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On the focusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space

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Abstract. In this paper, we consider the focusing mass-critical nonlinear fourth-order Schrödinger equation. We prove that blowup solutions to this equation with initial data in $H^\gamma(\mathbb{R}^d), 5 \leq d \leq 7, \frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$ concentrate at least the mass of the ground state at the blowup time. This extends the work in [**35**] where Zhu-Yang-Zhang studied the formation of singularity for the equation with rough initial data in $\mathbb{R}^4$. We also prove that the equation is globally well-posed with initial data $u_0 \in H^\gamma(\mathbb{R}^d), 5 \leq d \leq 7, \frac{8d}{3d+8} < \gamma < 2$ satisfying $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$, where Q is the solution to the ground state equation.

Contents

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1. Introduction

Consider the focusing mass-critical nonlinear fourth-order Schrödinger equation, namely

$$\text{(NL4S)} \qquad \left\{ \begin{array}{rcl} i\partial_t u(t,x) + \Delta^2 u(t,x) &=& (|u|^{\frac{8}{d}} u)(t,x), \quad t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) &=& u_0(x) \in H^\gamma(\mathbb{R}^d), \end{array} \right.$$

where $u(t,x)$ is a complex valued function in $\mathbb{R}^+ \times \mathbb{R}^d$. The fourth-order Schrödinger equation was introduced by Karpman [**20**] and Karpman-Shagalov [**21**] taking into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such a fourth-order Schrödinger equation is of the form

$$i\partial_t u + \Delta^2 u + \varepsilon \Delta u + \mu |u|^{\nu-1} u = 0, \quad u(0) = u_0, \tag{1.1}$$

where $\varepsilon \in \{0, \pm 1\}, \mu \in \{\pm 1\}$ and $\nu > 1$. The (NL4S) is a special case of (1.1) with $\varepsilon = 0$ and $\mu = -1$.

The (NL4S) enjoys a natural scaling invariance, that is if u solves (NL4S), then for any $\lambda > 0$,

$$u_\lambda(t,x) := \lambda^{-\frac{d}{2}} u(\lambda^{-4} t, \lambda^{-1} x) \tag{1.2}$$

solves the same equation with initial data $u_\lambda(0,x) = \lambda^{-\frac{d}{2}} u_0(\lambda^{-1} x)$. This scaling also preserves the L^2-norm, i.e. $\|u_\lambda(0)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$. It is known (see [**11, 12**]) that the (NL4S) is locally well-posed in $H^\gamma(\mathbb{R}^d)$ for $\gamma \geq 0$ satisfying for $d \neq 1,2,4$,

$$\lceil \gamma \rceil \leq 1 + \frac{8}{d}, \tag{1.3}$$

where $\lceil \gamma \rceil$ is the smallest integer greater than or equal to γ. This condition ensures the nonlinearity to have enough regularity. Moreover, the unique solution enjoys mass conservation, i.e.

$$M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)} = \|u_0\|^2_{L^2(\mathbb{R}^d)},$$

and H^2-solution has conserved energy, i.e.

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2} |\Delta u(t,x)|^2 - \frac{d}{2d+8} |u(t,x)|^{\frac{2d+8}{d}} dx = E(u_0).$$

In the sub-critical regime, i.e. $\gamma > 0$, the time of existence depends only on the H^γ-norm of the initial data. Let T^* be the maximal time of existence. The local well-posedness gives the following blowup alternative criterion: either $T^* = \infty$ or

$$T^* < \infty, \quad \lim_{t \to T^*} \|u(t)\|_{H^\gamma(\mathbb{R}^d)} = \infty.$$

The study of blowup solutions for the focusing nonlinear fourth-order Schrödinger equation has been attracted a lot of interest in a past decay (see e.g. [**15**], [**3**], [**34**], [**35**], [**4**] and references therein). It is closely related to ground states Q of (NL4S) which are solutions to the elliptic equation

$$\Delta^2 Q(x) - Q(x) + |Q(x)|^{\frac{8}{d}} Q(x) = 0. \tag{1.4}$$

The equation (1.4) is obtained by considering solitary solutions (standing waves) of (NL4S) of the form $u(t,x) = Q(x)e^{-it}$. The existence of solutions to (1.4) is proved in [**34**], but the uniqueness of the solution is still an open problem. In the case

$\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$, using the sharp Gagliardo-Nirenberg inequality (see [**15**] or [**34**]), namely

$$\|u\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}(\mathbb{R}^d)} \leq C(d)\|u\|^{\frac{8}{d}}_{L^2(\mathbb{R}^d)}\|\Delta u\|^2_{L^2(\mathbb{R}^d)}, \quad C(d) := \frac{1+\frac{4}{d}}{\|Q\|^{\frac{8}{d}}_{L^2(\mathbb{R}^d)}}, \tag{1.5}$$

together with the energy conservation, Fibich-Ilan-Papanicolaou in [**15**] (see also [**3**]) proved that the (NL4S) is globally well-posed in $H^2(\mathbb{R}^d)$. Moreover, the authors in [**15**] also provided some numerical observations showing that the H^2-solution to (NL4S) may blowup if the initial data satisfies $\|u_0\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$. Baruch-Fibich-Mandelbaum in [**3**] proved some dynamical properties of the radially symmetric blowup solution such as blowup rate, L^2-concentration. Later, Zhu-Yang-Zhang in [**34**] removed the radially symmetric assumption and established the profile decomposition, the existence of the ground state of elliptic equation (1.4) and the following concentration compactness property for the (NL4S).

THEOREM 1.1 (Concentration compactness [**34**]). *Let $(v_n)_{n\geq 1}$ be a bounded family of $H^2(\mathbb{R}^d)$ functions such that*

$$\limsup_{n\to\infty} \|\Delta v_n\|_{L^2(\mathbb{R}^d)} \leq M < \infty \quad \text{and} \quad \limsup_{n\to\infty} \|v_n\|_{L^{2+\frac{8}{d}}(\mathbb{R}^d)} \geq m > 0.$$

Then there exists a sequence $(x_n)_{n\geq 1}$ of $\mathbb{R}^d$ such that up to a subsequence

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^2(\mathbb{R}^d) \text{ as } n \to \infty,$$

with $\|V\|^{\frac{8}{d}}_{L^2(\mathbb{R}^d)} \geq \frac{\|Q\|^{\frac{8}{d}}_{L^2(\mathbb{R}^d)} m^{2+\frac{8}{d}}}{\left(1+\frac{4}{d}\right)M^2}$, where Q is the solution to the ground state equation (1.4).

Consequently, the authors in [**35**] established the limiting profile and the L^2-concentration for (NL4S) with initial data $u_0 \in H^\gamma(\mathbb{R}^4)$, $\frac{9+\sqrt{721}}{20} < \gamma < 2$. Recently, Boulenger-Lenzmann in [**4**] proved a general result on finite-time blowup for the focusing generalized nonlinear fourth-order Schrödigner equation(i.e. (1.1) with $\mu = 1$) with radial data in $H^2(\mathbb{R}^d)$.

The goal of this paper is to extend the results of [**35**] to higher dimensions $d \geq 5$ and to prove the global existence of (NL4S) for initial data $u_0 \in H^\gamma(\mathbb{R}^d), 0 < \gamma < 2$ satisfying $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$. Since we are working with low regularity data, the energy argument does not work. In order to overcome this problem, we make use of the I-method. Due to the high-order term $\Delta^2 u$, we requires the nonlinearity to have at least two orders of derivatives in order to successfully establish the almost conservation law. We thus restrict ourself in spatial space of dimensions $d = 5, 6, 7$. Our main results are as follows.

THEOREM 1.2. *Let $d = 5, 6, 7$ and $u_0 \in H^\gamma(\mathbb{R}^d)$ with $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$. If the corresponding solution to the* (NL4S) *blows up in finite time $0 < T^* < \infty$, then there exists a function $U \in H^2(\mathbb{R}^d)$ such that $\|U\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$ and there exist sequences $(t_n, \lambda_n, x_n)_{n\geq 1} \in \mathbb{R}^+ \times \mathbb{R}^+_* \times \mathbb{R}^d$ satisfying*

$$t_n \nearrow T^* \text{ as } n \to \infty \quad \text{and} \quad \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}, \quad \forall n \geq 1$$

such that

$$\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^{a(d,\gamma)-}(\mathbb{R}^d) \text{ as } n \to \infty,$$

where

$$a(d,\gamma) := \frac{4d\gamma^2 + (2d+48)\gamma + 16d}{16d + (56-3d)\gamma - 16\gamma^2},$$

and Q is the solution of the ground state equation (1.4).

The proof of the above theorem is based on the combination of the I-method and the concentration compactness property given in Theorem 1.1 which is similar to those given in [**32**] and [**35**]. The I-method was first introduced by I-Team in [**7**] in order to treat the nonlinear Schrödinger equation at low regularity. It then becomes a useful way to address the low regularity problem for the nonlinear dispersive equations. The idea is to replace the non-conserved energy $E(u)$ when $\gamma < 2$ by an "almost conserved" variance $E(Iu)$ with I a smoothing operator which is the identity at low frequency and behaves like a fractional integral operator of order $2-\gamma$ at high frequency. Since Iu is not a solution of (NL4S), we may expect an energy increment. The key is to show that on intervals of local well-posedness, the modified energy $E(Iu)$ is an "almost conserved" quantity and grows much slower than the modified kinetic energy $\|\Delta Iu\|^2_{L^2(\mathbb{R}^d)}$. To do so, we need delicate estimates on the commutator between the I-operator and the nonlinearity. Note that when $d=4$, the nonlinearity is algebraic, one can use the Fourier transform technique to write the commutator explicitly and then control it by multi-linear analysis. In our setting, the nonlinearity is not algebraic. Thus we can not apply the Fourier transform technique. Fortunately, thanks to a special Strichartz estimate (2.5), we are able to apply the technique given in [**32**] to control the commutator. The concentration compactness property given in Theorem 1.1 is very useful to study the dynamical properties of blowup solutions for the nonlinear fourth-order Schrödinger equation. With the help of this property, Zhu-Yang-Zhang proved in [**34**] the L^2-concentration of blowup solutions and the limiting profile of minimal-mass blowup solutions with non-radial data in $H^2(\mathbb{R}^d)$. In [**35**], they extended these results for non-radial data below the energy space in the fourth dimensional space.

As a consequence of Theorem 1.2, we have the following mass concentration property.

THEOREM 1.3. *Let $d = 5, 6, 7$ and $u_0 \in H^\gamma(\mathbb{R}^d)$ with $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$. Assume that the corresponding solution u to the* (NL4S) *blows up in finite time $0 < T^* < \infty$. If $\alpha(t) > 0$ is an arbitrary function such that*

$$\lim_{t \nearrow T^*} \frac{(T^*-t)^{\frac{\gamma}{8}}}{\alpha(t)} = 0,$$

then there exists a function $x(t) \in \mathbb{R}^d$ such that

$$\limsup_{t \nearrow T^*} \int_{|x-x(t)| \leq \alpha(t)} |u(t,x)|^2 dx \geq \int_{\mathbb{R}^d} |Q(x)|^2 dx,$$

where Q is the solution to the ground state equation (1.4).

