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Backwards compact attractors for non-autonomous damped 3D Navier-Stokes equations

Jinyan Yin, Anhui Gu, and Yangrong Li

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Abstract. Both existence and backwards topological property of pullback attractors are discussed for 3D Navier-Stokes equations with a nonlinear damping and a non-autonomous force. A pullback attractor is obtained in a square integrable space if the order of damping is larger than three and further in a Sobolev space if the order belongs to $(3, 5)$, the latter of which improves the best range $[7/2, 5)$ given in literatures so far. The new hypotheses on the force used here are weaker than those given in literatures. More importantly, the obtained attractor is shown to be backwards compact, i.e. the union of attractors over the past time is pre-compact. This result is a successful application of some new abstract criteria on backwards compact attractors if an evolution process is backwards pullback limit-set compact or equivalently backwards pullback flattening.

Contents

1. Introduction 201
2. Theoretical results on backwards compact attractors 203
3. Preliminaries for the damped 3D NSE 208
4. Backwards compact attractors in H 211
5. Backwards compact attractors in V 213
Acknowledgments 217
References 217

1. Introduction

In this paper, we study the existence and backwards compactness of pullback attractors for the non-autonomous damped 3D Navier-Stokes equation (NSE) on a

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smooth bounded domain $\mathcal{O} \subset \mathbb{R}^3$:

$$(1.1)\qquad \begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = g(x,t), & x \in \mathcal{O},\ t \geq \tau, \\ \operatorname{div} u = 0, \quad u(x,\tau) = u_0, \quad u|_{\partial\mathcal{O}} = 0, & x \in \mathcal{O}, \end{cases}$$

where $\tau \in \mathbb{R}$, $\mu > 0$ is the kinematic viscosity, $\alpha > 0$ and $\beta \geq 1$ are constants in the nonlinear damping, u and p denote the velocity field and pressure field, g is a body force which will be special later.

The 3D NSE, arising from the flow of fluids, has been widely investigated in the non-damping case (i.e. $\alpha = 0$), see [**8, 11, 12, 13, 14, 18, 23**] for both deterministic and stochastic equations. In this case, the uniqueness of weak solutions and the global existence of strong solutions are still open questions, see [**9, 25, 28**].

Actually, the damping arises from the resistance to the motion of the flow and describes various physical phenomena, such as porous media flow, drag or friction effects, and some dissipative mechanisms, see [**1, 16**]. The appearance of the damping makes sure the global well-posedness of weak solutions and strong solutions for the 3D NSE.

Since the force is time-dependent, the dynamics is non-autonomous, which is described by a concept of pullback attractor. For the theoretical results of pullback attractors, we refer to [**4, 5, 6, 7, 19**]. For some applications, especially on non-Newtonian fluids related to Eq. (1.1), we refer to [**2, 15, 31**].

This paper will discuss a relatively new subject about *backwards compactness* of a pullback attractor $\mathcal{A}(\cdot)$, which means $\cup_{s\leq t}\mathcal{A}(s)$ is pre-compact for all $t \in \mathbb{R}$.

Specifically, for the problem (1.1), this article deals with the following 3 aspects:

(1) The existence of a backwards compact attractor in H for all $\beta > 3$.

To discuss the existence of a pullback attractor in H, where H is a subspace of the square integrable space, we do not need to restrict the upper bound of β. This existence result can be regarded as a non-autonomous generalization of the corresponding result given in [**17**]. In fact, the authors in [**17**] shows that Eq. (1.1) (which is also termed as the convective Brinkman-Forchheimer (CBF) equation) has a global attractor in H for more general nonlinearity without the restriction of the upper bound.

To obtain the backwards compactness, we need only to assume that the force is *backwards tempered*. In this case, we can prove there is a backwards-uniform absorbing set which is bounded in the Sobolev space V. This fact simply proves the backwards compactness in H.

(2) The existence of a backwards compact attractor in V with $\beta \in (3, 5)$.

Since obtaining the existence of the attractor in V is more difficult, we will restrict the upper bound of the order by $\beta < 5$, which is similar to the range $\beta \in [7/2, 5]$ given in [**26, 27**]. However, our method can reduce the lower bound of the growth rate.

In fact, the existence of an attractor in V relates to the global well-posedness of strong solutions, which was proved by [**30**] for $\beta \in (3, 5]$ (improved the range in [**3**]). Only in a special case of $\mu = 1$ and $\alpha = 1$, the paper [**34**] treats the strong solution for $\beta \geq 5$. On the other hand, even for the simpler BF equation, the paper [**24**] restricted the order to obtain the attractor in V. This similar critical exponent case was also assumed in [**29**] for the stochastic wave equation.

To obtain the backwards compactness in V, we further assume that g is *backwards limiting* in the sense of (3.6). This assumption is weaker than global boundedness used in [**27**] and can be deduced by the *backwards absolutely continuity* (see (3.7)) together with the backward temperedness.

(3) Abstract results on backwards compactness of pullback attractors.

To prove the backwards compactness of the pullback attractor in V, we need to establish some theoretical criteria. We will prove that a non-autonomous system has a backwards compact attractor if it has an *increasing*, bounded and pullback absorbing set and it is *backwards pullback flattening* (see Theorem 2.7). By introducing several other backwards notions, we also establish some equivalent criteria for such backwards compactness (see Theorems 2.10 and 2.11), which can be conveniently applied to different non-autonomous models.

2. Theoretical results on backwards compact attractors

We review the theory of pullback attractors for an *evolution process*, and prove several equivalent results for the backwards compactness of pullback attractors.

Suppose $(X, \|\cdot\|_X)$ is a Banach space. A non-autonomous set in X is defined by a time-dependent family $\mathcal{D} = \{\mathcal{D}(t) \subset X : \ t \in \mathbb{R}\}$, and it is said to have a topological property (such as closedness, boundedness or compactness) if $\mathcal{D}(t)$ has this property for each $t \in \mathbb{R}$. We also say $\mathcal{D}$ is *increasing* if $\mathcal{D}(s) \subset \mathcal{D}(t)$ for $s \leq t$.

Definition 2.1. A non-autonomous set $\mathcal{D}$ is called backwards compact (resp. backwards bounded) if $\cup_{s\leq t}\mathcal{D}(s)$ is pre-compact (resp. bounded) in X for each $t \in \mathbb{R}$.

Definition 2.2. An evolution process S in X is a family of mappings $S(t,\tau) : X \to X$ with $t \geq \tau$, which satisfies

$$S(\tau,\tau) = id_X, \quad S(t,\tau) = S(t,s)S(s,\tau), \text{ for all } t \geq s \geq \tau \text{ with } t \in \mathbb{R},$$
$$\text{and } (t,\tau,x) \to S(t,\tau)x \text{ is continuous for } t \geq \tau \text{ and } x \in X.$$

Definition 2.3. A non-autonomous set $\mathcal{D}$ in X is called pullback attracting at time t with $t \in \mathbb{R}$ if, for each bounded subset B in X,

$$\lim_{\tau\to\infty} dist(S(t,t-\tau)B, \mathcal{D}(t)) = 0, \tag{2.1}$$

where the Hausdorff semi-distance is defined by $dist(A,B) := \sup_{a\in A}\inf_{b\in B}\|a - b\|_X$ for $A, B \subset X$.

