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Multiple nontrivial solutions for a class of nonlinear Schrödinger equations with linear coupling

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ABSTRACT. In this paper, we are concerned with the existence and multiplicity of nontrivial solutions of a class of nonlinear Schrödinger equations which arise from nonlinear optics. We prove that there are two families of semiclassical positive solutions, which concentrate on the minimal and maximum points of the associated potentials, respectively. We also investigate the relationship between the number of solutions and the topology of the set of the global minima of the potentials by the minimax theorem. The novelty is that it might be the first attempt to explore multiplicity and concentration of positive solutions for such kind of coupled Schrödinger equations.

CONTENTS

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1. Introduction and Main Results

Consider the nonlinear Schrödinger equations

$$(\mathscr{K}_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u = g(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\varepsilon^2\Delta v + M(x)v = f(v) + \lambda u, & x \in \mathbb{R}^N, \\ u(x), v(x) \to 0, \ \text{as } |x| \to \infty, \end{cases}$$

where ε is a small positive parameter, $\lambda > 0$, $V, M \in C(\mathbb{R}^N, \mathbb{R})$ are positive functions, and f and g are superlinear and subcritical nonlinearity.

The system $(\mathscr{K}_\varepsilon)$ is initially generated from the following time-dependent nonlinear vector Schrödinger equations

$$(1.1) \qquad \begin{cases} i\hbar\frac{\partial\phi_1}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi_1 + V(x)\phi_1 + g(\phi_1) + \lambda\phi_2, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ i\hbar\frac{\partial\phi_2}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi_2 + M(x)\phi_2 + f(\phi_2) + \lambda\phi_1, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \phi_j = \phi_j(t,x) \in \mathbb{C}, \ \phi_j(t,x) \to 0, \ \text{as } |x| \to \infty, \ j = 1, 2, \end{cases}$$

where i is the imaginary unit, Δ is the Laplacian operator, and $\hbar > 0$ is the Planck constant. System (1.1) arises in nonlinear optics (cf. [**3, 5**]). As we know, nonlinear Schrödinger equations (NLS) have been broadly investigated in many aspects, especially in standing wave solutions [**2, 3, 35, 36**]. A standing wave solution of system (1.1) takes the form

$$(\phi_1(t,x), \, \phi_2(t,x)) = \left(u(x)e^{-\frac{iEt}{\hbar}}, \ v(x)e^{-\frac{iEt}{\hbar}}\right), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

If we assume that $f(e^{i\theta}u) = e^{i\theta}f(u)$ and $g(e^{i\theta}u) = e^{i\theta}g(u)$ for all $u \in \mathbb{R}$. Then, $(\phi_1, \, \phi_2)$ solves system (1.1) if and only if (u, v) solves the system

$$(1.2) \qquad \begin{cases} -\frac{\hbar^2}{2m}\Delta u + (V(x) - E)u = g(u) + \lambda v, \ x \in \mathbb{R}^N, \\ -\frac{\hbar^2}{2m}\Delta v + (M(x) - E)v = f(v) + \lambda u, \ x \in \mathbb{R}^N, \\ u(x), v(x) \to 0, \ \text{as } |x| \to \infty. \end{cases}$$

In this study, we are concerned with positive solutions of system $(\mathscr{K}_\varepsilon)$ for small $\hbar > 0$. In this case, the standing waves are referred to as semiclassical states. Let $\varepsilon^2 = \frac{\hbar}{2m}$. For our convenience, by replacing $(V(x) - E)$ and $(M(x) - E)$ by $V(x)$ and $M(x)$ respectively, it turns out to be system $(\mathscr{K}_\varepsilon)$. In this framework, our interests focus on the existence of solutions and their asymptotic behaviors as $\varepsilon \to 0$. Typically, solutions tend to concentrate around critical points of V or M (which are called spikes). In order to study the concentration phenomena of solutions of system $(\mathscr{K}_\varepsilon)$, we start with the constant coefficient problem (cf. [**1, 2, 4, 6, 9**] and the references therein):

$$(1.3) \qquad \begin{cases} -\Delta u + \mu u = g(u) + \lambda v, \ x \in \mathbb{R}^N, \\ -\Delta v + \sigma v = f(v) + \lambda u, \ x \in \mathbb{R}^N, \\ u(x), v(x) \to 0, \ \text{as } |x| \to \infty, \end{cases}$$

where μ and $\sigma > 0$ are constants. System (1.3) can be regarded as a limit problem after a suitable re-scaling of system (1.1). Before stating our results, let us briefly recall some definitions of solutions of system (1.3). It is well-known that a solution (u, v) of system (1.3) is called a bound state. If a bound state $(u, v) \neq (0, 0)$, we call it a nontrivial bound state. A solution is called a ground state solution if $(u, v) \neq (0, 0)$ and its energy is minimal among the energy of all the nontrivial bound states of system (1.3). A ground state solution which satisfies $u > 0$ and $v > 0$ is called a positive ground state solution. Ambrosetti et al considered the case where $f(s) = g(s) = s^3$, $\sigma = \rho = 1$, small $\lambda > 0$ and $N \leq 3$, and proved the existence of multi-bump solutions for system (1.3). When $g = (1 + a(x))|u|^{p-2}u$ and $f = (1 + b(x))|v|^{p-2}v$ ($2 < p < 2^*$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent), system (1.3) was investigated in [**1**] with the dimension $N = 1$ and in [**2**] with the dimension $N \geq 2$, respectively. The existence of nontrivial bound state was proved. Recently, Chen and Zou [**6**] dealt with the existence of positive ground state solution for system (1.3) with the critical growth. They also studied positive ground state solutions of system (1.3) with a more general nonlinearity in [**9**]. Based on these existing results, we will establish some properties of the least energy solutions for the limit problem (1.3) in Section 3.

Considerable attention is also dedicated to the semiclassical case [**7, 8, 14, 22**]. For example, Ikoma and Tanaka [**22**] considered the following system

$$(\mathscr{C}_\varepsilon^*) \qquad \begin{cases} -\varepsilon^2\Delta u + V_1(x)u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V_2(x)v = \mu_2 v^3 + \beta v u^2 & \text{in } \mathbb{R}^N, \end{cases}$$

where $N = 2, 3$, μ_1, μ_2, $\beta > 0$, and V_1 and V_2 are positive continuous functions. Let $P \in \mathbb{R}^N$ and $m(P)$ be the least energy level for nontrivial vector solutions of the problem

$$(\mathscr{C}_\varepsilon^{**}) \qquad \begin{cases} -\Delta u + V_1(P)u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(P)v = \mu_2 v^3 + \beta v u^2 & \text{in } \mathbb{R}^N. \end{cases}$$

Assume that there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that $\inf_{P\in\Lambda} m(P) < \inf_{P\in\partial\Lambda} m(P)$. By using the idea from [**20, 21**], it shows that there exists a sufficiently small $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, system $(\mathscr{C}_\varepsilon^*)$ possesses a family of positive solutions $\{(u_\varepsilon, v_\varepsilon)\}$, which concentrates, after extracting a subsequence $\varepsilon_n \to 0$, to a point $P_0 \in \Lambda$ with $m(P_0) = \inf_{P\in\Lambda} m(P)$. Moreover, $(u_\varepsilon, v_\varepsilon)$ converges to a least energy nontrivial solution of system $(\mathscr{C}_\varepsilon^{**})$ after a suitable re-scaling of x. The multiplicity and concentration of nontrivial solutions of system $(\mathscr{C}_\varepsilon^*)$ was presented in [**14**]. The existence and concentration of ground state solution of system $(\mathscr{K}_\varepsilon)$ with the subcritical growth was discussed in [**7**]. Very recently, Chen and Zou [**8**] considered the following equations with the critical growth

$$(1.4) \qquad \begin{cases} -\varepsilon^2\Delta u + a(x)u = u^{p-1} + \lambda v, \ x \in \mathbb{R}^N, \\ -\varepsilon^2\Delta v + b(x)v = v^{2^*-1} + \lambda u, \ x \in \mathbb{R}^N, \\ u > 0, v > 0 \text{ in } \mathbb{R}^N, \ u(x), v(x) \to 0, \text{ as } |x| \to \infty, \end{cases}$$

where $2 < p < 2^*$, $\lambda > 0$ and $\varepsilon > 0$ is sufficiently small. Assume that a has a local minimum. It shows that system $(\mathscr{K})_\varepsilon$ has a positive solution, which concentrate on the local minimum of a.

To the best of our knowledge, all these elegant results mentioned above are concerning the existence and concentration of positive solutions for system $(\mathscr{K})_\varepsilon$. It is quite natural to ask that: can one obtain the multiplicity and concentration of positive solutions for system $(\mathscr{K})_\varepsilon$ in the semiclassical case? Since the semiclassical state solutions describe a kind of transition from quantum mechanics to Newtonian mechanics from the point of view of physics [**10**], such phenomenon of concentration has been of interest to both mathematicians and physicists. It is also one of main motivations for this study.

Before stating assumptions for V and M, let us introduce the following notations:

$$V_0 = \min_{x\in\mathbb{R}^N} V(x), \quad V_{max} = \max_{x\in\mathbb{R}^N} V(x), \quad \text{and} \quad V_\infty = \liminf_{|x|\to\infty} V(x);$$

$$M_0 = \min_{x\in\mathbb{R}^N} M(x), \quad M_{max} = \max_{x\in\mathbb{R}^N} V(x), \quad \text{and} \quad M_\infty = \liminf_{|x|\to\infty} M(x);$$

$$\mathscr{M}_1 = \{x \in \mathbb{R}^N : V(x) = V_0\} \quad \text{and} \quad \mathscr{M}_2 = \{x \in \mathbb{R}^N : M(x) = M_0\}.$$

Assume that the bounded functions V and M satisfy the following three conditions:

($\mathscr{V}_0$) $V, M \in C(\mathbb{R}^N, \mathbb{R})$ such that $0 < V_0 < V_\infty < \infty$ and $0 < M_0 \le M_\infty < \infty$, and there exists $x_v \in \mathscr{M}_1$ such that $M(x_v) \le M_\infty$.

($\mathscr{V}_1$) $V, M \in C(\mathbb{R}^N, \mathbb{R})$ such that $0 < V_0 \le V_\infty < \infty$ and $0 < M_0 < M_\infty < \infty$, and there exists $x_m \in \mathscr{M}_2$ such that $V(x_m) \le V_\infty$.

($\mathscr{V}_2$) $V, M \in C(\mathbb{R}^N, \mathbb{R})$ satisfy

$$0 < \inf_{x\in\mathbb{R}^N} V(x) \le V^\infty = \limsup_{|x|\to\infty} V(x) \le V(x),$$

and

$$0 < \inf_{x\in\mathbb{R}^N} M(x) M^\infty = \limsup_{|x|\to\infty} M(x) \le M(x).$$

Moreover, there holds $|\mathscr{C}_1| > 0$ or $|\mathscr{C}_2| > 0$, where $\mathscr{C}_1 = \{x \in \mathbb{R}^N, V(x) > V^\infty\}$ and $\mathscr{C}_2 = \{x \in \mathbb{R}^N, M(x) > M^\infty\}$.

The hypothesis ($\mathscr{V}_0$) was first introduced by Rabinowitz [**36**] in the study of a nonlinear Schrödinger equation. For f and g, the following three conditions are imposed:

($\mathscr{F}_1$) Suppose that $f, g \in C(\mathbb{R}^N)$, $f(t) = o(t)$ and $g(t) = o(t)$ as $t \to 0$, and $f(t)t > 0$ and $g(t)t > 0$ for all $t > 0$. Moreover, one of the following cases holds: (1) $F(|t|) \ge F(t)$ ($\forall t \in \mathbb{R}$) and $g(t) = 0$ ($\forall t \le 0$); (2) $G(|t|) \ge G(t)$ ($\forall t \in \mathbb{R}$) and $f(t) = 0$ ($\forall t \le 0$), where $F(t) = \int_0^t f(s)ds$ and $G(t) = \int_0^t g(s)ds$.

($\mathscr{F}_2$) $\frac{f(t)}{|t|}$ and $\frac{g(t)}{|t|}$ are strictly increasing on interval $(0, \infty)$ for $f(t) \ne 0$ and $g(t) \ne 0$.

($\mathscr{F}_3$) Both $|f(t)| \le c(1 + |t|^{p-1})$ and $|g(t)| \le c(1 + |t|^{q-1})$ holds for some $c > 0$, where $2 < p$ and $q < \frac{2N}{N-2}$ if $N \ge 3$, or $2 < p$ and $q < \infty$ if $N = 1, 2$. Moreover, there holds $\frac{F(t)}{t^2} \to \infty$ and $\frac{G(t)}{t^2} \to \infty$, as $t \to \infty$.

From conditions of ($\mathscr{F}_1$)-($\mathscr{F}_2$), it is easy to see that

$$F(v) > 0, \ \ 2F(v) < f(v)v \ \ \text{and} \ \ G(u) > 0, \ \ 2G(u) < g(u)u, \ \ \forall\, u, v \ne 0, \tag{1.5}$$

where $F(u) = \int_0^u f(s)ds$. Following [**30, 31, 37**], in order to obtain some concentration phenomena for positive solutions, when condition ($\mathscr{V}_0$) holds, without loss of generality, we can assume $M(x_v) = \min_{x\in\mathscr{M}_1} M(x)$ and define the set

$$\mathscr{V} := \{x \in \mathscr{M}_1 : M(x) = M(x_v)\} \cup \{x \notin \mathscr{M}_1 : M(x) < M(x_v)\}.$$

Similarly, when condition ($\mathscr{V}_1$) holds, we assume $V(x_m) = \min_{x\in\mathscr{M}_2} V(x)$ and define the set

$$\mathscr{M} := \{x \in \mathscr{M}_2 : V(x) = V(x_m)\} \cup \{x \notin \mathscr{M}_2 : V(x) < V(x_m)\}.$$

Clearly, if $\mathscr{M}_1 \cap \mathscr{M}_2 \ne \emptyset$, one can deduce that $\mathscr{V} = \mathscr{M} = \mathscr{M}_1 \cap \mathscr{M}_2$.

Let

$$\mathscr{X}_{\varepsilon,\lambda}(u,v):=\int_{\mathbb{R}^N}(-\varepsilon^2|\nabla u|^2+|\nabla v|^2+V(x)|u|^2+V(x)|v|^2)-(G(u)+F(v))+\lambda\int_{\mathbb{R}^N}uv,$$

denote the energy function of system $(\mathscr{K}_\varepsilon)$.

Set

$$\gamma_\varepsilon=\inf\{\mathscr{X}_{\varepsilon,\lambda}(u,v):(u,v)\neq 0 \text{ is a solution of } (\mathscr{K}_\varepsilon)\}.$$

The solution $(\psi^0,\varphi^0)\neq 0$ with $\gamma_\varepsilon=\mathscr{X}_{\varepsilon,\lambda}(\psi^0,\varphi^0)$ is usually called a ground state solution. If $z^0=(\psi^0,\varphi^0)$ is a solution of system $(\mathscr{K}_\varepsilon)$ with $\psi^0>0$ and $\varphi^0>0$, we call z^0 a positive solution of system $(\mathscr{K}_\varepsilon)$. We say $z^0=(\psi^0,\varphi^0)>0$ (or ≥ 0) means that $\psi^0>0$ (or ≥ 0) and $\varphi^0>0$ (or ≥ 0). Let $\mathscr{L}'_\varepsilon$ denote the set of all positive ground state solutions of system $(\mathscr{K}_\varepsilon)$.

We summarize our main results as follows.

THEOREM 1.1. *Suppose that* $(\mathscr{V}_0)$ *and* $(\mathscr{F}_1)$-$(\mathscr{F}_3)$ *hold. Then for all sufficiently small* $\varepsilon>0$ *and* $0<\lambda<\delta:=\min\{1,V_0,M_0\}$, *there exists at least one ground state solution* $w_\varepsilon=(u_\varepsilon,v_\varepsilon)$ *in* $E=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ *to system* $(\mathscr{K}_\varepsilon)$. *In addition, if* V *and* M *are uniformly continuous functions, and* $g(t)=g_1(t)+c_1|t|^{q-2}t$ *and* $f(t)=f_1(t)+c_2|t|^{p-2}t$, *where* p *and* q *are given in* $(\mathscr{F}_3)$, $c_1,c_2>0$, $\frac{1}{2}g_1(t)t-G_1(t)\geq 0$ *and* $\frac{1}{2}f_1(t)t-F_1(t)\geq 0$, *and* $G_1(t)=\int_0^t g(s)ds$ *and* $F_1(t)=\int_0^t f(s)ds$, *then the following three statements are true.*

$(\mathscr{G}_1)$ $\mathscr{L}_\varepsilon$ *is compact in* $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

$(\mathscr{G}_2)$ *There exists a maximum point* x_ε *of* w_ε *such that*

$$\lim_{\varepsilon\to 0} dist(x_\varepsilon,\mathscr{V})=0.$$

For any sequences of such point $x_\varepsilon\to x_0$ *as* $\varepsilon\to 0$, $h_\varepsilon(x)=w_\varepsilon(\varepsilon x+x_\varepsilon)$ *uniformly converges to a positive ground state solution of the system*

$$(\mathscr{K}_{V_0})\qquad \begin{cases}-\Delta u+V(x_0)u=g(u)+\lambda v \ \text{in } \mathbb{R}^N,\\ -\Delta v+M(x_0)v=f(v)+\lambda u \ \text{in } \mathbb{R}^N,\\ u>0, v>0, \ \text{in } \mathbb{R}^N,\ u,v\in H^1(\mathbb{R}^N),\end{cases}$$

as $\varepsilon\to 0$.

$(\mathscr{G}_3)$ *There holds*

$$\lim_{|x|\to\infty} w_\varepsilon(x)=0 \quad \text{and} \quad \lim_{|x|\to\infty}|\nabla w_\varepsilon(x)|=0,$$

where $w_\varepsilon\in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ *with* $\sigma\in(0,1)$. *Furthermore, there exist two constants* C, $c>0$ *such that*

$$|w_\varepsilon(x)|\leq Ce^{-\frac{c}{\varepsilon}|x-x_\varepsilon|}$$

for all $x\in\mathbb{R}^N$.

THEOREM 1.2. *Suppose that* $(\mathscr{V}_1)$ *and* $(\mathscr{F}_1)$-$(\mathscr{F}_3)$ *hold. Then for all sufficiently small* $\varepsilon>0$ *and* $0<\lambda<\delta:=\min\{1,V_0,M_0\}$, *there is at least one ground state solution* $w_\varepsilon=(u_\varepsilon,v_\varepsilon)$ *in* E *to system* $(\mathscr{K}_\varepsilon)$. *Moreover, if* V *and* M *are uniformly continuous functions, and* $g(t)=g_1(t)+c_1t^{q-1}$ *and* $f(t)=f_1(t)+c_2t^{p-1}$, *where* p *and* q *are given in* $(\mathscr{F}_3)$, c_1, $c_2>0$, $\frac{1}{2}g_1(t)t-G_1(t)\geq 0$ *and* $\frac{1}{2}f_1(t)t-F_1(t)\geq 0$, *by replacing* $\mathscr{V}$ *with* $\mathscr{M}$ *in* $(\mathscr{G}_2)$, *then all three statements of* $(\mathscr{G}_1)$-$(\mathscr{G}_3)$ *remain true.*

From Theorems 1.1 and 1.2, we can obtain the following corollary immediately.

Corollary 1.3. Suppose that $(\mathscr{V}_0)$-$(\mathscr{V}_1)$ and $(\mathscr{F}_1)$-$(\mathscr{F}_3)$ hold. Then for all sufficiently small $\varepsilon > 0$ and $0 < \lambda < \delta := \min\{1, V_0, M_0\}$, system $(\mathscr{K}_\varepsilon)$ has two positive ground state solutions z_ε^1 and z_ε^2, which satisfy the concentration phenomena as described in Theorems 1.1 and 1.2 if V and M are uniformly continuous functions. Moreover, the two positive solutions z_ε^j $(j = 1, 2)$ are distinct if $\overline{\mathscr{V}} \cap \overline{\mathscr{M}} = \emptyset$.

Now, we shall state the existence of multiple positive solutions for system $(\mathscr{K}_\varepsilon)$. To do this, we recall that, if Y is a closed subset of a topological space X, the Ljusternik-Schnirelmann category $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. We assume that the following condition holds:

(K_1) If $\mathscr{M}_1 \cap \mathscr{M}_2 \neq \emptyset$, we set $\mathscr{O} = \mathscr{M}_1 \cap \mathscr{M}_2$.

In view of $(\mathscr{V}_0)$ or $(\mathscr{V}_1)$, the set $\mathscr{O}$ is compact. For any $\delta > 0$, we denote

$$\mathscr{O}_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \mathscr{O}) \leq \delta\}.$$

Theorem 1.4. *Suppose that $(\mathscr{V}_0)$ or $(\mathscr{V}_1)$, (K_1) and $(\mathscr{F}_1)$-$(\mathscr{F}_3)$ hold. If V and M are uniformly continuous functions, and $g(t) = g_1(t) + c_1 t^{q-1}$ and $f(t) = f_1(t) + c_2 t^{p-1}$, where p and q are given in $(\mathscr{F}_3)$, $c_1, c_2 > 0$, $\frac{1}{2} g_1(t)t - G_1(t) \geq 0$ and $\frac{1}{2} f_1(t)t - F_1(t) \geq 0$, then for each $\delta > 0$, there exist an $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$ and $0 < \lambda < \delta := \min\{1, V_0, M_0\}$, the following three statements are true.*

- $(\mathscr{I}_1)$ *There are at least $\mathrm{cat}_{\mathscr{O}_\delta}(\mathscr{O})$ positive solutions to system $(\mathscr{K}_\varepsilon)$.*
- $(\mathscr{I}_2)$ *If $w_\varepsilon = (u_\varepsilon, v_\varepsilon)$ denotes one of these positive solutions and $\sigma_\varepsilon \in \mathbb{R}^N$ such that $w_\varepsilon(\sigma_\varepsilon) = \max_{x \in \mathbb{R}^N} w_\varepsilon(x)$, then we have $\sigma_\varepsilon \to y_0 \in \mathscr{O}$.*
- $(\mathscr{I}_3)$ *There holds*

$$\lim_{|x| \to \infty} |w_\varepsilon(x)| = 0 \text{ and } u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N) \text{ with } \sigma \in (0, 1).$$

Furthermore, there exist two constants $C, c > 0$ such that

$$|w_\varepsilon(x)| \leq C e^{-\frac{c}{\varepsilon}|x - \sigma_\varepsilon|}$$

for all $x \in \mathbb{R}^N$.