When the mass of the initial data equals to the mass of the solution of the ground state equation (1.4), we have the following improvement of Theorem 1.2. Note that in the below result, we assume that there exists a unique solution to the ground state equation (1.4) which is a delicate open problem.

THEOREM 1.4. *Let $d = 5, 6, 7$ and $u_0 \in H^\gamma(\mathbb{R}^d)$ with $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$ be such that $\|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$. If the corresponding solution u*

to the (NL4S) *blows up in finite time* $0 < T^* < \infty$, *then there exist sequences* $(t_n, e^{i\theta_n}, \lambda_n, x_n)_{n\geq 1} \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+_* \times \mathbb{R}^d$ *satisfying*

$$t_n \nearrow T^* \text{ as } n \to \infty \quad \text{ and } \quad \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}, \quad \forall n \geq 1$$

such that

$$\lambda_n^{\frac{d}{2}} e^{i\theta_n} u(t_n, \lambda_n \cdot + x_n) \to Q \text{ strongly in } H^{a(d,\gamma)-}(\mathbb{R}^d) \text{ as } n \to \infty,$$

where

$$a(d,\gamma) := \frac{4d\gamma^2 + (2d+48)\gamma + 16d}{16d + (56-3d)\gamma - 16\gamma^2},$$

and Q *is the unique solution to the ground state equation* (1.4).

Our last result concerns with the global existence of (NL4S) with rough initial data u_0 satisfying $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$.

THEOREM 1.5. *Let* $d = 5, 6, 7$ *and* $u_0 \in H^\gamma(\mathbb{R}^d)$ *with* $\frac{8d}{3d+8} < \gamma < 2$ *be such that* $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$, *where* Q *is the solution to the ground state equation* (1.4). *Then the initial value problem* (NL4S) *is globally well-posed.*

The proof of this result is inspired by the argument of [**14**] which relies on the I-method and the sharp Gagliardo-Nirenberg inequality (1.5). Using the smallness assumption of the initial data, the sharp Gagliardo-Nirenberg inquality shows that the modified kinetic energy is controlled by the total energy. This allows us to establish the almost conservation law for the modified energy.

This paper is organized as follows. In Section 2, we introduce some notations and recall some results related to our problem. In Section 3, we recall some local existence results and prove the modified local well-posedness. In Section 4, we prove two types of modified energy increment. In Section 5, we give the proof of Theorem 1.2, Theorem 1.3 and Theorem 1.4. Finally, we prove the global well-posedness with small initial data in Section 6.

2. Preliminaries

In the sequel, the notation $A \lesssim B$ denotes an estimate of the form $A \leq CB$ for some constant $C > 0$. The notation $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$ if $A \leq cB$ for some small constant $c > 0$. We also use $\langle a \rangle := 1 + |a|$ and $a\pm := a \pm \epsilon$ for some universal constant $0 < \epsilon \ll 1$ and

2.1. Nonlinearity. Let $F(z) := |z|^{\frac{8}{d}} z, d = 5, 6, 7$ be the function that defines the nonlinearity in (NL4S). The derivative $F'(z)$ is defined as a real-linear operator acting on $w \in \mathbb{C}$ by

$$F'(z) \cdot w := w\partial_z F(z) + \overline{w}\partial_{\overline{z}} F(z),$$

where

$$\partial_z F(z) = \frac{2d+8}{2d}|z|^{\frac{8}{d}}, \quad \partial_{\overline{z}} F(z) = \frac{4}{d}|z|^{\frac{8}{d}} \frac{z}{\overline{z}}.$$

We shall identify $F'(z)$ with the pair $(\partial_z F(z), \partial_{\overline{z}} F(z))$, and define its norm by

$$|F'(z)| := |\partial_z F(z)| + |\partial_{\overline{z}} F(z)|.$$

It is clear that $|F'(z)| = O(|z|^{\frac{8}{d}})$. We also have the following chain rule

$$\partial_k F(u) = F'(u)\partial_k u,$$

for $k \in \{1, \cdots, d\}$. In particular, we have

$$\nabla F(u) = F'(u)\nabla u.$$

We next recall the fractional chain rule to estimate the nonlinearity.

LEMMA 2.1 (Fractional chain rule for C^1 functions [**6**], [**23**]). *Suppose that $G \in C^1(\mathbb{C}, \mathbb{C})$, and $\alpha \in (0,1)$. Then for $1 < q \leq q_2 < \infty$ and $1 < q_1 \leq \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,*

$$\||\nabla|^\alpha G(u)\|_{L^q_x} \lesssim \|G'(u)\|_{L^{q_1}_x} \||\nabla|^\alpha u\|_{L^{q_2}_x}.$$

We refer the reader to [**6**, Proposition 3.1] for the proof of the above estimate when $1 < q_1 < \infty$, and to [**23**, Theorem A.6] for the proof when $q_1 = \infty$. When G is no longer C^1, but Hölder continuous, we have the following fractional chain rule.

LEMMA 2.2 (Fractional chain rule for $C^{0,\beta}$ functions [**33**]). *Suppose that $G \in C^{0,\beta}(\mathbb{C}, \mathbb{C}), \beta \in (0,1)$. Then for every $0 < \alpha < \beta, 1 < q < \infty$, and $\frac{\alpha}{\beta} < \rho < 1$,*

$$\||\nabla|^\alpha G(u)\|_{L^q_x} \lesssim \||u|^{\beta - \frac{\alpha}{\rho}}\|_{L^{q_1}_x} \||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L^{\frac{\alpha}{\rho}q_2}_x},$$

provided $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\left(1 - \frac{\alpha}{\beta\rho}\right) q_1 > 1$.

We refer the reader to [**33**, Proposition A.1] for the proof of this result. We also need the following fractional Leibniz rule.

LEMMA 2.3 (Fractional Leibniz rule [**22**]). *Let $F \in C^k(\mathbb{C}, \mathbb{C}), k \in \mathbb{N}\backslash\{0\}$. Assume that there is $\nu \geq k$ such that*

$$|D^i F(z)| \lesssim |z|^{\nu - i}, \quad \forall z \in \mathbb{C}, i = 1, ..., k.$$

Then for $\gamma \in [0,k], 1 < q \leq q_2 < \infty$ and $1 < q_1 \leq \infty$ satisfying $\frac{1}{q} = \frac{\nu-1}{q_1} + \frac{1}{q_2}$,

$$\||\nabla|^\gamma F(u)\|_{L^q_x} \lesssim \|u\|^{\nu-1}_{L^{q_1}_x} \||\nabla|^\gamma u\|_{L^{q_2}_x}. \tag{2.1}$$

Moreover, if F is a homogeneous polynomial in u and $\overline{u}$, then (2.1) *holds true for any $\gamma \geq 0$.*

The reader can find the proof of this fractional Leibniz rule in [**22**, Appendix].

2.2. Strichartz estimates. Let $I \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the mixed norm

$$\|u\|_{L^p_t(I, L^q_x)} := \Big(\int_I \Big(\int_{\mathbb{R}^d} |u(t,x)|^q dx \Big)^{\frac{1}{q}} \Big)^{\frac{1}{p}}$$

with a usual modification when either p or q are infinity. When there is no risk of confusion, we may write $L^p_t L^q_x$ instead of $L^p_t(I, L^q_x)$. We also use $L^p_{t,x}$ when $p = q$.

DEFINITION 2.4. *A pair (p,q) is said to be* ***Schrödinger admissible****, for short $(p,q) \in S$, if*

$$(p,q) \in [2, \infty]^2, \quad (p,q,d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

Throughout this paper, we denote for $(p,q) \in [1, \infty]^2$,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{4}{p}. \tag{2.2}$$

DEFINITION 2.5. *A pair (p,q) is called **biharmonic admissible**, for short $(p,q) \in B$, if*

$$(p,q) \in S, \quad q < \infty, \quad \gamma_{p,q} = 0.$$

PROPOSITION 2.6 (Strichartz estimate for fourth-order Schrödinger equation [**11**]). *Let $\gamma \in \mathbb{R}$ and u be a (weak) solution to the linear fourth-order Schrödinger equation namely*

$$u(t) = e^{it\Delta^2}u_0 + \int_0^t e^{i(t-s)\Delta^2}F(s)ds,$$

for some data u_0, F. Then for all (p,q) and (a,b) Schrödinger admissible with $q < \infty$ and $b < \infty$,

$$\||\nabla|^\gamma u\|_{L^p_t(\mathbb{R},L^q_x)} \lesssim \||\nabla|^{\gamma+\gamma_{p,q}}u_0\|_{L^2_x} + \||\nabla|^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-4}F\|_{L^{a'}_t(\mathbb{R},L^{b'}_x)}. \tag{2.3}$$

Here (a,a') and (b,b') are conjugate pairs, and $\gamma_{p,q}, \gamma_{a',b'}$ are defined as in (2.2).

We refer the reader to [**11**, Proposition 2.1] for the proof of Proposition 2.6. The proof is based on the scaling technique instead of using a dedicate dispersive estimate of [**1**] for the fundamental solution of the homogeneous fourth-order Schrödinger equation. Note that the estimate (2.3) is exactly the one given in [**27**], [**28**] or [**29**] where the author considered (p,q) and (a,b) are either sharp Schrödinger admissible, i.e.

$$p,q \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

or biharmonic admissible.

The following result is a direct consequence of (2.3).

COROLLARY 2.7. *Let $\gamma \in \mathbb{R}$ and u a (weak) solution to the linear fourth-order Schrödinger equation for some data u_0, F. Then for all (p,q) and (a,b) biharmonic admissible,*

$$\||\nabla|^\gamma u\|_{L^p_t(\mathbb{R},L^q_x)} \lesssim \|\||\nabla|^\gamma u_0\|_{L^2_x} + \||\nabla|^\gamma F\|_{L^{a'}_t(\mathbb{R},L^{b'}_x)}, \tag{2.4}$$

and

$$\|\Delta u\|_{L^p_t(\mathbb{R},L^q_x)} \lesssim \|\Delta u_0\|_{L^2_x} + \|\nabla F\|_{L^2_t(\mathbb{R},L^{\frac{2d}{d+2}}_x)}. \tag{2.5}$$

2.3. Littlewood-Paley decomposition. Let φ be a radial smooth bump function supported in the ball $|\xi| \leq 2$ and equal to 1 on the ball $|\xi| \leq 1$. For $M = 2^k, k \in \mathbb{Z}$, we define the Littlewood-Paley operators

$$\begin{aligned}
\widehat{P_{\leq M}f}(\xi) &:= \varphi(M^{-1}\xi)\hat{f}(\xi), \\
\widehat{P_{>M}f}(\xi) &:= (1-\varphi(M^{-1}\xi))\hat{f}(\xi), \\
\widehat{P_M f}(\xi) &:= (\varphi(M^{-1}\xi) - \varphi(2M^{-1}\xi))\hat{f}(\xi),
\end{aligned}$$

where $\hat{\cdot}$ is the spatial Fourier transform. Similarly, we can define

$$P_{<M} := P_{\leq M} - P_M, \quad P_{\geq M} := P_{>M} + P_M,$$

and for $M_1 \leq M_2$,

$$P_{M_1 < \cdot \leq M_2} := P_{\leq M_2} - P_{\leq M_1} = \sum_{M_1 < M \leq M_2} P_M.$$

We recall the following standard Bernstein inequalities (see e.g. [**2**, Chapter 2] or [**31**, Appendix]).