Definition 2.4. A non-autonomous set $\mathcal{A}$ in X is a pullback attractor for an evolution process S if

(i) $\mathcal{A}(t)$ is compact for each $t \in \mathbb{R}$;
(ii) $\mathcal{A}$ is invariant, i.e. $S(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$;
(iii) $\mathcal{A}$ is pullback attracting in the sense of (2.1);
(iv) $\mathcal{A}$ is the minimal closed pullback attracting set.

Definition 2.5. A non-autonomous set $\mathcal{K}$ in X is called a pullback absorbing set at time $t \in \mathbb{R}$ for an evolution process S if, for each bounded subset B in X, there is $\tau_0 := \tau_0(t,B) > 0$ such that

$$S(t,t-\tau)B \subset \mathcal{K}(t), \text{ for all } \tau \geq \tau_0.$$

In order to study the backwards compactness of a pullback attractor, we generalize the concept of pullback flattening given in [**6, 19**].

DEFINITION 2.6. An evolution process S in X is called *backwards pullback flattening* if, given a bounded set $B \subset X$, $t \in \mathbb{R}$ and $\varepsilon > 0$, there are $\tau_0 := \tau_0(\varepsilon, t, B) > 0$, a finite-dimensional subspace X_ε of X and a mapping $P_\varepsilon : X \to X_\varepsilon$ such that

$$\cup_{\tau\geq\tau_0} \cup_{s\leq t} P_\varepsilon S(s, s-\tau)B \text{ is bounded, and}$$
$$\|(I - P_\varepsilon)(\cup_{\tau\geq\tau_0} \cup_{s\leq t} S(s, s-\tau)B)\|_X < \varepsilon.$$

THEOREM 2.7. *Let S be an evolution process in a Banach space X, then S has a backwards compact attractor $\mathcal{A}$ if*

(i) S has an increasing and bounded pullback absorbing set $\mathcal{K}$,

(ii) S is backwards pullback flattening.

PROOF. By (i), $\mathcal{K}$ is obviously a pullback attracting set, then for each $t \in \mathbb{R}$, $s \leq t$ and bounded subset B in X, by the increasing property, we have

$$\lim_{\tau\to\infty} dist(S(s, s-\tau)B, \mathcal{K}(t)) \leq \lim_{\tau\to\infty} dist(S(s, s-\tau)B, \mathcal{K}(s)) = 0.$$

This indicates S is strongly pullback bounded dissipative in the sense of [**6**, Definition 2.22]. By (ii), S is evidently pullback flattening. Hence, by [**6**, Theorem 2.27], S has a pullback attractor $\mathcal{A}$.

Since $\mathcal{A}$ is the minimal closed pullback attracting set, then

$$\mathcal{A}(s) \subset \overline{\mathcal{K}}(s) \subset \overline{\mathcal{K}}(t), \text{ for each } s \leq t \text{ with } t \in \mathbb{R},$$

where we use the fact that $\mathcal{K}$ is increasing. Then, $\cup_{s\leq t}\mathcal{A}(s) \subset \overline{\mathcal{K}}(t)$ and thus $\mathcal{A}$ is backwards bounded.

To obtain the backwards compactness of $\mathcal{A}$, we need to prove $\cup_{s\leq t}\mathcal{A}(s)$ is precompact for each fixed $t \in \mathbb{R}$. This is equivalent to prove $\kappa(\cup_{s\leq t}\mathcal{A}(s)) = 0$, where $\kappa(\cdot)$ denotes the Kuratowski measure of non-compactness, i.e. for $A \subset X$,

$$\kappa(A) := \inf\{d > 0 \mid A \text{ has a finite cover } \{A_i\} \text{ with } \mathrm{diam}_X(A_i) \leq d\}.$$

Since $\mathcal{A}$ is backwards bounded, there is a bounded subset B such that

$$\cup_{s\leq t}\mathcal{A}(s) \subset B.$$

It follows from the invariance of $\mathcal{A}$ that

$$\mathcal{A}(s) = S(s, s-\tau)\mathcal{A}(s-\tau) \subset S(s, s-\tau)B, \text{ for all } s \leq t \text{ and } \tau \geq 0.$$

Then, by the backwards pullback flattening property of S, for given $\varepsilon > 0$, there is a $\tau_0 = \tau_0(t, B, \varepsilon)$ such that

$$\begin{aligned}
&\kappa(\bigcup_{s\leq t}\mathcal{A}(s)) \leq \kappa(\bigcup_{\tau\geq\tau_0}\bigcup_{s\leq t} S(s, s-\tau)B) \\
&\leq \kappa(\bigcup_{\tau\geq\tau_0}\bigcup_{s\leq t} P_\varepsilon S(s, s-\tau)B) + \kappa((I - P_\varepsilon)(\bigcup_{\tau\geq\tau_0}\bigcup_{s\leq t} S(s, s-\tau)B)) \\
&\leq 0 + \kappa(B_X(0, \varepsilon)) \leq 2\varepsilon,
\end{aligned}$$

which implies $\mathcal{A}$ is backwards compact from [**20**, Lemma 2.7]. □

We also give two equivalent concepts, which relates to the past time.

DEFINITION 2.8. An evolution process S in X is called *backwards pullback limit-set compact* if, for each $t \in \mathbb{R}$ and bounded set $B \subset X$,

$$\lim_{\tau_0 \to \infty} \kappa(\cup_{\tau \geq \tau_0} \cup_{s \leq t} S(s, s-\tau)B) = 0,$$

where $\kappa(\cdot)$ is the Kuratowski measure mentioned above.

DEFINITION 2.9. An evolution process S in X is called *backwards pullback asymptotically compact* if, for each $t \in \mathbb{R}$,

$$\{S(s_n, s_n - \tau_n)x_n\}_{n \in \mathbb{N}} \text{ has a convergent subsequence in } X,$$

whenever $s_n \leq t$, $\tau_n \to \infty$ and $\{x_n\}$ is bounded in X.

In the above definition, we emphasize that $\{s_n\}$ is an arbitrary sequence from $(-\infty, t]$ and it may not be convergent or monotonous.

In an earlier work given in [**33**] (or see [**21, 32**] in the random case and [**22**] in both non-autonomous and random case), the authors proved that the limit-set compactness is equivalent to the asymptotic compactness with an additional assumption of bounded absorption, while Kloeden et.al. [**19**] proved the equivalence between pullback limit-set compactness and pullback asymptotic compactness without any extra assumptions. We generalize the latter assertion to the backwards case. In the proof, we use a new terminology of *backwards omega-limit sets*.

THEOREM 2.10. *Let S be an evolution process in a Banach space X, then S is backwards pullback limit-set compact if and only if it is backwards pullback asymptotically compact.*

PROOF. **Necessity.** Suppose S is backwards pullback limit-set compact. Let $t \in \mathbb{R}$, $s_n \leq t$, $\tau_n \to \infty$ and $x_n \in B$ with B bounded in X, then as $m \to \infty$,

$$\kappa\{S(s_n, s_n - \tau_n)x_n, n \geq m\} \leq \kappa(\cup_{n \geq m} \cup_{s \leq t} S(s, s-\tau_n)B) \to 0.$$

It follows from [**20**, Lemma 2.7] that $\{S(s_n, s_n - \tau_n)x_n\}$ has a convergent subsequence in X.

