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Analyticity and dynamics of a Navier-Stokes-Keller-Segel system on bounded domains

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Abstract. We consider a coupled chemotaxis-fluid model:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} - n\nabla\phi, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t n + \mathbf{u}\cdot\nabla n = D_n\Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ \partial_t c + \mathbf{u}\cdot\nabla c = D_c\Delta c - nf(c), \end{cases}$$

describing the interplay of hydrodynamics and chemotaxis in bacterial suspensions, on bounded domains in $\mathbb{R}^d$ $(d = 2, 3)$. In the first part of the paper, by assuming ϕ, χ and f are analytic functions, we show that solutions on periodic domains become instantaneously analytic with respect to spatial variables for rough data, due to the parabolic smoothing effect. The result holds for all space dimensions, $d \geq 2$, for short time when the initial data is large and for an arbitrarily long given time when initial data is small in a suitably strong regularity class. However, if the initial data belongs to a weaker regularity class, then the smoothing effect is shown to hold for only small data and large time. The second part of the paper is devoted to the study of long-time asymptotic behavior of classical solutions in two space dimensions when $\chi(c) \equiv 0 \equiv f(c)$ and $\phi = x_d$. By using L^p-based energy methods, it is shown that when $\nu > 0$ and $D_n > 0$, the equilibrium determined by the no-flow boundary condition for $\mathbf{u}$ and a naturally stabilizing boundary condition for n is globally asymptotically stable regardless of the magnitude of initial data. In addition, by developing a novel energy method, we show that when $\nu = 0$ and $D_n > 0$, the naturally stabilizing boundary datum for n is still globally asymptotically stable. These appear to be the first such results for the model on bounded domains with physical boundaries. In addition, the method for proving the last result can be of independent interest.

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Contents

1. Introduction

In this paper, we consider a mathematical model describing the so-called "chemotactic Boycott effect" associated with concentrating hydrodynamic flows arising from the interplay of chemotaxis and diffusion of nutrients or signaling chemicals in bacterial suspensions [26]. The model was originally proposed to describe the dynamics of swimming bacteria, *Bacillus subtilis*, as part of the effort to understand the interaction of viscous incompressible fluid flows, chemotaxis and oxygen diffusion and consumption. In dimensionless form, the chemotaxis-fluid model reads:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} - n\nabla\phi, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t n + \mathbf{u}\cdot\nabla n = D_n\Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ \partial_t c + \mathbf{u}\cdot\nabla c = D_c\Delta c - nf(c), \end{cases} \tag{1.1}$$

$\mathbf{x}\in\mathbb{R}^d$ $(d=2,3)$, $t>0$, where $\mathbf{u}, P, n, c$ denote the fluid velocity field, scalar pressure, bacteria density, and oxygen concentration, respectively. The constants ν, D_n, D_c are the kinematic viscosity coefficient, bacteria diffusion coefficient, and oxygen diffusion coefficient, respectively. The functions $\chi(c)\geq 0$ and $f(c)\geq f(0)=0$ are two given smooth functions of c denoting the chemotactic sensitivity and oxygen consumption rate, respectively. In the derivation of the model, the Boussinesq approximation was applied to reflect the effect due to heavy bacteria. The function ϕ denotes the potential function produced by various physical mechanisms, e.g., gravitational force ($\phi(\mathbf{x}) = x_d$) or centrifugal force ($\phi(\mathbf{x}) = g(|\mathbf{x}|)$).

The system (1.1) is closely connected with classic models in fluid dynamics and chemotaxis research. On one hand, when $n\equiv 0\equiv c$, one gets the well-known Navier-Stokes/Euler system:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u}, \quad \nu\geq 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

which gave birth to mathematical fluid dynamics.

On the other hand, when the hydrodynamic effect is ignored, the model reduces to the classic Keller-Segel model of chemotaxis:

$$\begin{cases} \partial_t n = D_n\Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ \partial_t c = D_c\Delta c - nf(c), \end{cases}$$

which has prominent applications in biological modeling and has been studied intensively in the mathematics literature [2]. The case when $\chi(c) = 1/c$ was originally proposed by Keller and Segel in [14], while the case when $f(c) = c$ was proposed by Levine, Sleeman, and Nilsen-Hamilton [17]. Extended applications of the case

$\chi(c) = 1/c$ can be found in [24]. The mathematical study of the case $D_c > 0$ was initiated by Li and Wang [19], and a summary of subsequent theoretical studies can be found in the recent works [2, 27]; for the particular case when $\chi(c) = f(c) = c$, one is referred to the review paper [2].

