfrac{\ell}{\omega} \in (0,1)$, and therefore

$$\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([a,T];X)} = 0.$$

From (5.10), we get

$$\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} = 0. \tag{5.11}$$

(II) Now, if $(t_m)_{m\in\mathbb{N}}$ has no limit, we proceed by contradiction. Namely, let us assume that

$$\limsup_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} > 0.$$

This means that that there exist $\varepsilon > 0$, and a subsequence of $(t_m)_{m\in\mathbb{N}}$ – denoted again by $(t_m)_{m\in\mathbb{N}}$ – such that

$$0 < \varepsilon \leq \|\mathcal{S}(h_m)(t_m) - \mathcal{S}(h)(t_m)\|$$

for each $m \in \mathbb{N}$. Next, we may assume without loss of generality that, on a new subsequence, at least

$$\lim_m t_m = t.$$

From now, on, repeating the arguments above we conclude that

$$0 < \varepsilon \leq \limsup_m \|\mathcal{S}(h_m)(t_m) - \mathcal{S}(h)(t_m)\| = 0,$$

which is a contradiction. This contradiction can be eliminated only if (5.11) holds true. But (5.11) yields

$$\lim_m \|\mathcal{S}(h_m) - \mathcal{S}(h)\|_{C([-\tau,T];X)} = 0,$$

and this completes the proof. □

Next, let us consider the nonlinear delay differential evolution problem with nonlocal initial datum

$$\begin{cases} u'(t) = Au(t) + f(t, v_t) + g(t, u_t), & t \in \mathbb{R}_+, \\ u(t) = H(u)(t), & t \in [-\tau, 0], \end{cases} \tag{5.12}$$

where $v \in C([-\tau,T];X)$. By Lemma 4.1, we know that, for each $v \in C([-\tau,T];X)$, the problem (5.12) has a unique mild solution $u \in C([-\tau,T];X)$. So, we can define the solution operator $\mathcal{R} : C([-\tau,T];X) \to C([-\tau,T];X)$ by

$$\mathcal{R}(v) := u, \tag{5.13}$$

where u is the unique mild solution of (5.12).

Lemma 5.2. *Let $\omega > 0$ and $\tau > 0$, and let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of type $(1, -\omega)$, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$. Let $f : [0,T] \times C([-\tau,0];X) \to X$ be a continuous and compact from $C([-\tau,T];X)$ mapping mapping – see Definition* 2.1 *– for which there exist $k \in \mathbb{R}_+$ and $m \in \mathbb{R}_+$ such that* (3.2) *holds true, let $g : [0,T] \times C([-\tau,0];X) \to X$ be continuous on $[0,T] \times C([-\tau,0];X)$ and Lipschitz with respect to its last argument whose Lipschitz constant ℓ satisfies* (3.3)*, and let $H : C([0,T];X) \to C([-\tau,0];X)$ be a $C([a,T];X)$-contraction – see Definition* 3.1*. Then the operator $\mathcal{R}$ defined by* (5.13) *is continuous and compact.*

Proof. We begin with the proof of the compactness. Let $B \subseteq C([-\tau,T];X)$ be any bounded set. Since f is compact from $C([-\tau,T];X)$ – see Definition 2.1 –, it follows that there exists a compact set $K \subseteq X$ such that

$$f(t, v_t) \in K$$

for each $v \in B$ and $t \in [0,T]$. In view of Lemma 2.1,

$$\left\{ \int_0^t S(t-s) f(s, v_s)\, ds\, ;\ v \in B \right\}$$

is relatively compact, for each $t \in [0, T]$. In addition, for each $0 \leq \widetilde{t} \leq t \leq T$, we have