Sufficiency. Suppose S is backwards pullback asymptotically compact. For each bounded subset B in X and $t \in \mathbb{R}$, we define a *backwards omega-limit set* by

$$\Omega(B,t) := \cap_{\tau_0 > 0} \overline{\cup_{\tau \geq \tau_0} \cup_{s \leq t} S(s, s-\tau)B}. \tag{2.2}$$

First, we prove $x \in \Omega(B,t) \Leftrightarrow$ there are $s_n \leq t$, $\tau_n \to \infty$ and $x_n \in B$ such that $x = \lim_{n \to \infty} S(s_n, s_n - \tau_n)x_n$. Indeed, if $x \in \Omega(B,t)$, we obtain from (2.2) that

$$x \in \overline{\cup_{\tau \geq n} \cup_{s \leq t} S(s, s-\tau)B}, \text{ for each } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, choose $s_n \leq t$, $\tau_n \geq n$ and $x_n \in B$ such that $\|S(s_n, s_n - \tau_n)x_n - x\|_X \leq \frac{1}{n}$. Taking $n \to \infty$, we have $S(s_n, s_n - \tau_n)x_n \to x$. Conversely, let $s_n \leq t$, $\tau_n \to \infty$, $x_n \in B$ and $x = \lim_{n \to \infty} S(s_n, s_n - \tau_n)x_n$. For every $\tau_0 > 0$, choosing $m > 0$ such that $\tau_m \geq \tau_0$, we obtain

$$x \in \overline{\{S(s_n, s_n - \tau_n)x_n : n \geq m\}} \subset \overline{\cup_{\tau \geq \tau_0} \cup_{s \leq t} S(s, s-\tau)B},$$

which implies $x \in \Omega(B,t)$.

Second, we show that $\Omega(B,t)$ is non-empty and compact. We claim that there is $\tau_0 > 0$ such that

$$\overline{\cup_{\tau \geq \tau_0} \cup_{s \leq t} S(s, s-\tau)B} \text{ is bounded.}$$

If this is not true, for any $\tau_0 > 0$, there are $s_n \le t$, $\tau_n \to \infty$ and $x_n \in B$ such that any subsequence of $\{S(s_n, s_n - \tau_n)x_n\}$ is unbounded, which contradicts the backwards pullback asymptotic compactness of the process. Since S is backwards pullback asymptotically compact, then there are $s_n \le t$, $\tau_n \to \infty$ and $x_n \in B$ such that, up to a subsequence,

$$\lim_{n\to\infty} S(s_n, s_n - \tau_n)x_n = x, \tag{2.3}$$

which implies $x \in \Omega(B,t)$ and thus $\Omega(B,t)$ is non-empty.

To prove the compactness of $\Omega(B,t)$, let $y_n \in \Omega(B,t)$, then there are $s_n \le t$, $\tau_n \to \infty$ and $x_n \in B$ such that

$$d(S(s_n, s_n - \tau_n)x_n, y_n) \to 0, \text{ as } n \to \infty.$$

By (2.3), we have

$$d(y_n, x) \le d(S(s_n, s_n - \tau_n)x_n, y_n) + d(S(s_n, s_n - \tau_n)x_n, x) \to 0, \text{ as } n \to \infty,$$

which implies $\Omega(B,t)$ is compact.

Next, we show that $\Omega(B,t)$ backwards pullback attracts B in the following sense:

$$\lim_{\tau\to\infty} \sup_{s\le t} \operatorname{dist}(S(s, s-\tau)B, \Omega(B,t)) = 0, \ \forall t \in \mathbb{R}. \tag{2.4}$$

Note that (2.4) is equivalent to $\lim_{\tau\to\infty} \operatorname{dist}(\cup_{s\le t} S(s, s-\tau)B, \Omega(B,t)) = 0$. If (2.4) is not true, then there are $\delta > 0$, $s_n \le t$, $\tau_n \to \infty$ and $x_n \in B$ such that

$$\operatorname{dist}(S(s_n, s_n - \tau_n)x_n, \Omega(B,t)) > \delta, \text{ for all } n \in \mathbb{N},$$

which is a contradiction, since the backwards asymptotic compactness implies that, passing to a subsequence, $S(s_n, s_n - \tau_n)x_n \to x \in \Omega(B,t)$.

Finally, we prove that the evolution process is backwards pullback limit-set compact. By (2.4), $\Omega(B,t)$ backwards pullback attracts B, then, for each $\varepsilon > 0$, there is $\tau_\varepsilon > 0$ such that

$$\operatorname{dist}(\cup_{s\le t} S(s, s-\tau)B, \Omega(B,t)) < \frac{\varepsilon}{4}, \text{ for all } \tau \ge \tau_\varepsilon,$$

which implies that

$$\cup_{\tau\ge\tau_\varepsilon} \cup_{s\le t} S(s, s-\tau)B \subset B_X(\Omega(B,t), \frac{\varepsilon}{4}). \tag{2.5}$$

Taking the Kuratowski measure on the both sides of (2.5), we find

$$\kappa(\cup_{\tau\ge\tau_\varepsilon} \cup_{s\le t} S(s, s-\tau)B) \le \kappa(B_X(\Omega(B,t), \frac{\varepsilon}{4})) < \frac{\varepsilon}{2},$$

where we use the compactness of $\Omega(B,t)$. Then, S is backwards limit-set compact. □

We also generalize the equivalence of pullback flattening and pullback asymptotic compact (see [**6**]) to the backwards case as follows.

THEOREM 2.11. *Let S be an evolution process in a uniformly convex Banach space X, then S is backwards pullback flattening if and only if it is backwards pullback asymptotically compact.*

PROOF. **Necessity.** Suppose that the process is backwards pullback flattening. Let $t \in \mathbb{R}$, $s_n \le t$, $\tau_n \to \infty$ and $x_n \in B$ with B bounded in X. For each $m \in \mathbb{N}$, there are $\tau_m > 0$, a finite-dimensional subspace X_m of X and a mapping $P_m : X \to X_m$ such that

$$\bigcup_{\tau \ge \tau_m} \bigcup_{s \le t} P_m S(s, s-\tau)B \text{ is bounded,}$$

$$\|(I - P_m)(\bigcup_{\tau \ge \tau_m} \bigcup_{s \le t} S(s, s-\tau)B)\|_X < \frac{1}{m}.$$

This shows that for all $n \ge n_m$ with some $n_m > 0$,

$$P_m S(s_n, s_n - \tau_n)x_n \text{ is bounded,}$$

$$\|(I - P_m)S(s_n, s_n - \tau_n)x_n\|_X < \frac{1}{m}.$$

Since the range of P_m has finite dimensions, then $\{P_m S(s_n, s_n - \tau_n)x_n\}$ is pre-compact in X. Hence, for any $m \in \mathbb{N}$, $\{S(s_n, s_n - \tau_n)x_n\}$ has a finite covering by balls of radius $\frac{2}{m}$. Let $m \to \infty$, then $\{S(s_n, s_n - \tau_n)x_n\}$ is pre-compact and thus it has a convergent subsequence in X.