Theorem 1.5. *Suppose that $(\mathscr{V}_2)$ and $(\mathscr{F}_1)$-$(\mathscr{F}_3)$ hold. Then for any $0 < \lambda < \delta := \min\{1, V_0, M_0\}$, there is no ground state solution for all $\varepsilon > 0$ to system $(\mathscr{K}_\varepsilon)$.*

Remark 1.6. In the present paper, we devote to the study of the multiplicity and concentration of positive solutions of system $(\mathscr{K}_\varepsilon)$, see Corollary 1.3 and Theorem 1.4. We wish that these results can fill in the gap of some existing results in the literature.

Remark 1.7. The conditions $(\mathscr{V}_0)$-$(\mathscr{V}_1)$ and the sets $\mathscr{V}$ and $\mathscr{M}$ were introduced in [**30, 31**], where they were used to study the concentration of solutions of Dirac and Schrödinger equations.

Making the change of variable $\varepsilon y = x$, we can reduce system $(\mathscr{K}_\varepsilon)$ to an equivalent system

$$(\mathscr{P}_\varepsilon) \qquad \begin{cases} -\Delta u + V_\varepsilon(x)u = g(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + M_\varepsilon(x)v = f(v) + \lambda u, & x \in \mathbb{R}^N, \\ u(x), v(x) \to 0, \text{ as } |x| \to \infty. \end{cases}$$

Throughout the paper we will often use the notation c to denote a generic positive constant. The value of c is allowed to vary from place to place.

The paper is organized as follows. In Section 2, we present some technical lemmas. In Section 3, we study some properties of the least energy solutions of the associated systems. Section 4 is dedicated to the concentration phenomena of ground state solutions of system $(\mathscr{P}_\varepsilon)$. In Section 5, we demonstrate the proofs of our main results on the existence and concentration of positive solutions for system $(\mathscr{P}_\varepsilon)$. In Section 6, we apply the Ljusternik-Schnirelmann category to discuss the multiplicity of positive solutions of system $(\mathscr{P}_\varepsilon)$.

2. Variational Setting and Nehari Manifolds

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Denote by $X\times Y$ the product space of X and Y with the norm $\|(x,y)\|_{X\times Y} := (\|x\|_X^2 + \|y\|_Y^2)^{\frac{1}{2}}$. If X and Y are Hilbert spaces with the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, then $X\times Y$ is also a Hilbert space with the inner product $((x,y),(w,z))_{X\times Y} = (x,w)_X + (y,z)_Y$.

Let $|\cdot|_q$ denote the usual L^q-norm and $(\cdot,\cdot)_2$ denote the usual $L^2 := L^2(\mathbb{R}^N)$-inner product. For the Hilbert space $H^1(\mathbb{R}^N)$, the inner product is denoted by

$$(u_1,u_2) = \int_{\mathbb{R}^N} (\nabla u_1 \nabla u_2 + u_1 u_2),$$

and correspondingly the norm is denoted by $\|u\| = (u,u)^{\frac{1}{2}}$.

Let $E = H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. For any $\varepsilon > 0$, let $H_\varepsilon^1 = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)u^2 < \infty\}$ denote the Hilbert space endowed with the inner product

$$(u,v)_\varepsilon = \int_{\mathbb{R}^N} \nabla u\nabla v + V(\varepsilon x)uv, \quad \text{for} \quad u,u \in H_\varepsilon,$$

and the induced norm is denoted by $\|u\|_\varepsilon^2 = (u,u)_\varepsilon$. Clearly, $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are equivalent norms for any $\varepsilon > 0$ and $V_\infty < \infty$. Similarly, one can define a Hilbert space by

$$H_\varepsilon^2 = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} M(\varepsilon x)u^2 < \infty \right\}.$$

Let $E_\varepsilon = H_\varepsilon^1 \times H_\varepsilon^2$. Clearly, $E = E_\varepsilon$ for each $\varepsilon > 0$. On E_ε we define a functional as

$$\begin{aligned}\mathscr{J}_{\varepsilon,\lambda}(z) &= \mathscr{J}_{\varepsilon,\lambda}(u,v)\\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 + V(\varepsilon x)|u|^2 + M(\varepsilon x)|v|^2\right)\\ &\quad - \int_{\mathbb{R}^N} (F(v)+G(u)) - \lambda \int_{\mathbb{R}^N} uv.\end{aligned}$$

for $u \in E_\varepsilon$. Obviously, $\mathscr{J}_{\varepsilon,\lambda} \in C^1(E_\varepsilon,\mathbb{R})$ and a standard argument shows that critical points of $\mathscr{J}_{\varepsilon,\lambda}$ are solutions of system $(\mathscr{P}_\varepsilon)$ (see [**2, 6, 7**]).

We may use the Nehari method to achieve our goal. This method has been widely used and developed in the past several decades, for instance, see [**2, 3, 16, 39, 40, 41**]. Following [**16**], we define the Nehari manifold corresponding to $\mathscr{J}_{\varepsilon,\lambda}$ by

$$\mathscr{N}_\varepsilon = \left\{(u,v) \in E_\varepsilon \setminus \{(0,0)\} : \mathscr{J}_{\varepsilon,\lambda}'(u,v)(u,v) = 0\right\}.$$

One can see that for $z=(u,v)\in\mathscr{N}_\varepsilon$, it has

$$\int_{\mathbb{R}^N}(|\nabla u|^2+V_\varepsilon(x)|u|^2+|\nabla v|^2+M_\varepsilon(x)|v|^2)=\int_{\mathbb{R}^N}(f(v)v+g(u)u)+2\lambda\int_{\mathbb{R}^N}uv, \tag{2.1}$$

where $V_\varepsilon(x)=V(\varepsilon x)$. This implies that for $u\in\mathscr{N}_\varepsilon$, there holds

$$\mathscr{J}_{\varepsilon,\lambda}|_{\mathscr{N}_\varepsilon}(u,v)=\int_{\mathbb{R}^N}\left[\left(\frac{1}{2}f(v)v-F(v)\right)+\left(\frac{1}{2}g(u)u-G(u)\right)\right] \tag{2.2}$$

Here, let us present some elementary properties for $\mathscr{N}_\varepsilon$.

Lemma 2.1. *Under the assumptions of Theorem 1.1, for* $0<\lambda<\min\{1,V_0,M_0\}$ *and* $\varepsilon>0$, *we have that for all* $z=(u,v)\in S_\varepsilon=\{z\in E_\varepsilon:\|z\|_\varepsilon=(\|u\|_\varepsilon^2+\|v\|_\varepsilon^2)^{\frac{1}{2}}=1\}$, *there exists a unique* $t_z>0$ *such that* $t_zz\in\mathscr{N}_\varepsilon$. *Moreover,* $m_\varepsilon(z)=t_zz$ *is the unique maximum of* $\mathscr{W}_{\varepsilon,\lambda}$ *on* E_ε. *Conversely, for each* $z\in\mathscr{N}_\varepsilon$, *we define*

$$\check{m}_\varepsilon(z)=m_\varepsilon^{-1}(z)=\frac{z}{\|z\|_\varepsilon}\in S_\varepsilon. \tag{2.3}$$

That is, the mapping m_ε *is a homeomorphism between* $S_\varepsilon\subset E_\varepsilon$ *and* $\mathscr{N}_\varepsilon$, *and* $\mathscr{N}_\varepsilon$ *is a Nehari manifold.*

Proof. Following [**11, 12**], for each $z=(u,v)\in S_\varepsilon$ and $t>0$, we define $k(t)=\mathscr{J}_{\varepsilon,\lambda}(tu,tv)$. It is easy to verify that $k(0)=0$ and $k(t)<0$ for the large $t>0$. Moreover, we claim that

$$k(t)>0 \quad \text{for} \quad t>0 \text{ small}. \tag{2.4}$$

Indeed, from the conditions $(\mathscr{F}_1)$-$(\mathscr{F}_3)$, we deduce that for each $\epsilon>0$ there exists a $C_\epsilon>0$ such that

$$\begin{aligned}&|f(v)|\le\epsilon|v|+C_\epsilon|v|^{p-1} \quad\text{and}\quad |F(v)|\le\epsilon|v|^2+C_\epsilon|v|^p,\\ &|g(u)|\le\epsilon|u|+C_\epsilon|u|^{q-1} \quad\text{and}\quad |G(u)|\le\epsilon|u|^2+C_\epsilon|u|^q,\end{aligned} \tag{2.5}$$

where p and q are given in $(\mathscr{F}_3)$. It follows that

$$\begin{aligned}k(t)=&\frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+V(\varepsilon x)|u|^2+M(\varepsilon x)|v|^2)\\ &-\int_{\mathbb{R}^N}(F(tv)+G(tu))-t^2\lambda\int_{\mathbb{R}^N}uv.\\ \ge&\frac{\delta_0t^2}{2}(\|u\|^2+\|v\|^2)-\epsilon t^2(|u|_2^2+|v|_2^2)-C_\epsilon(t^p|v|_p^p+t^q|u|_q^q)-\lambda t^2|u|_2|v|_2\\ \ge&\frac{t^2}{2}(\delta_0-\lambda)(\|u\|^2+\|v\|^2)-t^2(\|u\|^2+\|v\|^2)-cC_\epsilon(t^p\|v\|^p+t^q\|u\|^q)\\ =&\frac{t^2}{2}(\delta_0-\lambda-c\epsilon)(\|u\|^2+\|v\|^2)-cC_\epsilon(t^p\|v\|^p+t^q\|u\|^q),\end{aligned}$$

where $\delta_0=\min\{1,V_0,M_0\}$. Since $p,q>2$, if ϵ is small enough such that $\lambda+c\epsilon<\delta_0$, we find that $k(t)>0$ for the small $t>0$. Hence, $\max_{t>0}k(t)$ is attained at a $t=t_z>0$ so that $k'(t_z)=0$ and $t_zz\in\mathscr{N}_\varepsilon$. Suppose that there exist $t_{z,1}>t_{z,2}>0$

such that $t_{z,1}z,\ t_{z,2}z \in \mathscr{N}_\varepsilon$. Then we have

$$
(2.6)\qquad
\begin{aligned}
t_{z,1}^2\left(\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2\right) &= \int_{\mathbb{R}^N} (f(t_{z,1}v)t_{z,1}v + g(t_{z,1}u)t_{z,1}u) + \lambda t_{z,1}^2 \int_{\mathbb{R}^N} uv,\\
t_{z,2}^2(\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2) &= \int_{\mathbb{R}^N} (f(t_{z,2}v)t_{z,2}v + g(t_{z,2}u)t_{z,2}u) + \lambda t_{z,2}^2 \int_{\mathbb{R}^N} uv.
\end{aligned}
$$

A direct calculation gives

$$
0 = \int_{\mathbb{R}^N}\left(\frac{f(t_{z,1}v)}{t_{z,1}v} - \frac{f(t_{z,2}v)}{t_{z,2}v}\right)v^2 + \int_{\mathbb{R}^N}\left(\frac{g(t_{z,1}u)}{t_{z,1}u} - \frac{g(t_{z,2}u)}{t_{z,2}u}\right)u^2,
$$

which makes no sense in view of $(\mathscr{F}_2)$ and $t_{z,1} > t_{z,2} > 0$. Conversely, the inverse of m_ε is given by (2.3). That is, the mapping m_ε is a homeomorphism between $S_\varepsilon \subset E_\varepsilon$ and $\mathscr{N}_\varepsilon$. □

LEMMA 2.2. *Under the assumptions of Lemma 2.1, for* $0 < \lambda < \min\{1, V_0, M_0\}$ *and* $\varepsilon > 0$*, the following three properties are true.*

- (A_1) *The set* $\mathscr{N}_\varepsilon$ *is bounded away from* 0*. Furthermore,* $\mathscr{N}_\varepsilon$ *is closed in* E_ε*.*
- (A_2) *There is* $\alpha > 0$ *such that* $t_u \geq \alpha$ *for each* $u \in S_\varepsilon$ *and for each compact subset* $\mathcal{W} \subset S_\varepsilon$*, there exists* $C_{\mathcal{W}} > 0$ *such that* $t_u \leq C_{\mathcal{W}}$*, for all* $u \in \mathcal{W}$*.*
- (A_3) $c_\varepsilon = \inf_{\mathscr{N}_\varepsilon} \mathscr{J}_{\varepsilon,\lambda} \geq \rho > 0$ *and* $\mathscr{J}_{\varepsilon,\lambda}$ *is bounded below on* $\mathscr{N}_\varepsilon$*, where* $\rho > 0$ *is independent of* ε*.*

PROOF. (A_1) For $u \in \mathscr{N}_\varepsilon$, it follows from (2.1) and (2.5) that

$$
(2.7)\qquad
\begin{aligned}
\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2 &\leq c\epsilon(\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2) + cC_\epsilon(\|u\|_\varepsilon^q + \|v\|_\varepsilon^p) + \lambda|u|_2|v|_2\\
&\leq c\epsilon(\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2) + cC_\epsilon(\|u\|_\varepsilon^q + \|v\|_\varepsilon^p) + \frac{\lambda}{2V_0}(\|u\|_\varepsilon + \|v\|_\varepsilon).
\end{aligned}
$$

From (2.7), we have

$$
\left(1 - c\epsilon - \frac{\lambda}{2V_0}\right)(\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2) \leq cC_\epsilon(\|u\|_\varepsilon^q + \|v\|_\varepsilon^p).
$$

Since $\lambda < V_0$, we can choose ϵ small enough such that $1 - c\epsilon - \frac{\lambda}{2V_0} > 0$. There is a $\sigma > 0$ (independent of ε) such that

$$
\|u\|_\varepsilon^{q-2} + \|v\|_\varepsilon^{p-2} \geq \sigma.
$$

It further gives

$$
(2.8)\qquad \|z\|_\varepsilon^{p-2} + \|z\|_\varepsilon^{q-2} \geq \|u\|_\varepsilon^{q-2} + \|v\|_\varepsilon^{p-2} \geq \sigma.
$$

That is, the set $\mathscr{N}_\varepsilon$ is bounded away from 0. To prove that the set $\mathscr{N}_\varepsilon$ is closed in E_ε, we let $\{z_n\} \subset \mathscr{N}_\varepsilon$ such that $z_n \to z$ in E_ε. We know that $\mathscr{J}'_{\varepsilon,\lambda}(z_n)$ is bounded, and then from

$$
\begin{aligned}
\mathscr{J}'_{\varepsilon,\lambda}(z_n)z_n - \mathscr{J}'_{\varepsilon,\lambda}(z)z &= (\mathscr{J}'_{\varepsilon,\lambda}(z_n) - \mathscr{J}'_{\varepsilon,\lambda}(z))z + \mathscr{J}'_{\varepsilon,\lambda}(z_n)(z_n - z)\\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}
$$

we see that $\mathscr{J}'_{\varepsilon,\lambda}(z)z = 0$. It follows from (2.7) that

$$
\|z\|_\varepsilon^2 = \lim_{n\to\infty} \|z_n\|_\varepsilon^2 \geq \delta > 0.
$$

So we arrive at $z \in \mathscr{N}_\varepsilon$.

(A_2) For $\{z_n\} \subset E_\varepsilon \setminus \{0\}$, there exist t_{z_n} such that $t_{z_n}z_n \in \mathscr{N}_\varepsilon$. From Part (A_1), one can see that $\|t_{z_n}z_n\|_\varepsilon = t_{z_n}\|z_n\|_\varepsilon \geq \sigma > 0$. It is impossible to have that

$t_{z_n} \to 0$, as $n \to \infty$. To prove $t_z \le C_{\mathcal{W}}$ for all $z \in \mathcal{W} \subset S_\varepsilon$, by way of contradiction, we suppose that there exists $\{z_n\} \subset \mathcal{W} \subset S_\varepsilon$ such that $t_n = t_{z_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists $z = (u,v) \in \mathcal{W}$ such that $z_n = (u_n, v_n) \to z = (u,v)$ in E_ε and $z_n(x) \to z(x)$ a.e. on $\mathbb{R}^N$ after passing to a subsequence. Then, it follows from $(\mathscr{F}_3)$ that

$$\begin{aligned}
&\mathscr{J}_{\varepsilon,\lambda}(z_n)\\
=&\frac{t_n^2}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2+V(\varepsilon x)|u_n|^2+M(\varepsilon x)|v_n|^2\right)\\
&-\int_{\mathbb{R}^N}(F(t_nv_n)+G(t_nu_n))-\lambda t_n^2\int_{\mathbb{R}^N}u_nv_n\\
\le& t_n^2\left[\frac{1}{2}(\|u_n\|_\varepsilon^2+\|v_n\|_\varepsilon^2)+\lambda\|u_n\|_\varepsilon\|v_n\|_\varepsilon-\int_{\mathbb{R}^N}\left(\frac{F(t_nv_n)}{(t_nv_n)^2}v_n^2+\frac{G(t_nu_n)}{(t_nu_n)^2}u_n^2\right)\right]\\
\to&-\infty,\quad \text{as}\quad n\to\infty.
\end{aligned}$$

However, from (2.2) we know that $\mathscr{J}_{\varepsilon,\lambda}(t_nz_n) \ge 0$. This yields a contradiction.

(A_3) For $\varepsilon > 0$, $\lambda > 0$, $s > 0$ and $z = (u,v) \in E_\varepsilon \setminus \{0\}$, similar to the proof of Lemma 2.1, we can see that for each $\epsilon > 0$, there exists a $C_\epsilon > 0$ such that

$$\begin{aligned}
\mathscr{J}_{\varepsilon,\lambda}(sz) =& \frac{s^2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+V(\varepsilon x)|u|^2+M(\varepsilon x)|v|^2)\\
&-\int_{\mathbb{R}^N}(F(sv)+G(su))-\lambda s^2\int_{\mathbb{R}^N}uv\\
\ge& \frac{s^2}{2}(\delta_0-\lambda-c\epsilon)\left(\|u\|^2+\|v\|^2\right)-cC_\epsilon\left(s^p\|v\|^p+s^q\|u\|^q\right),
\end{aligned}$$

where ϵ is small enough such that $\lambda + c\epsilon < \delta_0$. So, there is a $\rho > 0$ such that $\mathscr{J}_{\varepsilon,\lambda}(sz) \ge \rho > 0$ for small $s > 0$. On the other hand, it follows from Lemma 2.1 that

$$c_\varepsilon = \inf_{\mathscr{N}_\varepsilon}\mathscr{J}_{\varepsilon,\lambda}(u) = \inf_{w\in E_\varepsilon\setminus\{0\}}\max_{s>0}\mathscr{J}_{\varepsilon,\lambda}(sw) = \inf_{w\in S_\varepsilon}\max_{s>0}\mathscr{J}_{\varepsilon,\lambda}(sw).$$

Hence, we obtain $c_\varepsilon \ge \rho > 0$ and $\mathscr{J}_{\varepsilon,\lambda}|_{\mathscr{N}_\varepsilon} \ge \rho > 0$. □

Let us move to the functionals $\hat{\mathscr{U}}_{\varepsilon,\lambda} : E_\varepsilon \setminus \{0\} \to \mathbb{R}$ and $\mathscr{U}_{\varepsilon,\lambda} : S_\varepsilon \to \mathbb{R}$ defined by

$$\hat{\mathscr{U}}_{\varepsilon,\lambda}(z) = \mathscr{J}_{\varepsilon,\lambda}(\hat{m}_\varepsilon(z)) \quad\text{and}\quad \mathscr{U}_{\varepsilon,\lambda} = \hat{\mathscr{J}}_{\varepsilon,\lambda}|_{S_\varepsilon},$$

respectively, where $\hat{m}_\varepsilon(z) = t_zz = t_z(u,v)$ is the unique maximum of $\mathscr{J}_{\varepsilon,\lambda}$ on E_ε.

