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The limit behavior of relaxation time for full compressible magnetohydrodynamic flows with Cattaneo's law

Guowei Liu and Xin Xu

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ABSTRACT. In this paper, we discuss the system of full compressible magnetohydrodynamic equations with replacing the Fourier's law by Cattaneo's law in $\mathbb{R}^3$. First, local existence of solutions for general initial data and global existence of solutions for small initial data are shown. Then we obtain the uniform convergence of solutions of the relaxed system to that of the classical system when the relaxation time ϵ goes to 0.

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1. Introduction

In this paper, we investigate the Cauchy problem of full system of partial differential equations for three dimensional viscous compressible magnetohydrodynamic

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(MHD) flows in the Eulerian coordinates:

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla p = (\nabla\times B)\times B + \mathrm{div}\Psi, \tag{1.2}$$

$$\varepsilon_t + \nabla\cdot(u(\varepsilon' + p)) + \nabla\cdot q = \nabla\cdot((u\times B)\times B + \nu B\times(\nabla\times B) + u\Psi), \tag{1.3}$$

$$B_t - \nabla\times(u\times B) = -\nabla\times(\nu\nabla\times B), \quad \nabla\cdot B = 0, \tag{1.4}$$

$$\epsilon q_t + q + \kappa\nabla\theta = 0, \tag{1.5}$$

with initial condition

$$(\rho, u, \theta, B, q)(x, 0) = (\rho_0, u_0, \theta_0, B_0, q_0) \tag{1.6}$$

where ρ denotes the density, $u = (u_1, u_2, u_3)$ is the velocity, θ is the absolute temperature, $B = (B_1, B_2, B_3)$ is the magnetic field, $q = (q_1, q_2, q_3)$ is the heat flux. Ψ is the viscous stress tensor given by

$$\Psi = \mu(\nabla u + \nabla u^T) + \mu'(\nabla\cdot u)I,$$

and ε is the total energy given by

$$\varepsilon = \rho(e + \frac{1}{2}|u|^2) + \frac{1}{2}|B|^2, \quad \varepsilon' = \rho(e + \frac{1}{2}|u|^2),$$

with e the internal energy, $\frac{1}{2}|u|^2$ the kinetic energy, and $\frac{1}{2}|B|^2$ the magnetic energy. The equations of state $p = p(\rho, \theta), e = e(\rho, \theta)$ relate the pressure and the internal energy to the density and the temperature of the flow, I is the 3×3 identity matrix, and ∇u^T is the transpose of the matrix ∇u. The viscosity coefficients μ, μ' of the flow satisfy

$$\mu > 0, \quad 2\mu + 3\mu' > 0,$$

$\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, $\epsilon > 0$ is the constant relaxation time and $\kappa > 0$ is the heat conductivity. In this paper, we consider the general equations of state and assume that the pressure $p = p(\rho, \theta)$ and $e = e(\rho, \theta)$ are smooth functions of (ρ, θ) satisfying

$$\rho^2 e_\rho(\rho, \theta) = p(\rho, \theta) - \theta p_\theta(\rho, \theta). \tag{1.7}$$

In particular, the case of a polytropic gas $p = R\rho\theta, e = c_v\theta$ is included here where $R > 0$ is the gas constant, $c_v = R/(\gamma - 1)$ is the heat capacity of the gas at constant volume, and $\gamma > 1$ is the adiabatic exponent.

If $\epsilon = 0$, then the system (1.1)-(1.7) becomes the classical MHD system satisfying Fourier law $q = -\kappa\nabla\theta$, that is

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{1.8}$$

$$(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla p = (\nabla\times B)\times B + \mathrm{div}\Psi, \tag{1.9}$$

$$\varepsilon_t + \nabla\cdot(u(\varepsilon' + p)) = \nabla\cdot(\kappa\nabla\theta + (u\times B)\times B + \nu B\times(\nabla\times B) + u\Psi), \tag{1.10}$$

$$B_t - \nabla\times(u\times B) = -\nabla\times(\nu\nabla\times B), \quad \nabla\cdot B = 0. \tag{1.11}$$

Due to the wide applications in the real world, there is a large literature on the mathematical theory of the classical MHD system (1.8)-(1.11), see [**2, 3, 4, 5, 6, 7, 8, 14, 10, 11, 12, 15**] and the references cited therein. For example, the local theory is classical and can be obtained through Kato's method [**14**]. Also, many global results have been obtained. In one dimension, we refer to [**2, 3**] for the global large solutions. In three dimension, we refer to [**6, 7**] for the existence of variational weak solutions and refer to [**8, 15**] for the global existence and large time behaviors.

However, as far as we concerned, there is little literatures on the MHD with Cattaneo's law. Cattaneo's law is among one of the physical laws describing the finite speed of heat conduction in contrast to the Fourier' law possessing an inherent infinite propagation speed. It has been widely used in thermoelasticity which results in the second sound phenomenon, see [**16, 17**] and the references cited therein.

The purpose of this paper is twofold. The first one is to establish existence of solutions for the systems (1.1)-(1.7). We obtain the local existence for general initial data(Theorem 2.1) and the global existence for some small initial data (Theorem 2.2) by the Kawashima's theorem about existence of solutions for systems of hyperbolic-parabolic composite type, respectively. The second one is to use the energy method to obtain the uniform convergence of solutions of the relaxed system (1.1)-(1.7) to that of the classical system (1.8)-(1.11) when the relaxation time ϵ goes to 0 (Theorem 3.1).

Throughout this paper, the character α is multi-indexes, $\nabla^\alpha := \partial_{\mathrm{x}_1}^{\alpha_1}\partial_{\mathrm{x}_2}^{\alpha_2}\cdots\partial_{\mathrm{x}_\mathrm{n}}^{\alpha_\mathrm{n}}$ and $|\alpha| := \sum_{i=1}^{n}|\alpha_i|$. We denote by $\mathbb{R}$ the set of real numbers. We also use C to represent the generic constant which may take different values in different places. Let $\mathrm{L}^\mathrm{p}(\Omega)$ and $\mathrm{W}^{\mathrm{m},\mathrm{p}}(\Omega)$ be the usual Lebesgue space and Sobolev space endowed with norms $\|\cdot\|_{\mathrm{L}^\mathrm{p}}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{m},\mathrm{p}}}$ respectively (see e.g. [**1**]), where

$$\|\varphi\|_{\mathrm{L}^\mathrm{p}} := \Big(\int_\Omega |\varphi|^\mathrm{p}\mathrm{dx}\Big)^{1/\mathrm{p}} \quad \text{and} \quad \|\varphi\|_{\mathrm{W}^{\mathrm{m},\mathrm{p}}} := \Big(\sum_{|\beta|\leqslant \mathrm{m}}\int_\Omega |\mathrm{D}^\beta\varphi|^\mathrm{p}\mathrm{dx}\Big)^{1/\mathrm{p}}.$$

In particular, we take $\mathrm{H}^\mathrm{s} = \mathrm{W}^{\mathrm{s},2}$ with norm $\|\cdot\|_s$. We also let $\mathcal{C}(I;E)$ be the space of continuous functions on the interval I, with values in the Banach space E, endowed with the usual norm.

The rest of this paper is organized as follows. In the second section, we prove the existence of solutions for the system (1.1)-(1.7). In the third section, we devote to the limit behavior of solutions for the system (1.1)-(1.7) when the relaxation time ϵ goes to 0.

2. Existence of solutions

In this section, we consider local existence of solutions for general initial data and global existence of solutions for small initial data for the system (1.1)-(1.7). First, we give the following assumption.

Assumption 2.1. (1) *The initial data $(\rho_0, u_0, \theta_0, B_0, q_0)$ satisfy*

$$(\rho_0, u_0, \theta_0, B_0, q_0)$$
$$\subset [\rho_*, \rho^*]\times[-C_1, C_1]^3\times[\theta_*, \theta^*]\times[-C_1, C_1]^3\times[-C_1, C_1]^3 := G_0,$$

where $C_1 > 0$ as well as $0<\rho_*<1<\rho^*<\infty$ and $0<\theta_*<1<\theta^*<\infty$ are constants.

(2) *For each given G_1 satisfying $\mathrm{G}_0 \subset\subset \mathrm{G}_1 \subset\subset \mathrm{G}, \forall\,(\rho, \mathrm{u}, \theta, \mathrm{B}, \mathrm{q}) \in \mathrm{G}_1$, the pressure p and the internal energy e satisfy*

$$\mathrm{p}(\rho,\theta), \mathrm{p}_\theta(\rho,\theta), \mathrm{p}_\rho(\rho,\theta), \mathrm{e}_\theta(\rho,\theta) > \mathrm{C}(\mathrm{G}_1) > 0,$$

where $\mathrm{C}(\mathrm{G}_1)$ is a positive constants depending on G_1 and $\mathrm{G} := \mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$.

Under the Assumption 2.1, we have the local existence theorem for general initial data for the system (1.1)-(1.7).