Sufficiency. Assume that the process is backwards pullback asymptotically compact. Let $\Omega(B, t)$ be the backwards omega-limit set as defined in (2.2), where $t \in \mathbb{R}$ and B is a bounded subset in X. It follows from the compactness of $\Omega(B, t)$, for any $\varepsilon > 0$, there are $i_\varepsilon \in \mathbb{N}$ and $x_1, x_2, \cdots, x_{i_\varepsilon} \in \Omega(B, t)$ such that

$$\Omega(B, t) \subset \bigcup_{j=1}^{i_\varepsilon} B_X(x_j, \frac{\varepsilon}{4}).$$

which, together with (2.5), implies that

$$\bigcup_{\tau \ge \tau_\varepsilon} \bigcup_{s \le t} S(s, s-\tau)B \subset B_X(\Omega(B, t), \frac{\varepsilon}{4}) \subset \bigcup_{j=1}^{i_\varepsilon} B_X(x_j, \frac{\varepsilon}{2}). \tag{2.6}$$

Since X is a uniformly convex Banach space, we can define $P_\varepsilon : X \to X_\varepsilon$ by the mapping onto the closest point in $X_\varepsilon := span\{x_1, x_2, \cdots, x_{i_\varepsilon}\}$ such that

$$\|x - P_\varepsilon x\|_X = dist(x, X_\varepsilon), \text{ for all } x \in X. \tag{2.7}$$

Since $x = P_\varepsilon x + (x - P_\varepsilon x)$ and $\|x - P_\varepsilon x\|_X \le \|x\|_X$, then $\|P_\varepsilon x\|_X \le 2\|x\|_X$. This, together with (2.6), shows

$$P_\varepsilon(\bigcup_{\tau \ge \tau_\varepsilon} \bigcup_{s \le t} S(s, s-\tau)B) \text{ is bounded.}$$

By $x_j \in X_\varepsilon$ and (2.6)-(2.7), we see

$$\|(I - P_\varepsilon)(\bigcup_{\tau \ge \tau_\varepsilon} \bigcup_{s \le t} S(s, s-\tau)B)\|_X < \frac{\varepsilon}{2}.$$

Then, S is backwards pullback flattening. □

By Theorems 2.7, 2.10 and 2.11, we have established three equivalent criteria for the backwards compactness of pullback attractors.

Remark. The backwards asymptotically compactness (Def.2.9) is also termed as strongly pullback asymptotically compactness in [**4**]. The authors proved the existence of a backwards bounded attractor for a backwards asymptotically compact process with point-dissipation (or bounded-dissipation). However, our results strengthen the conclusion from backwards boundedness to backwards compactness.

3. Preliminaries for the damped 3D NSE

3.1. The abstract equation on some function spaces. We take the function spaces used in this paper by

$$\mathcal{V} := \{u \in (C_0^\infty(\mathcal{O}))^3 : \text{div } u = 0\}, \quad H := cl_{(L^2(\mathcal{O}))^3}\mathcal{V}, \quad V := cl_{(H_0^1(\mathcal{O}))^3}\mathcal{V},$$

where cl_X denotes the closure taken in X.

It is well known that both H and V are separable Hilbert spaces with the inner products and norms given by, for $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in H$,

$$(u, v) = \sum_{i=1}^{3} \int_{\mathcal{O}} u_i \cdot v_i dx \quad \text{and} \quad |u|_2 = (u, u)^{\frac{1}{2}},$$

and, for $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in V$,

$$((u, v)) = \sum_{i=1}^{3} \int_{\mathcal{O}} \nabla u_i \cdot \nabla v_i dx \quad \text{and} \quad \|u\| = ((u, u))^{\frac{1}{2}}.$$

The injections $V \hookrightarrow H \equiv H' \hookrightarrow V'$ are dense and continuous. Besides, $\mathbf{H}^2 := (H^2(\mathcal{O}))^3$, $\mathbf{L}^p := (L^p(\mathcal{O}))^3$ with the norm $|\cdot|_p$, and $\langle \cdot, \cdot \rangle$ represents the duality pairing between V and V'.

Denote $\widetilde{P}$ by the Helmholtz-Leray orthogonal projection of $\mathbf{L}^2$ onto H, and apply it to Eq. (1.1) to obtain

$$\begin{cases} u_t + \mu A u + B(u) + F(u) = \widetilde{P} g(\cdot, t), & t \geq \tau, \\ u|_{\partial \mathcal{O}} = 0, \ t \geq \tau, \quad \text{and } u(x, \tau) = u_0(x), & x \in \mathcal{O}, \end{cases} \tag{3.1}$$

where $\widetilde{P}g(\cdot, t) = g(\cdot, t)$ since we will assume $g(\cdot, t) \in H$ for each $t \in \mathbb{R}$, the Stokes operator $A = -\widetilde{P}\Delta$ is defined by $\langle Au, v \rangle = ((u, v))$, the nonlinearity $F(u) = \widetilde{P}(\alpha |u|^{\beta-1} u)$ and the bilinear operator $B : V \times V \to V'$ satisfies $\langle B(u, v), w \rangle = b(u, v, w)$ with the trilinear form b,

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

We write $B(u) = B(u, u)$ shortly.

It follows from [**25**, Propositions 9.1 and 9.2] that b has the following properties: for $u \in H$ and $v, w \in V$,

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \tag{3.2}$$

and, for $u \in V, v \in D(A)$ and $w \in H$,

$$|b(u, v, w)| \leq c|u|_6 |\nabla v|_3 |w|_2 \leq c\|u\| \|v\|^{\frac{1}{2}} |Av|_2^{\frac{1}{2}} |w|_2. \tag{3.3}$$

3.2. Assumptions on the non-autonomous force. To obtain a backwards compact attractor in H for Eq.(3.1), we only assume that the force $g \in L^2_{loc}(\mathbb{R}, H)$ is *backwards tempered*, i.e.

$$\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}|g(\cdot,r)|_2^2 dr < \infty, \text{ for all } t \in \mathbb{R} \text{ and } \gamma > 0. \tag{3.4}$$

Remark. In fact, we can reduce the assumption $g \in L^2_{loc}(\mathbb{R}, H)$, replaced by $g \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O})^3)$. In this case, since $|\widetilde{P}g(\cdot,t)|_2 \le \|\widetilde{P}\||g(\cdot,t)|_2$ and $\|\widetilde{P}\|$ is a constant, it follows that (3.4) still holds true for $\widetilde{P}g$. However, we do not pursue this weaker assumption in this paper.

Next, we prove that (3.4) is equivalent to the *backwards translation boundedness*, which means

$$\sup_{s\le T}\int_{s-a}^{s} |g(\cdot,r)|_2^2 dr < \infty, \text{ for some } T \in \mathbb{R} \text{ and some } a > 0. \tag{3.5}$$

So, this assumption is obviously weaker than the translation boundedness (by taking the supremum over all time in (3.5)), which was assumed in [**27**].

PROPOSITION 3.1. *Let $g \in L^2_{loc}(\mathbb{R}, H)$, then g is backwards tempered if and only if g is backwards translation bounded.*

PROOF. Assume (3.5) holds for some $T \in \mathbb{R}$ and some $a > 0$. We prove it holds for all $t \in \mathbb{R}$. If $t > T$, by $g \in L^2_{loc}(\mathbb{R}, H)$, we have

$$\begin{aligned}\sup_{s\le t}\int_{s-a}^{s} |g(\cdot,r)|_2^2 dr &\le \sup_{s\le T}\int_{s-a}^{s} |g(\cdot,r)|_2^2 dr + \sup_{T<s\le t}\int_{s-a}^{s} |g(\cdot,r)|_2^2 dr \\ &\le \sup_{s\le T}\int_{s-a}^{s} |g(\cdot,r)|_2^2 dr + \int_{T-a}^{t} |g(\cdot,r)|_2^2 dr < \infty.\end{aligned}$$