Lemma 2.8 (Bernstein inequalities). *Let $\gamma \geq 0$ and $1 \leq p \leq q \leq \infty$. We have*

$$\begin{aligned} \|P_{\geq M} f\|_{L^p_x} &\lesssim M^{-\gamma} \||\nabla|^\gamma P_{\geq M} f\|_{L^p_x}, \\ \|P_{\leq M} |\nabla|^\gamma f\|_{L^p_x} &\lesssim M^\gamma \|P_{\leq M} f\|_{L^p_x}, \\ \|P_M |\nabla|^{\pm\gamma} f\|_{L^p_x} &\sim M^{\pm\gamma} \|P_M f\|_{L^p_x}, \\ \|P_{\leq M} f\|_{L^q_x} &\lesssim M^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq M} f\|_{L^p_x}, \\ \|P_M f\|_{L^q_x} &\lesssim M^{\frac{d}{p}-\frac{d}{q}} \|P_M f\|_{L^p_x}. \end{aligned}$$

2.4. *I*-operator. Let $0 \leq \gamma < 2$ and $N \gg 1$. We define the Fourier multiplier I_N by

$$\widehat{I_N f}(\xi) := m_N(\xi) \hat{f}(\xi),$$

where m_N is a smooth, radially symmetric, non-increasing function such that

$$m_N(\xi) := \begin{cases} 1 & \text{if } |\xi| \leq N, \\ (N^{-1}|\xi|)^{\gamma-2} & \text{if } |\xi| \geq 2N. \end{cases}$$

We shall drop the N from the notation and write I and m instead of I_N and m_N. We recall (see [**13**, Lemma 2.7]) some basic properties of the I-operator in the following lemma.

Lemma 2.9. *Let $0 \leq \sigma \leq \gamma < 2$ and $1 < q < \infty$. Then*

$$\|If\|_{L^q_x} \lesssim \|f\|_{L^q_x}, \tag{2.6}$$

$$\||\nabla|^\sigma P_{>N} f\|_{L^q_x} \lesssim N^{\sigma-2} \|\Delta I f\|_{L^q_x}, \tag{2.7}$$

$$\| \langle \nabla \rangle^\sigma f\|_{L^q_x} \lesssim \| \langle \Delta \rangle I f\|_{L^q_x}, \tag{2.8}$$

$$\|f\|_{H^\gamma_x} \lesssim \|If\|_{H^2_x} \lesssim N^{2-\gamma} \|f\|_{H^\gamma_x}, \tag{2.9}$$

$$\|If\|_{\dot{H}^2_x} \lesssim N^{2-\gamma} \|f\|_{\dot{H}^\gamma_x}. \tag{2.10}$$

When the nonlinearity $F(u)$ is algebraic, one can use the Fourier transform to write the commutator like $F(Iu) - IF(u)$ as a product of Fourier transforms of u and Iu, and then measure the frequency interactions. However, in our setting, the nonlinearity is no longer algebraic, we thus need the following rougher estimate which is a modified version of the Schrödinger context (see [**32**]).

Lemma 2.10. *Let $1 < \gamma < 2, 0 < \delta < \gamma - 1$ and $1 < q, q_1, q_2 < \infty$ be such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|I(fg) - (If)g\|_{L^q_x} \lesssim N^{-(2-\gamma+\delta)} \|If\|_{L^{q_1}_x} \| \langle \nabla \rangle^{2-\gamma+\delta} g\|_{L^{q_2}_x}. \tag{2.11}$$

We refer the reader to [**13**, Lemma 2.9] for the proof of this result. A direct consequence of Lemma 2.10 with the fact that

$$\nabla F(u) = \nabla u F'(u)$$

is the following commutator estimate.

Corollary 2.11. *Let $1 < \gamma < 2, 0 < \delta < \gamma - 1$ and $1 < q, q_1, q_2 < \infty$ be such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|\nabla I F(u) - (I\nabla u)F'(u)\|_{L^q_x} \lesssim N^{-(2-\gamma+\delta)} \|\nabla I u\|_{L^{q_1}_x} \| \langle \nabla \rangle^{2-\gamma+\delta} F'(u)\|_{L^{q_2}_x}. \tag{2.12}$$

3. Modified local well-posedness

We firstly recall the local theory for (NL4S) in Sobolev spaces (see [**11, 12**]).

PROPOSITION 3.1 (Local well-posedness in Sobolev spaces). *Let* $5 \leq d \leq 7, 0 < \gamma < 2$ *and* $u_0 \in H^\gamma(\mathbb{R}^d)$. *Then the equation* (NL4S) *is locally well-posed on* $[0, T_{\text{lwp}}]$ *with*

$$T_{\text{lwp}} \sim \|u_0\|_{H^\gamma_x}^{-\frac{4}{\gamma}}.$$

Moreover,

$$\sup_{(a,b)\in B} \|u\|_{L^a_t([0,T_{\text{lwp}}],W^{\gamma,b}_x)} \lesssim \|u_0\|_{H^\gamma_x}.$$

The implicit constants depend only on the dimension d and the regularity γ.

PROOF. Let us introduce

$$p = \frac{2(d+4)}{d-2\gamma}, \quad q = \frac{2d(d+4)}{d^2+8\gamma}.$$

It is easy to check that (p,q) is biharmonic admissible. We next choose (m,n) so that

$$\frac{1}{p'} = \frac{\frac{8}{d}}{m} + \frac{1}{p}, \quad \frac{1}{q'} = \frac{\frac{8}{d}}{n} + \frac{1}{q}, \tag{3.1}$$

or

$$m = \frac{4(d+4)}{d(2+\gamma)}, \quad n = \frac{2(d+4)}{d-2\gamma}.$$

With this choice of n, we have the Sobolev embedding $\dot{W}^{\gamma,q}_x \hookrightarrow L^n_x$.

Now, we consider

$$X := \left\{u \in L^p_t([0,T], W^{\gamma,q}_x) \mid \|u\|_{L^p_t([0,T],W^{\gamma,q}_x)} \leq M\right\}$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^p_t([0,T],L^q_x)},$$

where $T, M > 0$ to be chosen later. By Duhamel's formula, it suffices to prove that the functional

$$\Phi(u)(t) := e^{it\Delta^2}u_0 - i\int_0^t e^{i(t-s)\Delta^2}|u(s)|^{\nu-1}u(s)ds$$

is a contraction on (X,d). By Strichartz estimate (2.4),

$$\|\Phi(u)\|_{L^p_t([0,T],W^{\gamma,q}_x)} \lesssim \|u_0\|_{H^\gamma_x} + \|F(u)\|_{L^{p'}_t([0,T],W^{\gamma,q'}_x)},$$
$$\|\Phi(u)-\Phi(v)\|_{L^p_t([0,T],L^q_x)} \lesssim \|F(u)-F(v)\|_{L^{p'}_t([0,T],L^{q'}_x)},$$

where $F(u) = |u|^{\frac{8}{d}}u$ and similarly for $F(v)$. Using (3.1), we apply Lemma 2.3 with $k = 2, \gamma \in (0,2), \nu = 1 + \frac{8}{d}$ to have

$$\|F(u)\|_{W^{\gamma,q'}_x} \lesssim \|u\|^{\frac{8}{d}}_{L^n_x}\|u\|_{W^{\gamma,q}_x} \lesssim \|u\|^{\frac{8}{d}}_{\dot{W}^{\gamma,q}_x}\|u\|_{W^{\gamma,q}_x}.$$

Note that $\nu \geq k$ since $5 \leq d \leq 7$. Using again (3.1), the Hölder inequality and Sobolev embedding then imply

$$\|F(u)\|_{L^{p'}_t([0,T],W^{\gamma,q'}_x)} \lesssim \|u\|^{\frac{8}{d}}_{L^m_t([0,T],\dot{W}^{\gamma,q}_x)}\|u\|_{L^p_t([0,T],W^{\gamma,q}_x)} \lesssim T^{\frac{2\gamma}{d}}\|u\|^{1+\frac{8}{d}}_{L^p_t([0,T],W^{\gamma,q}_x)}.$$

Similarly, we have

$$\begin{aligned}\|F(u)-F(v)\|_{L^{p'}_t([0,T],L^{q'}_x)} &\lesssim T^{\frac{2\gamma}{d}}\Big(\|u\|^{\frac{8}{d}}_{L^p_t([0,T],\dot W^{\gamma,q}_x)}+\|v\|^{\frac{8}{d}}_{L^p_t([0,T],\dot W^{\gamma,q}_x)}\Big)\|u-v\|_{L^p_t([0,T],L^q_x)}\\ &\lesssim T^{\frac{2\gamma}{d}}\Big(\|u\|^{\frac{8}{d}}_{L^p_t([0,T],W^{\gamma,q}_x)}+\|v\|^{\frac{8}{d}}_{L^p_t([0,T],W^{\gamma,q}_x)}\Big)\|u-v\|_{L^p_t([0,T],L^q_x)}.\end{aligned}$$

This shows that for all $u,v\in X$, there exists $C>0$ independent of T and $u_0\in H^\gamma_x$ so that

$$\begin{aligned}\|\Phi(u)\|_{L^p_t([0,T],W^{\gamma,q}_x)} &\lesssim C\|u_0\|_{H^\gamma_x}+CT^{\frac{2\gamma}{d}}M^{1+\frac{8}{d}},\\ d(\Phi(u),\Phi(v)) &\lesssim CT^{\frac{2\gamma}{d}}M^{\frac{8}{d}}d(u,v).\end{aligned} \tag{3.2}$$

If we set $M=2C\|u_0\|_{H^\gamma_x}$ and choose

$$T\sim\|u_0\|^{-\frac{4}{\gamma}}_{H^\gamma_x},$$

then X is stable by Φ and Φ is a contraction on (X,d). The fixed point argument proves the local existence. Moreover, by Strichartz estimate (2.4),

$$\sup_{(a,b)\in B}\|u\|_{L^a_t([0,T],W^{\gamma,b}_x)}\lesssim\|u_0\|_{H^\gamma_x}+\|F(u)\|_{L^{p'}_t([0,T],W^{\gamma,q'}_x)}\lesssim\|u_0\|_{H^\gamma_x}.$$

The proof is complete. □

Corollary 3.2 (Blowup criterion). *Let $5\le d\le 7$, $0<\gamma<2$ and $u_0\in H^\gamma(\mathbb{R}^d)$. Assume that the unique solution u to* (NL4S) *blows up at time $0<T^*<\infty$. Then,*

$$\|u(t)\|_{H^\gamma_x}\gtrsim(T^*-t)^{-\frac{\gamma}{4}}, \tag{3.3}$$

for all $0<t<T^*$.