$$\begin{aligned}
&\left\| \int_0^t S(t-s)f(s,v_s)\,ds - \int_0^{\widetilde{t}} S(\widetilde{t}-s)f(s,v_s)\,ds \right\| \\
&\leq \left\| \int_0^t S(t-s)f(s,v_s)\,ds - \int_0^t S(\widetilde{t}-s)f(s,v_s)\,ds \right\| \\
&\qquad + \left\| \int_{\widetilde{t}}^t S(\widetilde{t}-s)f(s,v_s)\,ds \right\| \\
&\leq \int_0^t \sup_{x\in K} \|S(t-\widetilde{t})x - x\| ds + \int_{\widetilde{t}}^t \sup_{v\in B} \|f(s,v_s)\|\,ds \\
&\leq T \sup_{x\in K} \|S(t-\widetilde{t})x - x\| + |t-\widetilde{t}| \sup_{(s,v)\in[0,T]\times B} \|f(s,v_s)\|\,ds.
\end{aligned}$$

Since $\{S(t) : X \to X;\ t \in \mathbb{R}_+\}$ is a C_0-semigroup and K is compact, it follows that

$$\lim_{(t-\widetilde{t})\downarrow 0} \sup_{x\in K} \|S(t-\widetilde{t})x - x\| = 0.$$

By observing that $\{f(s,v_s);\ v \in K,\ s \in [0,T]\}$ is bounded – see (3.2) – , from the above inequality, we conclude that the set

$$\mathcal{F}(B) := \{t \mapsto \mathcal{F}(v)(t);\ v \in B\},$$

where, for each $v \in B$ and $t \in [0,T]$,

$$\mathcal{F}(v)(t) := \int_a^t S(t-s)f(s,v_s)\,ds,$$

is equicontinuous.

From Arzełà-Ascoli Theorem 2.2, we conclude that $\mathcal{F}(B)$ is relatively compact in $C([0,T];X)$. At this point, let us observe that

$$\mathcal{R}(v) = \mathcal{S}(h),$$

where $\mathcal{S}$ is defined in Lemma 5.1, and

$$h(t) := f(t, v_t).$$

From Lemma 5.1, it follows that $\mathcal{R}(B)$ is relatively compact. Since the continuity of $\mathcal{R}$ follows also from Lemma 5.1 and the continuity of f, the proof is complete. □

We can now proceed to the proof of Theorem 3.1 which is somehow inspired from the proof of the main result in Vrabie [**31**]. See also Burlică, Necula, Roşu and Vrabie [**9**, Section 4.5.3, pp. 152–154].

Proof. We show that the operator $\mathcal{R}$ satisfies the hypotheses of the Schaefer Fixed Point Theorem 2.1. In view of Lemma 5.2, we know that $\mathcal{R}$ is continuous and compact. Hence it remains merely to check out that the set

$$\mathcal{E}(\mathcal{R}) := \{u \in C([-\tau,T];X);\ \exists\lambda \in [0,1] \text{ such that } \lambda\mathcal{R}(u) = u\}$$

is bounded.

Let $u \in C([-\tau,T];X)$ be such that there exists $\lambda \in [0,1]$ such that

$$\lambda\mathcal{R}(u) = u.$$

Then u is a mild solution of the problem

$$\begin{cases} u'(t) = Au(t) + f(t, \lambda u_t) + g(t, u_t), & t \in \mathbb{R}_+, \\ u(t) = H(u)(t), & t \in [-\tau, 0], \end{cases}$$

From (3.2) and the Lipschitz condition on g, we deduce that, for each $t \in [0, T]$,

$$\|f(t, \lambda u_t)\| + \|g(t, u_t)\|$$

$$\leq k\lambda \|u_t\|_{C([-\tau,0];X)} + m + \ell \|u_t\|_{C([-\tau,0];X)} + \|g(t, 0)\|.$$