If $t \le T$, then it is obvious that the result holds. This shows that for all $t \in \mathbb{R}$ and $\gamma > 0$,

$$\begin{aligned}\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}|g(\cdot,r)|_2^2 dr &= \sup_{s\le t}\sum_{n=0}^{\infty}\int_{s-(n+1)a}^{s-na} e^{\gamma(r-s)}|g(\cdot,r)|_2^2 dr \\ &\le \sup_{s\le t}\sum_{n=0}^{\infty} e^{-\gamma na}\int_{s-(n+1)a}^{s-na} |g(\cdot,r)|_2^2 dr \\ &\le \frac{1}{1-e^{-\gamma a}}\sup_{\delta\le t}\int_{\delta-a}^{\delta} |g(\cdot,r)|_2^2 dr < \infty.\end{aligned}$$

The inverse is similar to that given in [**10**, Proposition 4.1]. □

To prove backwards compactness in the Sobolev space V, we assume further g is *backwards limiting*, i.e.

$$\lim_{\gamma\to+\infty}\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}|g(\cdot,r)|_2^2 dr = 0, \text{ for all } t \in \mathbb{R}. \tag{3.6}$$

In fact, this condition can be deduced from the backwards temperedness together with an easy-verified condition. We say g is *backwards absolutely continuous* if

$$\lim_{a\to 0^+}\sup_{s\le t}\int_{s-a}^{s} |g(\cdot,r)|_2^2 dr = 0, \text{ for all } t \in \mathbb{R}. \tag{3.7}$$

PROPOSITION 3.2. *Let g be backwards translation bounded and backwards absolutely continuous, then g is backwards limiting.*

PROOF. Fix $t \in \mathbb{R}$. Let $\{\gamma_i\}$ be a sequence ordered $\gamma_{i+1} \geq \gamma_i$ such that $\gamma_{i+1} \to \infty$ as $i \to \infty$. Define

$$H(i) := \sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma_i (r-s)} |g(\cdot, r)|_2^2 dr, \text{ for all } i \in \mathbb{N}.$$

It follows from Proposition 3.1 that $H(i) < \infty$. Obviously, $H(i)$ is non-negative decreasing, and thus $\lim_{i \to \infty} H(i) = \delta \geq 0$. To this end, we prove δ must be zero by contradiction.

If $\delta > 0$, then there exists $a > 0$ such that, since g is backwards absolutely continuous,

$$\sup_{s \leq t} \int_{s-a}^{s} |g(\cdot, r)|_2^2 dr < \frac{\delta}{4},$$

which implies

$$\sup_{s \leq t} \int_{s-a}^{s} e^{\gamma_i (r-s)} |g(\cdot, r)|_2^2 dr \leq \sup_{s \leq t} \int_{s-a}^{s} |g(\cdot, r)|_2^2 dr < \frac{\delta}{4}. \tag{3.8}$$

Besides, by Proposition 3.1, we have as $i \to \infty$,

$$\begin{aligned} \sup_{s \leq t} \int_{-\infty}^{s-a} e^{\gamma_i (r-s)} |g(\cdot, r)|_2^2 dr &= \sup_{s \leq t-a} \int_{-\infty}^{s} e^{\gamma_i (r-s-a)} |g(\cdot, r)|_2^2 dr \\ &\leq e^{-\gamma_i a} \sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma_i (r-s)} |g(\cdot, r)|_2^2 dr \\ &\leq e^{-\gamma_i a} \sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma_1 (r-s)} |g(\cdot, r)|_2^2 dr \to 0. \end{aligned} \tag{3.9}$$

It yields from (3.8)-(3.9) that for a large i,

$$H(i) = \sup_{s \leq t} \int_{s-a}^{s} e^{\gamma_i (r-s)} |g(\cdot, r)|_2^2 dr + \sup_{s \leq t} \int_{-\infty}^{s-a} e^{\gamma_i (r-s)} |g(\cdot, r)|_2^2 dr < \frac{\delta}{2} < \delta.$$

This contradicts with $\lim_{i \to \infty} H(i) = \delta > 0$. □

3.3. The deduced evolution process.

PROPOSITION 3.3. *Assume $\beta \geq 1$ and $g \in L^2_{loc}(\mathbb{R}, H)$, then for every $\tau \in \mathbb{R}$, $u_0 \in H$ and given $T > \tau$, there is a weak solution of Eq. (3.1) such that*

$$u(t, \tau, u_0) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{\beta+1}(\tau, T; \boldsymbol{L}^{\beta+1}),$$

with $u(\tau, \tau, u_0) = u_0$. Moreover, when $\beta > 3$, the weak solution is unique and the map $u_0 \mapsto u(t, \tau, u_0)$ is continuous in H. If $\beta > 3$ and $u_0 \in V \cap \boldsymbol{L}^{\beta+1}$, then the weak solution becomes a strong solution (unique) such that

$$\begin{aligned} &u(t, \tau, u_0) \in L^\infty(\tau, T; V) \cap L^\infty(\tau, T; L^{\beta+1}) \cap L^2(\tau, T; \boldsymbol{H}^2), \\ &u_t(t, \tau, u_0) \in L^2(\tau, T; H), \end{aligned}$$

and in fact $u(t, \tau, u_0) \in C([\tau, T]; V)$.

PROOF. For the existence and uniqueness of weak solutions, we refer to [**3**, Theorem 1] and [**17**, Lemma 4.1] respectively. For the existence of strong solutions, we refer to [**3**, Theorem 2] for $\beta \geq 7/2$ and [**30**, Theorem 3.1] with $\beta \in (3, 7/2)$. The readers can also see Lemmas 4.1 and 4.2 in the next section.

We then prove the continuity of strong solutions $u(t, \tau, u_0)$ from $[\tau, T]$ into V. Since $u(t, \tau, u_0) \in L^2(\tau, T; \mathbf{H}^2)$ and $u_t(t, \tau, u_0) \in L^2(\tau, T; H)$, it follows from [**25**, Corollary 7.3] that $u(t, \tau, u_0) \in C([\tau, T]; V)$. □

Note that, we restrict $\beta \in (3, 5)$ to prove the backwards compactness of pullback attractors in V. If $\beta \in (3, 5)$, then $V \hookrightarrow \mathbf{L}^6 \hookrightarrow \mathbf{L}^{\beta+1}$ since $\mathcal{O} \subset \mathbb{R}^3$ is regular enough. From now on, $u_0 \in V \cap \mathbf{L}^{\beta+1}$ is substituted by $u_0 \in V$ because of the equivalence of V and $V \cap \mathbf{L}^{\beta+1}$.

By Proposition 3.3, we can use the unique weak solution to define an evolution process $S(t, \tau) : H \to H$ by

$$S(t, \tau)u_0 := u(t, \tau, u_0), \text{ for all } t \geq \tau \text{ and } u_0 \in H. \tag{3.10}$$

4. Backwards compact attractors in H

We first prove the evolution process has an increasing bounded pullback absorbing set in H for $\beta \geq 1$. The letter $c > 0$ represents a universal constant which may vary in different places.