Proof. We follow the argument of [**5**]. Let $0<t<T^*$. If we consider (NL4S) with initial data $u(t)$, then it follows from (3.2) the fixed point argument that if for some $M>0$

$$C\|u(t)\|_{H^\gamma_x}+C(T-t)^{\frac{2\gamma}{d}}M^{1+\frac{8}{d}}\le M,$$

then $T<T^*$. Thus,

$$C\|u(t)\|_{H^\gamma_x}+C(T^*-t)^{\frac{2\gamma}{d}}M^{1+\frac{8}{d}}>M,$$

for all $M>0$. Choosing $M=2C\|u(t)\|_{H^\gamma_x}$, we see that

$$(T^*-t)^{\frac{2\gamma}{d}}\|u(t)\|^{\frac{8}{d}}_{H^\gamma_x}>C.$$

This proves (3.3) and the proof is complete. □

We next define for any spacetime slab $J\times\mathbb{R}^d$,

$$Z_I(J):=\sup_{(p,q)\in B}\|\langle\Delta\rangle Iu\|_{L^p_t(J,L^q_x)}.$$

We have the following commutator estimates.

LEMMA 3.3. *Let* $5 \le d \le 7, 1 < \gamma < 2, 0 < \delta < \gamma - 1$ *and* J *a compact interval. Then*

$$\|IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+4}})} \lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{1+\frac{8}{d}}, \tag{3.4}$$

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim N^{-(2-\gamma+\delta)} (Z_I(J))^{1+\frac{8}{d}}, \tag{3.5}$$

$$\|\nabla IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{1+\frac{8}{d}} + N^{-(2-\gamma+\delta)} (Z_I(J))^{1+\frac{8}{d}}, \tag{3.6}$$

$$\|\nabla IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+4}})} \lesssim (Z_I(J))^{1+\frac{8}{d}}. \tag{3.7}$$

PROOF. We firstly note that the estimates (3.5) and (3.7) are given in [**13**, Lemma 3.1]. Let us consider (3.4). By (2.6) and Hölder's inequality,

$$\begin{aligned} \|IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+4}})} &\lesssim \|F(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+4}})} \\ &\lesssim \|u\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \|F'(u)\|_{L_t^{\frac{d+8}{2(2+\gamma)}}(J, L_x^{\frac{d(d+8)}{4d+16-8\gamma}})}. \end{aligned}$$

Since $F'(u) = O(|u|^{\frac{8}{d}})$, the Sobolev embedding implies

$$\begin{aligned} \|IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+4}})} &\lesssim \|u\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \|u\|^{\frac{8}{d}}_{L_t^{\frac{4(d+8)}{d(2+\gamma)}}(J, L_x^{\frac{2(d+8)}{d+4-2\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} \|u\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \|u\|^{\frac{8}{d}}_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2(d+8)}{d+4-2\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} \|u\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \||\nabla|^\gamma u\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} \|\langle \nabla \rangle^\gamma u\|^{1+\frac{8}{d}}_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{1+\frac{8}{d}}. \end{aligned}$$

Here we use (2.8) and the fact $\left(\frac{2(d+8)}{d-4\gamma}, \frac{2d(d+8)}{d^2+4d+16\gamma}\right)$ is biharmonic admissible to get the last estimate.

It remains to prove (3.6). We have from (3.5) and the triangle inequality that

$$\|\nabla IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim \|(\nabla Iu)F'(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} + N^{-(2-\gamma+\delta)} (Z_I(J))^{1+\frac{8}{d}}. \tag{3.8}$$

By Hölder's inequality,

$$\|(\nabla Iu)F'(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim \|\nabla Iu\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+2d+16(\gamma-1)}})} \|F'(u)\|_{L_t^{\frac{d+8}{2(2+\gamma)}}(J, L_x^{\frac{d(d+8)}{4d+16-8\gamma}})}. \tag{3.9}$$

We use the Sobolev embedding to estimate

$$\|\nabla Iu\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+2d+16(\gamma-1)}})} \lesssim \|\Delta Iu\|_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \lesssim Z_I(J). \tag{3.10}$$

Here $\left(\frac{2(d+8)}{d-4\gamma}, \frac{2d(d+8)}{d^2+4d+16\gamma}\right)$ is biharmonic admissible. Since $F'(u) = O(|u|^{\frac{8}{d}})$, the Sobolev embedding again gives

$$\begin{aligned}
\|F'(u)\|_{L_t^{\frac{d+8}{2(2+\gamma)}}(J, L_x^{\frac{d(d+8)}{4d+16-8\gamma}})} &\lesssim \|u\|^{\frac{8}{d}}_{L_t^{\frac{4(d+8)}{d(2+\gamma)}}(J, L_x^{\frac{2(d+8)}{d+4-2\gamma}})} \\
&\lesssim |J|^{\frac{2\gamma}{d}} \|u\|^{\frac{8}{d}}_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2(d+8)}{d+4-2\gamma}})} \\
&\lesssim |J|^{\frac{2\gamma}{d}} \||\nabla|^{\gamma} u\|^{\frac{8}{d}}_{L_t^{\frac{2(d+8)}{d-4\gamma}}(J, L_x^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \\
&\lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{\frac{8}{d}}. \qquad (3.11)
\end{aligned}$$

Collecting (3.8) – (3.11), we obtain (3.6). The proof is complete. □

Proposition 3.4 (Modified local well-posedness). *Let $5 \leq d \leq 7, 1 < \gamma < 2, 0 < \delta < \gamma - 1$ and $u_0 \in H^\gamma(\mathbb{R}^d)$. Let*

$$\widetilde{T}_{\mathrm{lwp}} := c\|Iu_0\|^{-\frac{4}{\gamma}}_{H^2_x},$$

for a small constant $c = c(d,\gamma) > 0$. Then (NL4S) *is locally well-posed on $[0, \widetilde{T}_{\mathrm{lwp}}]$. Moreover, for N sufficiently large,*

$$Z_I([0, \widetilde{T}_{\mathrm{lwp}}]) \lesssim \|Iu_0\|_{H^2_x}. \qquad (3.12)$$

Proof. By (2.9), $\|u_0\|_{H^\gamma_x} \lesssim \|Iu_0\|_{H^2_x}$. Thus,

$$\widetilde{T}_{\mathrm{lwp}} = c\|Iu_0\|^{-\frac{4}{\gamma}}_{H^2_x} \lesssim c\|u_0\|^{-\frac{4}{\gamma}}_{H^\gamma_x} \leq T_{\mathrm{lwp}},$$

provided c is small enough. Here T_{lwp} is as in Proposition 3.1. This shows that (NL4S) is locally well-posed on $[0, \widetilde{T}_{\mathrm{lwp}}]$. It remains to prove (3.12). Denote $J = [0, \widetilde{T}_{\mathrm{lwp}}]$. By Strichartz estimates (2.4) and (2.5),

$$\begin{aligned}
Z_I(J) &\lesssim \sup_{(p,q)\in B} \|Iu\|_{L^p_t(J, L^q_x)} + \sup_{(p,q)\in B} \|\Delta Iu\|_{L^p_t(J,L^q_x)} \\
&\lesssim \|Iu_0\|_{L^2_x} + \|IF(u)\|_{L^2_t(J, L_x^{\frac{2d}{d+4}})} + \|\Delta Iu_0\|_{L^2_x} + \|\nabla IF(u)\|_{L^2_t(J, L_x^{\frac{2d}{d+2}})} \\
&\lesssim \|Iu_0\|_{H^2_x} + \|IF(u)\|_{L^2_t(J, L_x^{\frac{2d}{d+4}})} + \|\nabla IF(u)\|_{L^2_t(J, L_x^{\frac{2d}{d+2}})}.
\end{aligned}$$

We next use (3.4) and (3.6) to have

$$Z_I(J) \lesssim \|Iu_0\|_{H^2_x} + \left(|J|^{\frac{2\gamma}{d}} + N^{-(2-\gamma+\delta)}\right)(Z_I(J))^{1+\frac{8}{d}}.$$

By taking $c = c(d,\gamma)$ small enough (or $|J|$ is small) and N large enough, the continuity argument shows (3.12). The proof is complete. □

4. Modified energy increment

In this section, we will derive two types of the modified energy increment. The first one is to show that the modified energy of u, namely $E(Iu)$ grows much slower than the modified kinetic of u, namely $\|\Delta Iu\|^2_{L^2_x}$. It is crucial to prove the limiting profile for blowup solutions given in Theorem 1.2. The second one is the "almost" conservation law for initial data whose mass is smaller than mass of the solution to the ground state equation (1.4). With the help of this "almost" conservation law, we are able to prove the global well-posedness given in Theorem 1.5.

LEMMA 4.1 (Local increment of the modified energy). *Let* $5 \leq d \leq 7, \max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2, 0 < \delta < \gamma + \frac{8}{d} - 3$ *and* $u_0 \in H^\gamma(\mathbb{R}^d)$. *Let*

$$\widetilde{T}_{\rm lwp} := c\|Iu_0\|_{H^2_x}^{-\frac{4}{\gamma}},$$

for some small constant $c = c(d, \gamma) > 0$. *Then, for* N *sufficiently large,*

$$\sup_{t \in [0, \widetilde{T}_{\rm lwp}]} |E(Iu(t)) - E(Iu_0)| \lesssim N^{-(2-\gamma+\delta)}\Big(\|Iu_0\|_{H^2_x}^{2+\frac{8}{d}} + \|Iu_0\|_{H^2_x}^{2+\frac{16}{d}}\Big). \tag{4.1}$$

Here the implicit constant depends only on γ *and* $\|u_0\|_{H^\gamma_x}$.

PROOF. By Proposition 3.4, the equation (NL4S) is locally well-posed on $[0, \widetilde{T}_{\rm lwp}]$ and the unique solution u satisfies

$$Z_I([0, \widetilde{T}_{\rm lwp}]) \lesssim \|Iu_0\|_{H^2_x}. \tag{4.2}$$

Next, we have from a direct computation that

$$\partial_t E(Iu(t)) = \mathrm{Re} \int \overline{I\partial_t u}(\Delta^2 Iu - F(Iu))dx.$$

The Fundamental Theorem of Calculus gives

$$E(Iu(t)) - E(Iu_0) = \int_0^t \partial_s E(Iu(s))ds = \mathrm{Re} \int_0^t \int \overline{I\partial_s u}(\Delta^2 Iu - F(Iu))dxds.$$

As $I\partial_t u = i\Delta^2 Iu - iIF(u)$, we have

$$\begin{aligned} E(Iu(t)) - E(Iu_0) &= \mathrm{Re} \int_0^t \int \overline{I\partial_s}(IF(u) - F(Iu)dxds \\ &= \mathrm{Im} \int_0^t \int \overline{\Delta^2 Iu - IF(u)}(IF(u) - F(Iu)dxds \\ &= \mathrm{Im} \int_0^t \int \overline{\Delta Iu}\Delta(IF(u) - F(Iu))dxds \\ &\quad - \mathrm{Im} \int_0^t \int \overline{IF(u)}(IF(u) - F(Iu))dxds. \end{aligned}$$

We next write

$$\begin{aligned} \Delta(IF(u) - F(Iu)) &= I(\Delta u F'(u) + |\nabla u|^2 F''(u)) - \Delta Iu F'(Iu) - |\nabla Iu|^2 F''(Iu) \\ &= \Delta Iu(F'(u) - F'(Iu)) + |\nabla Iu|^2(F''(u) - F''(Iu)) \\ &\quad + \nabla Iu \cdot (\nabla u - \nabla Iu)F''(u) + I(\Delta F'(u)) - (\Delta Iu)F'(u) \\ &\quad + I(\nabla u \cdot \nabla u F''(u)) - (\nabla Iu) \cdot \nabla u F''(u). \end{aligned}$$