Denoting by

$$m_0 := m + \sup_{t \in [0,T]} \|g(t, 0)\|,$$

we get

$$\|f(t, \lambda u_t)\| + \|g(t, u_t)\| \leq (k\lambda + \ell)\|u_t\|_{C([-\tau,0];X)} + m_0$$

for each $t \in [0, T]$. Let us observe that

$$\|\mathcal{R}(u)(t)\| \leq e^{-\omega t}\|\mathcal{R}(u)(0)\| + \frac{k\lambda + \ell}{\omega}\left(1 - e^{-\omega t}\right)\|u\|_{C([-\tau,T];X)} + \frac{m_0}{\omega}\left(1 - e^{-\omega t}\right)$$

for each $t \in [0, T]$ and

$$u(t) = H(u)(t)$$

for each $t \in [-\tau, 0]$. Since $u = \lambda \mathcal{R}(u)$, multiplying both sides the last inequality by λ, after some trivial computations, we get

$$\|u(t)\| \leq \frac{\lambda(k\lambda + \ell)}{\omega}\|u\|_{C([-\tau,T];X)} + \frac{\lambda m_0}{\omega} \leq \frac{k + \ell}{\omega}\|u\|_{C([-\tau,T];X)} + \frac{m_0}{\omega}.$$

Recalling that H is a $C([a, T]; X)$-contraction, from this inequality and (3.3), we conclude that the set $\mathcal{E}(\mathcal{R})$ is bounded. From Theorem 2.1, it follows that $\mathcal{R}$ has at least one fixed point $u \in C([-\tau, T]; X)$. Obviously, u is a mild solution of the problem (1.1), and this completes the proof. □

6. A damped semilinear wave equation with delay

We analyze an example showing that Theorem 3.1 handles as particular cases second order semilinear hyperbolic problems on bounded domains in $\mathbb{R}^d$, $d \geq 1$.

For previous results on second order linear hyperbolic equations without delay subjected to nonlocal initial conditions see A. Avalishvili and M. Avalishvili [**3**]. Specific mathematical models described by nonlocal problems can be found in Shelukhin [**26**].

Let Ω be a nonempty bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^1 boundary Σ, let $Q_+ = \mathbb{R}_+ \times \Omega$, let $\tau \geq 0$, $\omega > 0$, $Q_\tau = [-\tau, 0] \times \Omega$, $\Sigma_+ = \mathbb{R}_+ \times \Sigma$ and let us consider the non-local initial-value problem for the damped wave equation with delay:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \mathcal{L}u + h_1\left(t, \displaystyle\int_{-\tau}^{0} u(t+s, x)\, ds\right) + h_2\left(t, \left(\dfrac{\partial u}{\partial t}\right)_t\right) & \text{in } Q_+, \\ u(t, x) = 0, & \text{on } \Sigma_+, \\ u(t, x) = [H_1(u)(t)](x), \quad \dfrac{\partial u}{\partial t}(t, x) = \left[H_2\left(\dfrac{\partial u}{\partial t}\right)(t)\right](x), & \text{in } Q_\tau, \end{cases} \tag{6.14}$$

where

$$[\mathcal{L}u](t,x) := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(t,x) - 2\omega \frac{\partial u}{\partial t}(t,x) - \omega^2 u(t,x). \tag{6.15}$$

The problem (6.14) represents a closed loop system of a control problem associated to the damped wave propagation in Ω obeying some nonlocal initial conditions in the case in which the synthesis operator $h_1 + h_2$ reacts after some delay τ.

For $\mathbb{I}$ an interval and $p \in [1, +\infty)$, whenever necessary, we will identify a function u from $\mathbb{I} \times \Omega$ to $\mathbb{R}$ with a function from $\mathbb{I}$ to $L^p(\Omega)$, and we denote it, for simplicity, again by u. By virtue of this identification, for $t \in \mathbb{I}$, we will write $u(t)(x)$ instead of $u(t,x)$ a.e. for $x \in \Omega$, or even $u(t)$.

Theorem 6.1. *Let Ω be a nonempty bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^1 boundary Σ, let $T > 0$, let $\tau \geq 0$ and let $a \in (0, T]$. Let us assume that the functions h_1, h_2, H_1 and H_2 satisfy*:

(*i*) *$h_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist $k \in \mathbb{R}_+$ and $m \in \mathbb{R}_+$ such that*
$$|h_1(t,y)| \leq k|y| + m$$
for all $(t,y) \in [0,T] \times \mathbb{R}$.