LEMMA 4.1. *Let $\beta \geq 1$ and g be backwards tempered, then for each $t \in \mathbb{R}$ and $R > 0$, there exists $\tau_0 := \tau_0(R) > 0$ such that for all $\tau \geq \tau_0$ and $|u_0|_2 \leq R$,*

$$\sup_{s \leq t} |u(s, s-\tau, u_0)|_2^2 \leq c(1 + G(t)), \tag{4.1}$$

$$\sup_{s \leq t} \int_{s-\tau}^{s} e^{\mu\lambda_1(r-s)} (\|u(r, s-\tau, u_0)\|^2 + |u(r, s-\tau, u_0)|_{\beta+1}^{\beta+1}) dr \leq c(1 + G(t)), \tag{4.2}$$

where λ_1 is the first eigenvalue of A and G is a non-negative increasing function defined by

$$G(t) := \sup_{s \leq t} \int_{-\infty}^{s} e^{\mu\lambda_1(r-s)} |g(\cdot, r)|_2^2 dr < \infty. \tag{4.3}$$

PROOF. Let $t \in \mathbb{R}$ be fixed. For each $s \leq t$, we multiply Eq. (3.1) with $u := u(s, s-\tau, u_0)$ and integrate it over $\mathcal{O}$ to obtain

$$\frac{1}{2}\frac{d}{ds}|u|_2^2 + \mu\|u\|^2 + \alpha|u|_{\beta+1}^{\beta+1} = (g(\cdot, s), u) \leq \frac{\mu\lambda_1}{4}|u|_2^2 + \frac{1}{\mu\lambda_1}|g(\cdot, s)|_2^2,$$

where we have used (3.2). Since $\|u\|^2 \geq \lambda_1|u|_2^2$, then

$$\frac{d}{ds}|u|_2^2 + \mu\lambda_1|u|_2^2 + \frac{\mu}{2}\|u\|^2 + 2\alpha|u|_{\beta+1}^{\beta+1} \leq c|g(\cdot, s)|_2^2. \tag{4.4}$$

Multiplying (4.4) by $e^{\mu\lambda_1 s}$ and integrating it over $[s-\tau, s]$, we have

$$\begin{aligned} &|u(s)|_2^2 + \int_{s-\tau}^{s} e^{\mu\lambda_1(r-s)} (\frac{\mu}{2}\|u(r)\|^2 + 2\alpha|u(r)|_{\beta+1}^{\beta+1}) dr \\ &\leq e^{-\mu\lambda_1\tau}|u_0|_2^2 + c\int_{s-\tau}^{s} e^{\mu\lambda_1(r-s)}|g(\cdot, r)|_2^2 dr \\ &\leq c(1 + \int_{-\infty}^{s} e^{\mu\lambda_1(r-s)}|g(\cdot, r)|_2^2 dr), \end{aligned} \tag{4.5}$$

for all $\tau \geq \tau_0$ with some $\tau_0 := \tau_0(R) > 0$.

Taking the supremum with respect to the past time $s \leq t$ in (4.5), we obtain (4.1) and (4.2). By the hypothesis (3.4), $G(t)$ is obviously finite and increasing. $\square$

We then give the backwards estimates in V. In this case, we need to assume $\beta > 3$.

LEMMA 4.2. *Let $\beta > 3$ and g be backwards tempered, then, for each $t \in \mathbb{R}$ and $R > 0$, there exists $\tau_0 := \tau_0(R) \geq 2$ such that for all $\tau \geq \tau_0$ and $|u_0|_2 \leq R$,*

$$\sup_{s\leq t}\sup_{r\in[s-1,s]}(\|u(r,s-\tau,u_0)\|^2 + |u(r,s-\tau,u_0)|_{\beta+1}^{\beta+1}) \leq c(1+G(t)). \tag{4.6}$$

where $G(t)$ is given in (4.3).

PROOF. Let $t \in \mathbb{R}$ be fixed and $s \leq t$. We multiply Eq. (3.1) with Au and u_s respectively and integrate them over $\mathcal{O}$, then the results are

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|u\|^2 + \mu|Au|_2^2 + \alpha\beta\int_{\mathcal{O}}|u|^{\beta-1}|\nabla u|^2dx \\
&\quad = (g(\cdot,s),Au) - \int_{\mathcal{O}}(u\cdot\nabla)uAudx \\
&\quad \leq \frac{\mu}{2}|Au|_2^2 + c|g(\cdot,s)|_2^2 + c|(u\cdot\nabla)u|_2^2,
\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}
|u_s|_2^2 + \frac{\mu}{2}\frac{d}{ds}\|u\|^2 + \frac{\alpha}{\beta+1}\frac{d}{ds}|u|_{\beta+1}^{\beta+1} &= (g(\cdot,s),u_s) - \int_{\mathcal{O}}(u\cdot\nabla)uu_sdx \\
&\leq \frac{1}{2}|u_s|_2^2 + c|g(\cdot,s)|_2^2 + c|(u\cdot\nabla)u|_2^2.
\end{aligned}\tag{4.8}$$

Since $\beta > 3$, we have $\beta - 1 > 2$. By the Young inequality, we deduce

$$c|(u\cdot\nabla)u|_2^2 \leq c\int_{\mathcal{O}}|u|^2|\nabla u|^2dx \leq \frac{\alpha\beta}{2}\int_{\mathcal{O}}|u|^{\beta-1}|\nabla u|^2dx + c\|u\|^2. \tag{4.9}$$

Let $\delta = \min\{\mu+1, \frac{2\alpha}{\beta+1}\}$, then it yields from (4.7)-(4.9) that for all $s \leq t$,

$$\delta\frac{d}{ds}(\|u\|^2 + |u|_{\beta+1}^{\beta+1}) + \mu|Au|_2^2 \leq c\|u\|^2 + c|g(\cdot,s)|_2^2. \tag{4.10}$$

Integrate (4.10) over $[r_1, r]$ with $r_1 \in [s-2, s-1]$ and $r \in [s-1, s]$ to obtain

$$\begin{aligned}
&\|u(r,s-\tau,u_0)\|^2 + |u(r,s-\tau,u_0)|_{\beta+1}^{\beta+1} \\
&\quad \leq c\|u(r_1)\|^2 + |u(r_1)|_{\beta+1}^{\beta+1} + c\int_{r_1}^{r}\|u(r_2)\|^2dr_2 + c\int_{r_1}^{r}|g(\cdot,r_2)|_2^2dr_2.
\end{aligned}$$

We then integrate the above inequality on $r_1 \in [s-2, s-1]$ with $s \leq t$, it follows from (4.2) that

$$\begin{aligned}
&\sup_{r\in[s-1,s]}(\|u(r,s-\tau,u_0)\|^2 + |u(r,s-\tau,u_0)|_{\beta+1}^{\beta+1}) \\
&\leq c\int_{s-2}^{s}(\|u(r)\|^2 + |u(r)|_{\beta+1}^{\beta+1})dr + c\int_{s-2}^{s}|g(\cdot,r)|_2^2dr \\
&\leq c(1+G(t)),
\end{aligned}\tag{4.11}$$

whenever $\tau \geq \tau_0$ with some $\tau_0 := \tau_0(R) \geq 2$. Hence, we get (4.6) by taking the supremum in (4.11) with respect to $s \in (-\infty, t]$. $\square$

Finally, we derive a backwards compact attractor in H for $\beta > 3$.