THEOREM 2.1. *Let* $\mathrm{s} \geqslant \mathrm{s}_0 + 1$ *with* $\mathrm{s}_0 \geqslant 2$ *be integers, suppose that Assumption 2.1 hold and the initial data* $(\rho_0 - 1, \mathrm{u}_0, \theta_0 - 1, \mathrm{B}_0, \mathrm{q}_0) \in \mathrm{H}^{\mathrm{s}}$ *and* $\nabla \cdot \mathrm{B}_0 = 0$. *Then for each convex open subset* G_1 *satisfying* $\mathrm{G}_0 \subset\subset \mathrm{G}_1 \subset\subset \mathrm{G}$, *there exists* $\mathrm{T} > 0$ *such that system (1.1)-(1.7) has a unique classical solution* $(\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{B}^\epsilon, \mathrm{q}^\epsilon)$ *satisfying*

$$(\rho^\epsilon - 1, \theta^\epsilon - 1, \mathrm{q}^\epsilon) \in \mathcal{C}([0, \mathrm{T}]; \mathrm{H}^{\mathrm{s}}) \cap \mathcal{C}^1([0, \mathrm{T}]; \mathrm{H}^{\mathrm{s}-1})$$

and

$$\mathrm{u}^\epsilon, \mathrm{B}^\epsilon \in \mathcal{C}([0, \mathrm{T}]; \mathrm{H}^{\mathrm{s}}) \cap \mathcal{C}^1([0, \mathrm{T}]; \mathrm{H}^{\mathrm{s}-2})$$

and

$$(\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{B}^\epsilon, \mathrm{q}^\epsilon) \in \mathrm{G}_1, \forall\, (\mathrm{x}, \mathrm{t}) \in \mathbb{R}^3 \times [0, \mathrm{T}].$$

PROOF. At first, we rewrite the system (1.1)-(1.7) into a symmetric form

$$\rho_{\mathrm{t}} + \nabla \cdot (\rho \mathrm{u}) = 0, \tag{2.1}$$

$$\rho(\mathrm{u}_{\mathrm{t}} + (\mathrm{u} \cdot \nabla)\mathrm{u}) + \nabla \mathrm{p} - (\nabla \times \mathrm{B}) \times \mathrm{B} = \mu \Delta \mathrm{u} + (\mu + \mu')\nabla(\nabla \cdot \mathrm{u}), \tag{2.2}$$

$$\begin{aligned}\rho \mathrm{e}_\theta(\theta_{\mathrm{t}} + \mathrm{u} \cdot \nabla\theta) + \theta \mathrm{p}_\theta(\nabla \cdot \mathrm{u}) + \nabla \cdot \mathrm{q} = \frac{\mu}{2}|\nabla \mathrm{u} + \nabla \mathrm{u}^{\mathrm{T}}|^2 \\ + \mu'(\mathrm{div} \mathrm{u})^2 + \nu|\nabla \times \mathrm{B}|^2,\end{aligned} \tag{2.3}$$

$$\mathrm{B}_{\mathrm{t}} + (\mathrm{u} \cdot \nabla)\mathrm{B} + \mathrm{B}(\nabla \cdot \mathrm{u}) - (\mathrm{B} \cdot \nabla)\mathrm{u} = \nu \Delta \mathrm{B}, \quad \nabla \cdot \mathrm{B} = 0, \tag{2.4}$$

$$\epsilon \mathrm{q}_{\mathrm{t}} + \mathrm{q} + \kappa \nabla\theta = 0. \tag{2.5}$$

Put $\mathrm{w} = (\rho, \mathrm{u}, \theta, \mathrm{B}, \mathrm{q})^{\mathrm{T}}$. Then the system (2.1)-(2.5) is written in the form

$$\mathrm{A}^0(\mathrm{w})\mathrm{w}_{\mathrm{t}} + \sum_{\mathrm{j}=1}^{3} \mathrm{A}^{\mathrm{j}}(\mathrm{w})\mathrm{w}_{\mathrm{x_j}} - \sum_{\mathrm{j,k}=1}^{3} \mathrm{B}^{\mathrm{jk}}(\mathrm{w})\mathrm{w}_{\mathrm{x_j x_k}} + \mathrm{L}(\mathrm{w})\mathrm{w} = \mathrm{g}(\mathrm{w}, \mathrm{D_x w}),$$

where $\mathrm{A}^0(\mathrm{w}), \mathrm{A}^{\mathrm{j}}(\mathrm{w})$ and $\mathrm{B}^{\mathrm{jk}}(\mathrm{w})$ are square matrices of order 11, and $\mathrm{g}(\mathrm{w}, \mathrm{D_x w})$ is a $\mathbb{R}^{11}$-valued function. They are given explicitly by

$$\mathrm{A}^0(\mathrm{w}) = \begin{bmatrix} \frac{\mathrm{p}_\rho}{\rho} & & & & \\ & \rho \mathrm{I} & & & \\ & & \frac{\rho \mathrm{e}_\theta}{\theta} & & \\ & & & \mathrm{I} & \\ & & & & \frac{\epsilon}{\kappa\theta} \end{bmatrix},$$

$$\sum_{\mathrm{j}} \mathrm{A}^{\mathrm{j}}(\mathrm{w})\xi_{\mathrm{j}} = \begin{bmatrix} \frac{\mathrm{p}_\rho}{\rho}(\mathrm{u} \cdot \xi) & \mathrm{p}_\rho \xi & 0 & 0 & 0 \\ \mathrm{p}_\rho \xi^{\mathrm{T}} & \rho(\mathrm{u} \cdot \xi)\mathrm{I} & \mathrm{p}_\theta \xi^{\mathrm{T}} & \xi^{\mathrm{T}} \cdot \mathrm{B} - (\mathrm{B} \cdot \xi)\mathrm{I} & 0 \\ 0 & \mathrm{p}_\theta \xi & \frac{\rho \mathrm{e}_\theta}{\theta}(\mathrm{u} \cdot \xi) & 0 & \frac{\xi}{\theta} \\ 0 & \mathrm{B}^{\mathrm{T}} \cdot \xi - (\mathrm{B} \cdot \xi)\mathrm{I} & 0 & (\mathrm{u} \cdot \xi)\mathrm{I} & 0 \\ 0 & 0 & \frac{\xi^{\mathrm{T}}}{\theta} & 0 & 0 \end{bmatrix},$$

$$\sum_{\mathrm{j,k}} \mathrm{B}^{\mathrm{jk}}(\mathrm{w})\xi_{\mathrm{j}}\xi_{\mathrm{k}} = \begin{bmatrix} 0 & & & & \\ & \mu \mathrm{I} + (\mu + \mu')\xi^{\mathrm{T}}\xi & & & \\ & & 0 & & \\ & & & \nu \mathrm{I} & \\ & & & & 0 \end{bmatrix},$$

$$\mathrm{L(w)} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \frac{1}{\kappa\theta}\mathrm{I} \end{bmatrix},$$

$$\mathrm{g(w, D_x w)} = \begin{bmatrix} 0 \\ 0 \\ \frac{\mu}{2\theta}|\nabla \mathrm{u} + \nabla \mathrm{u}^{\mathrm{T}}|^2 + \frac{\mu'}{\theta}(\nabla\cdot \mathrm{u})^2 + \nu|\nabla\times \mathrm{B}|^2 \\ 0 \\ 0 \end{bmatrix}.$$

From the Assumption 2.1, it is seen that

(i) $\mathrm{A^0(w)}$ is real symmetric and positive definite for all $\mathrm{w} \in \mathrm{G_1}$;
(ii) $\mathrm{A^j(w)}$ and $\mathrm{B^{jk}(w)}$ are real symmetric and $\mathrm{B^{jk}(w)} = \mathrm{B^{kj}(w)}$ for all $\mathrm{w} \in \mathrm{G_1}$;
(iii) $\sum\limits_{\mathrm{j,k}} \mathrm{B^{jk}(w)}\xi_{\mathrm{i}}\xi_{\mathrm{j}}$ is real symmetric and semi-positive definite for all $\mathrm{w} \in \mathrm{G_1}$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{S}^2$;
(iv) $\mathrm{g(w, D_x w)}$ can be regarded as a lower order term, $\mathrm{g}(\overline{\mathrm{w}}, 0) = 0$ holds for the constant state $\overline{\mathrm{w}} = (1, 0, 1, 0, 0)$.

By the Kawashima's local existence theorem [**9**, Theorem 2.9], we can get the result in Theorem 2.1 immediately. □

To end this section, we establish global existence of solutions for the small initial data for the system (1.1)-(1.7). In the following, we take constant state $\overline{\mathrm{w}} = (1, 0, 1, 0, 0)$ and set $\overline{\mathrm{p}}_\rho = \mathrm{p}_\rho(1, 1) > 0, \overline{\mathrm{p}}_\theta = \mathrm{p}_\theta(1, 1) > 0, \overline{\mathrm{e}}_\theta = \mathrm{e}_\theta(1, 1) > 0$.