(*ii*) *$h_2 : [0,T] \times C([-\tau, 0]; L^2(\Omega)) \to L^2(\Omega)$ is continuous, and there exists $\gamma \in \mathbb{R}_+$ such that*
$$\|h_2(t,y) - h_2(t,z)\|_{L^2(\Omega)} \leq \gamma \|y - z\|_{C([-\tau,0];L^2(\Omega))}$$
for all $(t,y),(t,z) \in [0,T] \times C([-\tau,0]; L^2(\Omega))$.

(*iii*) *$H_1 : C([a,T]; H_0^1(\Omega)) \to C([-\tau,0]; H_0^1(\Omega))$, and there exists $a \in (0,T]$ such that*
$$\|H_1(u) - H_1(\widetilde{u})\|_{C([-\tau,0];H_0^1(\Omega))} \leq \|u - \widetilde{u}\|_{C([a,T];H_0^1(\Omega))}$$
for all $u, \widetilde{u} \in C([-\tau,0]; H_0^1(\Omega))$.

(*iv*) *$H_2 : C([a,T]; L_0^2(\Omega)) \to C([-\tau,0]; L^2(\Omega))$, and with the same $a \in (0,T]$ given by* (*iii*), *we have*
$$\|H_2(v) - H_2(\widetilde{v})\|_{C([-\tau,0];L^2(\Omega))} \leq \|u - v\|_{C([a,T];L^2(\Omega))}$$
for all $v, \widetilde{v} \in C([-\tau,0]; L^2(\Omega))$.

(*v*) *With λ_1 the first eigenvalue of $-\Delta$, we have $k + \gamma(1 + \omega\lambda_1^{-1}) < \omega$.*

Then, there exists at least one mild solution, u, of the problem (6.14), *satisfying $u \in C([-\tau,T]; H_0^1(\Omega))$, and $\dfrac{\partial u}{\partial t} \in C([-\tau,T]); L^2(\Omega))$.*

Before passing to the proof of Theorem 6.1, we discuss two significant examples of functions H_i, $i = 1, 2$, satisfying the hypotheses (*iii*) and (*iv*).

Example 6.1. Let $b \leq t_0 < t_1 < \cdots < t_n \leq T$ and $b \leq s_0 < s_1 < \cdots < s_m \leq T$, with $n, m \in \mathbb{N}^*$, let $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots \beta_m$ be real numbers satisfying
$$\sum_{p=0}^{n} \alpha_p \leq 1, \quad \sum_{q=0}^{m} \beta_q \leq 1,$$
and let us define
$$H_1(u)(t) := \sum_{p=0}^{n} \alpha_p u(t_p + t), \qquad H_2(v)(t) := \sum_{q=0}^{m} \beta_q v(s_q + t),$$

for each $u \in C([-\tau,T];H_0^1(\Omega))$, $v \in C([-\tau,T];L^2(\Omega))$ and $t \in [-\tau,0]$. One can easily check that, if $\tau < b < T$, then H_1 and H_2, defined as before, satisfy *(iii)* and *(iv)* in Theorem 6.1 with $a = b-\tau$.

Example 6.2. Let us define

$$H_1(u)(t) := u(T+t), \quad H_2(v)(t) := v(T+t),$$

for each $u \in C([-\tau,T];H_0^1(\Omega))$, $v \in C([-\tau,T];L^2(\Omega))$ and $t \in [-\tau,0]$. Clearly, this case, which corresponds to the usual T-periodic conditions, is a specific form of the preceding one, with $n = m = 1$, $t_0 = s_0 = T$ and $\alpha_0 = \beta_0 = 1$. Of course, if $\tau < b < T$, then H_1 and H_2 satisfy *(iii)* and *(iv)* in Theorem 6.1 with $a = b-\tau$