Thus,

$$E(Iu(t)) - E(Iu_0) = \mathrm{Im} \int_0^t \int \overline{\Delta Iu}\Delta Iu(F'(u) - F'(Iu))dxds \tag{4.3}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}|\nabla Iu|^2(F''(u) - F''(Iu))dxds \tag{4.4}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}\nabla Iu \cdot (\nabla u - \nabla Iu)F''(u)dxds \tag{4.5}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}[I(\Delta uF'(u)) - (\Delta Iu)F'(u)]dxds \tag{4.6}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}[I(\nabla u \cdot \nabla uF''(u)) - (\nabla Iu) \cdot \nabla uF''(u)]dxds \tag{4.7}$$

$$-\mathrm{Im} \int_0^t \int \overline{IF(u)}(IF(u) - F(Iu))dxds. \tag{4.8}$$

Let $J = [0, \widetilde{T}_{\mathrm{lwp}}]$. By Hölder's inequality, we estimate

$$\begin{aligned}
|(4.3)| &\lesssim \|\Delta Iu\|^2_{L^4_t(J,L_x^{\frac{2d}{d-2}})} \|F'(u) - F'(Iu)\|_{L^2_t(J,L_x^{\frac{d}{2}})} \\
&\lesssim (Z_I(J))^2 \||u - Iu|(|u| + |Iu|)^{\frac{8}{d}-1}\|_{L^2_t(J,L_x^{\frac{d}{2}})} \\
&\lesssim (Z_I(J))^2 \|P_{>N}u\|_{L_t^{\frac{16}{d}}(J,L^4_x)} \|u\|^{\frac{8}{d}-1}_{L_t^{\frac{16}{d}}(J,L^4_x)}. \qquad (4.9)
\end{aligned}$$

By (2.7),

$$\|P_{>N}u\|_{L_t^{\frac{16}{d}}(J,L^4_x)} \lesssim N^{-2}\|\Delta Iu\|_{L_t^{\frac{16}{d}}(J,L^4_x)} \lesssim N^{-2}Z_I(J). \tag{4.10}$$

Here $\left(\frac{16}{d}, 4\right)$ is biharmonic admissible. Similarly, by (2.8),

$$\|u\|_{L_t^{\frac{16}{d}}(J,L^4_x)} \lesssim Z_I(J). \tag{4.11}$$

Collecting (4.9) – (4.11), we get

$$|(4.3)| \lesssim N^{-2}(Z_I(J))^{2+\frac{8}{d}}. \tag{4.12}$$

Next, we bound

$$\begin{aligned}
|(4.4)| &\lesssim \|\Delta Iu\|_{L^4_t(J,L_x^{\frac{2d}{d-2}})} \||\nabla Iu|^2\|_{L_t^{\frac{16}{11}}(J,L_x^{\frac{4d}{4d-11}})} \|F''(u) - F''(Iu)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \\
&\lesssim \|\Delta Iu\|_{L^4_t(J,L_x^{\frac{2d}{d-2}})} \|\nabla Iu\|^2_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \|F''(u) - F''(Iu)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \\
&\lesssim (Z_I(J))^3 \||u - Iu|^{\frac{8}{d}-1}\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \\
&\lesssim (Z_I(J))^3 \|P_{>N}u\|^{\frac{8}{d}-1}_{L_t^{\frac{16(8-d)}{d}}(J,L_x^{\frac{4(8-d)}{15-2d}})} \\
&\lesssim N^{-2\left(\frac{8}{d}-1\right)}(Z_I(J))^{2+\frac{8}{d}}. \qquad (4.13)
\end{aligned}$$

The third line follows by dropping the I-operator and applying (2.8) with the fact $\gamma > 1$. We also use the fact

$$|F''(z) - F''(\zeta)| \lesssim |z - \zeta|^{\frac{8}{d}-1}, \quad \forall z, \zeta \in \mathbb{C},$$

for $5 \leq d \leq 7$. The last estimate uses (4.10). Note that

$$\Big(\frac{32}{11}, \frac{8d}{4d-11}\Big) \text{ and } \Big(\frac{16(8-d)}{d}, \frac{4(8-d)}{15-2d}\Big)$$

are biharmonic admissible. Similarly, we estimate

$$\begin{aligned}|(4.5)| &\lesssim \|\Delta Iu\|_{L^4_t(J,L_x^{\frac{2d}{d-2}})} \|\nabla Iu\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \\ &\quad \|\nabla u - \nabla Iu\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \|F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \\ &\lesssim (Z_I(J))^2 \|\nabla P_{>N} u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \|F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})}.\end{aligned}$$

Using (2.7), we have

$$\|\nabla P_{>N} u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \lesssim N^{-1} \|\Delta Iu\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \lesssim N^{-1} Z_I(J).$$

As $F''(u) = O(|u|^{\frac{8}{d}-1})$, the estimate (2.8) gives

$$\|F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \lesssim \|u\|^{\frac{8}{d}-1}_{L_t^{\frac{16(8-d)}{d}}(J,L_x^{\frac{4(8-d)}{15-2d}})} \lesssim (Z_I(J))^{\frac{8}{d}-1}. \tag{4.14}$$

We thus obtain

$$|(4.5)| \lesssim N^{-1}(Z_I(J))^{2+\frac{8}{d}}. \tag{4.15}$$

By Hölder's inequality,

$$|(4.6)| \lesssim \|\Delta Iu\|_{L^2_t(J,L_x^{\frac{2d}{d-4}})} \|I(\Delta u F'(u)) - (\Delta Iu)F'(u)\|_{L^2_t(J,L_x^{\frac{2d}{d+4}})}. \tag{4.16}$$

We then apply Lemma 2.10 with $q = \frac{2d}{d+4}, q_1 = \frac{2d(d-3)}{d^2-7d+16}$ and $q_2 = \frac{d(d-3)}{2(2d-7)}$ to get

$$\|I(\Delta u F'(u)) - (\Delta Iu)F'(u)\|_{L_x^{\frac{2d}{d+4}}} \lesssim N^{-\alpha} \|\Delta Iu\|_{L_x^{\frac{2d(d-3)}{d^2-7d+16}}} \| \langle \nabla \rangle^\alpha F'(u)\|_{L_x^{\frac{d(d-3)}{2(2d-7)}}},$$

where $\alpha = 2 - \gamma + \delta$. The Hölder inequality then implies

$$\begin{aligned}\|I(\Delta u F'(u)) - (\Delta Iu)F'(u)\|_{L^2_t(J,L_x^{\frac{2d}{d+4}})} &\lesssim N^{-\alpha} \|\Delta Iu\|_{L_t^{\frac{2(d-3)}{d-4}}(J,L_x^{\frac{2d(d-3)}{d^2-7d+16}})} \\ &\quad \times \| \langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})}.\end{aligned} \tag{4.17}$$

We have

$$\begin{aligned}\| \langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})} &\lesssim \|F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})} \\ &\quad + \||\nabla|^\gamma F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})}.\end{aligned} \tag{4.18}$$

As $F'(u) = O(|u|^{\frac{8}{d}})$, the estimate (2.8) implies

$$\|F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})} \lesssim \|u\|^{\frac{8}{d}}_{L_t^{\frac{16(d-3)}{d}}(J,L_x^{\frac{4(d-3)}{2d-7}})} \lesssim (Z_I(J))^{\frac{8}{d}}. \tag{4.19}$$

Here $\left(\frac{16(d-3)}{d}, \frac{4(d-3)}{2d-7}\right)$ is biharmonic admissible. In order to treat the second term in (4.18), we apply Lemma 2.1 with $q = \frac{d(d-3)}{2(2d-7)}, q_1 = \frac{2d(d-3)}{-d^2+11d-26}$ and $q_2 = \frac{2d(d-3)}{d^2-3d-2}$ to get

$$\||\nabla|^\alpha F'(u)\|_{L_x^{\frac{d(d-3)}{2(2d-7)}}} \lesssim \|F''(u)\|_{L_x^{\frac{2d(d-3)}{-d^2+11d-26}}} \||\nabla|^\alpha u\|_{L_x^{\frac{2d(d-3)}{d^2-3d-2}}}. \tag{4.20}$$

Hölder's inequality then gives

$$\||\nabla|^\alpha F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})} \lesssim$$
$$\|F''(u)\|_{L_t^{4(d-3)}(J,L_x^{\frac{2d(d-3)}{-d^2+11d-26}})} \||\nabla|^\alpha u\|_{L_t^{4(d-3)}(J,L_x^{\frac{2d(d-3)}{d^2-3d-2}})}.$$

As $F''(u) = O(|u|^{\frac{8}{d}-1})$, we have

(4.21)
$$\|F''(u)\|_{L_t^{4(d-3)}(J,L_x^{\frac{2d(d-3)}{-d^2+11d-26}})} \lesssim \|u\|^{\frac{8}{d}-1}_{L_t^{\frac{4(8-d)(d-3)}{d}}(J,L_x^{\frac{2(8-d)(d-3)}{-d^2+11d-26}})} \lesssim (Z_I(J))^{\frac{8}{d}-1}.$$

Here $\left(\frac{4(8-d)(d-3)}{d}, \frac{2(8-d)(d-3)}{-d^2+11d-26}\right)$ is biharmonic admissible. Since $\left(4(d-3), \frac{2d(d-3)}{d^2-3d-2}\right)$ is also a biharmonic admissible, we have from (2.8) that

$$\||\nabla|^\alpha u\|_{L_t^{4(d-3)}(J,L_x^{\frac{2d(d-3)}{d^2-3d-2}})} \lesssim Z_I(J). \tag{4.22}$$

Note that $\alpha < 1 < \gamma$. Collecting $(4.18) - (4.22)$, we show

$$\| \langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)}(J,L_x^{\frac{d(d-3)}{2(2d-7)}})} \lesssim (Z_I(J))^{\frac{8}{d}}. \tag{4.23}$$

Combining (4.16), (4.17) and (4.23), we get

$$|(4.6)| \lesssim N^{-(2-\gamma+\delta)}(Z_I(J))^{2+\frac{8}{d}}. \tag{4.24}$$

Similarly, we bound

(4.25)
$$|(4.7)| \lesssim \|\Delta I u\|_{L_t^4(J,L_x^{\frac{2d}{d-2}})} \|I(\nabla u \cdot \nabla u F''(u)) - (I\nabla u)\cdot \nabla u F''(u)\|_{L_t^{\frac{4}{3}}(J,L_x^{\frac{2d}{d+2}})}.$$

Applying Lemma 2.10 with $q = \frac{2d}{d+2}, q_1 = \frac{8d}{4d-11}$ and $q_2 = \frac{8d}{19}$ and using Hölder inequality, we have

(4.26)
$$\|I(\nabla u \cdot \nabla u F''(u)) - (I\nabla u)\cdot \nabla u F''(u)\|_{L_t^{\frac{4}{3}}(J,L_x^{\frac{2d}{d+2}})} \lesssim N^{-\alpha}\|I\nabla u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})}$$
$$\times \| \langle \nabla \rangle^\alpha (\nabla u F''(u))\|_{L_t^{\frac{8}{5}}(J,L_x^{\frac{8d}{19}})}.$$