Lemma 2.3. *(See* [**16**, Corollary 3.3]*) Under the assumptions of Lemma 2.3, for $\lambda \ne 0$ and $\varepsilon > 0$ we have that*

(B_1) *$\mathscr{U}_{\varepsilon,\lambda} \in C^1(S_\varepsilon, \mathbb{R})$, and*

$$\mathscr{U}'_{\varepsilon,\lambda}(w)z = \|m_\varepsilon(w)\|_\varepsilon\mathscr{J}'_{\varepsilon,\lambda}(m_\varepsilon(w))z \quad \text{for} \quad z \in \mathcal{T}_wS_\varepsilon.$$

(B_2) *$\{w_n\}$ is a Palais-Smale sequence for $\mathscr{U}_{\varepsilon,\lambda}$ if and only if $\{m_\varepsilon(w_n)\}$ is a Palais-Smale sequence for $\mathscr{J}_{\varepsilon,\lambda}$. If $\{u_n\} \subset \mathscr{N}_\varepsilon$ is a bounded Palais-Smale sequence for $\mathscr{J}_{\varepsilon,\lambda}$, then $\check{m}_\varepsilon(z_n)$ is a Palais-Smale sequence for $\mathscr{U}_{\varepsilon,\lambda}$, where $\check{m}_\varepsilon(z) = m_\varepsilon^{-1}(z) = \frac{z}{\|z\|_\varepsilon}$.*

(B_3) *$\inf_{S_\varepsilon}\mathscr{U}_{\varepsilon,\lambda} = \inf_{\mathscr{N}_\varepsilon}\mathscr{J}_{\varepsilon,\lambda} = c_\varepsilon$. Moreover, $z \in S_\varepsilon$ is a critical point of $\mathscr{U}_{\varepsilon,\lambda}$ if and only if $m_\varepsilon(z)$ is a critical point of $\mathscr{J}_{\varepsilon,\lambda}$, and the corresponding critical values coincide.*

REMARK 2.4. By Lemma 2.1, we note that the infimum of $\mathscr{J}_{\varepsilon,\lambda}$ over $\mathscr{N}_\varepsilon$ has the following minimax characterization:

$$c_\varepsilon = \inf_{\mathscr{N}_\varepsilon} \mathscr{J}_{\varepsilon,\lambda}(u) = \inf_{w\in E_\varepsilon\setminus\{0\}} \max_{s>0} \mathscr{J}_{\varepsilon,\lambda}(sw) = \inf_{w\in S_\varepsilon} \max_{s>0} \mathscr{J}_{\varepsilon,\lambda}(sw). \tag{2.9}$$

3. Ground State Solutions for Autonomous Equations

In this section, we shall prove some properties of the least energy solutions of the autonomous equations. Specifically, for each $\mu \geq V_0$, $\sigma \geq M_0$ and $\lambda \neq 0$, we consider the following system

$$(\mathcal{P}_{\mu\sigma,\lambda}) \qquad \begin{cases} -\Delta u + \mu u = g(u) + \lambda v & \text{in } \mathbb{R}^N, \\ -\Delta v + \sigma v = f(v) + \lambda u & \text{in } \mathbb{R}^N, \\ u(x) \to 0, v(x) \to 0, & \text{as } |x| \to \infty. \end{cases}$$

For any $\mu > 0$, let

$$H_\mu = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \mu u^2 < \infty \right\}$$

be the Sobolev space endowed with the norm $\|u\|_\mu^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \mu|u|^2$. Clearly, the norms $\|\cdot\|_\mu$ and $\|\cdot\|$ are equivalent. Let $E_{\mu\sigma} = H_\mu \times H_\sigma$. It is easy to see $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) = E_{\mu\sigma}$ for each $\mu, \sigma > 0$. We define the functional on $E_{\mu\sigma}$ by

$$\begin{aligned} \mathscr{J}_{\mu\sigma,\lambda}(z) = \mathscr{J}_{\mu\sigma,\lambda}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + \mu|u|^2 + \sigma|v|^2) \\ &\quad - \int_{\mathbb{R}^N} (F(v) + G(u)) - \lambda \int_{\mathbb{R}^N} uv \end{aligned}$$

for $u \in E_{\mu\sigma}$. Obviously, $\mathscr{J}_{\mu\sigma,\lambda} \in C^1(E_{\mu\sigma}, \mathbb{R})$ and a standard argument shows that critical points of $\mathscr{J}_{\mu\sigma,\lambda}$ are solutions of system $(\mathscr{P}_{\mu\sigma,\lambda})$ (see [**2, 6, 7**]). In order to find critical points of the functional $\mathscr{J}_{\mu\sigma,\lambda}$, we will employ the Nehari manifold methods. We define the Nehari manifold associated with $\mathscr{J}_{\mu\sigma,\lambda}$ by

$$\mathscr{N}_{\mu\sigma} = \{u \in E_{\mu\sigma} \setminus \{0\} : \mathscr{J}'_{\mu\sigma,\lambda}(u,v)(u,v) = 0\}.$$

For $u \in \mathscr{N}_{\mu\sigma}$, one can see that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + \mu|u|^2) + \int_{\mathbb{R}^N} (|\nabla v|^2 + \sigma|v|^2) = \int_{\mathbb{R}^N} (f(v)v + g(u)u) + 2\lambda \int_{\mathbb{R}^N} uv. \tag{3.1}$$

This implies that for $u \in \mathscr{N}_{\mu\sigma}$, there holds

$$\mathscr{J}_{\mu\sigma,\lambda}|_{\mathcal{N}_{\mu\sigma}} = \int_{\mathbb{R}^N} \left[(\frac{1}{2}f(v)v - F(v)) + (\frac{1}{2}g(u)u - G(u)) \right]. \tag{3.2}$$

Similar to Lemmas 2.1 and 2.2, we know that $\mathscr{N}_{\mu\sigma}$ has the following elementary properties.

LEMMA 3.1. *Under the assumptions of Theorem 1.1, for* $0 < \lambda \leq \delta_0 = \min\{1, V_0, M_0\}$ *and* $\mu, \sigma > 0$, *the following statements are true.*

- (C_1) *For all* $z \in S_{\mu\sigma} = \{z = (u,v) \in E_{\mu\sigma} : \|u\|_\mu^2 + \|v\|_\sigma^2 = 1\}$, *there exists a unique* $t_u > 0$ *such that* $t_z z \in \mathscr{N}_{\mu\sigma}$. *Moreover,* $m_{\mu\sigma}(z) = t_z z$ *is the unique maximum of* $\mathscr{J}_{\mu\sigma,\lambda}$ *on* $E_{\mu\sigma}$.
- (C_2) *The set* $\mathscr{N}_{\mu\sigma}$ *is bounded away from* 0. *Furthermore,* $\mathscr{N}_{\mu\sigma}$ *is closed in* $E_{\mu\sigma}$

(C_3) *There is* $\delta > 0$ *such that* $t_z \geq \delta$ *holds for each* $u \in S_{\mu\sigma}$ *and for each compact subset* $\mathcal{W} \subset S_{\mu\sigma}$, *there exists* $C_{\mathcal{W}} > 0$ *such that* $t_z \leq C_{\mathcal{W}}$ *holds for all* $z \in \mathcal{W}$

(C_4) $\mathscr{N}_{\mu\sigma}$ *is a regular manifolds diffeomorphic to the sphere of* $E_{\mu\sigma}$.

(C_5) $c_{\mu\sigma} = \inf_{\mathscr{N}_{\mu\sigma}} \mathscr{J}_{\mu\sigma,\lambda} > 0$, *and* $\mathscr{J}_{\mu\sigma,\lambda}|_{\mathscr{N}_{\mu\sigma}}$ *is bounded below by some positive constant.*

From (C_1) of Lemma 3.1, we know that for each $z \in E_{\mu\sigma} \setminus \{0\}$, there exists a unique $t_z > 0$ such that $t_z z \in \mathscr{N}_{\mu\sigma}$. So we can define the mapping: $\hat{m}_{\mu\sigma}$: $E_{\mu\sigma} \setminus \{0\} \to \mathscr{N}_{\mu\sigma}$ by

$$\hat{m}_{\mu\sigma}(z) = t_z z.$$

Clearly, $m_{\mu\sigma} = \hat{m}_{\mu\sigma}|_{S_{\mu\sigma}}$.

Let

$$\begin{aligned}
&\hat{\mathscr{U}}_{\mu\sigma,\lambda} : E_{\mu\sigma} \setminus \{0\} \to \mathbb{R},\\
&\hat{\mathscr{Y}}_{\mu\sigma,\lambda}(w) := \mathscr{J}_{\mu\sigma,\lambda}(\hat{m}_{\mu\sigma}(w)),\\
&\mathscr{U}_{\mu\sigma,\lambda} := \hat{\mathscr{U}}_{\mu\sigma,\lambda}|_{S_{\mu\sigma}}.
\end{aligned}$$

If the inverse of the mapping $m_{\mu\sigma}$ to $S_{\mu\sigma}$ is given by

$$\check{m}_{\mu\sigma} = m_{\mu\sigma}^{-1} : \mathscr{N}_{\mu\sigma} \to S_{\mu\sigma} \quad \text{and} \quad \check{m}_{\mu\sigma} = \frac{u}{\|u\|_{\mu\sigma}},$$

then by using the same arguments as in [**16**, Corollary 10] we have the following lemma.

LEMMA 3.2. *Under the assumptions of Lemma 3.1, for*

$$0 < \lambda \leq \delta_0 = \min\{1, V_0, M_0\}$$

and $\mu, \sigma > 0$ *and* $\varepsilon > 0$, *the following statements are true.*

(i) $\mathscr{U}_{\mu\sigma,\lambda} \in C^1(S_{\mu\sigma}, \mathbb{R})$, *and*

$$\mathscr{U}'_{\mu\sigma,\lambda}(w)z = \|m_{\mu\sigma}(w)\|_{\mu\sigma} \mathscr{J}'_{\mu\sigma,\lambda}(m_{\mu\sigma}(w))z, \ \textit{for } z \in \mathcal{T}_w S_{\mu\sigma}.$$

(ii) $\{w_n\}$ *is a Palais-Smale sequence for* $\mathscr{U}_{\mu\sigma,\lambda}$ *if and only if* $\{m_{\mu\sigma}(w_n)\}$ *is a Palais-Smale sequence for* $\mathscr{J}_{\mu\sigma,\lambda}$. *If* $\{z_n\} \subset \mathscr{N}_{\mu\sigma}$ *is a bounded Palais-Smale sequence for* $\mathscr{J}_{\mu\sigma,\lambda}$, *then* $\check{m}_{\mu\sigma}(z_n)$ *is a Palais-Smale sequence for* $\mathscr{J}_{\mu\sigma,\lambda}$, *where* $\check{m}_{\mu\sigma}(z) = m_{\mu\sigma}^{-1}(z) = \frac{z}{\|z\|_{\mu\sigma}}$.

(iii) There holds

$$\inf_{S_{\mu\sigma}} \mathscr{U}_{\mu\sigma,\lambda} = \inf_{\mathscr{N}_{\mu\sigma}} \mathscr{J}_{\mu\sigma,\lambda} = c_{\mu\sigma}.$$

Moreover, $z \in S_{\mu\sigma}$ *is a critical point of* $\mathscr{U}_{\mu\sigma,\lambda}$ *if and only if* $m_{\mu\sigma}(z)$ *is a critical point of* $\mathscr{J}_{\mu\sigma,\lambda}$, *and the corresponding critical values coincide.*

REMARK 3.3. By Lemma 3.1, we note that the infimum of $\mathscr{J}_{\mu\sigma,\lambda}$ over $\mathscr{N}_{\mu\sigma}$ has the following minimax characterization:

(3.3)
$$0 < c_{\mu\sigma} = \inf_{z \in \mathscr{N}_{\mu\sigma}} \mathscr{J}_{\mu\sigma,\lambda}(z) = \inf_{w \in E_{\mu\sigma} \setminus \{0\}} \max_{s>0} \mathscr{J}_{\mu\sigma,\lambda}(sw) = \inf_{w \in S_{\mu\sigma}} \max_{s>0} \mathscr{J}_{\mu\sigma,\lambda}(sw).$$

To prove compactness of minimizing sequences for $\mathscr{J}_{\mu\sigma,\lambda}$, we need the following technical lemma [**18**].

Lemma 3.4. *Let $r > 0$ and $q \in [2, 2^*]$. If $\{z_n\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |z_n|^q = 0,$$

then we have $z_n \to 0$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. Moreover, if $q = 2^*$, then $z_n \to 0$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $p \in (2, 2^*]$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$.

The following lemma is regarding the property for minimizing sequences.

Lemma 3.5. *Let $\{z_n\} \subset \mathscr{N}_{\mu\sigma}$ be a minimizing sequence for $\mathscr{J}_{\mu\sigma,\lambda}$. Then $\{z_n\}$ is bounded. Moreover, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \geq \delta > 0,$$

where $B_r(y_n) = \big\{y \in \mathbb{R}^N : |y - y_n| \leq r\big\}$.

Proof. We first prove the boundedness of $\{z_n\}$. By way of contradiction, we suppose that there exists a sequence $\{z_n\} \subset \mathscr{N}_{\mu\sigma}$ such that $\|z_n\|_{\mu\sigma} \to \infty$ and $\mathscr{J}_{\mu\sigma,\lambda}(u_n) \to c_{\mu\sigma}$. Let

$$w_n = \frac{z_n}{\|z_n\|_{\mu\sigma}} \quad \text{and} \quad z_n = (z_n^1, z_n^2).$$

Then $w_n \rightharpoonup w$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. We have two cases: either $\{w_n\}$ is vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |w_n|^2 = 0, \tag{3.4}$$

or it is non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |w_n|^2 \geq \delta > 0. \tag{3.5}$$

As shown in [**19, 23**], we will reveal that neither (3.4) nor (3.5) takes place, which leads to the desired contradiction.

If $\{w_n\}$ is vanishing, it follows Lemma 3.4 that

$$w_n \to 0 \text{ in } L^t = L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$$

for $t \in \Big(2, \frac{2N}{N-2}\Big)$. From (2.5), we see that

$$|F(v) + G(u)| \leq \epsilon(|u|^2 + |v|^2) + C'_\epsilon(|u|^t + |v|^t),$$

where $t = \max\{p, q\}$ and $C'_\epsilon > 0$. This gives

$$\int_{\mathbb{R}^N} (G(rw_n^1) + F(hw_n^2)) \to 0 \text{ as } n \to \infty$$

for each $h, r \in \mathbb{R}$, where $w_n = (w_n^1, w_n^2)$. Making use of inequalities $0 < \lambda < \min\{1, V_0, M_0\}$, $\mu \geq V_0 > 0$ and $\sigma \geq M_0 > 0$, we have

$$
\begin{aligned}
c_{\mu\sigma} + o(1) &\geq \mathscr{J}_{\mu\sigma,\lambda}(z_n) \geq \mathscr{J}_{\mu\sigma,\lambda}(hw_n) \\
&= \frac{h^2}{2}\int_{\mathbb{R}^N}(|\nabla w_n^1|^2 + |\nabla w_n^2|^2 + \mu|w_n^1|^2 + \sigma|w_n^1|^2) - \int_{\mathbb{R}^N}(F(hw_n^2) + G(hw_n^1)) \\
&\quad - \lambda h^2 \int_{\mathbb{R}^N} w_n^1 w_n^2 \\
&\geq \frac{1}{2}h^2(1-\lambda) - \int_{\mathbb{R}^N}(F(hw_n^2) + G(hw_n^1)) \\
&\to \frac{1}{2}h^2(1-\lambda), \text{ as } n \to \infty.
\end{aligned}
$$

This yields a contradiction when h is large enough.

On the other hand, if non-vanishing occurs, we set $v_n(x) = w_n(x + y_n)$. Since $\|v_n\|_{\mu\sigma} = \|w_n\|_{\mu\sigma}$, v_n is bounded in $E_{\mu\sigma}$. Then we extract a subsequence again, and then there holds $v_n(x) \rightharpoonup v_0(x)$ in E. By (3.5) we know that $v_0 \not\equiv 0$. In particular, we can find a set $\mathcal{O} \subset \mathbb{R}^N \times \mathbb{R}^N$ such that

$$
\text{meas}(\mathcal{O}) > 0 \quad \text{and} \quad v_n(x) \to v_0(x) \neq 0 \quad \text{for } x \in \mathcal{O}. \tag{3.6}
$$

From condition $(\mathscr{F}_3)$, for the large n we deduce that

$$
\begin{aligned}
0 &\leq \frac{\mathscr{J}_{\mu\sigma,\lambda}(z_n)}{\|z_n\|_{\mu\sigma}^2} \\
&= \frac{1}{2} - \lambda \int_{\mathbb{R}^N} w_n^1 w_n^2 - \frac{\int_{\mathbb{R}^N}(F(z_n^2) + G(z_n^1))}{\|z_n\|_{\mu\sigma}^2} \\
&\leq -\frac{\int_{\mathbb{R}^N}(F(z_n^2) + G(z_n^1))}{\|z_n\|_{\mu\sigma}^2} \\
&= -\int_{\mathcal{O}} \frac{(F(z_n^2(x+y_n)) + G(z_n^1(x+y_n)))}{|z_n^2(x+y_n)|^2 + |z_n^1(x+y_n)|^2}|w_n(x+y_n)|^2 \\
&< 0.
\end{aligned}
$$

This is, obviously, a contradiction.

Note that $\{z_n\}$ is bounded. If

$$
\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |z_n|^2 = 0,
$$

from Lemma 3.4 we deduce that

$$
u_n \to 0 \quad \text{in} \quad L^t = L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)
$$

for $t \in \left(2, \frac{2N}{N-2}\right)$. Since $\mathscr{J}'_{\mu\sigma,\lambda}(z_n)z_n = 0$, we get

$$
\int_{\mathbb{R}^N}(|\nabla z_n^1|^2 + |\nabla z_n^2|^2 + \mu|z_n^1|^2 + \sigma|z_n^2|^2) = \int_{\mathbb{R}^N} f(z_n^2)z_n^2 + g(z_n^1)z_n^1 + 2\lambda\int_{\mathbb{R}^N} z_n^1 z_n^2,
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla z_n^1|^2 + |\nabla z_n^2|^2) + (\mu - \lambda)\int_{\mathbb{R}^N}|z_n^1|^2 + (\sigma - \lambda)\int_{\mathbb{R}^N}|z_n^2|^2 \\
&\leq \int_{\mathbb{R}^N} f(z_n^2)z_n^2 + g(z_n^1)z_n^1 \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Consequently, we see that $z_n \to 0$ in $E_{\mu\sigma}$ as $n \to \infty$. This is a contradiction with the fact that $\mathscr{N}_{\mu\sigma}$ is bounded away from 0. $\square$

THEOREM 3.6. *Under the assumptions described in Theorem 1.1, the following three statements are true.*

(A_1) *There is at least one positive ground state solution* $z_{\mu\sigma,\lambda} = (u_{\mu\sigma\kappa}, v_{\mu\sigma\kappa})$ *in* $E_{\mu\sigma}$ *to system* $(\mathscr{P}_{\mu\sigma,\lambda})$ *when* $0 < \lambda \le \delta_0 = \min\{1, V_0, M_0\}$ *and* $\mu, \sigma > 0$.

(A_2) *When* $0 < \lambda \le \delta_0 = \min\{1, V_0, M_0\}$ *and* $\mu, \sigma > 0$, *there holds*

$$\lim_{|x|\to\infty} |u_{\mu\sigma,\lambda}(x)| = \lim_{|x|\to\infty} |v_{\mu\sigma,\lambda}(x)| = 0,$$

and

$$\lim_{|x|\to\infty} |\nabla u_{\mu\sigma,\lambda}(x)| = \lim_{|x|\to\infty} |\nabla v_{\mu\sigma,\lambda}(x)| = 0,$$

where $u_{\mu\sigma,\lambda}, v_{\mu\sigma,\lambda} \in C^{1,\sigma}_{loc}$ *with* $\sigma \in (0,1)$. *Furthermore, there exist* $C, c > 0$ *such that*

$$u_{\mu\sigma\kappa}(x) + u_{\mu\sigma\kappa}(x) \le Ce^{-c|x - x_{\mu\sigma,\lambda}|},$$

where

$$|z_{\mu\sigma,\lambda}(x_{\mu\sigma,\lambda})| = \max_{x\in\mathbb{R}^N} |z_{\mu\sigma,\lambda}(x)|.$$

(A_3) *If* $g(t) = g_1(t) + c_1 t^{q-1}$ *and* $f(t) = f_1(t) + c_2 t^{p-1}$ *(p and q are given in* $(\mathscr{F}_3)$*),* $c_1, c_2 > 0$, $\frac{1}{2}g_1(t)t - G_1(t) \ge 0$ *and* $\frac{1}{2}f_1(t)t - F_1(t) \ge 0$, *then* $\mathscr{L}_{\mu\sigma,\lambda}$ *is compact in* $E_{\mu\sigma}$ *when* $0 < \lambda \le \delta_0 = \min\{1, V_0, M_0\}$ *and* $\lambda < \mu, \sigma$, *where* $F_1(t) = \int_0^t f_1(s)ds$, $G_1(t) = \int_0^t g_1(s)ds$, *and* $\mathscr{L}_{\mu\sigma,\lambda}$ *denotes the set of all least energy solutions of system* $(\mathscr{P}_{\mu\sigma,\lambda})$.

PROOF. (A_1) We follow the idea of [**16**]. From (C_5) of Lemma 3.1 we know that $c_{\mu\sigma} > 0$ for $\mu, \sigma > 0$. If $z_0 \in \mathscr{N}_{\mu\sigma}$ satisfies $\mathscr{J}_{\mu\sigma,\lambda}(z_0) = c_{\mu\sigma}$, then $\check{m}_{\mu\sigma}(z_0)$ is a minimizer of $\mathscr{U}_{\mu\sigma,\lambda}$ and a critical point of $\mathscr{U}_{\mu\sigma,\lambda}$. According to Lemma 3.2, z_0 is a critical point of $\mathscr{J}_{\mu\sigma,\lambda}$. It remains to show that there exists a minimizer z of $\mathscr{J}_{\mu\sigma,\lambda}|_{\mathcal{N}_{\mu\sigma}}$. By Ekeland's variational principle [**17**], there exists a sequence $\{\nu_n\} \subset S_{\mu\sigma}$ such that

$$\mathscr{U}_{\mu\sigma,\lambda}(\nu_n) \to c_{\mu\sigma} \text{ and } \mathscr{U}'_{\mu\sigma,\lambda}(\nu_n) \to 0, \text{ as } n \to \infty.$$

Set

$$z_n = m_{\mu\sigma}(\nu_n) = (u_n, v_n) \in \mathscr{N}_{\mu\sigma}$$

for all $n \in \mathbb{N}$. We find

$$\mathscr{J}_{\mu\sigma,\lambda}(u_n) \to c_{\mu\sigma} \text{ and } \mathscr{J}'_{\mu\sigma,\lambda}(u_n) \to 0, \text{ as } n \to \infty.$$

By Lemma 3.4, we know that $\{z_n\}$ is bounded and there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \ge \delta > 0.$$

So we can choose a $r' > r > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\lim_{n\to\infty} \int_{B_{r'}(y_n)} |u_n|^2 \ge \frac{\delta}{2} > 0. \tag{3.7}$$

Note that $\mathscr{J}_{\mu\sigma,\lambda}$ and $\mathscr{N}_{\mu\sigma}$ are invariant under translations. We may take $\{y_n\}$ being bounded in $\mathbb{Z}^3$. So $z_n \rightharpoonup z = (u,v) \neq 0$ and $\mathscr{J}'_{\mu\sigma,\lambda}(z) = 0$. To prove $\mathscr{J}_{\mu\sigma,\lambda}(z) = c_{\mu\sigma}$, using Fatou's lemma and (1.5), we get

$$\begin{aligned} c_{\mu\sigma} &= \lim_{n\to\infty}\inf\left(\mathscr{J}_{\mu\sigma,\lambda}(z_n) - \frac{1}{2}\mathscr{J}'_{\mu\sigma,\lambda}(z_n)z_n\right) \\ &= \lim_{n\to\infty}\inf\int_{\mathbb{R}^N}\left(\frac{1}{2}f(v_n)v_n - F(v_n) + \frac{1}{2}g(u_n)u_n - G(u_n)\right) \\ &\geq \int_{\mathbb{R}^N}\left(\frac{1}{2}f(v)v - F(v) + \frac{1}{2}g(u)u - G(u)\right) \\ &= \mathscr{J}_{\mu\sigma,\lambda}(z) - \frac{1}{2}\mathscr{J}'_{\mu\sigma,\lambda}(z)z \\ &= \mathscr{J}_{\mu\sigma,\lambda}(z). \end{aligned}$$

That is, $\mathscr{J}_{\mu\sigma,\lambda}(z) \leq c_{\mu\sigma}$. However, the reverse inequality follows from the definition of $c_{\mu\sigma}$ due to $z = (u,v) \in \mathcal{N}_{\mu\sigma}$. So we get $\mathscr{J}_{\mu\sigma,\lambda}(z) = c_{\mu\sigma}$.