In addition, when the chemotactic movement becomes inactive, i.e., when $\chi(c) \equiv 0 \equiv f(c)$, the model reduces to the following system of equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} - n \nabla \phi, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t n + \mathbf{u} \cdot \nabla n = D_n \Delta n, \\ \partial_t c + \mathbf{u} \cdot \nabla c = D_c \Delta c. \end{cases} \tag{1.2}$$

In particular, when the potential function is produced by gravity, the above system contains the incompressible Boussinesq equations for buoyancy driven fluid flows:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} - n \mathbf{e}_d, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t n + \mathbf{u} \cdot \nabla n = D_n \Delta n, \end{cases} \tag{1.3}$$

which have been frequently used to model atmospheric and oceanographic turbulences, as well as other astrophysical situations where rotation and stratification play a dominant role [22, 25]. Mathematically, when $d = 2$, the model shares a similar vortex stretching effect as in three-dimensional fluid flows, and when $\nu = 0$ and $D_n = 0$, it is formally identical to the three-dimensional incompressible Euler equations for axisymmetric swirling flows [23]. Due to its close connection with classic models in fluid dynamics, the 2D Boussinesq equations have attracted considerable attention in recent years. A large body of literatures has been contributed to the mathematical analysis of the model under various conditions on the system parameters, see, for example, [1, 15, 18, 29], just to mention a few.

Following standard terminology, we shall call (1.1) the "Navier-Stokes-Keller-Segel" (NSKS) system. To the authors' knowledge, the NSKS system is among the first generation of such models bridging classic models in mathematical fluid dynamics and chemotaxis research. Not only does the coupled system find interests and applications in biosciences, it also brings challenging questions into the mathematical community. For example, it is a well known result that the two-dimensional Navier-Stokes/Euler equations are globally well-posed in the regime of large-amplitude classical solutions. However, such a property is not easily enjoyed by (1.1), due to additional nonlinearities in the chemotactic component and coupling through potential forcing and advection.

In this paper, we study (1.1) on bounded domains in $\mathbb{R}^d$. In the first half, we study the relation between the structure of the system (1.1) and its well-posedness, and thus, consider the domain $\Omega = [-\pi, \pi]^d$, where $d \geq 2$, with periodic boundary conditions. In the second half, due to physical considerations, we restrict ourselves to the case of bounded domains in $d = 2, 3$ subject to the appropriate boundary conditions. To put things into perspective, we briefly survey the literature in connection with this work.

In the direction of numerical studies, Chertock *et. al.* [5] simulated (1.1) on a two-dimensional rectangular domain with mixed type (stress-free, no-flow, no-flux, periodic) boundary conditions and produced plumes for the same species of bacteria in [10], assuming $\chi(c)$ and $f(c)$ are constant multiples of each other (i.e., $\chi(c) = \mu f(c)$). Results produced in [5] showed significant consistency with the experimental observations reported in [10]. For rigorous analysis, recently, Liu-Lorz [20], Lorz [21], and Winkler [28] studied the existence of weak solutions to (1.1) on bounded domains in two and three space dimensions subject to no-flow and no-flux boundary conditions. A search in the database shows that so far these are the only rigorous mathematical results concerning the qualitative behavior of (1.1) on bounded domains with physical boundaries, while many important features of the model have not been explored yet.

Our contributions to this contemporary body of knowledge are to establish spatial analyticity and determine long-time asymptotic behavior of solutions to (1.1) on bounded spatial domains subject to various boundary conditions. We would like to mention that the study of spatial analyticity is motivated by the numerical studies in [5], since it is well known that such a property can inform numerical computations, see Section 5 for more discussion. On the other hand, the study of long-time behavior of solutions is also strongly motivated by the numerical evidence observed in [5]. Indeed, it was observed in [5] that in two space dimensions, in the region where the chemotactic movement becomes inactive (i.e., $\chi(c) \equiv 0 \equiv f(c)$), bacteria stop directed swimming and continue (along with the fluid) to sink down into the bottom part of the domain and a pile of bacteria is formed, due to gravity. Motivated by such an observation, we prescribe a special type boundary condition for the bacteria density function to corroborate the numerical result obtained in [5], and study the stability of equilibrium states determined by such boundary condition.