THEOREM 2.2. *Let* $\mathrm{s} \geqslant \mathrm{s_0} + 1$ *with* $\mathrm{s_0} \geqslant 2$ *be integers, suppose that*

$$(\rho_0 - 1, \mathrm{u_0}, \theta_0 - 1, \mathrm{B_0}, \mathrm{q_0}) \in \mathrm{H^s}$$

and $\nabla \cdot \mathrm{B_0} = 0$. *If there exists a positive constant* ϵ_0 *which is small enough such that*

$$\|(\rho_0 - 1, \mathrm{u_0}, \theta_0 - 1, \mathrm{B_0}, \mathrm{q_0}\|_{\mathrm{s}} \leqslant \epsilon_0,$$

then there exists a global unique solution $(\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{B}^\epsilon, \mathrm{q}^\epsilon)$ *of system (1.1)-(1.7) satisfying*

$$(\rho^\epsilon - 1, \theta^\epsilon - 1, \mathrm{q}^\epsilon) \in \mathcal{C}([0, \infty]; \mathrm{H^s}) \cap \mathcal{C}^1([0, \infty]; \mathrm{H^{s-1}})$$

and

$$\mathrm{u}^\epsilon, \mathrm{B}^\epsilon \in \mathcal{C}([0, \infty]; \mathrm{H^s}) \cap \mathcal{C}^1([0, \infty]; \mathrm{H^{s-2}}).$$

PROOF. Linearizing system (2.1)-(2.5) around the equilibrium state $\overline{\mathrm{w}}$, one has

$$\mathrm{A^0}(\overline{\mathrm{w}})\mathrm{w_t} + \sum_{\mathrm{j}=1}^{3} \mathrm{A^j}(\overline{\mathrm{w}})\mathrm{w_{x_j}} - \sum_{\mathrm{j,k}=1}^{3} \mathrm{B^{jk}}(\overline{\mathrm{w}})\mathrm{w_{x_j x_k}} + \mathrm{L}(\overline{\mathrm{w}})\mathrm{w} = 0 \tag{2.6}$$

where

$$\mathrm{A^0}(\overline{\mathrm{w}}) = \begin{bmatrix} \overline{\mathrm{p}}_\rho & & & & \\ & I & & & \\ & & \overline{\mathrm{e}}_\theta & & \\ & & & \mathrm{I} & \\ & & & & \frac{\epsilon}{\kappa} \end{bmatrix},$$

$$\sum_{\mathrm{j}} \mathrm{A}^{\mathrm{j}}(\overline{\mathrm{w}})\xi_{\mathrm{j}} = \begin{bmatrix} 0 & \overline{\mathrm{p}}_\rho \xi & 0 & 0 & 0 \\ \overline{\mathrm{p}}_\rho \xi^{\mathrm{T}} & 0 & \overline{\mathrm{p}}_\theta \xi^{\mathrm{T}} & 0 & 0 \\ 0 & \overline{\mathrm{p}}_\theta \xi & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^{\mathrm{T}} & 0 & 0 \end{bmatrix},$$

$$\sum_{\mathrm{j,k}} \mathrm{B}^{\mathrm{jk}}(\overline{\mathrm{w}})\xi_{\mathrm{j}}\xi_{\mathrm{k}} = \begin{bmatrix} 0 & & & & \\ & \mu \mathrm{I} + (\mu+\lambda)\xi^{\mathrm{T}}\xi & & & \\ & & 0 & & \\ & & & \nu \mathrm{I} & \\ & & & & 0 \end{bmatrix},$$

$$\mathrm{L}(\overline{\mathrm{w}}) = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \frac{1}{\kappa}\mathrm{I} \end{bmatrix},$$

It is seen that

(i) $\mathrm{A}^0(\overline{\mathrm{w}})$ is real symmetric and positive definite;
(ii) $\mathrm{A}^{\mathrm{j}}(\overline{\mathrm{w}})(\mathrm{j}=1,2,3)$ are real symmetric;
(iii) $\mathrm{B}^{\mathrm{jk}}(\overline{\mathrm{w}})(\mathrm{j,k}=1,2,3)$ are real symmetric and satisfy $\mathrm{B}^{\mathrm{jk}}(\overline{\mathrm{w}}) = \mathrm{B}^{\mathrm{kj}}(\overline{\mathrm{w}})$, moreover, $\sum_{\mathrm{j,k}} \mathrm{B}^{\mathrm{jk}}(\overline{\mathrm{w}})\xi_{\mathrm{j}}\xi_{\mathrm{k}}$ is real symmetric and positive semi-definite for any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{S}^2$;
(iv) $\mathrm{L}(\overline{\mathrm{w}})$ is real symmetric and positive semi-definite.

In order to apply the Kawashima's global existence theory, we should check there is a compensating function for the system (2.6). Note that for any $\hat{\mathrm{w}} = (\hat{\rho}, \hat{\mathrm{u}}, \hat{\theta}, \hat{\mathrm{B}}, \hat{\mathrm{q}})^{\mathrm{T}} \in \mathbb{R}^{11}$, we obtain

$$\langle (\sum_{\mathrm{j,k}} \mathrm{B}^{\mathrm{jk}}(\overline{\mathrm{w}})\xi_{\mathrm{j}}\xi_{\mathrm{k}})\hat{\mathrm{w}}, \hat{\mathrm{w}} \rangle \geqslant \min\{\mu, 2\mu+\mu'\}|\hat{\mathrm{u}}|^2 + \nu|\hat{\mathrm{B}}|^2, \tag{2.7}$$

$$\langle \mathrm{L}(\overline{\mathrm{w}})\hat{\mathrm{w}}, \hat{\mathrm{w}} \rangle \geqslant \frac{1}{\kappa}|\hat{\mathrm{q}}|^2. \tag{2.8}$$

If let $\Psi = (\hat{\rho}, \hat{\mathrm{u}}, \hat{\theta}, \hat{\mathrm{B}}, \hat{\mathrm{q}})^{\mathrm{T}} \in \mathbb{R}^{11} \setminus \{0\}$ be such that for some $\xi \in \mathcal{S}^2$

$$\sum_{\mathrm{j,k}} \mathrm{B}^{\mathrm{jk}}(\overline{\mathrm{w}})\xi_{\mathrm{j}}\xi_{\mathrm{k}}\Psi = 0,$$
$$\mathrm{L}(\overline{\mathrm{w}})\Psi = 0.$$

Then, by (2.7) and (2.8), one gets that $\hat{\mathrm{u}} = \hat{\mathrm{B}} = \hat{\mathrm{q}} = 0$. Hence, $\Psi = (\hat{\rho}, 0, \hat{\theta}, 0, 0)$ where $\hat{\rho} = 0$ and $\hat{\theta} = 0$ can not occur simultaneously. It follows that, for any $\lambda \in \mathbb{R}$,

$$\lambda \mathrm{A}^0(\overline{\mathrm{w}})\Psi + \sum_{\mathrm{j}} \mathrm{A}^{\mathrm{j}}(\overline{\mathrm{w}})\xi_{\mathrm{j}}\Psi = (\lambda \overline{\mathrm{p}}_\theta \hat{\rho}, (\overline{\mathrm{p}}_\rho \hat{\rho} + \overline{\mathrm{p}}_\theta \hat{\theta})\xi, \overline{\mathrm{e}}_\theta \hat{\theta}, 0, \hat{\theta}\xi)^{\mathrm{T}} \neq 0.$$

By [**13**, Theorem 1.1], there exists a compensating function for the linear system (2.6). Therefore, by applying [**13**, Theorem 4.1], we obtain the results of Theorem 2.2. □

REMARK 2.1. *It should be remarked that it is possible to find out the compensating function for the system (2.6) directly. In contrast to the above equivalent condition of existence of compensating function, the process is much more complex and in addition, need the smallness of ϵ. For the reader's convenience, we describe the details of how to search for the compensating function immediately as follows.*

One can choose that

$$\sum_{j} K^{j}\xi_{j} = \alpha \begin{bmatrix} 0 & \overline{p}_{\rho}\xi & 0 & 0 & 0 \\ -\xi^{T} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\kappa}{\epsilon}\xi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\overline{e}_{\theta}}\xi^{T} & 0 & 0 \end{bmatrix},$$

where $\alpha > 0$ *will be specified in the later. By some simple calculations imply*

$$\sum_{j} K^{j}\xi_{j}A^{0}(\overline{w}) = \alpha \begin{bmatrix} 0 & \overline{p}_{\rho}\xi & 0 & 0 & 0 \\ -\overline{p}_{\rho}\xi^{T} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\xi^{T} & 0 & 0 \end{bmatrix},$$

$$\sum_{j,k} K^{j}A^{k}(\overline{w})\xi_{j}\xi_{k} = \alpha \begin{bmatrix} \overline{p}_{\rho}^{2} & 0 & \overline{p}_{\theta}\overline{p}_{\rho} & 0 & 0 \\ 0 & -\overline{p}_{\rho}\xi^{T}\xi & 0 & 0 & 0 \\ 0 & 0 & \frac{\kappa}{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\overline{p}_{\theta}}{\overline{e}_{\theta}}\xi^{T}\xi & 0 & 0 & -\frac{1}{\overline{e}_{\theta}}\xi^{T}\xi \end{bmatrix}.$$

Thus, the symmetric part of matrix $\sum_{j,k} K^{j}A^{k}(\overline{w})\xi_{j}\xi_{k} + B^{jk}(\overline{w})\xi_{j}\xi_{k} + L(\overline{w})$ *is*

$$\begin{aligned} M &:= \frac{1}{2}\Big\{\sum K^{j}A^{k}\xi_{j}\xi_{k} + (K^{j}A^{k}\xi_{j}\xi_{k})^{T}\Big\} + \sum B^{jk}\xi_{j}\xi_{k} + L(\overline{w}) \\ &= \begin{bmatrix} \alpha\overline{p}_{\rho}^{2} & 0 & \frac{1}{2}\alpha\overline{p}_{\theta}\overline{p}_{\rho} & 0 & 0 \\ 0 & \mu I + ((\mu+\mu') - \alpha\overline{p}_{\rho})\xi^{T}\xi & 0 & 0 & -\frac{\alpha\overline{p}_{\theta}}{2\overline{e}_{\theta}}\xi^{T}\xi \\ \frac{1}{2}\alpha\overline{p}_{\theta}\overline{p}_{\rho} & 0 & \frac{\alpha\kappa}{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & \nu I & 0 \\ 0 & -\frac{\alpha\overline{p}_{\theta}}{2\overline{e}_{\theta}}\xi^{T}\xi & 0 & 0 & \frac{1}{\kappa}I - \frac{\alpha}{\overline{e}_{\theta}}\xi^{T}\xi \end{bmatrix}. \end{aligned} \tag{2.9}$$