To find a positive ground state solution for system $(\mathscr{P}_{\mu\sigma,\lambda})$, we know that, for each $z = (u,v) \in E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, there exists a $t > 0$ such that

$$t|z| = (t|u|, t|v|) \in \mathscr{N}_{\mu\sigma}.$$

It follows from condition $(\mathscr{F}_1)$ that

$$\mathscr{J}_{\mu\sigma,\lambda}(t|u|, t|v|) \leq \mathscr{J}_{\mu\sigma,\lambda}(tu, tv).$$

Since $z \in \mathscr{N}_{\mu\sigma}$, there holds $\mathscr{J}_{\mu\sigma,\lambda}(tu,tv) \leq \mathscr{J}_{\mu\sigma,\lambda}(u,v)$. This leads to

$$\mathscr{J}_{\mu\sigma,\lambda}(t|u|, t|v|) \leq \mathscr{J}_{\mu\sigma,\lambda}(u,v)$$

and $(t|u|, t|v|)$ is a nonnegative ground state solution. It follows from Harnack's inequality [**24**] that

$$z_{\mu\sigma,\lambda} := (u_{\mu\sigma,\lambda}, v_{\mu\sigma,\lambda}) = (t|u|, t|v|) > 0$$

for all $x \in \mathbb{R}^N$.

(A_2) Suppose that $z_{\mu\sigma,\lambda}(x) = (u_{\mu\sigma,\lambda}(x), v_{\mu\sigma,\lambda}(x))$ is a positive ground state solution of system $(\mathcal{P}_{\mu\sigma,\lambda})$. By the standard arguments as shown in [**25, 26**], we find $u_b, v_b \in L^q(\mathbb{R}^N)$ for all $q \in [2,\infty]$. Using a similar proof to that of [**26**, Theorem 2.1], one can see that

$$\lim_{|x|\to\infty} v_{\mu\sigma,\lambda}(x) = 0, \qquad \lim_{|x|\to\infty} u_{\mu\sigma,\lambda}(x) = 0,$$

$$\lim_{|x|\to\infty} |\nabla v_{\mu\sigma,\lambda}(x)| = 0 \ \text{ and } \ \lim_{|x|\to\infty} |\nabla u_{\mu\sigma,\lambda}(x)| = 0,$$

where $u_{\mu\sigma,\lambda}, v_{\mu\sigma,\lambda} \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ for some $\sigma \in (0,1)$.

To prove

$$|z_{\mu\sigma,\lambda}(x)| \leq Ce^{-c|x - x_{\mu\sigma,\lambda}|},$$

where

$$|z_{\mu\sigma,\lambda}(x)| = \max_{x\in\mathbb{R}^N} |z_{\mu\sigma,\lambda}(x)|,$$

by following [**26**] we choose a fixed number $\xi \in (0,\delta)$, where $\delta = \min\{\sqrt{\mu-\lambda}, \sqrt{\sigma-\lambda}\}$, and let $\eta = \delta^2 - \xi^2$. Since $z_{\mu\sigma,\lambda}(x) = (u_{\mu\sigma,\lambda}(x), v_{\mu\sigma,\lambda}(x)) \to 0$ as $|x| \to \infty$, there is a $r > 0$ such that

$$\frac{g(u_{\mu\sigma,\lambda}(x))}{u_{\mu\sigma,\lambda}(x)} \leq \eta \quad \text{and} \quad \frac{f(v_{\mu\sigma,\lambda}(x))}{v_{\mu\sigma,\lambda}(x)} \leq \eta, \quad \forall\ |x| \geq r. \tag{3.8}$$

Let
$$K(x) = Ge^{-\xi(|x-x_{\mu\sigma,\lambda}|-r)},$$
where
$$G = \max\{|u_{\mu\sigma,\lambda}(x)| : |x - x_b| = r\} + \max\{|v_{\mu\sigma,\lambda}(x)| : |x - x_{\mu\sigma,\lambda}| = r\}.$$
For $M > r$, we define the set Π_M by
$$\Pi_M = \{x \in \mathbb{R}^N : r < |x - x_{\mu\sigma,\lambda}| < M, \\ u_{\mu\sigma,\lambda}(x) + v_{\mu\sigma,\lambda}(x) > K(x),\ u_{\mu\sigma,\lambda}(x) > 0,\ v_{\mu\sigma,\lambda}(x) > 0\}.$$

We claim that Π_M is empty. Suppose, by contradiction, that $\Pi_M \neq \emptyset$. For $x \in \Pi_M$ we have
$$\Delta(K - u_{\mu\sigma,\lambda} - v_{\mu\sigma,\lambda}) = \left(\xi^2 - \frac{\xi(N-1)}{|x|}\right) K(x) + (g(u_{\mu\sigma,\lambda}) - (\mu - \lambda)u_{\mu\sigma,\lambda}) \\ + (f(v_{\mu\sigma,\lambda}) - (\sigma - \lambda)v_{\mu\sigma,\lambda}).$$
Using $(\mathcal{H}_2)$ and (3.8) yields
$$\begin{aligned} &\Delta(K - u_{\mu\sigma,\lambda} - v_{\mu\sigma,\lambda}) \\ \le&\xi^2 K(x) + u_{\mu\sigma,\lambda}\left[\frac{g(u_{\mu\sigma,\lambda})}{u_{\mu\sigma,\lambda}} - (\mu - \lambda)\right] + v_{\mu\sigma,\lambda}\left[\frac{f(v_{\mu\sigma,\lambda})}{v_{\mu\sigma,\lambda}} - (\mu - \lambda)\right] \\ \le&\xi^2 K(x) + u_{\mu\sigma,\lambda}(\eta - (\mu - \lambda)) + v_{\mu\sigma,\lambda})(\eta - (\sigma - \lambda)) \\ \le&\xi^2 K(x) + u_{\mu\sigma,\lambda}(\eta - \delta^2) + v_{\mu\sigma,\lambda})(\eta - \delta^2) \\ =&\xi^2 \left[K(x) - u_{\mu\sigma,\lambda} - v_{\mu\sigma,\lambda}\right]. \end{aligned} \tag{3.9}$$
From the definition of Π_M and (3.9), it has $\Delta(K - u_{\mu\sigma,\lambda}) - v_{\mu\sigma,\lambda})) < 0$ in Π_M. By the maximum principle, we find
$$K(x) - u_{\mu\sigma,\lambda}(x) - v_{\mu\sigma,\lambda}(x) \ge \min_{\partial\Pi_M}(K - u_{\mu\sigma,\lambda} - v_{\mu\sigma,\lambda}).$$

Since $|x - x_{\mu\sigma,\lambda}| = r$ does not belong to the boundary of Π_M, we have
$$K(x) - u_{\mu\sigma,\lambda}(x) - v_{\mu\sigma,\lambda}(x) \ge \min\left\{0, \min_{|x-x_{\mu\sigma,\lambda}|=M}(K(x) - u_{\mu\sigma,\lambda}(x) - v_{\mu\sigma,\lambda}(x))\right\}.$$
Let $M \to \infty$. Note that $u_{\mu\sigma,\lambda}$ and $v_{\mu\sigma,\lambda}$ decay to 0 at infinity. So for each fixed $|x - x_{\mu\sigma,\lambda}| > r$, there holds
$$K(x) - u_{\mu\sigma,\lambda}(x) - v_{\mu\sigma,\lambda}(x) \ge 0.$$
This obviously contradicts the definition of Π_M. So, the set Π_M is empty, i.e., for $|x - x_{\mu\sigma,\lambda}| > r$ such that $u_{\mu\sigma,\lambda} > 0$ and $v_{\mu\sigma,\lambda} > 0$, we have
$$v_{\mu\sigma,\lambda}(x) + v_{\mu\sigma,\lambda}(x) \le K(x).$$
That is, for $|x - x_{\mu\sigma,\lambda}| > r$, it has
$$|u_{\mu\sigma,\lambda}(x)| + |u_{\mu\sigma,\lambda}(x)| \le K(x) = Ge^{-\xi(|x-x_{\mu\sigma,\lambda}|-r)}.$$
Hence, there exist $C, c > 0$ such that
$$|z_{\mu\sigma,\lambda}(x)| \le Ce^{-c|x-x_{\mu\sigma,\lambda}|}.$$

(A_3) Take a bounded sequence $\{z^n_{\mu\sigma,\lambda}\} \subset \mathscr{L}_{\mu\sigma,\lambda} \cap \mathscr{N}_{\mu\sigma}$. Clearly, $\mathscr{J}_{\mu\sigma,\lambda}(z^n_{\mu\sigma,\lambda}) = c_{\mu\sigma}$ and $\mathscr{J}'_{\mu\sigma,\lambda}(z^n_{\mu\sigma,\lambda}) = 0$. Without loss of generality, we assume that $z^n_{\mu\sigma,\lambda} = (u^n_{\mu\sigma,\lambda}, v^n_{\mu\sigma,\lambda}) \rightharpoonup z_{\mu\sigma,\lambda} = (u_{\mu\sigma,\lambda}, v_{\mu\sigma,\lambda})$ in $E_{\mu\sigma}$. As shown in the proof of Lemma

3.4, one can easily see that $\{z^n_{\mu\sigma,\lambda}\}$ is non-vanishing. Namely, there exist $\{y_n\} \subset \mathbb{Z}^N$ and $\delta, r > 0$ such that

$$\lim_{n\to\infty}\int_{B_r(y_n)} |z^n_{\mu\sigma,\lambda}|^2 \geq \frac{\delta}{2} > 0.$$

By the invariance of $\mathscr{J}_{\mu\sigma,\lambda}$ and $\mathscr{N}_{\mu\sigma}$ under translations of the form $z \mapsto z(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that $\{y_n\}$ is bounded in $\mathbb{Z}^N$. So $z^n_{\mu\sigma,\lambda} \rightharpoonup z_{\mu\sigma,\lambda} \neq 0$ and $\mathscr{J}'_{\mu\sigma,\lambda}(z_{\mu\sigma,\lambda}) = 0$. Moreover, as shown in (A_1), we know that $\mathscr{J}_{\mu\sigma,\lambda}(z_{\mu\sigma,\lambda}) = c_{\mu\sigma}$. It follows that

$$\begin{aligned}
c_{\mu\sigma} &= \mathscr{J}_{\mu\sigma,\lambda}(z_{\mu\sigma,\lambda})\\
&= \mathscr{J}_{\mu\sigma,\lambda}(z_{\mu\sigma,\lambda}) - \frac{1}{2}(\mathscr{J}'_{\mu\sigma,\lambda}(z_{\mu\sigma,\lambda}), z_{\mu\sigma,\lambda})\\
&= \int_{\mathbb{R}^N} \frac{1}{2} g_1(u_{\mu\sigma,\lambda})u_{\mu\sigma,\lambda} - G_1(u_{\mu\sigma,\lambda}) + \frac{1}{2} f_1(v_{\mu\sigma,\lambda})v_{\mu\sigma,\lambda} - F_1(v_{\mu\sigma,\lambda})\\
&\quad + \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N} |v_{\mu\sigma,\lambda}|^p + \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^N} |u_{\mu\sigma,\lambda}|^q\\
&\leq \liminf_{n\to\infty}\left[\int_{\mathbb{R}^N} \frac{1}{2} g_1(u^n_{\mu\sigma,\lambda})u^n_{\mu\sigma,\lambda} - G_1(u^n_{\mu\sigma,\lambda}) + \frac{1}{2} f_1(v^n_{\mu\sigma,\lambda})v^n_{\mu\sigma,\lambda} - F_1(v^n_{\mu\sigma,\lambda})\right]\\
&\quad + \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N} |v^n_{\mu\sigma,\lambda}|^p + \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^N} |u^n_{\mu\sigma,\lambda}|^q\right]\\
&= \liminf_{n\to\infty}\left[\mathscr{J}_{\mu\sigma,\lambda}(z^n_{\mu\sigma,\lambda}) - \frac{1}{2}\mathscr{J}'_{\mu\sigma,\lambda}(z^n_{\mu\sigma,\lambda})z^n_{\mu\sigma,\lambda}\right]\\
&= c_{\mu\sigma}.
\end{aligned}$$

So, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u^n_{\mu\sigma,\lambda}|^q = \int_{\mathbb{R}^N} |u_{\mu\sigma,\lambda}|^q \quad\text{and}\quad \lim_{n\to\infty}\int_{\mathbb{R}^N} |v^n_{\mu\sigma,\lambda}|^p = \int_{\mathbb{R}^N} |v_{\mu\sigma,\lambda}|^p.$$

It follows from Brezis-Lieb's lemma [**17**] that $u^n_{\mu\sigma,\lambda} \to u_{\mu\sigma,\lambda}$ in $L^q(\mathbb{R}^N)$ and $v^n_{\mu\sigma,\lambda} \to v_{\mu\sigma,\lambda}$ in $L^p(\mathbb{R}^N)$. Note that $z^n_{\mu\sigma,\lambda}$ satisfies

$$\begin{cases} -\Delta u^n_{\mu\sigma,\lambda} + \mu u^n_{\mu\sigma,\lambda} = g_1(u^n_{\mu\sigma,\lambda}) + |u^n_{\mu\sigma,\lambda}|^{q-2}u^n_{\mu\sigma,\lambda} + \lambda v^n_{\mu\sigma,\lambda} & \text{in } \mathbb{R}^N,\\ -\Delta v^n_{\mu\sigma,\lambda} + \sigma v^n_{\mu\sigma,\lambda} = f_1(v^n_{\mu\sigma,\lambda}) + |v^n_{\mu\sigma,\lambda}|^{p-2}v^n_{\mu\sigma,\lambda} + \lambda u^n_{\mu\sigma,\lambda} & \text{in } \mathbb{R}^N. \end{cases} \tag{3.10}$$

Using $u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda}$ as a test function for the first equation of system (3.10), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\nabla u^n_{\mu\sigma,\lambda}\nabla(u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda}) + \mu u^n_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda})\right]\\
&= \int_{\mathbb{R}^N} \left[g_1(u^n_{\mu\sigma,\lambda})(u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda}) + \lambda v^n_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda})\right]\\
&\quad + \int_{\mathbb{R}^N} |u^n_{\mu\sigma,\lambda}|^{q-2}u^n_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda}).
\end{aligned} \tag{3.11}$$

Clearly, it follows from $(\mathscr{F}_1)$ and $(\mathscr{F}_3)$ that for each $\beta > 0$, there exists a $C_\beta > 0$ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}[g_1(u^n_{\mu\sigma,\lambda})(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})+\int_{\mathbb{R}^N}|u^n_{\mu\sigma,\lambda}|^{q-2}u^n_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})\right|\\ \le&\beta\int_{\mathbb{R}^N}|u^n_{\mu\sigma,\lambda}||u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}|+cC_\beta\int_{\mathbb{R}^N}|u^n_{\mu\sigma,\lambda}|^{q-1}|u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}|\\ \le&c\beta+cC_\beta|u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}|_{L^q(\mathbb{R}^N)}.\end{aligned}\tag{3.12}$$

Since $|u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}|_{L^q(\mathbb{R}^N)}\to 0$ as $n\to\infty$, by (3.11) and (3.12) we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\nabla u^n_{\mu\sigma,\lambda}\nabla(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})+\mu u^n_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})\right]\\ =&\int_{\mathbb{R}^N}\lambda v^n_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})+o_n(1),\end{aligned}\tag{3.13}$$

where $o_n(1)$ denotes a quantity approaching zero as $n\to\infty$. Similarly, using $v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda}$ as a test function for the second equation of (3.10), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}(\nabla v^n_{\mu\sigma,\lambda}\nabla(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda})+\sigma v^n_{\mu\sigma,\lambda}(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda}))\\ =&\lambda\int_{\mathbb{R}^N}u^n_{\mu\sigma,\lambda}(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda})+o_n(1).\end{aligned}\tag{3.14}$$

Furthermore, since $z_{\mu\sigma,\lambda}$ satisfies the system

$$\begin{cases}-\Delta u_{\mu\sigma,\lambda}+\mu u_{\mu\sigma,\lambda}=g_1(u_{\mu\sigma,\lambda})+|u_{\mu\sigma,\lambda}|^{q-2}u_{\mu\sigma,\lambda}+\lambda v_{\mu\sigma,\lambda} & \text{in } \mathbb{R}^N,\\ -\Delta v_{\mu\sigma,\lambda}+\sigma v_{\mu\sigma,\lambda}=f_1(v_{\mu\sigma,\lambda})+|v_{\mu\sigma,\lambda}|^{p-2}v_{\mu\sigma,\lambda}+\lambda u_{\mu\sigma,\lambda} & \text{in } \mathbb{R}^N.\end{cases}\tag{3.15}$$

Using the same arguments as for (3.13) and (3.14), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N}(\nabla u_{\mu\sigma,\lambda}\nabla(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})+\mu u_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}))\\ =&\lambda\int_{\mathbb{R}^N}v_{\mu\sigma,\lambda}(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})+o_n(1),\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}(\nabla v_{\mu\sigma,\lambda}\nabla(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda})+\sigma v_{\mu\sigma,\lambda}(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda}))\\ =&\lambda\int_{\mathbb{R}^N}u_{\mu\sigma,\lambda}(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda})+o_n(1).\end{aligned}\tag{3.17}$$

Combining (3.13), (3.14), (3.16) and (3.17), we infer that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})|^2+\mu\int_{\mathbb{R}^N}|u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}|^2\\ &+\int_{\mathbb{R}^N}|\nabla(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda})|^2+\sigma\int_{\mathbb{R}^N}|v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda}|^2\\ =&2\lambda\int_{\mathbb{R}^N}(u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda})(v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda})+o_n(1)\\ \le&\lambda\int_{\mathbb{R}^N}|u^n_{\mu\sigma,\lambda}-u_{\mu\sigma,\lambda}|^2+\lambda\int_{\mathbb{R}^N}|v^n_{\mu\sigma,\lambda}-v_{\mu\sigma,\lambda}|^2+o_n(1).\end{aligned}\tag{3.18}$$

Since $\lambda < \mu, \sigma$, we can deduce that $\|u^n_{\mu\sigma,\lambda} - u_{\mu\sigma,\lambda}\|_{H^1(\mathbb{R}^N)} \to 0$ and $\|v^n_{\mu\sigma,\lambda} - v_{\mu\sigma,\lambda}\|_{H^1(\mathbb{R}^N)} \to 0$ as $n \to \infty$. That is, $\|z^n_{\mu\sigma,\lambda} - z_{\mu\sigma,\lambda}\|_{E_{\mu\sigma}} \to 0$ as $n \to \infty$. □

REMARK 3.7. We point out that our arguments in this section can be applied to the case of periodic potentials. That is,

$$(\hat{\mathcal{P}}_{VM,\lambda}) \qquad \begin{cases} -\Delta u + V(x)u = g(u) + \lambda v & \text{in } \mathbb{R}^N, \\ -\Delta v + M(x)v = f(v) + \lambda u & \text{in } \mathbb{R}^N. \end{cases}$$

where $V(x)$ and $M(x)$ are positive continues functions and periodic in x. By an analogous argument, the conclusions of Theorem 3.6 still hold.

4. Some Useful Results

In this section we shall present some preliminary results which will be used in the next section.