NOTATION 1.1. *Throughout this paper,* $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ *and* $\|\cdot\|_{W^{s,p}}$ *denote the norms of the usual Lebesgue measurable spaces* $L^p(\Omega)$, $L^\infty(\Omega)$ *and the usual Sobolev space* $W^{s,p}(\Omega)$*, respectively. For* $p = 2$*, we denote the norms* $\|\cdot\|_{L^2}$ *and* $\|\cdot\|_{W^{s,2}}$ *by* $\|\cdot\|$ *and* $\|\cdot\|_{H^s}$*, respectively. Unless specified,* C *denotes a generic constant which is independent of the unknown functions and time, but may depend on* Ω*, the system parameters, and initial data. The value of the constant may vary line by line according to the context.*

1.1. Spatial Analyticity. The first part of this paper is devoted to the study of the parabolic smoothing effect of (1.1). For this, we consider (1.1) on the domain $\Omega = [-\pi, \pi]^d$, where $d \geq 2$, with periodic boundary conditions. Then, under the notation defined in Section 2, we have

THEOREM 1.1. *Suppose that* χ, f *are real analytic on* $\mathbb{R}$ *with majorants* g_χ, g_f*, respectively, and that* ϕ *is analytic on* Ω*. Let* $\sigma \in (-1, 0]$, $\epsilon \in (0, 1]$*, and* $R > 0$*. Suppose* $(\mathbf{u}_0, n_0, c_0)$ *is given, each periodic over* Ω*, with* $\mathbf{u}_0, n_0$ *having zero spatial mean over* Ω *such that*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathbf{k}|^\sigma |\hat{\mathbf{u}}_0(\mathbf{k})| + \sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathbf{k}|^\sigma |\hat{n}_0(\mathbf{k})| + \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + |\mathbf{k}|^2)^{\epsilon/2} |\hat{c}_0(\mathbf{k})| \leq R < \infty, \tag{1.4}$$

where $\hat{\mathbf{u}}_0(\mathbf{k}), \hat{n}_0(\mathbf{k}), \hat{c}_0(\mathbf{k})$ *denote the* $\mathbf{k}^{th}$ *Fourier coefficient of the corresponding Fourier series of* $\mathbf{u}_0, n_0, c_0$*, respectively.*

(1) *($\epsilon > 0$) There exist* $T = T(R) > 0$*, an absolute constant* $C > 0$*, and a solution* $(\mathbf{u}, n, c)$ *of (1.1) for* $t \in (0, T]$ *that is real analytic with uniform spatial analyticity radius bounded below by* $\geq C\sqrt{t}$*.*
(2) *($\epsilon > 0$) Given* $T > 0$*, there exists* $R = R(T) > 0$*, an absolute constant* $C > 0$*, and a real analytic solution* $(\mathbf{u}, n, c)$ *of (1.1) for* $t \in (0, T]$ *with uniform spatial analyticity radius bounded below by* $\geq C\sqrt{t}$*.*

(3) *($\epsilon = 0$) Given $T > 0$, there exists $R = R(T) > 0$, $C > 0$, and a real analytic solution $(\mathbf{u}, n, c)$ of (1.1) for $t \in (0, T]$ with uniform spatial analyticity radius bounded below by $\geq C\sqrt{t}$.*
(4) *($\epsilon = 0$) There exist $R, T > 0$ sufficiently small such that a real analytic solution $(\mathbf{u}, n, c)$ of (1.1) exists for $t \in (0, T]$ with uniform spatial analyticity radius bounded below by $\geq C\sqrt{t}$.*

Remark 1.1. *We explicitly identify the restrictions on T and R in Section 2.5. In particular, we require $T, R > 0$ to satisfy (2.116) and (2.120).*

Theorem 1.1 ensures that as long as the initial data belongs to sufficiently strong regularity class, then the corresponding solution instantaneously regularizes to become not only smooth, but *analytic* in space. Such results for the Navier-Stokes equations are well-known and, by now, classical (cf. for instance [7]). In fact, it was shown by Ferrari and Titi [6] that solutions to a very general class of parabolic equations enjoy such a property provided that its nonlinearity is given as a *real analytic* function in its arguments and the initial data belongs to a suitable Sobolev class, i.e., $H^s(\Omega)$, for $s > d/2$. Although for convenience the analysis in [6] was performed on a scalar equation, the analogous statement for systems of such equations also holds. However, if we denote the space whose norm is given by the condition (1.4) imposed on, for instance, the initial velocity, $\mathbf{u}_0$, by $\mathcal{V}_\sigma$, then $H^s \hookrightarrow \mathcal{V}_\sigma$ for $s > d/2 + \sigma$, where $\sigma \in (-1, 0]$. Thus, our result can be viewed as an extension of the work [6] in the particular case of (1.1).