THEOREM 4.3. *Assume $\beta > 3$ and g is backwards tempered, then the evolution process S generated by the non-autonomous damped 3D NSE possesses a pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in H. Moreover, the attractor is backwards compact in H.*

PROOF. Define a non-autonomous set by

$$\mathcal{K}(t) := \{w \in V; \|w\|^2 \le c(1 + G(t))\}, \text{ for each } t \in \mathbb{R},$$

where $G(t)$ is given in (4.3). By the compactness of the Sobolev embedding, $\mathcal{K}(t)$ is compact in H. By (4.6), $\mathcal{K}$ is a (backward uniformly) pullback absorbing set in H. Hence, there is a pullback attractor $\mathcal{A}$. On the other hand, since $\mathcal{K}$ is increasing, compact and uniformly pullback absorbing set, it follows easily that the process is backwards pullback asymptotically compact. Then Theorem 2.7 and 2.11 imply that $\mathcal{A}$ is backwards compact. □

5. Backwards compact attractors in V

We show in this section the pullback attractor in H is also a backwards compact attractor in V but we restrict $\beta \in (3, 5)$. To do this, let $\{e_i\}_{i=1}^{\infty}$ be the orthonormal basis consisted of the eigenfunctions of A in the space H and let $\{\lambda_i\}$ be the corresponding eigenvalues ordered $\lambda_{i+1} \ge \lambda_i$ such that $\lambda_{i+1} \to \infty$ as $i \to \infty$. Denote $P_i : H \to H_i$ by the orthogonal projection with $H_i = \text{span}\{e_1, e_2, \cdots, e_i\}$. Note that $\{e_i\}_{i=1}^{\infty} \subset V \cap D(A)$ and it is also orthogonal in $V \cap D(A)$.

For any solution $u \in V$ of Eq. (3.1), we consider the unique orthogonal decomposition given by

$$u = P_i u \oplus (I - P_i)u = u_{1,i} \oplus u_{2,i}, \quad \forall i \in \mathbb{N}.$$

We require a Poincaré-type inequality about $u_{2,i}$, and present its proof for completeness although it may be known.

LEMMA 5.1. *For each $w \in (I - P_i)(V \cap D(A)) \subset V \cap D(A)$, we have*

$$\|w\|^2 \ge \lambda_{i+1}|w|_2^2 \quad \textit{and} \quad |Aw|_2^2 \ge \lambda_{i+1}\|w\|^2.$$

PROOF. Since $w \in H_i^{\perp} = \text{span}\{e_{i+1}, e_{i+2}, \cdots\}$, then

$$w = \sum_{j=i+1}^{\infty} (w, e_j)e_j \quad \textit{and} \quad |w|_2^2 = \sum_{j=i+1}^{\infty} |(w, e_j)|^2.$$

Since $A : D(A) \subset H \to H$ is a closed operator, it follows that

$$Aw = A \sum_{j=i+1}^{\infty} (w, e_j)e_j = \sum_{j=i+1}^{\infty} (w, e_j)Ae_j = \sum_{j=i+1}^{\infty} \lambda_j (w, e_j)e_j.$$

Therefore, we have for $w \in (I - P_i)(V \cap D(A))$

$$\|w\|^2 = \langle Aw, w\rangle = \sum_{j=i+1}^{\infty} \lambda_j |(w, e_j)|^2 \ge \lambda_{i+1}|w|_2^2.$$

Similarly,

$$|Aw|_2^2 = (Aw, Aw) = \sum_{j=i+1}^{\infty} \lambda_j^2 |(w, e_j)|^2 \ge \lambda_{i+1} \sum_{j=i+1}^{\infty} \lambda_j |(w, e_j)|^2 = \lambda_{i+1}\|w\|_2^2.$$

□

We next show the following assertion in order to prove the backwards pullback flattening property in V.

LEMMA 5.2. *Under the same assumptions of Lemma 4.1, for each $t \in \mathbb{R}$ and $R > 0$, there exists $\tau_0 := \tau_0(R) \geq 2$ such that for all $\tau \geq \tau_0$ and $\|u_0\| \leq R$,*

$$\sup_{s \leq t} \int_{s-1}^{s} |Au(r, s-\tau, u_0)|_2^2 dr \leq c(1 + G(t)), \tag{5.1}$$

where $G(t)$ is given in (4.3).

PROOF. By (4.7) and (4.9), we have for all $s \leq t$,

$$\frac{d}{ds}\|u\|^2 + \mu |Au|_2^2 \leq c\|u\|^2 + c|g(\cdot, s)|_2^2. \tag{5.2}$$

Integrating (5.2) over $[s-1, s]$, we deduce from (4.11) that for all $s \leq t$,

$$\begin{aligned} \mu \int_{s-1}^{s} |Au(r)|_2^2 dr \qquad \qquad \qquad \qquad \\ \leq \|u(s-1)\|^2 + c\int_{s-1}^{s} \|u(r)\|^2 dr + c\int_{s-1}^{s} |g(\cdot, r)|_2^2 dr \leq c(1 + G(t)). \end{aligned} \tag{5.3}$$

Therefore, we obtain (5.1) by taking the supremum in (5.3) over all the past time $s \leq t$. □

We also need the following *Gagliardo-Nirenberg* inequality.

Let $\mathcal{O} \subset \mathbb{R}^n$ be bounded (or unbounded) and $u \in (C_0^\infty(\mathcal{O}))^n$, then there is $c > 0$ such that

$$|D^j u|_p \leq c|D^m u|_r^a |u|_q^{1-a},$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$, $1 \leq p, q, r \leq \infty$, $0 \leq j < m$, $\frac{j}{m} \leq a \leq 1$ and c only depends on n, m, j, q, r, a. If $m - j - \frac{n}{r}$ is a nonnegative integer, then the above inequality holds for $\frac{j}{m} \leq a < 1$.

LEMMA 5.3. *Let $\beta \in (3, 5)$ and let g be backwards tempered and backwards limiting, then for each $\varepsilon > 0$, $t \in \mathbb{R}$ and $R > 0$, there exist $\tau_1 := \tau_1(\varepsilon, R) \geq 2$ and $N := N(\varepsilon, R) > 0$ such that for all $\tau \geq \tau_1$, $i \geq N$ and $\|u_0\| \leq R$,*

$$\sup_{s \leq t} \|(I - P_i)u(s, s-\tau, u_0)\|^2 < \varepsilon. \tag{5.4}$$

PROOF. Let $i \in \mathbb{N}$ be fixed. Multiply Eq. (3.1) with $Au_{2,i}$ to find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{2,i}\|^2 + \mu|Au_{2,i}|_2^2 \\ &= (g(\cdot, t), Au_{2,i}) - \int_{\mathcal{O}} (u \cdot \nabla)uAu_{2,i}dx - \alpha \int_{\mathcal{O}} |u|^{\beta-1}uAu_{2,i}dx \\ &\leq c\|u\|\|u\|^{\frac{1}{2}}|Au|_2^{\frac{1}{2}}|Au_{2,i}|_2 + \frac{\mu}{4}|Au_{2,i}|_2^2 + c|u|_{2\beta}^{2\beta} + c|g(\cdot, t)|_2^2 \\ &\leq \frac{\mu}{2}|Au_{2,i}|_2^2 + c\|u\|^3|Au|_2 + c|g(\cdot, t)|_2^2 + c|u|_{2\beta}^{2\beta}, \end{aligned} \tag{5.5}$$

where we have used (3.3) in the second step. Let $p = 2\beta$, $q = \beta + 1$, $j = 0$, $r = m = 2$ and $n = 3$ in the Gagliardo-Nirenberg inequality, then $a = \frac{3(\beta-1)}{\beta(\beta+7)}$. We know $0 < a < 1$ if $\beta > 3$. Hence, the last term in (5.5) is bounded by

$$|u|_{2\beta}^{2\beta} \leq c|Au|_2^{\frac{6(\beta-1)}{\beta+7}} |u|_{\beta+1}^{\frac{2(\beta^2+4\beta+3)}{\beta+7}}.$$

Note that, in what follows, we need $\beta < 5$ to ensure the exponent $\frac{6(\beta-1)}{\beta+7} < 2$.