LEMMA 4.1. *Suppose that assumptions of $(\mathscr{F}_1) - (\mathscr{F}_3)$ are satisfied. If $k = \min\{\mu_1 - \mu_2, \sigma_2 - \sigma_1\} \geq 0$, then $c_{\mu_1\sigma_1} \geq c_{\mu_2\sigma_2}$ holds for all $\lambda > 0$. Moreover, if $k > 0$, then $c_{\mu_1\sigma_1} < c_{\mu_2\sigma_2}$ holds for all $\lambda > 0$. In particular, we have $c_{\mu_1\sigma_i} > c_{\mu_2\sigma_i}$ if $\mu_1 > \mu_2$, and $c_{\mu_i\sigma_1} > c_{\mu_i\sigma_2}$ if $\sigma_1 < \sigma_2$ $(i = 1, 2)$.*

PROOF. For $\mu_1, \mu_2, \sigma_1, \sigma_2 > 0$, one has that $E_{\mu_i\sigma_j} = E$ $(i, j = 1, 2)$. Let $z_1 = (u_1, v_1) \in \mathscr{N}_{\mu_1\sigma_1}$ satisfy

$$c_{\mu_1\sigma_1} = \mathscr{I}_{\mu_1\sigma_1,\lambda}(z_1) = \max_{w \in E_{\mu_1\sigma_1}} \mathscr{I}_{\mu_1\sigma_1,\lambda}(w).$$

On the other hand, let $z_2 = (u_2, v_2) \in E_{\mu_2\sigma_2}$ satisfy

$$\mathscr{I}_{\mu_2\sigma_2,\lambda}(z_2) = \max_{w \in E_{\mu_2\sigma_2}} \mathscr{I}_{\mu_2\sigma_2,\lambda}(w).$$

So we see that

$$\begin{aligned} c_{\mu_1\sigma_1} &\geq \mathscr{I}_{\mu_1\sigma_1,\lambda}(z_2) \\ &= \mathscr{I}_{\mu_2\sigma_2,\lambda}(z_2) + (\mu_1 - \mu_2)\int_{\mathbb{R}^N} u_2^2 + (\sigma_2 - \sigma_1)\int_{\mathbb{R}^N} v_2^2 \\ &\geq c_{\mu_2\sigma_2} + (\mu_1 - \mu_2)\int_{\mathbb{R}^N} u_2^2 + (\sigma_2 - \sigma_1)\int_{\mathbb{R}^N} v_2^2. \end{aligned}$$

□

To prove the concentration phenomena of ground state solutions of system $(\mathscr{P}_\varepsilon)$, we start with an auxiliary system

$$(\hat{\mathscr{P}}_\varepsilon) \qquad \begin{cases} -\Delta u + \hat{V}_\varepsilon(x)u = g(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \hat{M}_\varepsilon(x)v = f(v) + \lambda u, & x \in \mathbb{R}^N, \\ u(x), v(x) \to 0, \text{ as } |x| \to \infty, \end{cases}$$

where $\hat{V}_\varepsilon(x) = \hat{V}(\varepsilon x)$ and $\hat{M}_\varepsilon(x) = \hat{M}(\varepsilon x)$. Correspondingly, the energy functional is given by

$$\begin{aligned} \hat{\mathscr{I}}_{\varepsilon,\lambda}(z) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \hat{V}_\varepsilon(x)u^2) + \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + \hat{M}_\varepsilon(x)v^2) \\ - \int_{\mathbb{R}^N} (F(v) + G(u)) - \lambda \int_{\mathbb{R}^N} uv, \end{aligned}$$

where

$$z = (u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).$$

As in the first section, we can define

$$\hat{c}_\varepsilon = \inf_{w \in \hat{E}_\varepsilon} \hat{\mathscr{J}}_\varepsilon(w),$$

where

$$\hat{E}_\varepsilon = \hat{H}^1_\varepsilon \times \hat{H}^2_\varepsilon, \quad \hat{H}^1_\varepsilon = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \hat{V}(x) u^2 < \infty\}$$

and

$$\hat{H}^2_\varepsilon = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \hat{M}(x) u^2 < \infty\}.$$

LEMMA 4.2. *Under assumptions of* $(\mathscr{V}_0)-(\mathscr{V}_1)$ *and* $(\mathscr{F}_1)-(\mathscr{F}_3)$, *The following two statements are true.*

(i) There holds

$$\limsup_{\varepsilon \to 0} c_\varepsilon \le c_{V(y_0)M(y_0)}$$

for $y_0 \in \mathbb{R}$. *In particular,*

$$\limsup_{\varepsilon \to 0} c_\varepsilon \le c_{V(0)M(0)}$$

holds for all $\varepsilon > 0$. *Moreover, if* $\mathscr{M}_1 \cap \mathscr{M}_2 \neq \emptyset$, *then* $\lim_{\varepsilon \to 0} c_\varepsilon = c_{V_0 M_0}$.
(ii) If $\hat{V}_\varepsilon(x) \to \vartheta_1$ *and* $\hat{M}_\varepsilon(x) \to \vartheta_2$ *uniformly on bounded sets of* x *as* $\varepsilon \to 0$, *then there holds*

$$\lim_{\varepsilon \to 0} \hat{c}_\varepsilon \le c_{\vartheta_1 \vartheta_2}.$$

PROOF. (i) Since V and M are bounded functions, for each $\varepsilon > 0$ and $\mu, \sigma > 0$, it has $E_\varepsilon = E_{\mu\sigma} = E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Let $z = (u, v) \in \mathscr{N}_{V(y_0)M(y_0)}$ satisfy

$$c_{V(y_0)M(y_0)} = \mathscr{J}_{V(y_0)M(y_0)}(z) = \inf_{w \in E_{V(y_0)M(y_0)} \setminus \{0\}} \max_{t > 0} \mathscr{J}_{V(y_0)M(y_0)}(tw).$$

For $w_\varepsilon(x) = z(x - \frac{y_0}{\varepsilon}) = (u_\varepsilon, v_\varepsilon) = (u(x - \frac{y_0}{\varepsilon}), v(x - \frac{y_0}{\varepsilon}))$, there exists a $t_\varepsilon > 0$ such that $t_\varepsilon w_\varepsilon \in \mathcal{N}_\varepsilon$. It is not difficult to see that t_ε is bounded for small $\varepsilon > 0$. Conversely, if $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, there holds

$$\begin{aligned}
0 < c_\varepsilon &\le \mathscr{J}_\varepsilon(t_\varepsilon w_\varepsilon) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla(t_\varepsilon u_\varepsilon)|^2 + V(\varepsilon x) |t_\varepsilon u_\varepsilon|^2) + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla(t_\varepsilon v_\varepsilon)|^2 + M(\varepsilon x) |t_\varepsilon v_\varepsilon|^2) \\
&\quad - \int_{\mathbb{R}^N} (F(t_\varepsilon v_\varepsilon) + G(t_\varepsilon u_\varepsilon)) - \lambda t_\varepsilon^2 \int_{\mathbb{R}^N} u_\varepsilon v_\varepsilon \\
&\le t_\varepsilon^2 [\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla(u_\varepsilon)|^2 + (V_{max} + \lambda) |u_\varepsilon|^2) + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla(v_\varepsilon)|^2 + (M_{max} + \lambda) |v_\varepsilon|^2) \\
&\quad - \int_{\mathbb{R}^N} (\frac{F(t_\varepsilon v_\varepsilon)}{t_\varepsilon^2 v_\varepsilon^2} v_\varepsilon^2 + \frac{G(t_\varepsilon u_\varepsilon)}{t_\varepsilon^2 u_\varepsilon^2} u_\varepsilon^2)] \\
&= t_\varepsilon^2 [\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla(u)|^2 + (V_{max} + \lambda) |u|^2) + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla(v)|^2 + (M_{max} + \lambda) |v_\varepsilon|^2) \\
&\quad - \int_{\mathbb{R}^N} (\frac{F(t_\varepsilon v)}{t_\varepsilon^2 v^2} v^2 + \frac{G(t_\varepsilon u)}{t_\varepsilon^2 u^2} u^2)] \\
&\to -\infty, \text{ as } \varepsilon \to 0,
\end{aligned}$$

where

$$V_{max} = \max_{x\in\mathbb{R}^N} V(x) \quad \text{and} \quad M_{max} = \max_{x\in\mathbb{R}^N} M(x).$$

This is certainly impossible.

Now we claim that

$$t_\varepsilon \to 1 \text{ as } \varepsilon \to 0. \tag{4.1}$$

Since $z = (u, v) \in \mathscr{N}_{V(y_0)M(y_0)}$, we know that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(y_0)|u|^2) + \int_{\mathbb{R}^N} (|\nabla v|^2 + M(y_0)|v|^2) = \int_{\mathbb{R}^N} (f(v)v + g(u)u) + \lambda \int_{\mathbb{R}^N} uv. \tag{4.2}$$

Since $t_\varepsilon w_\varepsilon \in \mathscr{N}_\varepsilon$, it gives

$$\begin{aligned} &t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + V(\varepsilon x)|u_\varepsilon|^2) + t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + M(\varepsilon x)|v_\varepsilon|^2) \\ &= \int_{\mathbb{R}^N} (f(t_\varepsilon v_\varepsilon)t_\varepsilon v_\varepsilon + g(t_\varepsilon u_\varepsilon)t_\varepsilon u_\varepsilon) + t_\varepsilon^2 \lambda \int_{\mathbb{R}^N} u_\varepsilon v_\varepsilon. \end{aligned} \tag{4.3}$$

From (4.3) we further have

$$\begin{aligned} &t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\varepsilon x + y_0)|u|^2) + t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla v|^2 + M(\varepsilon x + y_0)|v|^2) \\ &= \int_{\mathbb{R}^N} (f(t_\varepsilon v)t_\varepsilon v + g(t_\varepsilon u)t_\varepsilon u) + t_\varepsilon^2 \lambda \int_{\mathbb{R}^N} uv. \end{aligned} \tag{4.4}$$

For each $\epsilon > 0$, there exists a $R > 0$ such that

$$\int_{|x|>R} (K(\varepsilon x + y_0) - K(y_0))|u|^2 < c\epsilon,$$

and

$$\int_{|x|\le R} (K(\varepsilon x + y_0) - K(y_0))|u|^2 \to 0, \text{ as } \varepsilon \to 0.$$

That is,

$$\int_{\mathbb{R}^N} K(\varepsilon x + y_0)|u|^2 = \int_{\mathbb{R}^N} K(y_0)|u|^2 + o_\varepsilon(1), \tag{4.5}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$.

Similarly, we can deduce that

$$\int_{\mathbb{R}^N} M(\varepsilon x + y_0)|u|^2 = \int_{\mathbb{R}^N} M(y_0)|u|^2 + o_\varepsilon(1). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) yields

$$\begin{aligned} &t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + K(y_0)|u|^2) + t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla v|^2 + M(y_0)|v|^2) \\ &= \int_{\mathbb{R}^N} (f(t_\varepsilon v)t_\varepsilon v + g(t_\varepsilon u)t_\varepsilon u) + t_\varepsilon^2 \lambda \int_{\mathbb{R}^N} uv + o_\varepsilon(1). \end{aligned} \tag{4.7}$$

From (4.2) and (4.7), we find

$$0 = \int_{\mathbb{R}^N} (\frac{f(t_\varepsilon v)}{(t_\varepsilon v)} - \frac{f(v)}{v})v^2 + \int_{\mathbb{R}^N} (\frac{g(t_\varepsilon u)}{(t_\varepsilon u)} - \frac{g(u)}{u})u^2 + o_\varepsilon(1).$$

So it follows from $(\mathscr{F}_2)$ that we arrive at (4.1).

Since $t_\varepsilon v_\varepsilon \in \mathcal{N}_\varepsilon$, it gives

$$
\begin{aligned}
c_\varepsilon &\le \mathscr{J}_{\varepsilon,\lambda}(t_\varepsilon w_\varepsilon)\\
&= \mathscr{J}_{V(y_0)M(y_0),\lambda}(t_\varepsilon w_\varepsilon) + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(V(\varepsilon x)-V(y_0))u_\varepsilon^2\\
&\quad + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(M(y_0)-M(\varepsilon x))v_\varepsilon^2\\
&= \mathscr{J}_{V(y_0)M(y_0)}(t_\varepsilon w_\varepsilon) + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(V(\varepsilon x+y_0)-V(y_0))u^2\\
&\quad + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(M(y_0)-M(\varepsilon x+y_0))v^2\\
&= \mathscr{J}_{V(y_0)M(y_0)}(t_\varepsilon w_\varepsilon) + o_\varepsilon(1).
\end{aligned}
\tag{4.8}
$$

It follows that

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} c_\varepsilon &\le \limsup_{\varepsilon\to 0} \mathscr{J}_{V(y_0)M(y_0)}(t_\varepsilon w_\varepsilon) + \limsup_{\varepsilon\to 0} o_\varepsilon(1)\\
&= \mathscr{J}_{V(y_0)M(y_0)}(u,v)\\
&= c_{V(y_0)M(y_0)}.
\end{aligned}
$$

In particular, we take $y_0 = 0$, it has

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V(0)M(0)}.$$

For the case of $\mathcal{V}\cap\mathcal{M}\neq\emptyset$, we can assume that $0\in\mathcal{V}\cap\mathcal{M}$ and use an indirect argument to prove $c_\varepsilon \ge c_{V(0)M(0)}$. Namely, assume that $c_\varepsilon < c_{V(0)M(0)}$ for some $\varepsilon > 0$. By the definition of c_ε (see (2.9)), we can choose an $w\in E_\varepsilon\setminus\{0\}$ such that

$$\max_{s>0} \mathscr{J}_{\varepsilon,\lambda}(sw) < c_{V(0)M(0)}.$$

By the definition of $c_{V(0)M(0)}$ (see (3.3)), we know that

$$c_{V(0)M(0)} \le \max_{s>0} \mathscr{J}_{V(0)M(0)}(sw).$$

Since $V_\varepsilon(x)\ge V(0)$, $M_\varepsilon(x)\ge M(0)$ and $\mathscr{J}_{\varepsilon,\lambda}(z)\ge \mathscr{J}_{V(0)M(0)}(z)$ for all $z\in E_\varepsilon$, we have

$$c_{V(0)M(0)} > \max_{s>0} \mathscr{J}_{\varepsilon,\lambda}(sw) \ge \max_{s>0} \mathscr{J}_{V(0)M(0)}(sw) \ge c_{V(0)M(0)}.$$

This is, obviously, a contradiction. So we have

$$c_{V(0)M(0)} \le \lim_{\varepsilon\to 0} c_\varepsilon \le \limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V(0)M(0)}.$$

That is,

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0M_0}.$$

(ii) Take $z\in\mathscr{N}_{\vartheta_1\vartheta_2}$ such that

$$c_{\vartheta_1\vartheta_2} = \mathscr{J}_{\vartheta_1\vartheta_2}(u) = \max_{w\in E_{\vartheta_1\vartheta_2}\setminus\{0\}} \mathscr{J}_{\vartheta_1\vartheta_2}(w).$$

Then we take $z_1 = (u_1, v_1) \in \hat{E}_\varepsilon \setminus \{0\}$ such that

$$\begin{aligned}\hat{c}_\varepsilon &\le \hat{\mathscr{J}}_\varepsilon(z_1)\\ &= \max_{s>0} \hat{\mathscr{J}}_\varepsilon(sz)\\ &= \mathscr{J}_{\vartheta_1\vartheta_2,\lambda}(z_1) + \int_{\mathbb{R}^N}(\hat{V}_\varepsilon(x)-\vartheta_1)u_1^2 + \frac{1}{2}\int_{\mathbb{R}^N}(\hat{M}_\varepsilon(x)-\vartheta_2)v_1^2.\end{aligned}$$

Using the same argument as that of Part (i), one can easily check that

$$\int_{\mathbb{R}^N}(\hat{V}_\varepsilon(x)-\vartheta_1)u_1^2 + \frac{1}{p}\int_{\mathbb{R}^N}(\hat{M}_\varepsilon(x)-\vartheta_2)|v_1|^2 \to 0, \text{ as } \varepsilon \to 0,$$

and

$$\hat{c}_\varepsilon \le \max_{w\in E_{\vartheta_1\vartheta_2}\setminus\{0\}} \mathscr{J}_{\vartheta_1\vartheta_2,\lambda}(w) + o(1) = \mathscr{J}_{\vartheta_1\vartheta_2}(u) + o(1) = c_{\vartheta_1\vartheta_2} + o(1).$$

Consequently, we have completed the proof. □

5. Positive Solutions for System $(\mathscr{P}_\varepsilon)$

5.1. A Compactness Condition. In order to obtain the existence of positive solutions for system $(\mathscr{P}_\varepsilon)$, we should prove some lemmas on compactness. So, the main purpose of this subsection is to present the Palais-Smale sequences properties for the functional $\mathscr{J}_{\varepsilon,\lambda}$. Since $V_0 < V_\infty$ and $M_0 \le M_\infty$, we can choose $\mu, \sigma > 0$ such that

$$V_0 \le \mu < V_\infty \ \text{ and } \ M_0 \le \sigma \le M_\infty. \tag{5.1}$$

LEMMA 5.1. *Assume that assumptions $(\mathscr{V}_0)$ and $(\mathscr{F}_1)-(\mathscr{F}_3)$ hold. Let $\{z_n\} \subset \mathscr{N}_\varepsilon$ satisfy $\mathscr{J}_{\varepsilon,\lambda}(z_n) \to c$, $0 < c \le c_{\mu\sigma} < c_{V_\infty M_\infty}$ (μ and σ are given in (5.1)) and $z_n \rightharpoonup 0$ in E_ε. Then one of the following statements is true.*

(i) $z_n \to 0$ in E_ε;

(ii) there exist a sequence $y_n \in \mathbb{R}^N$ and two constants $r, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} z_n^2 \ge \delta.$$

PROOF. Suppose that Case (ii) does not occur, i.e., there exists a $r > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} u_n^2 = 0.$$

By Lemma 3.4, we can derive that $z_n \to 0$ in $L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$. From $\mathscr{J}'_{\varepsilon,\lambda}(z_n)z_n = 0$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}(|u_n|^2 + V_\varepsilon(x)|u_n|^2) + \int_{\mathbb{R}^N}(|v_n|^2 + V_\varepsilon(x)|v_n|^2)\\ =&\int_{\mathbb{R}^N}(f(u_n)u_n + g(v_n)v_n) + \lambda\int_{\mathbb{R}^N} u_nv_n\\ \to& 0, \text{ as } n \to \infty.\end{aligned}$$

Since V and M are positive bounded functions, it follows that $z_n \to 0$ in E_ε as $n \to \infty$. □

As in [**28**, Lemma 5.2] (also see [**27, 38**]), we have the following results.

LEMMA 5.2. *Under the assumptions of Lemma 5.1, Let* $\{z_n\} \subset \mathscr{N}_\varepsilon$ *satisfy* $\mathscr{J}_{\varepsilon,\lambda}(u_n) \to c$, $0 < c \leq c_{\mu\sigma} < c_{V_\infty M_\infty}$ *and* $z_n \rightharpoonup 0$ *in* E_ε. *Then we have* $z_n \to 0$ *in* E_ε *for small* $\varepsilon > 0$.

The next lemma is regarding the functional $\mathscr{U}_{\varepsilon,\lambda}$ and the Palais-Smale condition.

LEMMA 5.3. *Under the assumptions of* $(\mathscr{V}_0)$ *and* $(\mathscr{F}_1) - (\mathscr{F}_3)$, *for* $0 < \lambda < \delta = \min\{V_0, M_0, 1\}$, *if* $\{w_n\} \subset S_\varepsilon$ *satisfy* $\mathscr{U}_{\varepsilon,\lambda}(w_n) \to c$ *and* $\mathscr{U}'_{\varepsilon,\lambda}(w_n) \to 0$ *with* $0 < c \leq c_{\mu\sigma} < c_{V_\infty M_\infty}$, *then* $\{w_n\}$ *has a convergent subsequence in* E_ε.

PROOF. Let

$$z_n = m_\varepsilon(w_n) = t_n w_n.$$

Then it follows from Lemma 2.3 that $\{t_n w_n\} \subset \mathscr{N}_\varepsilon$, and

$$\mathscr{J}_{\varepsilon,\lambda}(z_n) \to c, \quad \mathscr{J}'_{\varepsilon,\lambda}(z_n) \to 0 \quad \text{and} \quad \mathscr{J}'_{\varepsilon,\lambda}(z_n)z_n = 0. \tag{5.2}$$

To prove the boundedness of $\{z_n\}$. By way of contradiction, we assume that $\|z_n\|_\varepsilon \to \infty$, as $n \to \infty$. Let

$$k_n = \frac{z_n}{\|z_n\|_\varepsilon} = (k_n^1, k_n^2).$$

Then $k_n \rightharpoonup k = (k^1, k^2)$ and $k_n(x) \to k(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. There are two cases: $\{k_n\}$ is either vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |k_n|^2 = 0, \tag{5.3}$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |k_n|^2 \geq \delta > 0. \tag{5.4}$$

As shown in [**19**], we will demonstrate that neither (5.3) nor (5.4) occurs and then arrive at the desired result.

If $\{k_n\}$ is vanishing, by Lemma 3.4 we have

$$k_n \to 0 in L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \text{ for } p \in (2, 2^*).$$

It follows from (2.5) that

$$\int_{\mathbb{R}^N} (F(Rk_n^2) + G(Rk_n^1)) \to 0 \text{ as } n \to \infty$$

for each $R \in \mathbb{R}$. For each $\varepsilon > 0$, from the boundedness of V and M, we know that two norms $\|\cdot\|$ and $\|\cdot\|_\varepsilon$ are equivalent. From the equality $\|k_n\|_\varepsilon = 1$, we see that there exists a $d > 0$ such that $\|k_n\| \geq d$, and

$$\begin{aligned} c + o(1) &\geq \mathscr{J}_{\varepsilon,\lambda}(z_n) \\ &\geq \mathscr{J}_{\varepsilon,\lambda}(Rk_n) \\ &= \frac{R^2}{2}\|k_n\|_\varepsilon^2 - \int_{\mathbb{R}^N}(F(Rk_n^2) + G(Rk_n^1)) - \lambda R^2 \int_{\mathbb{R}^N} k_n^1 k_n^2 \\ &\geq \frac{\delta R^2}{2}\|k_n\|^2 - \frac{\lambda R^2}{2}\|k_n\|^2 - \int_{\mathbb{R}^N}(F(Rk_n^2) + G(Rk_n^1)) \\ &\geq \frac{R^2}{2}(\delta - \lambda)d - \int_{\mathbb{R}^N}(F(Rk_n^2) + G(Rk_n^1)) \\ &\to \frac{dR^2}{2} \text{ as } n \to \infty. \end{aligned}$$

This yields a contradiction if R is large enough.

Let

$$\hat{k}_n = k_n(x + y_n) = \left(\hat{k}_n^1, \hat{k}_n^2\right).$$

This means $\hat{k}_n \rightharpoonup \hat{k} = (\hat{k}^1, \hat{k}^2)$ in E. From (5.4) we know that there exists a subset $\tilde{\Omega}$ in $\mathbb{R}^N \times \mathbb{R}^N$ with the positive measure such that $\hat{k} \neq 0$ a.e. in $\tilde{\Omega}$. For the large n, it has

$$\begin{aligned} 0 &\leq \frac{\mathscr{J}_{\mu\sigma\kappa,\lambda}(z_n)}{\|z_n\|_\varepsilon^2} \\ &= \frac{1}{2} - \frac{\int_{\mathbb{R}^N}(G(u_n(x+y_n)) + F(v_n(x+y_n))}{\|z_n(x+y_n)\|_\varepsilon^2} - \lambda \int_{\mathbb{R}^N} k_n^1 k_n^2 \\ &\leq \frac{1 + c_1\lambda}{2} - \int_{\mathbb{R}^N} \frac{(G(u_n(x+y_n)) + F(v_n(x+y_n))}{|z_n(x+y_n)|^2}|k_n(x+y_n)|^2, \end{aligned} \tag{5.5}$$

where $c_1 > 0$.

Set

$$\hat{z}(x) = (\hat{u}(x), \hat{v}(x)) = z(x + y_n).$$

By the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_\varepsilon$, we know that $\|z_n\| \to \infty$ as $n \to \infty$. Due to the fact $\|\hat{z}\|_\varepsilon = \|z(x + y_n)\|_\varepsilon \geq c\|z_n\|$, there holds $\|\hat{z}_n\| \to \infty$ as $n \to \infty$. Since $|\hat{z}_n| = \|\hat{z}_n\|_\varepsilon|\hat{k}_n| \to \infty$ if $k \neq 0$ as $n \to \infty$, it follows from $(\mathscr{F}_3)$ that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{(G(\hat{u}_n(x)) + F(\hat{v}_n(x))}{|\hat{z}_n(x)|^2}|\hat{k}_n(x)|^2 &\geq \int_{\tilde{\Omega}} \frac{(G(\hat{u}_n(x)) + F(\hat{v}_n(x))}{|\hat{u}_n(x) + \hat{v}_n(x)|^2}|\hat{k}_n(x)|^2 \\ &\to \infty, \text{ as } n \to \infty. \end{aligned} \tag{5.6}$$

Substituting (5.6) into (5.5) leads to another contradiction.