The fractional chain rule implies

(4.27)
$$\| \langle \nabla \rangle^\alpha (\nabla u F''(u))\|_{L_t^{\frac{8}{5}}(J,L_x^{\frac{8d}{19}})} \lesssim \| \langle \nabla \rangle^{\alpha+1} u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \|F''(u)\|_{L_t^{16}(J,L_x^{\frac{4d}{15-2d}})}$$
$$+ \|\nabla u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \| \langle \nabla \rangle^\alpha F''(u)\|_{L_t^{16}(J,L_x^{\frac{4d}{15-2d}})}.$$

By our assumptions on γ and δ, we see that $\alpha + 1 < \gamma$. Thus, using (2.8) (and dropping the I-operator if necessary) and (4.14), we have

(4.28)
$$\|I\nabla u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})}, \|\nabla u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})}, \| \langle \nabla \rangle^{\alpha+1} u\|_{L_t^{\frac{32}{11}}(J,L_x^{\frac{8d}{4d-11}})} \lesssim Z_I(J),$$

$$\|F''(u)\|_{L_t^{16}(J,L_x^{\frac{4d}{15-2d}})} \lesssim (Z_I(J))^{\frac{8}{d}-1}. \tag{4.29}$$

Here $\left(\frac{32}{11}, \frac{8d}{4d-11}\right)$ is biharmonic admissible. To bound $\| \langle \nabla \rangle^\alpha F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})}$, we use

(4.30)
$$\| \langle \nabla \rangle^\alpha F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \lesssim \|F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} + \||\nabla|^\alpha F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})}.$$

We next use Lemma 2.2 with $\beta = \frac{8}{d}-1$, $\alpha = 2-\gamma+\delta$, $q = \frac{4d}{15-2d}$ and q_1, q_2 satisfying

$$\Big(\frac{8}{d}-1-\frac{\alpha}{\rho}\Big)q_1 = \frac{\alpha}{\rho}q_2 = \frac{4(8-d)}{15-2d},$$

and $\frac{\alpha}{\frac{8}{d}-1} < \rho < 1$. Note that the choice of ρ is possible since $\alpha < \frac{8}{d}-1$ by our assumptions. With these choices, we have

$$\Big(1-\frac{\alpha}{\beta\rho}\Big)q_1 = \frac{4d}{15-2d} > 1,$$

for $5 \leq d \leq 7$. Then,

$$\||\nabla|^\alpha F''(u)\|_{L_x^{\frac{4d}{15-2d}}} \lesssim \||u|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}\|_{L_x^{q_1}} \||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L_x^{\frac{\alpha}{\rho}q_2}} \lesssim \|u\|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}_{L_x^{(\frac{8}{d}-1-\frac{\alpha}{\rho})q_1}} \||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L_x^{\frac{\alpha}{\rho}q_2}}.$$

By Hölder's inequality,

$$\begin{aligned}\||\nabla|^\alpha F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} &\lesssim \|u\|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}_{L_t^{(\frac{8}{d}-1-\frac{\alpha}{\rho})p_1}(J,L_x^{(\frac{8}{d}-1-\frac{\alpha}{\rho})q_1})} \||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L_t^{\frac{\alpha}{\rho}p_2}(J,L_x^{\frac{\alpha}{\rho}q_2})} \\ &= \|u\|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}_{L_t^{\frac{16(8-d)}{d}}(J,L_x^{\frac{4(8-d)}{15-2d}})} \||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L_t^{\frac{16(8-d)}{d}}(J,L_x^{\frac{4(8-d)}{15-2d}})},\end{aligned}$$

provided

$$\Big(\frac{8}{d}-1-\frac{\alpha}{\rho}\Big)p_1 = \frac{\alpha}{\rho}p_2 = \frac{16(8-d)}{d}.$$

Since $\left(\frac{16(8-d)}{d}, \frac{4(8-d)}{15-2d}\right)$ is biharmonic admissible, we have from (2.8) with the fact $0 < \rho < 1 < \gamma$ that

$$\||\nabla|^\alpha F''(u)\|_{L^{16}_t(J,L_x^{\frac{4d}{15-2d}})} \lesssim (Z_I(J))^{\frac{8}{d}-1}. \tag{4.31}$$

Collecting (4.25) – (4.31), we get

$$|(4.7)| \lesssim N^{-(2-\gamma+\delta)}(Z_I(J))^{2+\frac{8}{d}}. \tag{4.32}$$

Finally, we consider (4.8). We bound

$$\begin{aligned}|(4.8)| &\lesssim \||\nabla|^{-1}IF(u)\|_{L^2_t(J,L_x^{\frac{2d}{d-2}})} \|\nabla(IF(u)-F(Iu))\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})} \\ &\lesssim \|\nabla IF(u)\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})} \|\nabla(IF(u)-F(Iu))\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})}.\end{aligned} \tag{4.33}$$

By (3.7),

$$\|\nabla IF(u)\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})} \lesssim (Z_I(J))^{1+\frac{8}{d}}.$$

By the triangle inequality, we estimate

$$\begin{aligned}\|\nabla(IF(u)-F(Iu))\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})} &\lesssim \|(\nabla Iu)(F'(u)-F'(Iu))\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})} \\ &\quad + \|\nabla IF(u) - (\nabla Iu)F'(u)\|_{L^2_t(J,L_x^{\frac{2d}{d+2}})}.\end{aligned}$$

We firstly use Hölder's inequality and estimate as in (4.9) to get

$$\begin{aligned}\|(\nabla Iu)(F'(u)-F'(Iu))\|_{L^2_t(J,L^{\frac{2d}{d+2}}_x)} &\lesssim \|\nabla Iu\|_{L^\infty_t(J,L^{\frac{2d}{d-2}}_x)}\|F'(u)-F'(Iu)\|_{L^2_t(J,L^{\frac{d}{2}}_x)}\\ &\lesssim \|\Delta Iu\|_{L^\infty_t(J,L^2_x)}\|P_{>N}u\|_{L^{\frac{16}{d}}_t(J,L^4_x)}\|u\|^{\frac{8}{d}-1}_{L^{\frac{16}{d}}_t(J,L^4_x)}\\ &\lesssim N^{-2}(Z_I(J))^{1+\frac{8}{d}}. \end{aligned} \tag{4.34}$$

By (3.5),

$$\|\nabla IF(u)-(\nabla Iu)F'(u)\|_{L^2_t(J,L^{\frac{2d}{d+2}}_x)} \lesssim N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}}. \tag{4.35}$$

Combining (4.33) – (4.35), we get

$$\begin{aligned}|(4.8)| &\lesssim (Z_I(J))^{1+\frac{8}{d}}(N^{-2}(Z_I(J))^{1+\frac{8}{d}}+N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}})\\ &\lesssim N^{-(2-\gamma+\delta)}(Z_I(J))^{2+\frac{16}{d}}.\end{aligned} \tag{4.36}$$

Combining (4.12), (4.13), (4.15), (4.24), (4.32), (4.36) and using (4.2), we prove (4.1). The proof is complete. □

We next introduce some notations. We define

$$\Lambda(t) := \sup_{0\leq s\leq t}\|u(s)\|_{H^\gamma_x}, \quad \Sigma(t) := \sup_{0\leq s\leq t}\|I_N u(s)\|_{H^2_x}. \tag{4.37}$$

Proposition 4.2 (Increment of the modified energy). *Let $5\leq d\leq 7$ and $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)}<\gamma<2$. Let $u_0\in H^\gamma(\mathbb{R}^d)$ be such that the corresponding solution u to* (NL4S) *blows up at time $0<T^*<\infty$. Let $0<T<T^*$. Then for*

$$N(T)\sim \Lambda(T)^{\frac{a(\gamma)}{2(2-\gamma)}}, \tag{4.38}$$

we have

$$|E(I_{N(T)}u(T))|\lesssim \Lambda(T)^{a(\gamma)}.$$

Here the implicit constants depend only on γ, T^* and $\|u_0\|_{H^\gamma_x}$, and $0<a(\gamma)<2$ is given by

$$a(\gamma) := \frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{\left[\frac{8}{d}-1-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)\right]-}. \tag{4.39}$$

Proof. Let $\tau := c\Sigma(T)^{-\frac{4}{\gamma}}$ for some constant $c=c(d,\gamma)>0$ small enough. For $N(T)$ sufficiently large, Proposition 3.4 shows the local existence and the unique solution satisfies

$$Z_{I_{N(T)}}([t,t+\tau])\lesssim \|I_{N(T)}u(t)\|_{H^2_x}\lesssim \Sigma(T),$$

uniformly in t provided that $[t,t+\tau]\subset[0,T]$. We next split $[0,T]$ into $O(T/\tau)$ subintervals and apply Lemma 4.1 on each of these intervals to have

(4.40)

$$\sup_{t\in[0,T]}|E(I_{N(T)}u(t))| \lesssim |E(I_{N(T)}u_0)|+\frac{T}{\tau}N(T)^{-(2-\gamma+\delta)}\left(\Sigma(T)^{2+\frac{8}{d}}+\Sigma(T)^{2+\frac{16}{d}}\right)$$

$$\lesssim |E(I_{N(T)}u_0)|+N(T)^{-(2-\gamma+\delta)}\left(\Sigma(T)^{2+\frac{8}{d}+\frac{4}{\gamma}}+\Sigma(T)^{2+\frac{16}{d}+\frac{4}{\gamma}}\right), \tag{4.41}$$

for $\max\left\{3-\frac{8}{d},\frac{8}{d}\right\}<\gamma<2$ and $0<\delta<\gamma+\frac{8}{d}-3$. Next, by (2.9), we have

$$\Sigma(T) \lesssim N(T)^{2-\gamma}\Lambda(T). \tag{4.42}$$

Moreover, the Gagliardo-Nirenberg inequality (1.5) together with (2.10) imply

$$\begin{aligned}
|E(I_{N(T)}u_0)| &\lesssim \|\Delta I_{N(T)}u_0\|_{L^2_x}^2 + \|I_{N(T)}u_0\|_{L_x^{2+\frac{8}{d}}}^{2+\frac{8}{d}} \\
&\lesssim \|\Delta I_{N(T)}u_0\|_{L^2_x}^2 + \|I_{N(T)}u_0\|_{L^2_x}^{\frac{8}{d}}\|\Delta I_{N(T)}u_0\|_{L^2_x}^2 \\
&\lesssim N(T)^{2(2-\gamma)}\left(\|u_0\|_{H^\gamma_x}^2 + \|u_0\|_{H^\gamma_x}^{2+\frac{8}{d}}\right) \\
&\lesssim N^{2(2-\gamma)}.
\end{aligned} \tag{4.43}$$

Substituting (4.42) and (4.43) to (4.41), we get

$$\begin{aligned}
\sup_{t\in[0,T]}|E(I_{N(T)}u(t))| \lesssim{}& N(T)^{2(2-\gamma)} + N(T)^{-(2-\gamma+\delta)+(2-\gamma)\left(2+\frac{8}{d}+\frac{4}{\gamma}\right)}\Lambda(T)^{2+\frac{8}{d}+\frac{4}{\gamma}} \\
&+ N(T)^{-(2-\gamma+\delta)+(2-\gamma)\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)}\Lambda(T)^{2+\frac{16}{d}+\frac{4}{\gamma}}.
\end{aligned} \tag{4.44}$$