So, Theorem 6.1 also handles T-periodic problems of the form:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \mathcal{L}u + h_1\left(t, \displaystyle\int_{-\tau}^0 u(t+s,x)\,ds\right) + h_2\left(t, \left(\dfrac{\partial u}{\partial t}\right)_t\right) & \text{in } Q_+, \\ u(t,x) = 0, & \text{on } \Sigma_+, \\ u(t,x) = u(t+T,x), \quad \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial u}{\partial t}(t+T,x), & \text{in } Q_\tau, \end{cases} \tag{6.16}$$

where $\mathcal{L}$ is defined by (6.15). More precisely, from Theorem 6.1, we obtain:

Theorem 6.2. *Let Ω be a nonempty bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^1 boundary Σ, let $T > 0$, let $\tau \geq 0$ and let $a \in (0,T]$. Let us assume that the functions h_1 and h_2 satisfy:*

- *(i)* *$h_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist $k \in \mathbb{R}_+$ and $m \in \mathbb{R}_+$ such that*
$$|h_1(t,y)| \leq k|y| + m$$
for all $(t,y) \in [0,T] \times \mathbb{R}$.
- *(ii)* *$h_2 : \mathbb{R}_+ \times C([-\tau,0];L^2(\Omega)) \to L^2(\Omega)$ is continuous, and there exists $\gamma \in \mathbb{R}_+$ such that*
$$\|h_2(t,y) - h_2(t,z)\|_{L^2(\Omega)} \leq \gamma \|y-z\|_{C([-\tau,0];L^2(\Omega))}$$
for all $(t,y),(t,z) \in [0,T] \times C([-\tau,0];L^2(\Omega))$.
- *(iii′)* *h_1 and h_2 are T-periodic with respect to their first argument, and $T > \tau$.*
- *(v)* *With λ_1 the first eigenvalue of $-\Delta$, we have $k + \gamma(1+\omega\lambda_1^{-1}) < \omega$.*

Then, there exists at least one T-periodic mild solution, u, of the problem (6.16), *satisfying $u \in C([-\tau,T];H_0^1(\Omega))$, and $\dfrac{\partial u}{\partial t} \in C([-\tau,T];L^2(\Omega))$.*

We can now proceed to the proof of Theorem 6.1.

Proof. The problem (6.14) can be rewritten as a first-order system of partial differential equations of the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = v - \omega u, & \text{in } Q_+, \\ \dfrac{\partial v}{\partial t} = \Delta u - \omega v + h_1\left(t, \displaystyle\int_{-\tau}^0 u(t+s,x)\,ds\right) + h_2(t, -\omega u_t + v_t)) & \text{in } Q_+, \\ u(t,x) = 0, & \text{on } \Sigma_+, \\ u(t,x) = [H_1(u)(t)](x), \quad v(t,x) = [H_2(v)(t)]\,(x), & \text{in } Q_\tau. \end{cases} \tag{6.17}$$

The product space X, defined by

$$X = \begin{pmatrix} H_0^1(\Omega) \\ \times \\ L^2(\Omega) \end{pmatrix}, \tag{6.18}$$

endowed with the natural inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \right\rangle = \int_\Omega \nabla u(x) \cdot \nabla \widetilde{u}(x)\, dx + \int_\Omega v(x)\widetilde{v}(x)\, dx$$

for each $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \in X$, is a real Hilbert space. Clearly, (6.17) can be rewritten as an abstract evolution equation subjected to nonlocal initial conditions of the form: (1.1) in the space X given by (6.18), where A, f, g and H are defined as follows.

First, let us define the linear operator $A : D(A) \subseteq X \to X$ by

$$D(A) = \begin{pmatrix} H_0^1(\Omega) \cap H^2(\Omega) \\ \times \\ H_0^1(\Omega) \end{pmatrix}, \quad A \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} -\omega u + v \\ \Delta u - \omega v \end{pmatrix}$$

for each $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$.