Thus, there exists $z = (u, v) \in E_\varepsilon$ such that $z_n \rightharpoonup z$ in E_ε, and z is a critical point of $\mathscr{J}'_{\varepsilon,\lambda}$. Set $h_n = z_n - z$. By Brezis-Lieb's Lemma (see [**17**]) we have

$$\int_{\mathbb{R}^N} |\nabla h_n|^2 = \int_{\mathbb{R}^N} |\nabla z_n|^2 - \int_{\mathbb{R}^N} |\nabla z|^2 + o(1).$$

Moreover, as shown in [**29**], one can easily check that

$$\mathscr{J}_{\varepsilon,\lambda}(h_n) = \mathscr{J}_{\varepsilon,\lambda}(z_n) - \mathscr{J}_\varepsilon(z) + o(1) \text{ and } \mathscr{J}'_{\varepsilon,\lambda}(h_n) \to 0, \text{ as } n \to \infty.$$

Thus, it follows from $\mathscr{J}'_{\varepsilon,\lambda}(z) = 0$ and (1.5) that

$$\mathscr{J}_{\varepsilon,\lambda}(z) = \mathscr{J}_{\varepsilon,\lambda}(u,v) = \int_{\mathbb{R}^N} (\frac{1}{2}f(u)u - F(u) + \frac{1}{2}g(v)v - G(v)) \geq 0. \tag{5.7}$$

That is,

$$\mathscr{J}_{\varepsilon,\lambda}(h_n) = \mathscr{J}_{\varepsilon,\lambda}(z_n) - \mathscr{J}_{\varepsilon,\lambda}(z) + o(1) \to c - d_1, \text{ as } n \to \infty,$$

where $d_1 = \mathscr{J}_{\varepsilon,\lambda}(z) \geq 0$. Using Lemma 5.2 and $d_2 = c - d_1 \leq c \leq c_{\mu\sigma} < c_{V_\infty M_\infty}$, we see that $h_n = z_n - z \to 0$ in E_ε. Apparently, $z \in \mathscr{N}_\varepsilon$. Since $z_n = t_n w_n$ and t_n is bounded, it gives $t_n \to t \neq 0$ (if $t = 0$, then $z = 0$). From the boundedness of $\{w_n\}$, we infer that there exists a w such that $w_n \rightharpoonup w$ in E_ε. Consequently, it follows from $t_n \to t$ and $z_n \to z$ that $w_n \to w$ and $z = tw$. □

5.2. Existence and Concentration of Positive Solutions. We now are in a position to present the proof of the existence of positive ground state solutions of system $(\mathscr{P}_\varepsilon)$.

LEMMA 5.4. *Under the assumptions of Theorem 1.1, for small $\varepsilon > 0$, c_ε is attained by the positive function z_ε.*

PROOF. From Lemma 2.2, we know that $c_\varepsilon \geq \rho > 0$ for each $\varepsilon > 0$. If $z_\varepsilon \in \mathscr{N}_\varepsilon$ satisfies $\mathscr{J}_{\varepsilon,\lambda}(z_\varepsilon) = c_\varepsilon$, according to Lemma 2.3, $\check{m}_\varepsilon(z_\varepsilon)$ is a minimizer of $\mathscr{U}_{\varepsilon,\lambda}$ and thus a critical point of $\mathscr{U}_{\varepsilon,\lambda}$, so that z_ε is a critical point of $\mathscr{J}_{\varepsilon,\lambda}$. It remains to show that there exists a minimizer z_ε of $\mathscr{J}_\varepsilon|_{\mathscr{N}_\varepsilon}$. To this end, by Ekeland's variational principle [**17**], there exists a sequence $\{w_n\} \subset S_\varepsilon$ such that $\mathscr{U}_{\varepsilon,\lambda}(w_n) \to c_\varepsilon$ and $\mathscr{U}'_{\varepsilon,\lambda}(w_n) \to 0$ as $n \to \infty$.

Set

$$z_n = m_\varepsilon(w_n) \in \mathscr{N}_\varepsilon$$

for all $n \in \mathbb{N}$. Using Lemma 2.3 again, we derive that $\mathscr{J}_{\varepsilon,\lambda}(z_n) \to c_\varepsilon$, $\mathscr{J}'_{\varepsilon,\lambda}(z_n)z_n = 0$ and $\mathscr{J}'_{\varepsilon,\lambda}(z_n) \to 0$ as $n \to \infty$. As in the proof of Lemma 4.2, we let $y_0 = x_v$. By virtue of Lemma 5.3 and $\limsup_{\varepsilon\to 0} c_\varepsilon \leq c_{V(x_v)M(x_v)} < c_{V_\infty M_\infty}$, there is $z \in E_\varepsilon$ such that $h_n = z_n - z \to 0$ in E_ε. This implies that $z \in \mathscr{N}_\varepsilon$ and $\mathscr{J}_{\varepsilon,\lambda}(z) = c_\varepsilon$. Similar to the proof of Theorem 3.6, we can thus find a positive function z_ε such that $\mathscr{J}_{\varepsilon,\lambda}(z) = c_\varepsilon$. □

Let $\mathscr{L}'_\varepsilon$ denote the set of all positive ground state solutions of system $(\mathscr{P}_\varepsilon)$. The following lemma is regarding compactness of $\mathscr{L}'_\varepsilon$.

LEMMA 5.5. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then $\mathscr{L}'_\varepsilon$ is compact in $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for small $\varepsilon > 0$.*

PROOF. Let the bounded sequence $\{z_n\} \subset \mathscr{L}'_\varepsilon \cap \mathscr{N}_\varepsilon$ satisfy $\mathscr{J}_{\varepsilon,\lambda}(z_n) = c_\varepsilon$ and $\mathscr{J}'_{\varepsilon,\lambda}(z_n) = 0$. Without loss of generality, we may assume that $z_n \rightharpoonup z \in E_\varepsilon$. Since $\mathscr{J}'_{\varepsilon,\lambda}$ is weakly continuous, it gives $\mathscr{J}'_{\varepsilon,\lambda}(z) = 0$. Set $h_n = z_n - z$. By an analogous discussion in the proof of Lemma 5.3, we arrive at $h_n \to 0$ in E. □

LEMMA 5.6. *Under the assumptions of Theorem 1.1, there is a maximum point y_ε of $|z_\varepsilon|$ such that $dist(\varepsilon y_\varepsilon, \mathscr{V}) \to 0$, $\varepsilon y_\varepsilon \to y_0$, and $k_\varepsilon(x) = z_\varepsilon(x + y_\varepsilon)$ converges in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to a positive ground state solution of*

$$\begin{cases} -\Delta u + V(y_0)u = g(u) + \lambda v, & \text{in } \mathbb{R}^N, \\ -\Delta v + M(y_0)v = f(v) + \lambda u, & \text{in } \mathbb{R}^N, \\ u, v > 0 \text{ in } \mathbb{R}^N, \ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

as $\varepsilon \to 0$. In particular, if $\mathscr{M}_1 \cap \mathscr{M}_2 \neq \emptyset$, then $dist(\varepsilon y_\varepsilon, \mathscr{M}_1 \cap \mathscr{M}_2) \to 0$, and $k_\varepsilon(x) = z_\varepsilon(x+y_\varepsilon)$ converges in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to a positive ground state solution of

$$\begin{cases} -\Delta u + V_0 u = g(u) + \lambda v, & in\ \mathbb{R}^N, \\ -\Delta v + M_0 v = f(v) + \lambda u, & in\ \mathbb{R}^N, \\ u, v > 0\ in\ \mathbb{R}^N,\ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

as $\varepsilon \to 0$, where z_ε denotes the positive ground sate solution of system $(\mathscr{P}_\varepsilon)$.

Proof. Let $\varepsilon_j \to 0$ and $z_j \in \mathscr{L}'_{\varepsilon_j}$ such that $\mathscr{J}_{\varepsilon_j,\lambda}(z_j) = c_{\varepsilon_j}$ and $\mathscr{J}'_{\varepsilon_j,\lambda}(z_j) = 0$. Obviously, $\{z_j\} \subset \mathscr{N}_{\varepsilon_j}$. By the same arguments as shown in Lemma 5.3, it is easy to see that $\{z_j\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Assume that $z_j \rightharpoonup z$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Since $\mathscr{J}_{\varepsilon_j,\lambda}(z_j) = c_{\varepsilon_j} \le c_{V(y_0)M(y_0)}$ for each $y_0 \in \mathbb{R}$ with large j, for $y_0 = x_v$ (x_v is given in $(\mathscr{V}_0)$), we deduce that $\mathscr{J}_{\varepsilon_j,\lambda}(u_j) = c_{\varepsilon_j} \le c_{V(y_0)M(y_0)}$ for large j, according to Lemma 4.2. Then we have

$$\lim_{j\to\infty} c_{\varepsilon_j} \le c_{V(y_0)M(y_0)} < c_{V_\infty M_\infty}.$$

We now separate our discussions into four steps.

Step 1. To prove that $\{z_j\}$ is non-vanishing, from the proof of Lemma 5.3, there exist $r, \delta > 0$ and two sequences $\{y'_j\}$, $\{y_j\} \subset \mathbb{R}^N$ such that

$$\liminf_{j\to\infty} \int_{B_r(y'_j)} |z_j|^2 \ge \delta > 0, \tag{5.8}$$

and

$$|z_j(y_j)| = \max_{y\in\mathbb{R}^N} |z_j(y)|.$$

We claim that there is a $\varrho > 0$ (independent of j) such that

$$|z_j(y_j)| \ge \varrho > 0, \text{ uniformly for all } j \in \mathbb{N}. \tag{5.9}$$

Otherwise, we assume that $|z_j(y_j)| \to 0$ as $j \to \infty$. It follows from (5.8) that

$$0 < \delta \le \int_{B_r(y'_j)} |z_j|^2 \le c|z_j(y_j)|^2 \to 0 \text{ as } j \to \infty.$$

This is a contradiction. Furthermore, from (5.8)-(5.9), one can check that there exist $R > r > 0$ and $\delta' > 0$ such that

$$\liminf_{j\to\infty} \int_{B_R(y_j)} |z_j|^2 \ge \delta' > 0. \tag{5.10}$$

Step 2. The sequence $\{\varepsilon_j y_j\}$ is bounded. To accomplish this, we set

$$\begin{aligned} &w_j(x) = z_j(x+y_j) = (w_j^1(x), w_j^2(x)), \\ &\hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x+y_j)), \\ &\hat{M}_{\varepsilon_j}(x) = M(\varepsilon_j(x+y_j)). \end{aligned}$$

Then along a subsequence we have $w_j \rightharpoonup w = (w^1, w^2) \neq 0$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $w_j \to w$ in $L^p_{loc}(\mathbb{R}^N) \times L^p_{loc}(\mathbb{R}^N)$ ($p \in (2, 2^*)$). Apparently, w_j solves

$$\begin{cases} -\Delta w_j^1 + \hat{V}_{\varepsilon_j}(x) w_j^1 = g(w_j^1) + \lambda w_j^2, & \text{in } \mathbb{R}^N, \\ -\Delta w_j^2 + \hat{M}_{\varepsilon_j}(x) w_j^2 = f(w_j^2) + \lambda w_j^1, & \text{in } \mathbb{R}^N, \\ w_j^1, w_j^2 > 0 \text{ in } \mathbb{R}^N,\ w_j^1, w_j^2 \in H^1(\mathbb{R}^N). \end{cases} \tag{$\mathscr{P}_{\varepsilon,1}$}$$

The corresponding energy functional is denoted by

$$\begin{aligned}\mathscr{O}_{\varepsilon_j,\lambda}(v_j) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w_j^1|^2 + \hat{V}_{\varepsilon_j}(x)(w_j^1)^2) + (|\nabla w_j^2|^2 + \hat{M}_{\varepsilon_j}(x)(w_j^2)^2)\\ &\quad - \int_{\mathbb{R}^N}(F(w_j^2) + G(w_j^1)) - \lambda\int_{\mathbb{R}^N} w_j^1 w_j^2\\ &= \mathscr{J}_{\varepsilon_j,\lambda}(w_j)\\ &= c_{\varepsilon_j}.\end{aligned}$$

We are ready to show $\{\varepsilon_j y_j\}$ is bounded. Following [**30**], we assume by contradiction that $\varepsilon_j|y_j| \to \infty$. Without loss of generality, we assume $V(\varepsilon_j y_j) \to \tilde{V}^\infty$ and $M(\varepsilon_j y_j) \to \tilde{M}^\infty$. It follows from $(\mathscr{V}_0)$ that $V(y_0) < \tilde{V}^\infty$ and $M(y_0) \le \tilde{M}^\infty$. Since both V and M are uniformly continuous functions, we have

$$\begin{aligned}&|\hat{V}_{\varepsilon_j}(x) - \tilde{V}^\infty|\\ \le&|V(\varepsilon_j(x+y_j)) - V(\varepsilon_j y_j)| + |V(\varepsilon_j y_j) - \tilde{V}^\infty| \le c\varepsilon_j|x| + |V(\varepsilon_j y_j) - \tilde{V}^\infty|\\ \to&0,\ \text{as } j\to\infty,\end{aligned}$$

and

$$\begin{aligned}&|\hat{M}_{\varepsilon_j}(x) - \tilde{M}^\infty|\\ \le&|M(\varepsilon_j(x+y_j)) - \tilde{M}(\varepsilon_j y_j)| + |M(\varepsilon_j y_j) - \tilde{M}^\infty|\\ \le&c\varepsilon_j|x| + |M(\varepsilon_j y_j) - \tilde{M}^\infty| \to 0,\ \text{as } j\to\infty.\end{aligned}$$

In addition, for each $\phi = (\phi^1,\phi^2) \in C_0^\infty(\mathbb{R}^N)$, we deduce from $w_j \rightharpoonup w$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and $w_j \to w$ in $L^p_{loc}(\mathbb{R}^N)\times L^p_{loc}(\mathbb{R}^N)(\forall p\in(2,2^*))$ that

$$\begin{aligned}&\lim_{j\to\infty}\mathscr{O}'_{\varepsilon_j,\lambda}(w_j)\phi\\ =&\lim_{j\to\infty}\int_{\mathbb{R}^N}[(\nabla w_j^1\nabla\phi^1 + \hat{V}_{\varepsilon_j}(x)w_j^1\phi^1) + (\nabla w_j^2\nabla\phi^2 + \hat{M}_{\varepsilon_j}(x)w_j^2\phi^2)]\\ &-\lim_{j\to\infty}\int_{\mathbb{R}^N}(f(w_j^2)\phi^2 + g(w_j^1)\phi^1) - \lim_{j\to\infty}\lambda\int_{\mathbb{R}^N}(\phi^1 w_j^2 + \phi^2 w_j^1)\\ =&\int_{\mathbb{R}^N}[(\nabla w^1\nabla\phi^1 + \hat{V}_\infty w^1\phi^1) + (\nabla w^2\nabla\phi^2 + \hat{M}_\infty w^2\phi^2)]\\ &-\int_{\mathbb{R}^N}(f(w^2)\phi^2 + g(w^1)\phi^1) - \lambda\int_{\mathbb{R}^N}(\phi^1 w^2 + \phi^2 w^1)\\ =&0.\end{aligned}$$

Thus, $w = (w^1,w^2)$ solves

$$(\mathscr{P}_{\hat{V}_\infty\hat{M}_\infty})\qquad\begin{cases}-\Delta w^1 + \hat{V}_\infty(x)w^1 = g(w^1) + \lambda w^2, & \text{in } \mathbb{R}^N,\\ -\Delta w^2 + \hat{M}_\infty(x)w^2 = f(w^2) + \lambda w^1, & \text{in } \mathbb{R}^N,\\ w^1(x), w^2(x)\to 0,\ \text{as } |x|\to\infty.\end{cases}$$

Denote the associated energy functional by

$$\begin{aligned}\mathscr{O}_{\infty,\lambda}(v) &= \frac{1}{2}\|w^1\|^2_{\tilde{V}^\infty} + \frac{1}{2}\|w^2\|^2_{\tilde{M}^\infty} - \int_{\mathbb{R}^N}\left[f(w^2) + g(w^1)\right] - \lambda\int_{\mathbb{R}^N} w^1 w^2\\ &\ge c_{\tilde{V}^\infty\tilde{M}^\infty}.\end{aligned}$$

Notice that

$$V(y_0) < \tilde{V}^{\infty} \text{ and } M(y_0) \leq \tilde{M}^{\infty}.$$

By Lemma 4.1 we deduce that $c_{\tilde{V}^{\infty}\tilde{M}^{\infty}} > c_{V(y_0)M(y_0)}$. Since $\mathscr{O}'_{\varepsilon_j,\lambda}(w_j)w_j = \mathscr{J}'_{\varepsilon_j,\lambda}(z_j)z_j = 0$, it follows from Fatou's lemma and (1.5) that

(5.11)

$$\begin{aligned}
\lim_{j\to\infty} c_{\varepsilon_j} &= \lim_{j\to\infty} \mathscr{O}_{\varepsilon_j,\lambda}(w_j) \\
&= \lim_{j\to\infty}\left[\mathscr{O}_{\varepsilon_j,\lambda}(w_j) - \frac{1}{2}\mathscr{O}_{\varepsilon_j,\lambda}(w_j)'(w_j)w_j\right] \\
&\geq \liminf_{j\to\infty}\left\{\int_{\mathbb{R}^N}\left[\frac{1}{2}f\left(w_j^2\right)w_j^2 - F\left(w_j^2\right)\right] + \left[\frac{1}{2}g\left(w_j^1\right)w_j^1 - G\left(w_j^1\right)\right]\right\} \\
&\geq \left\{\int_{\mathbb{R}^N}\left[\frac{1}{2}f\left(w^2\right)w^2 - F\left(w^2\right)\right] + \left[\frac{1}{2}g\left(w^1\right)w^1 - G\left(w^1\right)\right]\right\} \\
&= \mathscr{O}_{\infty,\lambda}(w).
\end{aligned}$$

From (5.11) it gives

$$c_{V(y_0)M(y_0)} < c_{\tilde{V}^{\infty}\tilde{M}^{\infty}} \leq \mathscr{O}_{\infty,\lambda}(v) \leq \lim_{j\to\infty} c_{\varepsilon_j} \leq c_{V(y_0)M(y_0)}. \tag{5.12}$$

This is a contradiction, which implies that $\{\varepsilon_j y_j\}$ is bounded. Hence, we can assume $\tilde{y}_j = \varepsilon_j y_j \to \tilde{y}_0$. Then, $w = (w^1, w^2)$ solves

$$(\mathscr{P}_{V(\tilde{y}_0)M(\tilde{y}_0)}) \qquad \begin{cases} -\Delta w^1 + V(\tilde{y}_0)w^1 = g(w^1) + \lambda w^2, & \text{in } \mathbb{R}^N, \\ -\Delta w^2 + M(\tilde{y}_0)w^2 = f(w^2) + \lambda w^1, & \text{in } \mathbb{R}^N, \\ w^1(x), w^2(x) \to 0, \text{ as } |x| \to \infty. \end{cases}$$

Step 3. We claim that

$$\tilde{y}_0 \in \mathscr{V}. \tag{5.13}$$

We prove (5.13) by way of contradiction. Conversely, we assume that $\tilde{y}_0 \notin \mathscr{V}$. It is easy to check that $c_{V(y_0)M(y_0)} < c_{V(\tilde{y}_0)M(\tilde{y}_0)}$. Making use of the same derivation as that for (5.12) (with $\tilde{V}^{\infty}$ and $\tilde{M}^{\infty}$ replaced by $V(\tilde{y}_0)$ and $M(\tilde{y}_0)$, respectively), leads to

$$\lim_{j\to\infty} c_{\varepsilon_j} \leq c_{V(y_0)M(y_0)} < c_{V(\tilde{y}_0)M(\tilde{y}_0)} \leq \lim_{j\to\infty} c_{\varepsilon_j}. \tag{5.14}$$

This is, obviously, a contradiction.

To prove that (w^1, w^2) is a ground state solution of system $(\mathscr{P}_{V(\tilde{y}_0)M(\tilde{y}_0)})$, we choose $y_0 = \tilde{y}_0$. From Lemma 4.2, we find

$$\lim_{j\to\infty} c_{\varepsilon_j} \leq c_{V(\tilde{y}_0)M(\tilde{y}_0)}.$$

By using the same derivation as shown in (5.11), one can infer that

$$c_{V(\tilde{y}_0)M(\tilde{y}_0)} \leq \lim_{j\to\infty} c_{\varepsilon_j}.$$

This implies that

$$c_{V(\tilde{y}_0)M(\tilde{y}_0)} = \lim_{j\to\infty} c_{\varepsilon_j}$$

and (w^1, w^2) is a ground state solution of system $(\mathscr{P}_{V(\tilde{y}_0)M(\tilde{y}_0)})$. In particular, if $\mathscr{U}_1 \cap \mathscr{U}_2 \neq \emptyset$, then $\mathscr{V} = \mathscr{U} = \mathscr{U}_1 \cap \mathscr{U}_2$. It is easy to see that

$$\lim_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathscr{U}_1 \cap \mathscr{U}_2) = 0$$

and w_ε converges weakly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ (up to subsequences) to a least energy solution of

$$\begin{cases}-\Delta w^1+V_0w^1=g(w^1)+\lambda w^2, \quad \text{in } \mathbb{R}^N,\\ -\Delta w^2+M_0w^2=f(w^2)+\lambda w^1, \quad \text{in } \mathbb{R}^N,\\ w^1(x),w^2(x)\to 0, \text{ as } |x|\to\infty,\end{cases}$$

as $\varepsilon\to 0$.