Optimizing (4.44), we observe that if we take

$$N(T)^{2(2-\gamma)} \sim N(T)^{-(2-\gamma+\delta)+(2-\gamma)\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)}\Lambda(T)^{2+\frac{16}{d}+\frac{4}{\gamma}},$$

or

$$N(T)\sim \Lambda(T)^{\frac{2+\frac{16}{d}+\frac{4}{\gamma}}{(2-\gamma+\delta)-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)}},$$

then

$$\sup_{t\in[0,T]}|E(I_{N(T)}u(t))| \lesssim N(T)^{2(2-\gamma)} \sim \Lambda(T)^{\frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{(2-\gamma+\delta)-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)}}.$$

Denote

$$a(\gamma) := \frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{(2-\gamma+\delta)-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)}.$$

Since $2-\gamma+\delta<\frac{8}{d}-1$, we see that

$$a(\gamma) = \frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{\left[\frac{8}{d}-1-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)\right]-}.$$

In order to make $0<a(\gamma)<2$, we need

$$\left\{\begin{array}{rcl}
\frac{8}{d}-1-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right) & > & 0, \\
\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma) & < & \frac{8}{d}-1-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right).
\end{array}\right. \tag{4.45}$$

Solving (4.45), we obtain

$$\gamma > \frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)}.$$

This completes the proof. $\square$

Proposition 4.3 (Almost conservation law). *Let $5 \leq d \leq 7, \max\{3-\frac{8}{d}, \frac{8}{d}\} < \gamma < 2$ and $0 < \delta < \gamma + \frac{8}{d} - 3$. Let $u_0 \in H^\gamma(\mathbb{R}^d)$ satisfying $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$, where Q is the solution to the ground state equation (1.4). Assume in addition that $E(Iu_0) \leq 1$. Let*

$$\widetilde{T}_{\mathrm{lwp}} := c\|Iu_0\|_{H^2_x}^{-\frac{4}{\gamma}},$$

for some small constant $c = c(d,\gamma) > 0$. Then, for N sufficiently large,

$$\sup_{t\in[0,\widetilde{T}_{\mathrm{lwp}}]} |E(Iu(t)) - E(Iu_0)| \lesssim N^{-(2-\gamma+\delta)}.$$

Here the implicit constant depends only on γ and $E(Iu_0)$.

Remark 4.4. Using the sharp Gagliardo-Nirenberg inequality together with the conservation of mass, the modified energy is always positive for initial data satisfying $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$. Indeed,

$$\begin{aligned}
E(Iu(t)) &= \frac{1}{2}\|\Delta Iu(t)\|^2_{L^2_x} - \frac{1}{2+\frac{8}{d}}\|Iu(t)\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}_x} \\
&\geq \frac{1}{2}\|\Delta Iu(t)\|^2_{L^2_x} - \frac{1}{2}\Big(\frac{\|Iu(t)\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}}\|\Delta Iu(t)\|^2_{L^2_x} \\
&\geq \frac{1}{2}\|\Delta Iu(t)\|^2_{L^2_x} - \frac{1}{2}\Big(\frac{\|u(t)\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}}\|\Delta Iu(t)\|^2_{L^2_x} \\
&\geq \frac{1}{2}\|\Delta Iu(t)\|^2_{L^2_x} - \frac{1}{2}\Big(\frac{\|u_0\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}}\|\Delta Iu(t)\|^2_{L^2_x} \\
&> 0.
\end{aligned}$$

Here we use the fact that $\|Iu\|_{L^2_x} \leq \|u\|_{L^2_x}$ which follows from the functional calculus and that $\|I(\xi)\|_{L^\infty_\xi} \leq 1$.

Proof of Proposition 4.3. By Lemma 4.1, we have for N large enough,

$$\sup_{t\in[0,\widetilde{T}_{\mathrm{lwp}}]} |E(Iu(t)) - E(Iu_0)| \lesssim N^{-(2-\gamma+\delta)}\Big(\|Iu_0\|^{2+\frac{8}{d}}_{H^2_x} + \|Iu_0\|^{2+\frac{16}{d}}_{H^2_x}\Big).$$

We only need to control $\|Iu_0\|_{H^2_x}$. To do so, we use the sharp Gagliardo-Nirenberg inequality (1.5) and (2.6) to have

$$\begin{aligned}
\|Iu_0\|^2_{H^2_x} \sim \|\Delta Iu_0\|^2_{L^2_x} + \|Iu_0\|^2_{L^2_x} &= 2E(Iu_0) + \frac{1}{1+\frac{4}{d}}\|Iu_0\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}_x} + \|Iu_0\|^2_{L^2_x} \\
&\leq 2E(Iu_0) + \Big(\frac{\|Iu_0\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}}\|\Delta Iu_0\|^2_{L^2_x} + \|Iu_0\|^2_{L^2_x} \\
&\leq 2E(Iu_0) + \Big(\frac{\|u_0\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}}\|Iu_0\|^2_{H^2_x} + \|u_0\|^2_{L^2_x}.
\end{aligned}$$

Thus

$$\Big(1 - \Big(\frac{\|u_0\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}}\Big)\|Iu_0\|^2_{H^2_x} \leq 2E(Iu_0) + \|u_0\|^2_{L^2_x}.$$

By our assumptions $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$ and $E(Iu_0) \leq 1$, we obtain $\|Iu_0\|_{H^2_x} \lesssim 1$. The proof is complete. $\square$

5. Limiting profile

In this section, we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4.

5.1. Proof of Theorem 1.2. As the solution blows up at time $0 < T^* < \infty$, the blowup alternative allows us to choose a sequence of times $(t_n)_{n\geq 1}$ such that $t_n \to T^*$ as $n \to \infty$ and $\|u(t_n)\|_{H^\gamma_x} = \Lambda(t_n) \to \infty$ as $n \to \infty$ (see (4.37) for the notation). Denote

$$\psi_n(x) := \lambda_n^{\frac{d}{2}} I_{N(t_n)} u(t_n, \lambda_n x),$$

where $N(t_n)$ is given as in (4.38) with $T = t_n$ and the parameter λ_n is given by

$$\lambda_n^2 := \frac{\|\Delta Q\|_{L^2_x}}{\|\Delta I_{N(t_n)} u(t_n)\|_{L^2_x}}. \tag{5.1}$$

By (2.9) and the blowup criterion given in Corollary 3.2, we see that

$$\lambda_n^2 \lesssim \frac{\|\Delta Q\|_{L^2_x}}{\|u(t_n)\|_{H^\gamma_x}} \lesssim (T^* - t_n)^{\frac{\gamma}{4}} \text{ or } \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}.$$

On the other hand, $(\psi_n)_{n\geq 1}$ is bounded in $H^2(\mathbb{R}^d)$. Indeed,

$$\|\psi_n\|_{L^2_x} = \|I_{N(t_n)} u(t_n)\|_{L^2_x} \leq \|u(t_n)\|_{L^2_x} = \|u_0\|_{L^2_x},$$

$$\|\Delta \psi_n\|_{L^2_x} = \lambda_n^2 \|\Delta I_{N(t_n)} u(t_n)\|_{L^2_x} = \|\Delta Q\|_{L^2_x}. \tag{5.2}$$

By Proposition 4.2 with $T = t_n$, we have

$$E(\psi_n) = \lambda_n^4 E(I_{N(t_n)} u(t_n)) \lesssim \lambda_n^4 \Lambda(t_n)^{a(\gamma)} \lesssim \Lambda(t_n)^{a(\gamma)-2}.$$

As $0 < a(\gamma) < 2$ for $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$, we see that $E(\psi_n) \to 0$ as $n \to \infty$. Therefore, the expression of the modified energy and (5.2) give

$$\|\psi_n\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}_x} \to \Big(1 + \frac{4}{d}\Big) \|\Delta Q\|^2_{L^2_x}, \tag{5.3}$$

as $n \to \infty$. Applying Theorem 1.1 to the sequence $(\psi_n)_{n\geq 1}$ with $M = \|\Delta Q\|_{L^2_x}$ and $m = \left(\left(1 + \frac{4}{d}\right) \|\Delta Q\|^2_{L^2_x}\right)^{\frac{d}{2d+8}}$, there exist a sequence $(x_n)_{n\geq 1} \subset \mathbb{R}^d$ and a function $U \in H^2(\mathbb{R}^d)$ such that $\|U\|_{L^2_x} \geq \|Q\|_{L^2_x}$ and up to a subsequence,

$$\psi_n(\cdot + x_n) \rightharpoonup U \text{ weakly in } H^2(\mathbb{R}^d),$$

as $n \to \infty$. That is

$$\lambda_n^{\frac{d}{2}} I_{N(t_n)} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^2(\mathbb{R}^d), \tag{5.4}$$

as $n \to \infty$. To conclude Theorem 1.2, we need to remove $I_{N(t_n)}$ from (5.4). To do so, we consider for any $0 \leq \sigma < \gamma$,

$$\begin{aligned} \|\lambda_n^{\frac{d}{2}} (u - I_{N(t_n)} u)(t_n, \lambda_n \cdot + x_n)\|_{\dot{H}^\sigma_x} &= \lambda_n^\sigma \|P_{\geq N(t_n)} u(t_n)\|_{\dot{H}^\sigma_x} \\ &\lesssim \lambda_n^\sigma N(t_n)^{\sigma-\gamma} \|P_{\geq N(t_n)} u(t_n)\|_{\dot{H}^\gamma_x} \\ &\lesssim \Lambda(t_n)^{-\frac{\sigma}{2}} \Lambda(t_n)^{\frac{(\sigma-\gamma)a(\gamma)}{2(2-\gamma)}} \|P_{\geq N(t_n)} u(t_n)\|_{H^\gamma_x} \\ &\lesssim \Lambda(t_n)^{1-\frac{\sigma}{2}+\frac{(\sigma-\gamma)a(\gamma)}{2(2-\gamma)}}. \end{aligned} \tag{5.5}$$

Using the explicit expression of $a(\gamma)$ given in (4.39), we find that for

$$\sigma < a(d, \gamma) := \frac{4d\gamma^2 + (2d + 48)\gamma + 16d}{16d + (56 - 3d)\gamma - 16\gamma^2},$$

the exponent of $\Lambda(t_n)$ in (5.5) is negative. Note that an easy computation shows that the condition $a(d,\gamma) < \gamma$ requires

$$\frac{24-3d+\sqrt{9d^2+368d+576}}{32} < \gamma < 2,$$

which is satisfied by our assumption on γ. Thus,

$$\|\lambda_n^{\frac{d}{2}}(u - I_{N(t_n)}u)(t_n, \lambda_n \cdot + x_n)\|_{H_x^{a(d,\gamma)-}} \to 0, \tag{5.6}$$

as $n \to \infty$. Combining (5.4) and (5.6), we prove

$$\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^{a(d,\gamma)-}(\mathbb{R}^d),$$

as $n \to \infty$. The proof is complete. $\square$

5.2. Proof of Theorem 1.3. By Theorem 1.2, there exists a blowup profile $U \in H^2(\mathbb{R}^d)$ with $\|U\|_{L^2_x} \geq \|Q\|_{L^2_x}$ and there exist sequences $(t_n, \lambda_n, x_n)_{n\geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^d$ such that $t_n \to T^*$,

$$\frac{\lambda_n}{(T^*-t_n)^{\frac{\gamma}{8}}} \lesssim 1, \tag{5.7}$$

for all $n \geq 1$ and $\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U$ weakly in $H^{a(d,\gamma)-}(\mathbb{R}^d)$ (hence in $L^2(\mathbb{R}^d)$) as $n \to \infty$. Thus for any $R > 0$, we have

$$\liminf_{n\to\infty} \lambda_n^d \int_{|x|\leq R} |u(t_n, \lambda_n x + x_n)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