For any $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \in \mathbb{R}^{11}$ *where* $\eta_1, \eta_3 \in \mathbb{R}^1$ *and* $\eta_2, \eta_4, \eta_5 \in \mathbb{R}^3$. *Then we have*

$$\begin{aligned} \eta M\eta^{T} &= (\eta_1\alpha\overline{p}_{\rho}^{2} + \frac{1}{2}\eta_3\alpha\overline{p}_{\theta}\overline{p}_{\rho})\eta_1 + (\eta_2 N + \eta_5(-\frac{\alpha\overline{p}_{\theta}}{2\overline{e}_{\theta}}\xi^{T}\xi))\eta_2^{T} \\ &\quad + (\frac{1}{2}\eta_1\alpha\overline{p}_{\theta}\overline{p}_{\rho} + \eta_3\frac{\alpha\kappa}{\epsilon})\eta_3 + \eta_4 R\eta_4^{T} + (\eta_2(-\frac{\alpha\overline{p}_{\theta}}{2\overline{e}_{\theta}}\xi^{T}\xi) + \eta_5 Q)\eta_5^{T} \end{aligned} \tag{2.10}$$

where $N = \mu I + ((\mu+\mu') - \alpha\overline{p}_{\rho})\xi^{T}\xi, R = \nu I$ *and* $Q = \frac{1}{\kappa}I - \frac{\alpha}{\overline{e}_{\theta}}\xi^{T}\xi$. *It is obvious that for any* $0 < \epsilon < \frac{4\kappa}{\overline{p}_{\theta}^{2}}$,

$$\alpha\overline{p}_{\rho}^{2}\eta_1^{2} + \alpha\overline{p}_{\theta}p_{\rho}\eta_1\eta_3 + \frac{\alpha\kappa}{\epsilon}\eta_3^{2} = \alpha\overline{p}_{\rho}^{2}(\eta_1 + \frac{\overline{p}_{\theta}}{2\overline{p}_{\rho}}\eta_3)^2 + \frac{\alpha\kappa}{\epsilon}\eta_3^{2} - \frac{\alpha\overline{p}_{\theta}^{2}}{4}\eta_3^{2} > 0. \tag{2.11}$$

When α is small, the matrices N, Q are positive definite and we can choose small enough α such that

$$\eta_2 \mathrm{N} \eta_2^{\mathrm{T}} - \eta_5 \frac{\alpha \overline{\mathrm{p}}_\theta}{2\overline{\mathrm{e}}_\theta} \eta_2^{\mathrm{T}} - \eta_2 \frac{\alpha \overline{\mathrm{p}}_\theta}{2\overline{\mathrm{e}}_\theta} \eta_5^{\mathrm{T}} + \eta_5 \mathrm{Q} \eta_5^{\mathrm{T}} > 0. \tag{2.12}$$

From (2.9)-(2.12), if α is small enough, we deduce that $\eta \mathrm{M} \eta^{\mathrm{T}} > 0$ for any $\eta \in \mathbb{R}^{11}, \eta \neq 0$ and $\xi \in \mathcal{S}^2$, which implies that M is positive definite. Then the function $\mathrm{K}(\xi) = \sum \mathrm{K}^{\mathrm{j}} \xi_{\mathrm{j}}$ is the compensating function expected. Also, by the Kawashima's global existence theorem [**14**, Theorem 3.6]*, we complete the Theorem 2.2.*

3. The limit behavior of $\epsilon \to 0$

In this section, we are going to show the uniform convergence of solutions of the relaxed system ($\epsilon > 0$) (1.1)-(1.5) to that of the classical system ($\epsilon = 0$) (1.8)-(1.11) by the energy method. Firstly, we assume that the initial data is well-prepared and

$$\mathrm{q}_0 = -\kappa \nabla \theta_0.$$

Denote

$$\begin{aligned} \mathrm{T}_\epsilon = \sup\{\mathrm{T} > 0; (\rho^\epsilon - 1, \mathrm{u}^\epsilon, \theta^\epsilon - 1, \mathrm{B}^\epsilon, \mathrm{q}^\epsilon) \in \mathrm{C}([0, \mathrm{T}], \mathrm{H}^{\mathrm{s}}), \\ (\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{B}^\epsilon, \mathrm{q}^\epsilon) \in \mathrm{G}_1\}, \end{aligned}$$

where G_1 is a given set satisfying $\mathrm{G}_0 \subset\subset \mathrm{G}_1 \subset\subset \mathrm{G}$. Now, the main result in this section can be stated as

Theorem 3.1. *Let $(\rho, \mathrm{u}, \theta, \mathrm{B})$ be the classical solution of the system* (1.8)-(1.11) *with initial data $(\rho, \mathrm{u}, \theta, \mathrm{B})(\mathrm{x}, 0) = (\rho_0, \mathrm{u}_0, \theta_0, \mathrm{B}_0)$ satisfying*

$$\begin{aligned} \rho \in \mathrm{C}([0, \mathrm{T}_*], \mathrm{H}^{\mathrm{s}+3}) \cap \mathrm{C}^1([0, \mathrm{T}_*], \mathrm{H}^{\mathrm{s}+2}), \\ (\mathrm{u}, \theta, \mathrm{B}) \in \mathrm{C}([0, \mathrm{T}_*], \mathrm{H}^{\mathrm{s}+3}) \cap \mathrm{C}^1([0, \mathrm{T}_*], \mathrm{H}^{\mathrm{s}+1}), \end{aligned}$$

with $\mathrm{T}_* > 0$ and suppose the conditions of Theorem 2.1 hold. Then for any G_1 satisfying

$$\mathrm{G}_0 \cup \tilde{\mathrm{G}} \subset\subset \mathrm{G}_1 \subset\subset \mathrm{G},$$

where $\tilde{\mathrm{G}} = \{\cup(\rho, \mathrm{u}, \theta, \mathrm{B}, -\kappa\nabla\theta)(\mathrm{x}, \mathrm{t}), (\mathrm{x}, \mathrm{t}) \in \mathbb{R}^3 \times [0, \mathrm{T}_*]\}$, there are positive constants ϵ_0 and C which is independent of ϵ such that for $\epsilon \leqslant \epsilon_0$,

$$\begin{aligned} \|(\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{B}^\epsilon)(\mathrm{t}, \cdot) - (\rho, \mathrm{u}, \theta, \mathrm{B})(\mathrm{t}, \cdot)\|_{\mathrm{s}} \leqslant \mathrm{C}\epsilon, \\ \|(\mathrm{q}^\epsilon + \kappa\nabla\theta)(\mathrm{t}, \cdot)\|_{\mathrm{s}} \leqslant \mathrm{C}\epsilon^{\frac{1}{2}}, \end{aligned}$$

hold for $\mathrm{t} \in [0, \min\{\mathrm{T}_*, \mathrm{T}_\epsilon\})$.

Before the proceed proof of Theorem 3.1, we observe that the following theorem holds after Theorem 3.1 holds.

Theorem 3.2. *Under the same assumptions of Theorem 3.1, for any G_1 satisfying*

$$\mathrm{G}_0 \cup \tilde{\mathrm{G}} \subset\subset \mathrm{G}_1 \subset\subset \mathrm{G},$$

where $\tilde{\mathrm{G}} = \{\cup(\rho, \mathrm{u}, \theta, \mathrm{B}, -\kappa\nabla\theta)(\mathrm{x}, \mathrm{t}), (\mathrm{x}, \mathrm{t}) \in \mathbb{R}^3 \times [0, \mathrm{T}_*]\}$. Then we have $\mathrm{T}_\epsilon \geqslant \mathrm{T}_*$ holds for ϵ sufficiently small.

PROOF. We prove it by the contradiction method. Suppose that there exists a G_1 satisfying

$$\mathrm{G}_0 \cup \tilde{\mathrm{G}} \subset\subset \mathrm{G}_1 \subset\subset \mathrm{G},$$

and a sequence $\{\epsilon_k\}$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and $\mathrm{T}_{\epsilon_k} \leqslant \mathrm{T}_*$. Then we can choose some $\bar{\mathrm{G}}$ satisfying $\tilde{\mathrm{G}} \subset\subset \bar{\mathrm{G}} \subset\subset \mathrm{G}_1$. From Theorem 3.1, we have

$$|(\rho^\epsilon, u^\epsilon, \theta^\epsilon, B^\epsilon, q^\epsilon) - (\rho, u, \theta, B, -\kappa\nabla\theta)| \leqslant C\epsilon^{\frac{1}{2}}, \tag{3.1}$$

where the Sobolev embedding theorem is used. Thus, there is a ϵ_k such that $(\rho^{\epsilon_k}, u^{\epsilon_k}, \theta^{\epsilon_k}, B^{\epsilon_k}, q^{\epsilon_k}) \in \bar{\mathrm{G}}$ for all $(x,t) \in \mathbb{R}^3 \times [0, \mathrm{T}_{\epsilon_k})$. On the other hand,

$$\begin{aligned}
&\|(\rho^{\epsilon_k} - 1, u^{\epsilon_k}, \theta^{\epsilon_k} - 1, B^{\epsilon_k}, q^{\epsilon_k})\|_s \\
&\quad \leqslant \|(\rho^{\epsilon_k} - 1, u^{\epsilon_k}, \theta^{\epsilon_k} - 1, B^{\epsilon_k}, q^{\epsilon_k}) - (\rho, u, \theta, B, -\kappa\nabla\theta)\|_s \\
&\qquad\qquad\qquad + \|(\rho - 1, u, \theta - 1, B, -\kappa\nabla\theta)\|_s \\
&\quad \leqslant C\epsilon_k^{\frac{1}{2}} + \|(\rho - 1, u, \theta - 1, B, -\kappa\nabla\theta)\|_s. \qquad (3.2)
\end{aligned}$$

This implies that $\|(\rho^{\epsilon_k} - 1, u^{\epsilon_k}, \theta^{\epsilon_k} - 1, B^{\epsilon_k}, q^{\epsilon_k})\|_s$ is bounded. Then the local existence theory in Theorem 2.1 can be applied to deduce a contradiction to the definition of T_{ϵ_k}. □

Hence, from the results of Theorem 3.1 and Theorem 3.2, it is clear that the system (1.1)-(1.5) approximates to the classical system (1.8)-(1.11) as $\epsilon \to 0$. Our task is left to prove the Theorem 3.1.