Step 4. Since w_j and w satisfy systems $(\mathscr{P}_{\varepsilon,\lambda})$ and $(\mathscr{P}_{V(\tilde{y}_0)M(\tilde{y}_0)})$, respectively, we shall prove that $w_j\to w=(w^1,w^1)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. It follows from (5.10) that $w\neq 0$. By using the same argument as shown in the proof of (A_3) of Theorem 3.6, we have

$$\int_{\mathbb{R}^N}\nabla w_j^1\nabla\left(w_j^1-w^1\right)+\hat{V}_{\varepsilon_j}(x)\,w_j^1\left(w_j^1-w^1\right)=\lambda\int_{\mathbb{R}^N}w_j^2\left(w_j^1-w^1\right)+o(1), \tag{5.15}$$

$$\int_{\mathbb{R}^N}\nabla w_j^2\nabla\left(w_j^2-w^2\right)+\hat{V}_{\varepsilon_j}(x)\,w_j^2\left(w_j^2-w^2\right)=\lambda\int_{\mathbb{R}^N}w_j^1\left(w_j^2-w^2\right)+o(1), \tag{5.16}$$

$$\int_{\mathbb{R}^N}\nabla w^1\nabla\left(w_j^1-w^1\right)+V(\tilde{y}_0)\,w^1\left(w_j^1-w^1\right)=\lambda\int_{\mathbb{R}^N}w^2\left(w_j^1-w^1\right)+o(1), \tag{5.17}$$

$$\int_{\mathbb{R}^N}\nabla w^2\nabla\left(w_j^2-w^2\right)+M(\tilde{y}_0)\,w^2\left(w_j^2-w^2\right)=\lambda\int_{\mathbb{R}^N}w^2\left(w_j^2-w^2\right)+o(1). \tag{5.18}$$

Combining (5.15)-(5.18), we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\left|\nabla(w_j^1-w^1)\right|^2+\hat{V}_{\varepsilon_j}(x)\left|w_j^1-w^1\right|^2+(\hat{V}_{\varepsilon_j}(x)-V(\tilde{y}_0))w^1(w_j^1-w^1)\right]\\ =&\lambda\int_{\mathbb{R}^N}\left(w_j^1-w^1\right)\left(w_j^2-w^2\right)+o(1).\end{aligned} \tag{5.19}$$

From $\tilde{y}_j\to\tilde{y}_0$ and $\varepsilon_j\to 0$ as $j\to\infty$, we see that for each $\beta>0$ there exists a $R=R(\beta)>0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\hat{V}_{\varepsilon_j}(x)-V(\tilde{y}_0)\right)w^1(w_j^1-w^1)\\ &=\left(\int_{|x|>R}+\int_{|x|\le R}\right)(\hat{V}_{\varepsilon_j}(x)-V(\tilde{y}_0))w^1(w_j^1-w^1)\\ &\le c\left[\int_{|x|>R}(w^1)^2\right]^{\frac12}\left(\int_{|x|>R}\left|w_j^1-w^1\right|^2\right)^{\frac12}\\ &\quad +c\left|\hat{V}_{\varepsilon_j}(x)-V(\tilde{y}_0)\right|_{L^\infty(B_R(0))}\\ &\le c\beta+o(1).\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^N}(\hat{V}_{\varepsilon_j}(x)-V(\tilde{y}_0))w^1(w_j^1-w^1)=o(1)$$

as $j \to \infty$. So we have

(5.20)
$$\int_{\mathbb{R}^N} \left[\left|\nabla(w_j^1 - w^1)\right|^2 + \hat{V}_{\varepsilon_j}(x) \left|w_j^1 - w^1\right|^2 \right] = \lambda \int_{\mathbb{R}^N} (w_j^1 - w^1)(w_j^2 - w^2) + o(1).$$

Similarly, from (5.16) and (5.18) we get

(5.21)
$$\int_{\mathbb{R}^N} \left[\left|\nabla(w_j^2 - w^2)\right|^2 + \hat{M}_{\varepsilon_j}(x) \left|w_j^2 - w^2\right|^2 \right] = \lambda \int_{\mathbb{R}^N} (w_j^1 - w^1)(w_j^2 - w^2) + o(1).$$

Combining (5.20) and (5.21) leads to

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[|\nabla(w_j^2 - w^2)|^2 + M_0|w_j^2 - w^2|^2\right] \\
&\quad + \int_{\mathbb{R}^N} \left[|\nabla(w_j^1 - w^1)|^2 + V_0|w_j^1 - w^1|^2\right] \\
\leq &\int_{\mathbb{R}^N} \left[|\nabla(w_j^2 - w^2)|^2 + \hat{M}_{\varepsilon_j}(x)|w_j^2 - w^2|^2\right] \\
&\quad + \int_{\mathbb{R}^N} \left[|\nabla(w_j^1 - w^1)|^2 + \hat{V}_{\varepsilon_j}(x)|w_j^1 - w^1|^2\right] \\
=&2\lambda \int_{\mathbb{R}^N} (w_j^1 - w^1)(w_j^2 - w^2) + o(1) \\
\leq&\lambda \int_{\mathbb{R}^N} |w_j^1 - w^1|^2 + \lambda \int_{\mathbb{R}^N} |w_j^2 - w^2|^2 + o(1).
\end{aligned} \tag{5.22}$$

Since $\lambda < \min\{V_0, M_0\}$, we see that $w_j \to w$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Moreover, it follows from w_j^2, $w_j^1 > 0$ in $\mathbb{R}^N$ and (5.10) that w^1, $w^2 \geq 0$, $w^1 \not\equiv 0$ and $w^2 \not\equiv 0$. Consequently, it follows from Harnack's inequality (see [**24**]) that w^1, $w^2 > 0$ in $\mathbb{R}^N$. □

In order to obtain some exponent decay for the solution of system ($\mathscr{P}_\varepsilon$), we need the following regularization results. For the details of the proofs, one can see [**24, 32, 33**].

Lemma 5.7. *Let $z \in H^1(\mathbb{R}^N)$ satisfy*

$$-\Delta z + (Q(x) + H(x))z = f(x, z), \quad z \in H^1(\mathbb{R}^N),$$

where $H \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $Q(x) \geq 0$ in $\mathbb{R}^N$, $Q \in L^\infty_{loc}(\mathbb{R}^N, \mathbb{R}^+)$, and f is a Caratheodory function such that

$$0 \leq f(x, s) \leq C_f(s + s^{r-1}), \quad \forall s \geq 0,$$

where $2 < r < \frac{2N}{N-2}$. Then $z \in L^p(\mathbb{R}^N)$ for all $2 \leq p < \infty$. Furthermore, there is a positive constant C_p depending on p, C_f and Q such that $|z|_{L^p(\mathbb{R}^N)} \leq C_p\|z\|_{H^1(\mathbb{R}^N)}$. The dependence on Q of C_p can be given uniformly on Cauchy sequences Q_k in $L^{\frac{N}{2}}(\mathbb{R}^N)$.

Lemma 5.8. *Suppose that $t > N$, $k \in L^{\frac{t}{2}}(\Lambda)$ and $z \in H^1(\Lambda)$ satisfies*

$$-\Delta z \leq k(x),$$

in the weak sense, where Λ *is a open subset of* $\mathbb{R}^N$. *Then for any ball* $B_{2R}(y) \subset \Lambda$, *we have*

$$\sup_{B_R(y)} z \le C \left(|z^+|_{L^2(B_{2R}(y))} + |k|_{L^{\frac{t}{2}}(B_{2R}(y))} \right),$$

where C *depends on* N, t *and* R.

The following lemma is concerning the exponent decay of the positive ground state solution of system $(\mathscr{P}_\varepsilon)$.

LEMMA 5.9. *Under the assumptions of Theorem 1.1, if* $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ *is a positive ground state solution of system* $(\mathscr{P}_\varepsilon)$, *then for small* $\varepsilon > 0$ *we have*

$$\begin{aligned} &\lim_{|x|\to\infty} u_\varepsilon(x) = \lim_{|x|\to\infty} v_\varepsilon(x) = 0, \\ &\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = \lim_{|x|\to\infty} |\nabla v_\varepsilon(x)| = 0, \end{aligned} \tag{5.23}$$

where $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ *for* $\sigma \in (0,1)$. *Furthermore, there exist* $C, c > 0$ *such that*

$$u_\varepsilon(x) + v_\varepsilon(x) \le C e^{-c|x - y_\varepsilon|},$$

where $|z_\varepsilon(y_\varepsilon)| = \max_{x\in\mathbb{R}^N} |z_\varepsilon(x)|$.

PROOF. By the proof of (A_2) of Theorem 3.6, we know that for each small $\varepsilon > 0$, (5.23) holds and $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ for $\sigma \in (0,1)$. In the following we shall prove the exponent decay for the positive solution of $w_\varepsilon = u_\varepsilon + v_\varepsilon$. Let $\varepsilon_j \to 0$ and $z_j = (u_j, v_j) \in \mathscr{L}'_{\varepsilon_j}$ such that $\mathscr{U}_{\varepsilon_j,\lambda}(z_j) = c_{\varepsilon_j}$ and $\mathscr{U}'_{\varepsilon_j,\lambda}(z_j) = 0$. As in the proof of Lemma 5.6, we have $q_j(x) = (q_j^1(x), q_j^2(x)) = z_j(x + y_j) = (u_j(x + y_j), v_j(x + y_j))$ that solves

$$(\mathscr{P}_{\varepsilon,2}) \qquad \begin{cases} -\Delta q_j^1 + \hat{V}_{\varepsilon_j}(x) q_j^1 = g(q_j^1) + \lambda q_j^2, & \text{in } \mathbb{R}^N, \\ -\Delta q_j^2 + \hat{M}_{\varepsilon_j}(x) q_j^2 = f(q_j^2) + \lambda q_j^1, & \text{in } \mathbb{R}^N, \\ q_j^1, q_j^2 > 0 \text{ in } \mathbb{R}^N, \ q_j^1, q_j^2 \in H^1(\mathbb{R}^N). \end{cases}$$

From the first two equations, there holds

$$(\mathscr{P}_{\varepsilon,3}) \qquad -\Delta \tilde{w}_j + (\hat{M}_{\varepsilon_j}(x) + \hat{V}_{\varepsilon_j}(x) - \lambda)\tilde{w}_j = f(q_j^2) + g(q_j^1) + \hat{M}_{\varepsilon_j}(x) q_j^1 + \hat{V}_{\varepsilon_j}(x) q_j^2$$

in $\mathbb{R}^N$, where $\tilde{w}_j = q_j^1 + q_j^2$. Furthermore, we know that $q_j^1 \to q^1$ and $q_j^2 \to q^2$ in $H^1(\mathbb{R}^N)$, and $|z_j(y_j)| = \max_{y\in\mathbb{R}^N} |z_j(y)|$. So, we deduce from Lemma 5.7 that $\tilde{w}_j \in L^t(\mathbb{R}^N)$ for all $t \ge 2$ and

$$|\tilde{w}_j|_{L^t(\mathbb{R}^N)} \le N_t \|\tilde{w}_j\|_{H^1(\mathbb{R}^N)}, \tag{5.24}$$

where N_t does not depend on j. Clearly, for each $l \in (2, 2^*]$ there holds

$$\lim_{R\to\infty} \int_{|x|\ge R} \left[(q_j^1)^2 + (q_j^1)^l + (q_j^2)^2 + (q_j^2)^l \right] = 0, \text{ uniformly for } j \in \mathbb{N}. \tag{5.25}$$

Let $g_j(x) = f(q_j^2) + g(q_j^1)$. Then system $(\mathscr{P}_{\varepsilon,3})$ is equivalent to

$$(\mathscr{P}_{\varepsilon,4}) \qquad -\Delta \tilde{w}_j + (\hat{M}_{\varepsilon_j}(x) - \lambda) q_j^2 + (\hat{V}_{\varepsilon_j}(x) - \lambda) q_j^1 = g_j(x) \text{ in } \mathbb{R}^N.$$

This gives

$$(\mathscr{P}_{\varepsilon,5}) \qquad -\Delta \tilde{w}_j \le g_j(x) \text{ in } \mathbb{R}^N.$$

On the other hand, it deduces from (5.24) and (2.5) that for all $t \ge 2$, there exists a $C > 0$ such that

$$|g_j|_{L^t(\mathbb{R}^N)} \le C, \text{ for all } j \in \mathbb{N}.$$

By Lemma 5.8, for all $y \in \mathbb{R}^N$ we find

$$\sup_{B_1(y)} \tilde{w}_j \leq c(|\tilde{w}_j|_{L^2(B_2(y))} + |g_j|_{L^t(B_2(y))}). \tag{5.26}$$

This implies that $|\tilde{w}_j|_\infty$ is uniformly bounded. Combining (5.25) with (5.26) yields

$$\lim_{|x|\to\infty} \tilde{w}_j(x) = 0 \text{ uniformly for all } j \in \mathbb{N}.$$

Namely, there is an $\varepsilon_0 > 0$ such that

$$\lim_{|x|\to\infty} \tilde{w}_\varepsilon(x) = 0 \text{ uniformly for all } \varepsilon \in (0, \varepsilon_0].$$

Consequently, by using the same arguments as shown in the proof of (A_2) of Theorem 3.6, we know that there exist $C, \delta > 0$ (independent of ε) such that

$$\tilde{w}_\varepsilon(x) \leq Ce^{-\delta|x|},$$

where

$$\tilde{w}_\varepsilon = u_\varepsilon(x + y_\varepsilon) + v_\varepsilon(x + y_\varepsilon) \text{ and } |z_\varepsilon(y_\varepsilon)| = \max_{y\in\mathbb{R}^N} |z_\varepsilon(y)|.$$

□

Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. To prove Theorem 1.1, we go back to system $(\mathscr{K}_\varepsilon)$ with the variable substitution: $x \mapsto \frac{x}{\varepsilon}$. By Lemma 6.1 there is at least one positive ground state solution to system $(\mathscr{K}_\varepsilon)$ for small $\varepsilon > 0$, by Lemma 6.2 Part $(\mathscr{G}_1)$ holds, and by Lemmas 6.3 and 6.6 Parts $(\mathscr{G}_2)$ and $(\mathscr{G}_3)$ hold too.

To prove Theorem 1.2, we replace the condition $(\mathscr{V}_0)$ by $(\mathscr{V}_1)$, and the proof follows along the lines of the proof of Theorem 1.1. As stressed differences in the proofs of Lemmas 5.1-5.3, and 6.1-6.6, in (5.1) we take $\mu, \sigma > 0$ such that

$$V_0 \leq \mu \leq V_\infty \text{ and } M_0 \leq \sigma < M_\infty. \tag{5.27}$$

In addition, in the proofs of Lemmas 6.1 and 6.3, we take $y_0 = x_m$ and replace $\mathscr{V}$ by $\mathscr{U}$. □

6. Multiplicity of Positive Solutions of System $(\mathscr{P}_\varepsilon)$

In this section, we present the proof of the existence of multiple positive solutions of system $(\mathscr{P}_\varepsilon)$ by using the Ljusternik-Schnirelmann category theory. To accomplish this, we shall make good use of the ground state solution of $(\mathscr{P}_{V(y_0)M(y_0),\lambda})$ for $y_0 \in \mathbb{R}$. In the following discussion, we only consider the case of $V_0 < V_\infty$. Indeed, the proof of the case of $M_0 < M_\infty$ is almost the same as that of the case of $V_0 < V_\infty$. Let us consider $\delta > 0$ and $\mathscr{I} \in C_0^\infty(\mathbb{R}^+, [0, 1])$ denote a smooth nonincreasing function such that $\mathscr{I}(s) = 1$ if $0 \leq s \leq \frac{\delta}{2}$ and $\mathscr{I}(s) = 0$ if $s \geq \delta$. If $\mathscr{O} = \mathscr{M}_1 \cap \mathscr{M}_2 \neq \emptyset$, we let $z_1 = (u_1, u_2)$ denote a positive ground state solution of the problem $(\mathscr{P}_{V_0M_0})$. For any $y \in \mathscr{O}$, we define

$$\psi_{i,\varepsilon,y}(x) = \mathscr{I}(|\varepsilon x - y|)u_i\left(\frac{\varepsilon x - y}{\varepsilon}\right),\ i = 1, 2. \tag{6.1}$$

Apparently, $(\psi_{1,\varepsilon,y}(x), \psi_{2,\varepsilon,y}(x)) \in E_\varepsilon$. Then there exist a $t_{1,\varepsilon} > 0$ such that

$$t_{1,\varepsilon}(\psi_{1,\varepsilon,y}(x), \psi_{2,\varepsilon,y}(x)) \in \mathscr{N}_\varepsilon.$$

Define the mapping $\Psi_\varepsilon\colon \mathscr{O} \to \mathscr{N}_\varepsilon$ by

$$\Psi_\varepsilon(y) = t_{1,\varepsilon}(\psi_{1,\varepsilon,y}, \psi_{2,\varepsilon,y}). \tag{6.2}$$

We know that $\Psi_\varepsilon(y)$ has a compact support for any $y \in \mathscr{O}$. By using almost the same argument as described in [**27, 28, 34**], one can obtain the following results.

LEMMA 6.1. *Under the assumptions of* $(\mathscr{V}_0)$ *and* $(\mathscr{F}_1)$*-*$(\mathscr{F}_3)$*, we have*

$$\lim_{\varepsilon\to 0} \mathscr{J}_{\varepsilon,\lambda}(\Psi_\varepsilon(y)) = c_{V_0M_0}, \text{ uniformly for} y \in \mathscr{O}.$$

For each $\delta > 0$, let $\varrho = \varrho(\delta)$ satisfy $\mathscr{O}_\delta \subset B_\varrho(0)$. Define $\gamma\colon \mathbb{R}^N \to \mathbb{R}^N$ by

$$\gamma(x) = x \text{ for } |x| \le \varrho \text{ and } \gamma(x) = \frac{\varrho x}{|x|} \text{ for } |x| \ge \varrho.$$

Also, we define a barycenter type map $\beta_\varepsilon\colon \mathscr{N}_\varepsilon \to \mathbb{R}$ by

$$\beta_\varepsilon(u,v) = \frac{\int_{\mathbb{R}^N}\gamma(\varepsilon x)u^2}{2\int_{\mathbb{R}^N}u^2} + \frac{\int_{\mathbb{R}^N}\gamma(\varepsilon x)v^2}{2\int_{\mathbb{R}^N}v^2}.$$

As shown in Lemma 6.1, by using Lebesgue's Theorem it is easy to see that

$$\begin{aligned}
\beta_\varepsilon(\Psi_\varepsilon(y)) &= \frac{\int_{\mathbb{R}^N}\gamma(\varepsilon x)(\psi_{1,\varepsilon,y}(x))^2}{2\int_{\mathbb{R}^N}(\psi_{1,\varepsilon,y}(x))^2} + \frac{\int_{\mathbb{R}^N}\gamma(\varepsilon x)(\psi_{2,\varepsilon,y}(x))^2}{2\int_{\mathbb{R}^N}(\psi_{2,\varepsilon,y}(x))^2}\\
&= \frac{\int_{\mathbb{R}^N}\gamma(\varepsilon x+y)|u_1(x)\mathscr{I}(|\varepsilon x|)|^2}{2\int_{\mathbb{R}^N}|u_1(x)\mathscr{I}(|\varepsilon x|)|^2} + \frac{\int_{\mathbb{R}^N}\gamma(\varepsilon x+y)|u_2(x)\mathscr{I}(|\varepsilon x|)|^2}{2\int_{\mathbb{R}^N}|u_2(x)\mathscr{I}(|\varepsilon x|)|^2}\\
&= \frac{y}{2} + \frac{\int_{\mathbb{R}^N}(\gamma(\varepsilon x+y)-y)|u_1(x)\gamma(|\varepsilon x|)|^2}{2\int_{\mathbb{R}^N}|u_1(x)\gamma(|\varepsilon x|)|^2} + \frac{y}{2}\\
&\quad + \frac{\int_{\mathbb{R}^N}(\gamma(\varepsilon x+y)-y)|u_2(x)\gamma(|\varepsilon x|)|^2}{2\int_{\mathbb{R}^N}|u_2(x)\gamma(|\varepsilon x|)|^2}\\
&= y + o(1),
\end{aligned} \tag{6.3}$$

as $\varepsilon \to 0$ uniformly for $y \in \mathscr{O}$. Hence, we see that

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Psi_\varepsilon(y)) = y$$

uniformly for $y \in \mathscr{O}$.

LEMMA 6.2. *Suppose that the assumptions of* $(\mathscr{V}_0)$ *and* $(\mathscr{F}_1)-(\mathscr{F}_3)$ *hold. If* $\mathscr{O} = \mathscr{M}_1 \cap \mathscr{M}_2 \neq \emptyset$*, we take* $z_n = (u_n, v_n) \subset \mathscr{N}_{V_0M_0}$ *such that* $\mathscr{J}_{V_0M_0,\lambda}(z_n) \to c_{V_0M_0}$*. Then either* $\{z_n\}$ *has a subsequence strongly convergent in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *or there exists* $\{y_n\} \subset \mathbb{R}^N$ *such that the sequence* $w_n(x) = z_n(x+y_n)$ *converges strongly in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$*. In particular, there exists a minimizer of* $c_{V_0M_0}$*.*

PROOF. By Lemma 3.5, we know that $\{z_n\}$ is a bounded sequence. From Lemma 3.2, $w_n = \check{m}_{V_0M_0}(z_n)$ is a minimizer sequence of $\mathscr{U}_{V_0M_0,\lambda}$. By Ekeland's variational principle [**17**], we may assume that

$$\mathscr{M}_{V_0M_0,\lambda}(w_n) \to c_{V_0M_0} \text{ and } \mathscr{M}'_{V_0M_0,\lambda}(w_n) \to 0.$$

So it follows that

$$\mathscr{J}_{V_0M_0,\lambda}(z_n) \to c_{V_0M_0},\quad \mathscr{J}'_{V_0M_0,\lambda}(z_n) \to 0 \quad\text{and}\quad \mathscr{J}'_{V_0M_0,\lambda}(z_n)z_n = 0, \tag{6.4}$$

where $z_n = m_{V_0M_0}(w_n)$. For some subsequence, still denoted by $\{z_n\}$, we may assume that there exists $z = (u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that $z_n \rightharpoonup z$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. We divide our discussions into two cases.