Let us define $f : \mathbb{R}_+ \times C([-\tau, 0]; X) \to X$ and $g : \mathbb{R}_+ \times C([-\tau, 0]; X) \to X$ by

$$f\left(t, \begin{pmatrix} u \\ v \end{pmatrix}\right)(x) := \begin{pmatrix} 0 \\ h_1\left(t, \int_{-\tau}^0 u(s,x)\, ds\right) \end{pmatrix}$$

and respectively by

$$g\left(t, \begin{pmatrix} u \\ v \end{pmatrix}\right)(x) := \begin{pmatrix} 0 \\ h_2(t, -\omega u + v) \end{pmatrix}$$

for each $t \in \mathbb{R}_+$, each $\begin{pmatrix} u \\ v \end{pmatrix} \in C([-\tau, 0]; X)$ and a.e. for $x \in \Omega$. Let us further define $H : C([-\tau, T]; X) \to C([-\tau, 0]; X)$ by

$$\left[H \begin{pmatrix} u \\ v \end{pmatrix}(t)\right](x) := \left[\begin{pmatrix} H_1(u) \\ H_2(v) \end{pmatrix}(t)\right](x)$$

So, the problem (6.17) rewrites in the space X as:

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}'(t) = A \begin{pmatrix} u \\ v \end{pmatrix}(t) + f\left(t, \begin{pmatrix} u_t \\ v_t \end{pmatrix} ds\right) + g\left(t, \begin{pmatrix} u_t \\ v_t \end{pmatrix} ds\right), & t \in [0, T], \\ \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = H \begin{pmatrix} u \\ v \end{pmatrix}(t), & t \in [-\tau, 0], \end{cases}$$

where A, f, g and H are defined as above.

By Vrabie [**27**, Theorem 4.6.2, p. 93], we know that the linear operator $A + \omega I$ generates a C_0-group of unitary operators $\{G(t) : X \to X;\ t \in \mathbb{R}\}$. So, A is the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \in \mathbb{R}_+\}$, where $S(t) = e^{-\omega t}G(t)$, for each $t \in \mathbb{R}_+$ which, clearly, is of type $(1, -\omega)$.

From Burlică, Necula, Roşu and Vrabie [**9**, Lemma 2.2.1, p. 64], we conclude that f is continuous and compact from $C([-\tau, T]; X)$.

Using (ii) and Poincaré's Inequality – see Burlică, Necula, Roşu and Vrabie [**9**, Lemma 1.9.1, p. 37] –, we deduce that, for each $\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \in X$, we have

$$\left\| g\left(t, \begin{pmatrix} u \\ v \end{pmatrix}\right) - g\left(t, \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}\right)\right\|_X = \|h_2(t, -\omega u + v) - h_2(t, -\omega\widetilde{u} + \widetilde{v})\|_{L^2(\Omega)}$$

$$\leq \gamma\omega\|u - \widetilde{u}\|_{C([\,-\tau,0\,];L^2(\Omega))} + \gamma\|v - \widetilde{v}\|_{C([\,-\tau,0\,];L^2(\Omega))}$$

$$\leq \lambda_1^{-1}\gamma\omega\|u - \widetilde{u}\|_{C([\,-\tau,0\,];H_0^1(\Omega))} + \gamma\|v - \widetilde{v}\|_{C([\,-\tau,0\,];L^2(\Omega))}$$

$$\leq \gamma(1 + \lambda_1^{-1}\omega)\left(\|u - \widetilde{u}\|_{C([\,-\tau,0\,];H_0^1(\Omega))} + \|v - \widetilde{v}\|_{C([\,-\tau,0\,];L^2(\Omega))}\right)$$

$$= \gamma(1 + \lambda_1^{-1}\omega)\left\|\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix}\right\|_{C([\,-\tau,0\,]X)}.$$

Clearly, g is continuous on $[\,0,T\,] \times C([\,-\tau,0\,];X)$ and Lipschitz with respect to its second argument with Lipschitz constant

$$\ell = \gamma(1 + \lambda_1^{-1}\omega).$$

Finally, from (v), we conclude that ℓ, defined as above, satisfies (3.3). So, we are in the hypotheses of Theorem 3.1, wherefrom we get the conclusion. This completes the proof of Theorem 6.1. □

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