Proof of Theorem 3.1: Define $q = -\kappa\nabla\theta$ and

$$\rho^d = \frac{\rho^\epsilon - \rho}{\epsilon}, u^d = \frac{u^\epsilon - u}{\epsilon}, \theta^d = \frac{\theta^\epsilon - \theta}{\epsilon}, B^d = \frac{B^\epsilon - B}{\epsilon}, q^d = \frac{q^\epsilon - q}{\epsilon}.$$

From (1.1)-(1.5), the equations for $(\rho^d, u^d, \theta^d, B^d, q^d)$ can be written as

$$\begin{cases}
\rho^d_t + u^\epsilon \cdot \nabla\rho^d + u^d \cdot \nabla\rho + \rho^\epsilon \nabla \cdot u^d + \rho^d \nabla \cdot u = 0, \\
\rho^\epsilon u^d_t + \rho^d u_t + \rho^\epsilon u^\epsilon \cdot \nabla u^d + p^\epsilon_\rho \nabla\rho^d + p^\epsilon_\theta \nabla\theta^d + \frac{1}{\epsilon}(\rho^\epsilon u^\epsilon - \rho u) \cdot \nabla u \\
\qquad\qquad + \frac{1}{\epsilon}(p^\epsilon_\rho \nabla\rho - p_\rho \nabla\rho + p^\epsilon_\theta \nabla\theta - p_\theta \nabla\theta) \\
= \mu\Delta u^d + (\mu + \mu')\nabla\nabla \cdot u^d + (\nabla \times B^d) \times B^\epsilon + (\nabla \times B) \times B^d, \\
\rho^\epsilon e^\epsilon_\theta \theta^d_t + \frac{1}{\epsilon}(\rho^\epsilon e^\epsilon_\theta - \rho e_\theta)\theta_t + \rho^\epsilon e^\epsilon_\theta u^\epsilon \cdot \nabla\theta^d + \frac{1}{\epsilon}(\rho^\epsilon e^\epsilon_\theta u^\epsilon - \rho e_\theta u) \cdot \nabla\theta \\
\qquad\qquad + \theta^\epsilon p^\epsilon_\theta \nabla \cdot u^d + \frac{1}{\epsilon}(\theta^\epsilon p^\epsilon_\theta - \theta p_\theta)\nabla \cdot u + \nabla \cdot q^d \\
= \nu(\nabla \times B^\epsilon + \nabla \times B)\nabla \times B^d + \mu'(\nabla \cdot u^\epsilon + \nabla \cdot u)\nabla \cdot u^d, \\
\qquad + \mu\Big(\nabla u^\epsilon + (\nabla u^\epsilon)^T + \nabla u + (\nabla u)^T\Big) : \nabla u^d, \\
B^d_t + u^d \nabla B + u^\epsilon \cdot \nabla B^d - \nu\Delta B^d \\
= \frac{1}{\epsilon}\Big((B^\epsilon \cdot \nabla)u^\epsilon - (B \cdot \nabla)u - B^\epsilon(\nabla \cdot u^\epsilon) + B(\nabla \cdot u)\Big), \ \ \nabla \cdot B = 0, \\
\epsilon q^d_t + q^d + \kappa\nabla\theta^d = -q_t.
\end{cases}$$

The above equations are equivalent to

$$\begin{cases}
\rho^d_t + u^\epsilon \cdot \nabla\rho^d + \rho^\epsilon \nabla \cdot u^d = f_1, \\
u^d_t + u^\epsilon \cdot \nabla u^d + \frac{p^\epsilon_\rho}{\rho^\epsilon}\nabla\rho^d + \frac{p^\epsilon_\theta}{\rho^\epsilon}\nabla\theta^d - \frac{1}{\rho^\epsilon}\mu\Delta u^d - \frac{1}{\rho^\epsilon}(\mu + \mu')\nabla\nabla \cdot u^d = f_2, \\
\theta^d_t + u^\epsilon \cdot \nabla\theta^d + \frac{\theta^\epsilon p^\epsilon_\theta}{\rho^\epsilon e^\epsilon_\theta}\nabla \cdot u^d + \frac{1}{\rho^\epsilon e^\epsilon_\theta}\nabla \cdot q^d = f_3, \\
B^d_t + u^\epsilon \cdot \nabla B^d - \nu\Delta B^d = f_4, \qquad \nabla \cdot B = 0, \\
\epsilon q^d_t + q^d + \kappa\nabla\theta^d = f_5,
\end{cases}$$

where

$$
\begin{aligned}
\mathrm{f}_1 &= -\mathrm{u}^{\mathrm{d}}\cdot\nabla\rho-\rho^{\mathrm{d}}\nabla\cdot\mathrm{u},\\
\rho^\epsilon\mathrm{f}_2 &= -\rho^{\mathrm{d}}\mathrm{u}_{\mathrm{t}}-\frac{1}{\epsilon}\Big\{(\rho^\epsilon\mathrm{u}^\epsilon-\rho\mathrm{u})\cdot\nabla\mathrm{u}+(\mathrm{p}^\epsilon_\rho-\mathrm{p}_\rho)\nabla\rho+(\mathrm{p}^\epsilon_\theta-\mathrm{p}_\theta)\nabla\theta\Big\}\\
&\quad+(\nabla\times\mathrm{B}^{\mathrm{d}})\times\mathrm{B}^\epsilon+(\nabla\times\mathrm{B})\times\mathrm{B}^{\mathrm{d}},\\
\rho^\epsilon\mathrm{e}^\epsilon_\theta\mathrm{f}_3 &= -\frac{1}{\epsilon}\Big\{(\rho^\epsilon\mathrm{e}^\epsilon_\theta-\rho\mathrm{e}_\theta)\theta_{\mathrm{t}}-(\rho^\epsilon\mathrm{e}^\epsilon_\theta\mathrm{u}^\epsilon-\rho\mathrm{e}_\theta\mathrm{u})\cdot\nabla\theta-(\theta^\epsilon\mathrm{p}^\epsilon_\theta-\theta\mathrm{p}_\theta)\nabla\cdot\mathrm{u}\Big\}\\
&\quad+\nu(\nabla\times\mathrm{B}^\epsilon+\nabla\times\mathrm{B})\nabla\times\mathrm{B}^{\mathrm{d}}+\mu'(\nabla\cdot\mathrm{u}^\epsilon+\nabla\cdot\mathrm{u})\nabla\cdot\mathrm{u}^{\mathrm{d}}\\
&\quad+\mu\Big(\nabla\mathrm{u}^\epsilon+(\nabla\mathrm{u}^\epsilon)^{\mathrm{T}}+\nabla\mathrm{u}+(\nabla\mathrm{u})^{\mathrm{T}}\Big):\nabla\mathrm{u}^{\mathrm{d}},\\
\mathrm{f}_4 &= -\mathrm{u}^{\mathrm{d}}\nabla\mathrm{B}+\frac{1}{\epsilon}\Big\{(\mathrm{B}^\epsilon\cdot\nabla)\mathrm{u}^\epsilon-(\mathrm{B}\cdot\nabla)\mathrm{u}-\mathrm{B}^\epsilon(\nabla\cdot\mathrm{u}^\epsilon)+\mathrm{B}(\nabla\cdot\mathrm{u})\Big\},\\
\mathrm{f}_5 &= -\mathrm{q}_{\mathrm{t}}.
\end{aligned}
$$

Now let $\mathrm{E}^{\mathrm{d}}=\sup\limits_{0\leqslant\mathrm{t}\leqslant\mathrm{T}}\|(\rho^{\mathrm{d}},\mathrm{u}^{\mathrm{d}},\theta^{\mathrm{d}},\mathrm{B}^{\mathrm{d}},\sqrt{\epsilon}\mathrm{q}^{\mathrm{d}})\|_{\mathrm{s}}$. We have

$$
\|(\rho^\epsilon,\mathrm{u}^\epsilon,\theta^\epsilon,\mathrm{B}^\epsilon)\|_{\mathrm{s}}\leqslant\mathrm{C}+\epsilon\mathrm{E}^{\mathrm{d}},\qquad\|\mathrm{q}^\epsilon\|_{\mathrm{s}}\leqslant\mathrm{C}+\sqrt{\epsilon}\mathrm{E}^{\mathrm{d}}.
$$

Then, we have the following lemma.