By Lemma 5.1, $|Au_{2,i}|_2^2 \geq \lambda_{i+1}\|u_{2,i}\|^2$. Then (5.5) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\|u_{2,i}\|^2 + \mu\lambda_{i+1}\|u_{2,i}\|^2 \\ \leq c\|u\|^3|Au|_2 + c|Au|_2^{\frac{6(\beta-1)}{\beta+7}} |u|_{\beta+1}^{\frac{2(\beta^2+4\beta+3)}{\beta+7}} + c|g(\cdot,t)|_2^2.\end{aligned} \tag{5.6}$$

Multiplying (5.6) by $e^{\mu\lambda_{i+1}s}$ ($s \in [r,t]$), integrating the result in $s \in [r,t]$ and then integrating it once again in $r \in [t-1,t]$, we find

$$\begin{aligned}\|u_{2,i}(t)\|^2 \leq &\int_{t-1}^t e^{\mu\lambda_{i+1}(r-t)}\|u_{2,i}(r)\|^2 dr \\ &+ c\int_{t-1}^t e^{\mu\lambda_{i+1}(r-t)}\|u(r)\|^3|Au(r)|_2 dr \\ &+ c\int_{t-1}^t e^{\mu\lambda_{i+1}(r-t)}|Au(r)|_2^{\frac{6(\beta-1)}{\beta+7}} |u(r)|_{\beta+1}^{\frac{2(\beta^2+4\beta+3)}{\beta+7}} dr \\ &+ c\int_{t-1}^t e^{\mu\lambda_{i+1}(r-t)}|g(\cdot,r)|_2^2 dr.\end{aligned} \tag{5.7}$$

We now take into account the supremum of each term in (5.7) over the past time. For the first term on the right side, since $\|u_{2,i}\| \leq \|u\|$, we see from (4.6) and the increasing property of $G(\cdot)$ that

$$\begin{aligned}&\sup_{s\leq t}\int_{s-1}^s e^{\mu\lambda_{i+1}(r-s)}\|u_{2,i}(r,s-\tau)\|^2 dr \\ &\leq c(1+G(t))\sup_{s\leq t}\int_{s-1}^s e^{\mu\lambda_{i+1}(r-s)} dr \leq \frac{c}{\mu\lambda_{i+1}}(1+G(t)),\end{aligned} \tag{5.8}$$

for all $\tau \geq \tau_0$ with some $\tau_0 := \tau_0(R) \geq 2$. By (4.6) and (5.1), we obtain

$$\begin{aligned}&\sup_{s\leq t}\int_{s-1}^s e^{\mu\lambda_{i+1}(r-s)}\|u(r,s-\tau)\|^3|Au(r,s-\tau)|_2 dr \\ &\leq c(1+G(t))^{\frac{3}{2}}\sup_{s\leq t}\int_{s-1}^s e^{\mu\lambda_{i+1}(r-s)}|Au(r)|_2 dr \\ &\leq c(1+G(t))^{\frac{3}{2}}\sup_{s\leq t}\Big(\Big(\int_{s-1}^s |Au(r)|_2^2 dr\Big)^{\frac{1}{2}}\Big(\int_{s-1}^s e^{2\mu\lambda_{i+1}(r-s)} dr\Big)^{\frac{1}{2}}\Big) \\ &\leq \frac{c}{\sqrt{2\mu\lambda_{i+1}}}(1+G(t))^2,\end{aligned} \tag{5.9}$$

and, (since $\beta \in (3,5)$, then $\frac{6(\beta-1)}{\beta+7} < 2$)

(5.10)

$$\begin{aligned}
&\sup_{s\le t}\int_{s-1}^{s} e^{\mu\lambda_{i+1}(r-s)}|Au(r)|_2^{\frac{6(\beta-1)}{\beta+7}}|u(r)|_{\beta+1}^{\frac{2(\beta^2+4\beta+3)}{\beta+7}}dr\\
&\quad\le c(1+G(t))^{\frac{2(\beta+3)}{\beta+7}}\sup_{s\le t}\int_{s-1}^{s} e^{\mu\lambda_{i+1}(r-s)}|Au(r)|_2^{\frac{6(\beta-1)}{\beta+7}}dr\\
&\quad\le c(1+G(t))^{\frac{2(\beta+3)}{\beta+7}}\sup_{s\le t}\Big(\Big(\int_{s-1}^{s}|Au(r)|_2^2dr\Big)^{\frac{3(\beta-1)}{\beta+7}}\Big(\int_{s-1}^{s} e^{\frac{\beta+7}{2(5-\beta)}\mu\lambda_{i+1}(r-s)}dr\Big)^{\frac{2(5-\beta)}{\beta+7}}\Big)\\
&\quad\le \Big(\frac{c}{\mu\lambda_{i+1}}\Big)^{\frac{2(5-\beta)}{\beta+7}}(1+G(t))^{\frac{5\beta+3}{\beta+7}},
\end{aligned}$$

for all $\tau \ge \tau_1$ with some $\tau_1 \ge \tau_0$. By (3.6), we see

$$\lim_{i\to\infty}\sup_{s\le t}\int_{-\infty}^{s} e^{\mu\lambda_{i+1}(r-s)}|g(\cdot,r)|_2^2dr = 0. \tag{5.11}$$

Therefore, it follows from (5.7)-(5.11) that for all $\tau \ge \tau_1$,

$$\sup_{s\le t}\|(I-P_i)u(s,s-\tau,u_0)\|^2 \to 0, \text{ as } i\to\infty,$$

which finishes the proof. □

By using the previous auxiliary results, we now present the backwards compact attractor in V.

THEOREM 5.4. *Assume $\beta \in (3,5)$ and g are both backwards tempered and backwards limiting, then the evolution process S induced by the non-autonomous damped 3D NSE has a backwards compact attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in the Sobolev space V. Furthermore, the attractor is the minimal closed pullback attracting set in V.*

PROOF. We prove all conditions in Theorem 2.7 are fulfilled. By (4.6), the pullback absorbing set in V is given by

$$\mathcal{K}_1(t) := \{w \in V; \|w\|^2 \le c(1+G(t))\}, \text{ for every } t \in \mathbb{R},$$

where $G(t)$ is defined in (4.3). It is obvious, by (3.4), that $\mathcal{K}_1$ is bounded and increasing in V. By (4.6) and (5.4), the process S defined on V is backwards pullback flattening in V. Therefore, by Theorem 2.7, the process S related to Eq. (1.1) has a pullback attractor $\mathcal{A}$ in V, which is also backwards compact in V. □

REMARK 5.5. It seems that $\beta = 5$ is critical for the existence of attractors for the damped 3D NSE in the Sobolev space V. In this case, $\frac{6(\beta-1)}{\beta+7} = 2$ in (5.10), then the bound given in (5.10) is independent of i, and thus (5.10) does not tend to zero as $i \to \infty$. Actually, there exist similar problems in [**26, 27**] when the authors tried to prove the boundedness of $|Au|_2$. So they did not prove the case of $\beta = 5$ although claimed in their papers.

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School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: `yjy111@email.swu.edu.cn (J.Y. Yin)`

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: `gahui@swu.edu.cn (A.H. Gu)`

Corresponding author: Yangrong Li, School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail address: `liyr@swu.edu.cn (Y.R. Li)`