By change of variables, we get

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq R\lambda_n} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

Using the assumption $\frac{(T^*-t_n)^{\frac{\gamma}{8}}}{\alpha(t_n)} \to 0$ as $n \to \infty$, we have from (5.7) that $\frac{\lambda_n}{\alpha(t_n)} \to 0$ as $n \to \infty$. We thus obtain for any $R > 0$,

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

Let $R \to \infty$, we obtain

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \|U\|^2_{L^2_x}.$$

This implies

$$\limsup_{t\nearrow T^*} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t)} |u(t, x)|^2 dx \geq \|Q\|^2_{L^2_x}.$$

Sine for any fixed time t, the map $y \mapsto \int_{|x-y|\leq\alpha(t)} |u(t,x)|^2 dx$ is continuous and goes to zero as $|y| \to \infty$, there exists $x(t) \in \mathbb{R}^d$ such that

$$\sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t)} |u(t, x)|^2 dx = \int_{|x-x(t)|\leq \alpha(t)} |u(t, x)|^2 dx.$$

This shows

$$\limsup_{t\nearrow T^*} \int_{|x-x(t)|\leq \alpha(t)} |u(t, x)|^2 dx \geq \|Q\|^2_{L^2_x}.$$

The proof is complete. $\square$

5.3. Proof of Theorem 1.4**.** We firstly recall the following variational characterization of the solution to the ground state equation (1.4). Note that the uniqueness up to translations in space, phase and dilations of solution to this ground state equation is assumed here.

LEMMA 5.1 (Variation characterization of the ground state [**34**]). *If* $v \in H^2(\mathbb{R}^d)$ *is such that* $\|v\|_{L^2_x} = \|Q\|_{L^2_x}$ *and* $E(u) = 0$*, then* v *is of the form*

$$v(x) = e^{i\theta}\lambda^{\frac{d}{2}}Q(\lambda x + x_0),$$

for some $\theta \in \mathbb{R}, \lambda > 0$ *and* $x_0 \in \mathbb{R}^d$*, where* Q *is the unique solution to the ground state equation* (1.4)*.*

Using the notation in the proof of Theorem 1.2 and the assumption $\|u_0\|_{L^2_x} = \|Q\|_{L^2_x}$, we have

$$\|\psi_n\|_{L^2_x} \leq \|u_0\|_{L^2_x} = \|Q\|_{L^2_x} \leq \|U\|_{L^2_x}.$$

Sine $\psi_n(\cdot + x_n) \rightharpoonup U$ weakly in $L^2(\mathbb{R}^d)$, the semi-continuity of weak convergence implies

$$\|U\|_{L^2_x} \leq \liminf_{n\to\infty} \|\psi_n\|_{L^2_x} \leq \|Q\|_{L^2_x}.$$

Thus,

$$\|U\|_{L^2_x} = \|Q\|_{L^2_x} = \lim_{n\to\infty} \|\psi_n\|_{L^2_x}. \tag{5.8}$$

Hence up to a subsequence

$$\psi_n(\cdot + x_n) \to U \text{ strongly in } L^2(\mathbb{R}^d), \tag{5.9}$$

as $n \to \infty$. On the other hand, using (5.2), the Gagliardo-Nirenberg inequality (1.5) implies $\psi_n(\cdot + x_n) \to U$ strongly in $L^{2+\frac{8}{d}}(\mathbb{R}^d)$. Indeed, by (5.2),

$$\begin{aligned}\|\psi_n(\cdot + x_n) - U\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}_x} &\lesssim \|\psi(\cdot + x_n) - U\|^{\frac{8}{d}}_{L^2_x} \|\Delta(\psi_n(\cdot + x_n) - U\|^2_{L^2_x} \\ &\lesssim (\|\Delta Q\|_{L^2_x} + \|\Delta U\|_{L^2_x})^2 \|\psi(\cdot + x_n) - U\|^{\frac{8}{d}}_{L^2_x} \to 0,\end{aligned}$$

as $n \to \infty$. Moreover, using (5.3) and (5.8), the sharp Gagliardo-Nirenberg inequality (1.5) also gives

$$\|\Delta Q\|^2_{L^2_x} = \frac{1}{1+\frac{4}{d}} \|U\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}_x} \leq \Big(\frac{\|U\|_{L^2_x}}{\|Q\|_{L^2_x}}\Big)^{\frac{8}{d}} \|\Delta U\|^2_{L^2_x} = \|\Delta U\|^2_{L^2_x},$$

or $\|\Delta Q\|_{L^2_x} \leq \|\Delta U\|_{L^2_x}$. By the semi-continuity of weak convergence and (5.2),

$$\|\Delta U\|_{L^2_x} \leq \liminf_{n\to\infty} \|\Delta \psi_n\|_{L^2_x} = \|\Delta Q\|_{L^2_x}.$$

Therefore,

$$\|\Delta U\|_{L^2_x} = \|\Delta Q\|_{L^2_x} = \lim_{n\to\infty} \|\Delta \psi_n\|_{L^2_x}. \tag{5.10}$$

Combining (5.8), (5.10) and using the fact $\psi_n(\cdot + x_n) \rightharpoonup U$ weakly in $H^2(\mathbb{R}^d)$, we conclude that $\psi_n(\cdot + x_n) \to U$ strongly in $H^2(\mathbb{R}^d)$. In particular,

$$E(U) = \lim_{n\to\infty} E(\psi_n) = 0,$$

as $n \to \infty$. This shows that there exists $U \in H^2(\mathbb{R}^d)$ satisfying

$$\|U\|_{L^2_x} = \|Q\|_{L^2_x}, \quad \|\Delta U\|_{L^2_x} = \|\Delta Q\|_{L^2_x}, \quad E(U) = 0.$$

Applying the variational characterization given in Lemma 5.1, we have (taking $\lambda = 1$),

$$U(x) = e^{i\theta} Q(x + x_0),$$

for some $(\theta, x_0) \in \mathbb{R} \times \mathbb{R}^d$. Hence

$$\lambda_n^{\frac{d}{2}} I_{N(t_n)} u(t_n, \lambda_n \cdot + x_n) \to e^{i\theta} Q(\cdot + x_0) \text{ strongly in } H^2(\mathbb{R}^d),$$

as $n \to \infty$. Using (5.6), we prove

$$\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \to e^{i\theta} Q(\cdot + x_0) \text{ strongly in } H^{a(d,\gamma)-}(\mathbb{R}^d),$$

as $n \to \infty$. The proof is complete. □

6. Global well-posedness

In this section, we will give the proof of Theorem 1.5. By density argument, we assume that $u_0 \in C_0^\infty(\mathbb{R}^d)$. Let u be a global solution to (NL4S) with initial data u_0 satisfying $\|u_0\|_{H^\gamma_x} < \|Q\|_{L^2_x}$. In order to apply the almost conservation law given in Proposition 4.3, we need the absolute value of modified energy of initial data is small. Since $E(Iu_0)$ is not necessarily small, we will use the scaling (1.2) to make $E(Iu_\lambda(0))$ is small. We have

$$E(Iu_\lambda(0)) \leq \frac{1}{2}\|\Delta I u_\lambda(0)\|^2_{L^2_x} \lesssim N^{2(2-\gamma)}\|\Delta u_\lambda(0)\|^2_{L^2_x} = N^{2(2-\gamma)}\lambda^{-2\gamma}\|u_0\|^2_{\dot{H}^2_x}.$$

Thus, we can make $E(Iu_\lambda(0)) \leq \frac{1}{4}$ by taking

$$N \sim \lambda^{\frac{2-\gamma}{\gamma}}. \tag{6.1}$$

Moreover, since the scaling (1.2) preserves the L^2-norm, we have $\|u_\lambda(0)\|_{L^2_x} = \|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$. Thus, the assumptions of Proposition 4.3 are satisfied. Therefore, there exists $\tau > 0$ so that for N sufficiently large,

$$E(Iu_\lambda(t)) \leq E(Iu_\lambda(0)) + CN^{-(2-\gamma+\delta)},$$

for $t \in [0, \tau]$ where $\max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2$ and $0 < \delta < \gamma + \frac{8}{d} - 3$. We may reapply this proposition continuously so that $E(Iu_\lambda(t))$ reaches 1, that is at least $C_1 N^{2-\gamma+\delta}$ times. Therefore,

$$E(Iu_\lambda(C_1 \tau N^{2-\gamma+\delta})) \sim 1. \tag{6.2}$$

Now, given any $T \gg 1$, we choose $N \gg 1$ so that

$$T \sim C_1 \tau \frac{N^{2-\gamma+\delta}}{\lambda^4}.$$

Using (6.1), we have

$$T \sim N^{2-\gamma+\delta-\frac{4(2-\gamma)}{\gamma}}. \tag{6.3}$$

As $0 < \delta < \gamma + \frac{8}{d} - 3$ or $2 - \gamma + \delta < \frac{8}{d} - 1$, the exponent of N is positive provided that

$$\frac{8}{d} - 1 - \frac{4(2-\gamma)}{\gamma} > 0 \text{ or } \gamma > \frac{8d}{3d+8}.$$

Thus the choice of N makes sense for arbitrary $T \gg 1$. A direct computation and (6.1), (6.2) and (6.3) show

$$E(Iu(T)) = \lambda^4 E(Iu_\lambda(\lambda^4 T)) = \lambda^4 E(Iu_\lambda(C_1\tau N^{2-\gamma+\delta}) \sim \lambda^4$$
$$\leq N^{\frac{4(2-\gamma)}{\gamma}} \sim T^{\frac{4(2-\gamma)}{(2-\gamma+\delta)\gamma - 4(2-\gamma)}}.$$

This shows that there exists $C_2 = C_2(\tau, \|u_0\|_{H^\gamma_x})$ such that

$$E(Iu(T)) \leq C_2 T^{\frac{4(2-\gamma)}{(2-\gamma+\delta)\gamma-4(2-\gamma)}},$$

for any $T \gg 1$. Finally, by (2.9),

$$\begin{aligned}\|u(T)\|^2_{H^\gamma_x} &\lesssim \|Iu(T)\|^2_{H^2_x} \sim \|\Delta Iu(T)\|^2_{L^2_x} + \|Iu(T)\|^2_{L^2_x} \lesssim E(Iu(T)) + \|u_0\|^2_{L^2_x} \\ &\lesssim C_3 T^{\frac{4(2-\gamma)}{(2-\gamma+\delta)\gamma-4(2-\gamma)}} + C_4,\end{aligned}$$

where C_3, C_4 depends only on $\|u_0\|_{H^\gamma_x}$. The proof is complete. □

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