Lemma 3.1.

$$
\frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{E}^{\mathrm{d}})^2\leqslant\mathrm{C}\Big(1+(\mathrm{E}^{\mathrm{d}})^2+\epsilon(\mathrm{E}^{\mathrm{d}})^3+\epsilon^2(\mathrm{E}^{\mathrm{d}})^4\Big).\tag{3.3}
$$

Proof. Applying ∇^α with $0\leqslant|\alpha|\leqslant\mathrm{s}$ to above equations and multiply it by $\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}\nabla^\alpha\rho^{\mathrm{d}}$, $\rho^\epsilon\nabla^\alpha\mathrm{u}^{\mathrm{d}}$, $\frac{\rho^\epsilon\mathrm{e}^\epsilon_\theta}{\theta^\epsilon}\nabla^\alpha\theta^{\mathrm{d}}$, $\nabla^\alpha\mathrm{B}^{\mathrm{d}}$, $\frac{1}{\kappa\theta^\epsilon}\nabla^\alpha\mathrm{q}^{\mathrm{d}}$ respectively, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int\Big\{\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}(\nabla^\alpha\rho^{\mathrm{d}})^2+\rho^\epsilon(\nabla^\alpha\mathrm{u}^{\mathrm{d}})^2+\frac{\rho^\epsilon\mathrm{e}^\epsilon_\theta}{\theta^\epsilon}(\nabla^\alpha\theta^{\mathrm{d}})^2+(\nabla^\alpha\mathrm{H}^{\mathrm{d}})^2+\frac{\epsilon}{\kappa\theta^\epsilon}(\nabla^\alpha\mathrm{q}^{\mathrm{d}})^2\Big\}\mathrm{dx}\\
&\quad+\int\Big\{\mu(\nabla^{\alpha+1}u^d)^2+(\mu+\mu')(\nabla^\alpha\nabla\cdot u^d)^2+\frac{1}{\kappa\theta^\epsilon}(\nabla^\alpha q^d)^2+\nu(\nabla^{\alpha+1}B^d)^2\Big\}dx
\end{aligned}
$$

$$
=\sum_{\mathrm{i}=1}^{4}\mathrm{T}_{\mathrm{i}}+\sum_{\mathrm{i}=1}^{7}\mathrm{R}_{\mathrm{i}}+\mathrm{G}+\sum_{\mathrm{i}=1}^{5}\mathrm{F}_{\mathrm{i}},\tag{3.4}
$$

with

$$
\begin{aligned}
\mathrm{T}_1&=\int(\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon})_{\mathrm{t}}(\nabla^\alpha\rho^{\mathrm{d}})^2\mathrm{dx},\qquad \mathrm{T}_2=\int\rho^\epsilon_{\mathrm{t}}(\nabla^\alpha\mathrm{u}^{\mathrm{d}})^2\mathrm{dx},\\
\mathrm{T}_3&=\int(\frac{\rho^\epsilon\mathrm{e}^\epsilon_\theta}{\theta^\epsilon})_{\mathrm{t}}(\nabla^\alpha\theta^{\mathrm{d}})^2\mathrm{dx},\qquad \mathrm{T}_4=\int(\frac{\epsilon}{\kappa\theta^\epsilon})_{\mathrm{t}}(\nabla^\alpha\mathrm{q}^{\mathrm{d}})^2\mathrm{dx},\\
\mathrm{R}_1&=\int\nabla^\alpha(\mathrm{u}^\epsilon\cdot\nabla\rho^{\mathrm{d}})\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}\nabla^\alpha\rho^{\mathrm{d}}\mathrm{dx},\quad \mathrm{R}_2=\int\nabla^\alpha(\mathrm{u}^\epsilon\cdot\nabla\mathrm{u}^{\mathrm{d}})\rho^\epsilon\nabla^\alpha\mathrm{u}^{\mathrm{d}}\mathrm{dx},\\
\mathrm{R}_3&=\int\nabla^\alpha(\mathrm{u}^\epsilon\cdot\nabla\theta^{\mathrm{d}})\frac{\rho^\epsilon\mathrm{e}^\epsilon_\theta}{\theta^\epsilon}\nabla^\alpha\theta^{\mathrm{d}}\mathrm{dx},\quad \mathrm{R}_4=\int\nabla^\alpha(\mathrm{u}^\epsilon\cdot\nabla\mathrm{B}^{\mathrm{d}})\nabla^\alpha\mathrm{B}^{\mathrm{d}}\mathrm{dx},\\
\mathrm{R}_5&=\int\nabla^\alpha(\rho^\epsilon\nabla\cdot\mathrm{u}^{\mathrm{d}})\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}\nabla^\alpha\rho^{\mathrm{d}}\mathrm{dx}+\nabla^\alpha(\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}\nabla\rho^{\mathrm{d}})\rho^\epsilon\nabla^\alpha\mathrm{u}^{\mathrm{d}}\mathrm{dx},
\end{aligned}
$$

$$\begin{aligned}
\mathrm{R}_6 &= \int \nabla^\alpha(\frac{\mathrm{p}_\theta^\epsilon}{\rho^\epsilon}\nabla\theta^{\mathrm{d}})\rho^\epsilon\nabla^\alpha \mathrm{u}^{\mathrm{d}} + \nabla^\alpha(\frac{\theta^\epsilon \mathrm{p}_\theta^\epsilon}{\rho^\epsilon \mathrm{e}_\theta^\epsilon}\nabla\cdot \mathrm{u}^{\mathrm{d}})\frac{\rho^\epsilon \mathrm{e}_\theta^\epsilon}{\theta^\epsilon}\nabla^\alpha\theta^{\mathrm{d}}\mathrm{dx},\\
\mathrm{R}_7 &= \int \nabla^\alpha(\frac{1}{\rho^\epsilon \mathrm{e}_\theta^\epsilon}\nabla\cdot \mathrm{q}^{\mathrm{d}})\frac{\rho^\epsilon \mathrm{e}_\theta^\epsilon}{\theta^\epsilon}\nabla^\alpha\theta^{\mathrm{d}} + \nabla^\alpha(\kappa\nabla\theta^{\mathrm{d}})\frac{1}{\kappa\theta^\epsilon}\nabla^\alpha \mathrm{q}^{\mathrm{d}}\mathrm{dx},\\
\mathrm{G} &= \int [\nabla^\alpha, \frac{1}{\rho^\epsilon}](\mu\Delta \mathrm{u}^{\mathrm{d}} + (\mu+\mu')\nabla\nabla\cdot \mathrm{u}^{\mathrm{d}})\rho^\epsilon\nabla^\alpha \mathrm{u}^{\mathrm{d}}\mathrm{dx},\\
\mathrm{F}_1 &= \int \nabla^\alpha \mathrm{f}_1\frac{\mathrm{p}_\rho^\epsilon}{\rho^\epsilon}\nabla^\alpha\rho^{\mathrm{d}}\mathrm{dx}, \qquad \mathrm{F}_2 = \int \nabla^\alpha \mathrm{f}_2\rho^\epsilon\nabla^\alpha \mathrm{u}^{\mathrm{d}}\mathrm{dx},\\
\mathrm{F}_3 &= \int \nabla^\alpha \mathrm{f}_3\frac{\rho^\epsilon \mathrm{e}_\theta^\epsilon}{\theta^\epsilon}\nabla^\alpha\theta^{\mathrm{d}}\mathrm{dx}, \qquad \mathrm{F}_4 = \int \nabla^\alpha \mathrm{f}_4\nabla^\alpha \mathrm{B}^{\mathrm{d}}\mathrm{dx},\\
&\qquad\qquad \mathrm{F}_5 = \int \nabla^\alpha \mathrm{f}_5\frac{1}{\kappa\theta^\epsilon}\nabla^\alpha \mathrm{q}^{\mathrm{d}}\mathrm{dx}.
\end{aligned}$$

In the following, we will give the estimates of $\mathrm{T_i}, \mathrm{R_i}, \mathrm{F_i}, \mathrm{G}$ respectively. We should keep in mind the important fact that both $(\rho, \mathrm{u}, \theta, \mathrm{B})$ and $(\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{B}^\epsilon, \mathrm{q}^\epsilon)$ takes values in a convex compact subset of the state space. First of all, $\mathrm{T_i}(\mathrm{i} = 1, 2, 3, 4)$ can be estimated as

$$|\mathrm{T_i}| \leqslant \mathrm{C}\|(\rho_{\mathrm{t}}^\epsilon, \theta_{\mathrm{t}}^\epsilon)\|_{\mathrm{L}^\infty}(\mathrm{E}^{\mathrm{d}})^2 \leqslant \mathrm{C}(1 + \epsilon\|(\rho_{\mathrm{t}}^{\mathrm{d}}, \theta_{\mathrm{t}}^{\mathrm{d}})\|_{\mathrm{L}^\infty})(\mathrm{E}^{\mathrm{d}})^2. \tag{3.5}$$

By the first and the third equation in the system, we have

$$\|(\rho_{\mathrm{t}}^{\mathrm{d}}, \theta_{\mathrm{t}}^{\mathrm{d}})\|_{\mathrm{L}^\infty} \leqslant (\mathrm{C} + \epsilon \mathrm{E}^{\mathrm{d}})\mathrm{E}^{\mathrm{d}} + \mathrm{C}\|\mathrm{q}^{\mathrm{d}}\|_{\mathrm{s}}. \tag{3.6}$$