(a_1) If $z \neq 0$, then $z \in \mathscr{N}_{V_0 M_0}$ and

$$\begin{aligned} c_{V_0M_0} &\le \mathscr{J}_{V_0M_0,\lambda}(z) \\ &= \mathscr{J}_{V_0M_0,\lambda}(z) - \frac{1}{2}(\mathscr{J}'_{V_0M_0,\lambda}(z), z) \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2}g_1(u)u - G_1(u)\right) + \int_{\mathbb{R}^N} \left(\frac{1}{2}f_1(v)v - F_1(v)\right) \\ &\quad + c_1 \int_{\mathbb{R}^N} u^q + + c_2 \int_{\mathbb{R}^N} v^p \\ &\le \liminf_{n\to\infty} \left[\int_{\mathbb{R}^N} \left(\frac{1}{2}g_1(u_n)u_n - G_1(u_n)\right) + \int_{\mathbb{R}^N} \left(\frac{1}{2}f_1(v_n)v_n - F_1(v_n)\right) \right. \\ &\quad \left. + c_1 \int_{\mathbb{R}^N} u_n^q + + c_2 \int_{\mathbb{R}^N} v_n^p \right] \\ &= \liminf_{n\to\infty} \left[\mathscr{J}_{V_0M_0,\lambda}(z_n) - \frac{1}{2}\mathscr{J}'_{V_0M_0,\lambda}(z_n)z_n \right] \\ &= c_{V_0M_0}. \end{aligned}$$

So we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q = \int_{\mathbb{R}^N} |u|^q \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p = \int_{\mathbb{R}^N} |v|^p.$$

It follows from Brezis-Lieb's lemma and Sobolev's inequality that $u_n \to u$ in $L^t(\mathbb{R}^N)$ ($\forall t \in [2, q]$) and $v_n \to v$ in $L^t(\mathbb{R}^N)$ ($\forall t \in [2, p]$). Since

$$\mathscr{J}'_{V_0M_0}(z_n)(z_n - z) = o(1) \text{ and } \mathscr{J}'_{V_0M_0}(z)(z_n - z) = 0,$$

by using the same argument as shown in the proof of Theorem 3.6, we obtain

$$\|u_n - u\|_{H^1(\mathbb{R}^N)},\ \|v_n - v\|_{H^1(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty.$$

(a_2) If $z = 0$, according to Lemma 3.5, there exist $\{y_n\} \subset \mathbb{R}^N$ and $r, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \ge \delta. \tag{6.5}$$

Set $w_n(x) = z_n(x + y_n)$. We know that

$$\|w_n\|_{V_0M_0} = \|z_n\|_{V_0M_0},\ \mathscr{J}_{V_0M_0,\lambda}(w_n) \to c_{V_0M_0} \text{ and } \mathscr{J}'_{V_0M_0,\lambda}(w_n) \to 0.$$

Clearly, there exists $w \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with $w \neq 0$ such that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Then we arrive at the desired result by following the argument used in the case of $z \neq 0$. □

By using an analogous argument as shown in Lemma 5.6, we can obtain the following result immediately.

Lemma 6.3. *Under the assumptions of* $(\mathscr{V}_0)$ *and* $(\mathscr{F}_1) - (\mathscr{F}_3)$, *if* $\varepsilon_n \to 0$ *and* $\{z_n\} \subset \mathscr{N}_{\varepsilon_n}$ *such that* $\mathscr{J}_{\varepsilon_n,\lambda}(z_n) \to c_{V_0M_0}$, *then there exists a sequence* $\{y_n\} \subset \mathbb{R}^N$ *such that* $\tilde{y}_n = \varepsilon_n y_n \to y \in \mathscr{O}$.

Let $\mathscr{E}(\varepsilon)$ denote the positive function such that $\mathscr{E}(\varepsilon) \to 0$ as $\varepsilon \to 0$. We define the set:

$$\mathfrak{B}_\varepsilon = \{z \in \mathscr{N}_\varepsilon : \mathscr{J}_{\varepsilon,\lambda}(z) \le c_{V_0M_0} + \mathscr{E}(\varepsilon)\}.$$

In fact, for any $y \in \mathscr{O}$ we deduce from Lemma 6.1 that

$$\mathscr{E}(\varepsilon) = |\mathscr{J}_{\varepsilon,\lambda}(\Psi_\varepsilon(y)) - c_{V_0M_0}| \to 0 \text{ as } \varepsilon \to 0^+.$$

That is, $\Psi_\varepsilon(y) \in \mathfrak{B}_\varepsilon$ and $\mathfrak{B}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$.

LEMMA 6.4. *Suppose that the assumptions of Lemma 6.1 are satisfied. Then*

$$\lim_{\varepsilon\to 0} \sup_{z\in\mathfrak{B}_\varepsilon} dist(\beta_\varepsilon(z), \mathscr{O}_\delta) = 0$$

holds for any $\delta > 0$.

PROOF. Take $\{\varepsilon_n\} \subset \mathbb{R}^+$ such that $\varepsilon_n \to 0$. By the definition there exists $\{z_n\} \subset \mathfrak{B}_{\varepsilon_n}$ such that

$$dist(\beta_{\varepsilon_n}(z_n), \mathscr{O}_\delta) = \sup_{z\in\mathfrak{B}_{\varepsilon_n}} dist(\beta_{\varepsilon_n}(z), \mathscr{O}_\delta) + o(1).$$

It suffices to find a sequence $\{\tilde{y}_n\} \subset \mathscr{O}$ satisfying $|\beta_{\varepsilon_n}(z_n) - \tilde{y}_n| = o(1)$. Since $\mathscr{J}_{V_0M_0,\lambda}(tz_n) \le \mathscr{J}_{\varepsilon,\lambda}(tz_n)$ for $t \ge 0$ and $\{z_n\} \subset \mathfrak{B}_{\varepsilon_n} \subset \mathscr{N}_{\varepsilon_n}$, it gives

$$c_{V_0M_0} \le c_{\varepsilon_n} \le \mathscr{J}_{\varepsilon_n,\lambda}(z_n) \le c_{V_0M_0} + \mathscr{E}(\varepsilon_n),$$

which leads to

$$\mathscr{J}_{\varepsilon_n,\lambda}(z_n) \to c_{V_0M_0}.$$

By Lemma 6.3 there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\tilde{y}_n = y_n\varepsilon_n \in \mathscr{O}_\delta$ for the sufficiently large n. So we have

$$\begin{aligned}\beta_{\varepsilon_n}(z_n) = \tilde{y}_n &+ \frac{\int_{\mathbb{R}^N}(\gamma(\varepsilon_n z + \tilde{y}_n) - \tilde{y}_n)u_n^2(z+\tilde{y}_n)}{2\int_{\mathbb{R}^N} u_n^2(z+\tilde{y}_n)} \\ &+ \frac{\int_{\mathbb{R}^N}(\gamma(\varepsilon_n z + \tilde{y}_n) - \tilde{y}_n)v_n^2(z+\tilde{y}_n)}{2\int_{\mathbb{R}^N} v_n^2(z+\tilde{y}_n)}.\end{aligned}$$

Note that $\varepsilon_n z + \tilde{y}_n \to y \in \mathscr{O}$. Hence, we obtain $\beta_{\varepsilon_n}(z_n) = \tilde{y}_n + o(1)$ and then the sequence $\{\tilde{y}_n\}$ is the desired one. □

LEMMA 6.5. *Suppose that the assumptions of Theorem 1.4 are satisfied. If $z_n = (u_n, v_n)$ satisfies $\mathscr{J}_{\varepsilon_n,\lambda}(z_n) \to c_{V_0M_0}$, and there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |z_n|^2 \ge \delta > 0,$$

and $w_n(x) = (\tilde{u}_n, \tilde{v}_n) = z_n(x+y_n)$ satisfies the problem

$$(\mathscr{P}_{\varepsilon,5}) \qquad \begin{cases} -\Delta\tilde{u}_n + \hat{V}_{\varepsilon_j}(x)\tilde{u}_n = g(\tilde{u}_n) + \lambda\tilde{v}_n, & in\ \mathbb{R}^N, \\ -\Delta\tilde{v}_n + \hat{M}_{\varepsilon_j}(x)\tilde{v}_n = f(\tilde{v}_n) + \lambda\tilde{u}_n, & in\ \mathbb{R}^N, \\ \tilde{u}_n, \tilde{v}_n > 0\ in\ \mathbb{R}^N,\ \tilde{u}_n, \tilde{v}_n \in H^1(\mathbb{R}^N), \end{cases}$$

where $\hat{V}_{\varepsilon_n}(x) = V(\varepsilon_n x + \varepsilon_n y_n)$, $\hat{M}_{\varepsilon_n}(x) = M(\varepsilon_n x + \varepsilon_n y_n)$ and y_n is given in Lemma 6.4. Then we have that $w_n \to w$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with $w \neq 0$, $w_n \in L^\infty(\mathbb{R}^N)$ and $\|w_n\|_{L^\infty(\mathbb{R}^N)} \le C$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{|x|\to\infty} \tilde{u}_n(x) = \lim_{|x|\to\infty} \tilde{v}_n(x) = 0$$

holds uniformly for $n \in \mathbb{N}$ and $\tilde{u}_n(x) + \tilde{v}_n(x) \le ce^{-c|x-y_n|}$.

Since w_n satisfies system $(\mathscr{P}_\varepsilon^1)$, we know that $\mathscr{J}'_{\varepsilon_j,\lambda}(w_n) = 0$ and $\mathscr{J}_{\varepsilon_n,\lambda}(w_n) \to c_{V_0M_0}$. By using the same arguments as shown in the proofs of Lemmas 5.6 and 5.9, we can arrive at the desired result. So here we omit the details.

LEMMA 6.6. *Under the assumptions of Theorem 1.4, there exist at least* $cat_{\mathscr{O}_\delta}(\mathscr{O})$ *positive solutions to system* $(\mathscr{P}_\varepsilon)$ *for small* $\varepsilon > 0$.

PROOF. We prove the existence of at least $cat_{\mathscr{O}_\delta}(\mathscr{O})$ positive solutions to system $(\mathscr{P}_\varepsilon)$ by using the Ljusternik- Schnirelman category theory. Usually this theory needs $\mathscr{N}_\varepsilon$ to be a C^1-submanifold of E_ε. However, here $\mathscr{N}_\varepsilon$ is not a C^1-submanifold. So, we can not apply this theory to system $(\mathscr{P}_\varepsilon)$ directly. Fortunately, from Lemma 2.2, we know that the mapping m_ε is a homeomorphism between $\mathscr{N}_\varepsilon$ and S_ε, and S_ε is a C^1-submanifold of E_ε. So, we can apply this theory to the functional

$$\mathscr{U}_{\varepsilon,\lambda}(w) = \mathscr{J}_{\varepsilon,\lambda}(\hat{m}_\varepsilon(w))|_{S_\varepsilon} = \mathscr{J}_{\varepsilon,\lambda}(m_\varepsilon(w)),$$

where $\mathscr{U}_{\varepsilon,\lambda}$ is given in Lemma 2.3.

Define

$$\begin{aligned}\Upsilon_\varepsilon(y) &= m_\varepsilon^{-1}(t_{1,\varepsilon}(\psi_{1,\varepsilon,y},\psi_{2,\varepsilon,y})) = m_\varepsilon^{-1}(\Psi_\varepsilon(y))\\ &= t_{1,\varepsilon}\left(\frac{\psi_{1,\varepsilon,y}}{\|t_\varepsilon\psi_{\varepsilon,y}\|},\frac{\psi_{2,\varepsilon,y}}{\|t_\varepsilon\psi_{\varepsilon,y}\|}\right) = \left(\frac{\psi_{1,\varepsilon,y}}{\|\psi_{\varepsilon,y}\|},\frac{\psi_{2,\varepsilon,y}}{\|\psi_{\varepsilon,y}\|}\right)\end{aligned}$$

for $y \in \mathscr{O}$. It follows from Lemma 6.1 that

$$\lim_{\varepsilon\to 0}\mathscr{U}_{\varepsilon,\lambda}(\Upsilon_\varepsilon(y)) = \lim_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\Psi_\varepsilon(y)) = c_{V_0M_0}. \tag{6.6}$$

Set

$$\tilde{\mathfrak{B}}_\varepsilon := \{w \in S_\varepsilon : \mathscr{U}_{\varepsilon,\lambda}(w) \le c_{V_0M_0} + \mathscr{E}(\varepsilon)\}, \tag{6.7}$$

where $\mathscr{E}(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. It follows from (6.6) that

$$\mathscr{E}(\varepsilon) = |\mathscr{M}_{\varepsilon,\lambda}(\Upsilon_\varepsilon(y)) - c_{V_0M_0}| \to 0$$

as $\varepsilon \to 0^+$. Thus, $\Upsilon_\varepsilon(y) \in \tilde{\mathfrak{B}}_\varepsilon$ and $\tilde{\mathfrak{B}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$.

Recall that $\mathfrak{B}_\varepsilon := \{z \in \mathscr{N}_\varepsilon : \mathscr{J}_\varepsilon(z) \le c_{V_0M_0} + \mathscr{E}(\varepsilon)\}$. From Lemmas 2.1-2.3, and 6.1 and 6.4, we know that for sufficiently small $\varepsilon > 0$, the diagram

$$\mathscr{O} \xrightarrow{\Psi_\varepsilon} \mathfrak{B}_\varepsilon \xrightarrow{m_\varepsilon^{-1}} \tilde{\mathfrak{B}}_\varepsilon \xrightarrow{m_\varepsilon} \mathfrak{B}_\varepsilon \xrightarrow{\beta_\varepsilon} \mathscr{O}_\delta \tag{6.8}$$

is well-defined. By a similar derivation of (6.3), we get

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Psi_\varepsilon(y)) = y, \text{ uniformly in } y \in \mathscr{O}. \tag{6.9}$$

For small $\varepsilon > 0$, we denote $\beta_\varepsilon(\Psi_\varepsilon(y)) = y + \zeta(y)$ for $y \in \mathscr{O}$, where $|\zeta(y)| < \frac{\delta}{2}$ uniformly for $y \in \mathscr{O}$.

Denote

$$\eta(t, y) = y + (1 - t)\zeta(y).$$

Then η: $[0, 1] \times \mathscr{O} \to \mathscr{O}_\delta$ is continuous, and

$$\eta(0, y) = \beta_\varepsilon(\Psi_\varepsilon(y)), \quad \eta(1, y) = y, \text{ for all } y \in \mathscr{O}.$$

Let

$$\tilde{\Psi}_\varepsilon = m_\varepsilon^{-1} \circ \Psi_\varepsilon \text{ and } \tilde{\beta}_\varepsilon = \beta_\varepsilon \circ m_\varepsilon.$$

The composite mapping $\tilde{\beta}_\varepsilon \circ \tilde{\Psi}_\varepsilon = \beta_\varepsilon \circ \Psi_\varepsilon$ is homotopic to the inclusion mapping id: $\mathscr{O} \to \mathscr{O}_\delta$. So it follows from Lemma 2.2 of [**34**] that

$$cat_{\tilde{\mathfrak{B}}_\varepsilon}(\tilde{\mathfrak{B}}_\varepsilon) \geq cat_{\mathscr{O}_\delta}(\mathscr{O}). \tag{6.10}$$

Choose a function $\mathscr{E}(\varepsilon) > 0$ such that $\mathscr{E}(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $(c_{V_0 M_0} + \mathscr{E}(\varepsilon))$ is not a critical level for $\mathscr{U}_{\varepsilon,\lambda}$. For small $\varepsilon > 0$, from Lemma 5.3 one can see that $\mathscr{U}_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition in $\tilde{\mathfrak{B}}_\varepsilon$. So it follows from Theorem 2.1 of [**34**] that $\mathscr{U}_{\varepsilon,\lambda}$ has at least $cat_{\tilde{\mathfrak{B}}_\varepsilon}(\tilde{\mathfrak{B}}_\varepsilon)$ critical points on $\tilde{\mathfrak{B}}_\varepsilon$. By virtue of Lemma 2.3 and (6.10), we obtain that $\mathscr{J}_{\varepsilon,\lambda}$ has at least $cat_{\mathscr{O}_\delta}(\mathscr{O})$ critical points. □

LEMMA 6.7. *Suppose that the assumptions of Theorem 1.4 hold. Let* $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ *denote one of positive solutions of system* $(\mathscr{P}_\varepsilon)$ *and* k_ε *is a maximum of* $|z_\varepsilon|$. *Then we have*

$$\lim_{\varepsilon\to 0} V(\varepsilon k_\varepsilon) = V_0, \quad \lim_{\varepsilon\to 0} M(\varepsilon k_\varepsilon) = M_0,$$

$$\lim_{|x|\to\infty} u_\varepsilon(x) = \lim_{|x|\to\infty} v_\varepsilon(x) = 0 \text{ and } \lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = \lim_{|x|\to\infty} |\nabla v_\varepsilon(x)| = 0,$$

where $u_\varepsilon, v_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ *with* $\sigma \in (0,1)$. *Furthermore, there exist constants* $C, c > 0$ *(independent of* ε*) such that*

$$u_\varepsilon(x) + v_\varepsilon(x) \leq Ce^{-c|x-k_\varepsilon|}$$

for all $x \in \mathbb{R}^N$.

The proof of this lemma can be completed by a similar idea described in [**38**], so we omit the details here.

Now we are in the position to prove Theorem 1.4.

PROOF OF THEOREM 1.4. From Lemma 6.6, we know that there are at least $cat_{\mathscr{O}_\delta}(\mathscr{O})$ positive solutions to system $(\mathscr{P}_\varepsilon)$. Making the variable substitution: $x \mapsto \frac{x}{\varepsilon}$ to system $(\mathscr{K}_\varepsilon)$, we see that there are at least $cat_{\mathscr{O}_\delta}(\mathscr{O})$ positive solutions to system $(\mathscr{K}_\varepsilon)$. As shown in Lemma 6.7, the function $h_\varepsilon(x) = z_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of system $(\mathscr{K}_\varepsilon)$, then the maximum points σ_ε and k_ε for h_ε and z_ε respectively, satisfy the equality $\sigma_\varepsilon = \varepsilon k_\varepsilon$. Consequently, we arrive at

$$\lim_{n\to\infty} V(\sigma_n) = \lim_{n\to\infty} V(\varepsilon_n k_n) = V_0$$

and

$$\lim_{n\to\infty} M(\sigma_n) = \lim_{n\to\infty} M(\varepsilon_n k_n) = M_0.$$

□

LEMMA 6.8. *Under the assumptions of Theorem 1.5, there is no positive ground state solution to system* $(\mathscr{P}_\varepsilon)$.

PROOF. For each $\varepsilon > 0$, we know $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) = E_\varepsilon$. By following [**11**], it is easy to check that $c_\varepsilon = c_{V^\infty M^\infty}$. By way of contradiction, we assume that, for some $\varepsilon_0 > 0$ there exists a positive $\hat{z}$ such that $\hat{z} \in \mathscr{N}_{\varepsilon_0}$ and $c_{\varepsilon_0} = \mathscr{J}_{\varepsilon_0,\lambda}(\hat{z})$. From Lemma 2.2, there exists $\hat{e} = (\hat{e}_1, \hat{e}_2) \in S_{\varepsilon_0}$ such that $\hat{z} = m_{\varepsilon_0}(\hat{e}) = s_1\hat{e}$, where $s_1 > 0$. By virtue of Lemma 2.2 again, we can infer that $m_{\varepsilon_0}(\hat{e}) = \hat{m}_{\varepsilon_0}(\hat{e})$ is the unique global maximum of $\mathscr{J}_{\varepsilon_0,\lambda}$ on E. Note that

$$c_{V^\infty M^\infty} \leq \mathscr{J}_{V^\infty M^\infty,\lambda}(m_{V^\infty M^\infty}(\hat{e})) = \max_{z\in E} \mathscr{J}_{V^\infty M^\infty,\lambda}(z).$$

By $(\mathscr{V}_2)$, it follows that

$$V(x) \geq V^{\infty} \text{ and } M(x) \geq M^{\infty}$$

for all $x \in \mathbb{R}^N$, and

$$\mathscr{J}_{V^{\infty}M^{\infty}}(z) \leq \mathscr{J}_{\varepsilon_0,\lambda}(z)$$

for each $z \in E$. So we have

$$\begin{aligned} c_{V^{\infty}M^{\infty}} &\leq \mathscr{J}_{V^{\infty}M^{\infty}}(m_{V^{\infty}M^{\infty}}(\hat{e})) \\ &\leq \mathscr{J}_{\varepsilon_0,\lambda}(m_{V^{\infty}M^{\infty}}(\hat{e})) \\ &\leq \mathscr{J}_{\varepsilon_0,\lambda}(m_{\varepsilon_0}(\hat{e})) \\ &= c_{\varepsilon_0} \\ &= c_{V^{\infty}M^{\infty}}. \end{aligned}$$

That is,

$$c_{V^{\infty}M^{\infty}} = \mathscr{J}_{V^{\infty}M^{\infty}}(m_{V^{\infty}M^{\infty}}(\hat{e})) = \mathscr{J}_{\varepsilon_0,\lambda}(m_{V^{\infty}M^{\infty}}(\hat{e})).$$

Moreover, we can see that $z^{\infty} = m_{V^{\infty}M^{\infty}}(\hat{e}) = (u^{\infty}, v^{\infty})$ satisfies

$$(\mathscr{P}_{\infty,1}) \qquad \begin{cases} -\Delta u^{\infty} + V^{\infty}u^{\infty} = g(u^{\infty}) + \lambda v^{\infty}, & \text{in } \mathbb{R}^N, \\ -\Delta v^{\infty} + M^{\infty}v^{\infty} = f(v^{\infty}) + \lambda u^{\infty}, & \text{in } \mathbb{R}^N, \\ u^{\infty}, u^{\infty} > 0 \text{ in } \mathbb{R}^N, \ u^{\infty}, u^{\infty} \in H^1(\mathbb{R}^N). \end{cases}$$

However, we know that

(6.11)
$$\begin{aligned} &\mathscr{J}_{V^{\infty}M^{\infty}}(z^{\infty}) \\ =&\mathscr{J}_{\varepsilon_0,\lambda}(z^{\infty}) + \frac{1}{2}\int_{\mathbb{R}^N}(V^{\infty} - V(\varepsilon_0 x))(u^{\infty})^2 + \frac{1}{2}\int_{\mathbb{R}^N}(M^{\infty} - M(\varepsilon_0 x))(v^{\infty})^2. \end{aligned}$$

From $(\mathscr{V}_2)$ it gives

$$\frac{1}{2}\int_{\mathbb{R}^N}(V^{\infty} - V(\varepsilon_0 x))(u^{\infty})^2 + \frac{1}{2}\int_{\mathbb{R}^N}(M^{\infty} - M(\varepsilon_0 x))(v^{\infty})^2 < 0. \tag{6.12}$$

Combining (6.11) and (6.12) leads to $\mathscr{J}_{V^{\infty}M^{\infty}}(z^{\infty}) < \mathscr{J}_{\varepsilon_0,\lambda}(z^{\infty})$. This is a contradiction. □

Proof of Theorem 1.5. It is not difficult to see that, Theorem 1.5 exactly follows Lemma 6.8. □

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