Combining (3.5) and (3.6) gives

$$\begin{aligned}
|\mathrm{T_i}| &\leqslant \mathrm{C}(1 + \epsilon(\mathrm{C}\mathrm{E}^{\mathrm{d}} + \epsilon(\mathrm{E}^{\mathrm{d}})^2 + \mathrm{C}\|\mathrm{q}^{\mathrm{d}}\|_{\mathrm{s}}))(\mathrm{E}^{\mathrm{d}})^2\\
&\leqslant \mathrm{C}((\mathrm{E}^{\mathrm{d}})^2 + \epsilon(\mathrm{E}^{\mathrm{d}})^3 + \epsilon^2(\mathrm{E}^{\mathrm{d}})^4) + \delta\|\mathrm{q}^{\mathrm{d}}\|_{\mathrm{s}}^2,
\end{aligned} \tag{3.7}$$

where δ is a small constant which is independent of ϵ. Then we give the estimates of $\mathrm{R_i}(\mathrm{i} = 1, 2 \cdots, 7)$ as follows.

$$\begin{aligned}
\mathrm{R}_1 &= \int \mathrm{u}^\epsilon\nabla^\alpha\nabla\rho^{\mathrm{d}}\cdot\frac{\mathrm{p}_\rho^\epsilon}{\rho^\epsilon}\nabla^\alpha\rho^{\mathrm{d}}\mathrm{dx} + \int [\nabla^\alpha, \mathrm{u}^\epsilon]\nabla\rho^{\mathrm{d}}\cdot\frac{\mathrm{p}_\rho^\epsilon}{\rho^\epsilon}\nabla^\alpha\rho^{\mathrm{d}}\mathrm{dx}\\
&\leqslant \mathrm{C}\|\nabla(\frac{\mathrm{u}^\epsilon \mathrm{p}_\rho^\epsilon}{\rho^\epsilon})\|_{\mathrm{L}^\infty}\|\nabla^\alpha\rho^{\mathrm{d}}\|^2\\
&\qquad\qquad + \mathrm{C}(\|\nabla \mathrm{u}^\epsilon\|_{\mathrm{L}^\infty}\|\rho^{\mathrm{d}}\|_{\mathrm{s}} + \|\nabla\rho^{\mathrm{d}}\|_{\mathrm{L}^\infty}\|\mathrm{u}^\epsilon\|_{\mathrm{s}})\|\rho^{\mathrm{d}}\|_{\mathrm{s}}\\
&\leqslant \mathrm{C}(1 + \epsilon \mathrm{E}^{\mathrm{d}})(\mathrm{E}^{\mathrm{d}})^2\\
&\leqslant \mathrm{C}((\mathrm{E}^{\mathrm{d}})^2 + \epsilon(\mathrm{E}^{\mathrm{d}})^3),
\end{aligned} \tag{3.8}$$

where the symbol $[\cdot, \cdot]$ is the commutator. Similarly, for $\mathrm{i} = 2, 3, 4$,

$$|\mathrm{R_i}| \leqslant \mathrm{C}((\mathrm{E}^{\mathrm{d}})^2 + \epsilon(\mathrm{E}^{\mathrm{d}})^3). \tag{3.9}$$

Moreover,

$$
\begin{aligned}
\mathrm{R}_5 = &\int \mathrm{p}^\epsilon_\rho(\nabla^\alpha\nabla\cdot \mathrm{u}^\mathrm{d}\nabla^\alpha\rho^\mathrm{d}+\nabla^\alpha\nabla\rho^\mathrm{d}\nabla^\alpha\mathrm{u}^\mathrm{d})\\
&\qquad+\int[\nabla^\alpha,\rho^\epsilon]\nabla\cdot\mathrm{u}^\mathrm{d}\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}\nabla^\alpha\rho^\mathrm{d}\mathrm{dx}+\int[\nabla^\alpha,\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon}]\nabla\rho^\mathrm{d}\rho^\epsilon\nabla^\alpha\mathrm{u}^\mathrm{d}\mathrm{dx}\\
\leqslant{}& \mathrm{C}\|\nabla \mathrm{p}^\epsilon_\rho\|_{\mathrm{L}^\infty}\|\nabla^\alpha\mathrm{u}^\mathrm{d}\|\|\nabla^\alpha\rho^\mathrm{d}\|\\
&\qquad+\mathrm{C}(\|\nabla^\alpha\rho^\epsilon\|\|\nabla\mathrm{u}^\mathrm{d}\|_{\mathrm{L}^\infty}+\|\nabla\rho^\epsilon\|_{\mathrm{L}^\infty}\|\nabla^\alpha\mathrm{u}^\mathrm{d}\|)\|\nabla^\alpha\rho^\mathrm{d}\|\\
&\qquad+\mathrm{C}(\|\nabla^\alpha\rho^\mathrm{d}\|\|\nabla(\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon})\|_{\mathrm{L}^\infty}+\|\nabla\rho^\mathrm{d}\|_{\mathrm{L}^\infty}\|\nabla^\alpha(\frac{\mathrm{p}^\epsilon_\rho}{\rho^\epsilon})\|)\|\nabla^\alpha\mathrm{u}^\mathrm{d}\|\\
\leqslant{}& \mathrm{C}(1+\epsilon\mathrm{E}^\mathrm{d})(\mathrm{E}^\mathrm{d})^2\leqslant \mathrm{C}((\mathrm{E}^\mathrm{d})^2+\epsilon(\mathrm{E}^\mathrm{d})^3). \qquad (3.10)
\end{aligned}
$$

R_6 can also be estimated similarly. For R_7, one has

$$
\begin{aligned}
\mathrm{R}_7 &= \int \nabla^\alpha(\frac{1}{\rho^\epsilon \mathrm{e}^\epsilon_\theta}\nabla\cdot\mathrm{q}^\mathrm{d})\frac{\rho^\epsilon\mathrm{e}^\epsilon_\theta}{\theta^\epsilon}\nabla^\alpha\theta^\mathrm{d}+\nabla^\alpha(\kappa\nabla\theta^\mathrm{d})\frac{1}{\kappa\theta^\epsilon}\nabla^\alpha\mathrm{q}^\mathrm{d}\mathrm{dx}\\
&=\int\frac{1}{\theta^\epsilon}(\nabla^\alpha\nabla\cdot\mathrm{q}^\mathrm{d}\nabla^\alpha\theta^\mathrm{d}+\nabla^\alpha\nabla\theta^\mathrm{d}\nabla^\alpha\mathrm{q}^\mathrm{d})+\int[\nabla^\alpha,\frac{1}{\rho^\epsilon\mathrm{e}^\epsilon_\theta}]\nabla\cdot\mathrm{q}^\mathrm{d}\frac{\rho^\epsilon\mathrm{e}^\epsilon_\theta}{\theta^\epsilon}\nabla^\alpha\theta^\mathrm{d}\\
&\leqslant \mathrm{C}(\|\nabla(\frac{1}{\theta^\epsilon})\|_{\mathrm{L}^\infty}+\|\nabla(\frac{1}{\rho^\epsilon\mathrm{e}^\epsilon_\theta})\|_{\mathrm{L}^\infty})\|\nabla^\alpha\mathrm{q}^\mathrm{d}\|\|\nabla^\alpha\theta^\mathrm{d}\|\\
&\qquad\qquad+\|\nabla\cdot\mathrm{q}^\mathrm{d}\|_{\mathrm{L}^\infty}\|\nabla^\alpha(\frac{1}{\rho^\epsilon\mathrm{e}^\epsilon_\theta})\|\|\nabla^\alpha\theta^\mathrm{d}\|\\
&\leqslant \mathrm{C}(1+\epsilon\mathrm{E}^\mathrm{d})^2(\mathrm{E}^\mathrm{d})^2+\delta\|\mathrm{q}^\mathrm{d}\|_\mathrm{s}^2\\
&\leqslant \mathrm{C}((\mathrm{E}^\mathrm{d})^2+\epsilon(\mathrm{E}^\mathrm{d})^3+\epsilon^2(\mathrm{E}^\mathrm{d})^4)+\delta\|\mathrm{q}^\mathrm{d}\|_\mathrm{s}^2. \qquad (3.11)
\end{aligned}
$$

Similarly, we can give the estimate of G as

$$
\begin{aligned}
\mathrm{G}&\leqslant \mathrm{C}\Big(\|\nabla(\frac{1}{\rho^\epsilon})\|_{\mathrm{L}^\infty}\|\nabla^\alpha\nabla\mathrm{u}^\mathrm{d}\|+\|\nabla^2\mathrm{u}^\mathrm{d}\|_{\mathrm{L}^\infty}\|\nabla^\alpha(\frac{1}{\rho^\epsilon})\|\Big)\|\nabla^\alpha\mathrm{u}^\mathrm{d}\|\\
&\leqslant \mathrm{C}(1+\epsilon\mathrm{E}^\mathrm{d})^2(\mathrm{E}^\mathrm{d})^2+\delta\|\nabla^\alpha\nabla\mathrm{u}^\mathrm{d}\|^2\\
&\leqslant \mathrm{C}((\mathrm{E}^\mathrm{d})^2+\epsilon(\mathrm{E}^\mathrm{d})^3+\epsilon^2(\mathrm{E}^\mathrm{d})^4)+\delta\|\nabla^\alpha\nabla\mathrm{u}^\mathrm{d}\|^2. \qquad (3.12)
\end{aligned}
$$

Now, we only need to give the estimates of $\mathrm{F}_\mathrm{i}(\mathrm{i}=1,2,3,4,5)$. To this end, we need the following proposition which concerns the estimate of f_i.

PROPOSITION 3.1. *For $0\leqslant|\alpha|\leqslant \mathrm{s}$, we have*

$$
\begin{aligned}
\|\nabla^\alpha\mathrm{f}_1\|&\leqslant \mathrm{C}\mathrm{E}^\mathrm{d},\\
\|\nabla^\alpha\mathrm{f}_2\|&\leqslant \mathrm{C}(\mathrm{E}^\mathrm{d}+\epsilon(\mathrm{E}^\mathrm{d})^2)+\mathrm{C}(1+\epsilon\mathrm{E}^\mathrm{d})\|\nabla^\alpha\nabla\mathrm{B}^\mathrm{d}\|,\\
\|\nabla^\alpha\mathrm{f}_3\|&\leqslant \mathrm{C}(\mathrm{E}^\mathrm{d}+\epsilon(\mathrm{E}^\mathrm{d})^2+\epsilon^2(\mathrm{E}^\mathrm{d})^3)+\mathrm{C}(1+\epsilon\mathrm{E}^\mathrm{d})(\|\nabla^\alpha\nabla\mathrm{u}^\mathrm{d}\|+\|\nabla^\alpha\nabla\mathrm{B}^\mathrm{d}\|),\\
\|\nabla^\alpha\mathrm{f}_4\|&\leqslant \mathrm{C}(\mathrm{E}^\mathrm{d}+\epsilon(\mathrm{E}^\mathrm{d})^2)+\mathrm{C}(1+\epsilon\mathrm{E}^\mathrm{d})\|\nabla^\alpha\nabla\mathrm{u}^\mathrm{d}\|.
\end{aligned}
$$

The proof of this proposition will be postponed to the end of this section. With this proposition, we can give the estimates of $\mathrm{F}_\mathrm{i}(\mathrm{i}=1,2,3,4)$ as

$$
\mathrm{F}_\mathrm{i}\leqslant \mathrm{C}\Big((\mathrm{E}^\mathrm{d})^2+\epsilon(\mathrm{E}^\mathrm{d})^3+\epsilon^2(\mathrm{E}^\mathrm{d})^4\Big)+\delta(\|\nabla^\alpha\nabla\mathrm{u}^\mathrm{d}\|^2+\|\nabla^\alpha\nabla\mathrm{B}^\mathrm{d}\|^2), \qquad (3.13)
$$

while

$$
\begin{aligned}
|\mathrm{F}_5| &= |\int \nabla^\alpha \mathrm{q_t} \frac{1}{\kappa\theta^\epsilon} \nabla^\alpha \mathrm{q^d dx}| = |\int \nabla^\alpha \nabla \theta_\mathrm{t} \frac{1}{\kappa\theta^\epsilon} \nabla^\alpha \mathrm{q^d dx}| \\
&\leqslant \mathrm{C}\|\nabla^\alpha \mathrm{q^d}\| \leqslant \mathrm{C} + \delta\|\nabla^\alpha \mathrm{q^d}\|^2. \qquad (3.14)
\end{aligned}
$$

It follows (3.4)-(3.14) that (3.3). The proof of Lemma 3.1 is over. □

Continue the proof of Theorem 3.1: Denote $\Lambda(\mathrm{t}) = (\mathrm{E^d})^2$, from Lemma 3.1, we have

$$
\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(\mathrm{t}) \leqslant \mathrm{C}(1 + \Lambda(\mathrm{t}) + \epsilon^2 \Lambda(\mathrm{t})^2). \qquad (3.15)
$$

We assume a priori that $\Lambda(\mathrm{t}) \leqslant \mathrm{e^{2CT}} - 1$. We will show that $\Lambda(\mathrm{t}) \leqslant \frac{1}{2}(\mathrm{e^{2CT}} - 1)$ holds if we choose ϵ sufficiently small. Actually, if we set $\epsilon^2 \leqslant \frac{1}{\mathrm{e^{2CT}}-1}$, then we have $\Lambda(t) \leqslant \frac{1}{\epsilon^2}$. Hence, the inequality (3.15) turns into

$$
\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(\mathrm{t}) \leqslant \mathrm{C}(1 + \Lambda(\mathrm{t}) + \Lambda(\mathrm{t})),
$$

which can deduce that

$$
\Lambda(\mathrm{t}) \leqslant \frac{1}{2}(\mathrm{e^{2CT}} - 1).
$$

This completes the proof.

To end this section, we give the proof of the Proposition 3.1.

The proof of Proposition 3.1: By Moser-type inequalities and the Sobolev embedding theorem, we have

$$
\begin{aligned}
\|\nabla^\alpha \mathrm{f}_1\| &\leqslant \|\nabla^\alpha(\mathrm{u^d} \cdot \nabla\rho)\| + \|\nabla^\alpha(\rho^\mathrm{d} \nabla \cdot \mathrm{u})\| \\
&\leqslant \|\mathrm{u^d}\|_{\mathrm{L}^\infty}\|\nabla^\alpha\nabla\rho\| + \|\nabla\rho\|_{\mathrm{L}^\infty}\|\nabla^\alpha \mathrm{u^d}\| \\
&+ \|\rho^\mathrm{d}\|_{\mathrm{L}^\infty}\|\nabla^\alpha\nabla\cdot\mathrm{u}\| + \|\nabla\cdot\mathrm{u}\|_{\mathrm{L}^\infty}\|\nabla^\alpha\rho^\mathrm{d}\| \\
&\leqslant \mathrm{CE^d}.
\end{aligned}
$$

Recalling that both $(\rho^\epsilon, \mathrm{u}^\epsilon, \theta^\epsilon, \mathrm{H}^\epsilon, \mathrm{q}^\epsilon)$ and $(\rho, \mathrm{u}, \theta, \mathrm{H})$ take values in a convex compact subset of the state space. For f_2, we have

$$
\begin{aligned}
\|\nabla^\alpha(\frac{\rho^\mathrm{d}\mathrm{u_t}}{\rho^\epsilon})\| &\leqslant \|\mathrm{u_t}\|_{\mathrm{L}^\infty}\|\nabla^\alpha(\frac{\rho^\mathrm{d}}{\rho^\epsilon})\| + \|\frac{\rho^\mathrm{d}}{\rho^\epsilon}\|_{\mathrm{L}^\infty}\|\nabla^\alpha\mathrm{u_t}\| \\
&\leqslant \mathrm{CE^d} + \mathrm{C}\|\rho^\epsilon\|_\mathrm{s}\|\rho^\mathrm{d}\|_{\mathrm{L}^\infty} \\
&\leqslant \mathrm{C}(\mathrm{E^d} + \epsilon(\mathrm{E^d})^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|\frac{1}{\epsilon\rho^\epsilon}\{(\rho^\epsilon\mathrm{u}^\epsilon - \rho\mathrm{u})\cdot\nabla\mathrm{u} + (\mathrm{p}^\epsilon_\rho - \mathrm{p}_\rho)\nabla\rho + (\mathrm{p}^\epsilon_\theta - \mathrm{p}_\theta)\nabla\theta\}\|_\mathrm{s} \\
&\leqslant \mathrm{C}(\mathrm{E^d} + \epsilon(\mathrm{E^d})^2).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\|\nabla^\alpha((\nabla\times B^d)\times\frac{B^\epsilon}{\rho^\epsilon})\| \\
&\leqslant \|\nabla^\alpha(\nabla\times\mathrm{B^d})\|\|\frac{\mathrm{B}^\epsilon}{\rho^\epsilon}\|_{\mathrm{L}^\infty} + \|\nabla^\alpha(\frac{\mathrm{B}^\epsilon}{\rho^\epsilon})\|\|\nabla\times\mathrm{B^d}\|_{\mathrm{L}^\infty} \\
&\leqslant \mathrm{C}(1+\epsilon\mathrm{E^d})\|\nabla^\alpha\nabla\mathrm{B^d}\| + \mathrm{C}(\mathrm{E^d} + \epsilon(\mathrm{E^d})^2).
\end{aligned}
$$

For f_3, we only need to give the estimate of $\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}\nabla u^\epsilon \nabla u^d$ because other terms are similar. Actually,

$$\begin{aligned}
&\|\nabla^\alpha[\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}\nabla u^\epsilon \nabla u^d]\| = \|\nabla^\alpha[\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}(\nabla u + \epsilon\nabla u^d)\nabla u^d]\| \\
&\leqslant C\|\nabla u\|_{L^\infty}\|\nabla^\alpha[\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}\nabla u^d]\| + C\|\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}\nabla u^d\|_{L^\infty}\|\nabla^\alpha \nabla u\| \\
&\qquad + C\epsilon\|\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}\|_{L^\infty}\|\|\nabla^\alpha(\nabla u^d)^2\| + C\epsilon\|(\nabla u^d)^2\|_{L^\infty}\|\|\nabla^\alpha\frac{\mu}{\rho^\epsilon e_\theta^\epsilon}\| \\
&\leqslant C(E^d + \epsilon(E^d)^2 + \epsilon^2(E^d)^3 + (1+\epsilon E^d)\|\nabla^\alpha\nabla u^d\|).
\end{aligned}$$

Thus, we have get the desired estimate for f_3. And the estimate for f_4 is very similar to the estimate of f_1, f_2, f_3. So the proof is completed.

REMARK 3.1. *Here we point out that although our results hold in the dimension 3, they are also valid in any dimension* $n > 3$.

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School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, P. R. China

Institute of Applied Physics and Computational Mathematics, Beijing, 100088, P. R. China
E-mail address: xuxinaboy@126.com