While one expects a result in the generality of [6] to hold for this more general class of initial data, the above result demonstrates that the situation is clearly more delicate in the case of systems of equations. Indeed, Theorem 1.1 (and Theorem 2.24) show that structural obstructions to this smoothing effect become possible when the regularity of the initial data is taken to be sufficiently low. This phenomenon can be traced to the apparent "criticality" of the nonlinear term, $\nabla\cdot(n\chi(c)\nabla c)$, appearing in the n-equation, which has order equal to that of the dissipative term $-D_n\Delta n$. We show that as long one works in a "subcritical setting," i.e., $\epsilon > 0$, then the usual small-data/large-time and large-data/small-time dichotomies for well-posedness hold. This work, therefore, identifies a "critical setting," i.e., $\epsilon = 0$, in which the structure of (1.1) appears to preclude a "small-time/large-data" type result. We emphasize that the structural criticality in (1.1) appears only in the n-equation. Thus, a result analogous to Theorem 1.1, but in the generality of [6] for systems must necessarily address the issue of "anisotropies" in the regularity classes with respect to the components of the system.

Furthermore, it will be clear from our proof that if the interaction potential instead took the form $\nabla\cdot(n\chi(c)\,|\nabla|^\alpha\, c)$, where $0 < \alpha < 1$, then the desired small-time/large-data result can be recovered in the borderline case $\epsilon = 0$ (see Remarks 2.1, 2.2, and Theorem 2.24 below), which suggests the genuineness of the "criticality" we have identified. Nevertheless, this work leaves open the problem whether a small-time/large-data type well-posedness result in Gevrey classes can be established for (1.1) in the case $\epsilon = 0$.

The difficulties expressed by Theorem 1.1 appear to be consistent with those expressed by other results in the literature for the model (1.1). For instance, in [21] existence of *local weak solutions* is established in the case of no-flux boundary conditions when $d = 2, 3$, and for non-homogeneous Dirichlet boundary conditions when $d = 2$, *assuming that χ, f are positive constants*. On the other hand, under certain restrictions on χ, f, one can recover global existence of weak solutions [20]. We lastly point out that our proof of Theorems 1.1 and 2.24 combines ideas from [3, 4] and [6], and, to our knowledge, is the first such result for the system (1.1). Results concerning the spatial analyticity of solutions to a 2D Boussinesq system on a periodic domain are also available (cf. [8]).

1.2. Global Stability of Chemotactically Inactive Equilibria. The second part of this paper is devoted to the study of long-time asymptotic behavior of large-amplitude classical solutions to (1.1) on a two-dimensional bounded domain with physical boundary. We consider a special case in which the chemotactic movement is inactive and the flow is primarily influenced by hydrodynamic effects. It was mentioned before that in this case the coupled system (1.1) reduces to (1.2). Note that the first three equations in (1.2) form a closed system of equations. Hence, we shall focus on the long-time behavior of solutions to (1.3).

Motivated by the numerical evidence observed in [5], we consider a naturally stabilizing boundary condition for the bacteria density. Precisely, we consider the following initial-boundary value problem:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} - n\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t n + \mathbf{u}\cdot\nabla n = D_n\Delta n; \\ (\mathbf{u}, n)(\mathbf{x}, 0) = (\mathbf{u}_0, n_0)(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega}; \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad n|_{\partial\Omega} = \bar{n} - \alpha y, \end{cases} \tag{1.5}$$

$\mathbf{x} = (x, y)^{\mathrm{T}} \in \Omega,\ t > 0$, where $\mathbf{e}_2 = (0, 1)^{\mathrm{T}}$, $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$, and $\bar{n}, \alpha > 0$ are constants. The boundary condition for n indicates that the deeper the flow, the denser the bacteria population, which is a naturally stabilizing condition. To the authors' knowledge, this is the first time that such kind boundary condition is prescribed for the KSNS model or the 2D Boussinesq equations. Our goal is to prove that the equilibrium, $(\mathbf{0}, \bar{n} - \alpha y)$, is globally asymptotically stable under arbitrarily large perturbations. The result is recorded in the following theorem.

THEOREM 1.2. *Consider the initial-boundary value problem* (1.5). *Suppose that the initial data* $(\mathbf{u}_0, n_0) \in H^2(\Omega)$ *satisfy* $\nabla\cdot\mathbf{u}_0 = 0$ *and are compatible with the boundary conditions. Then there exists a unique solution* $(\mathbf{u}, n) \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^3(\Omega))$*, such that for any* $t > 0$*, it holds that*

$$\|\mathbf{u}(t)\|_{H^2}^2 + \|(n - \bar{n} + \alpha y)(t)\|_{H^2}^2 \le \beta e^{-\gamma t},$$

for some constants $\beta > 0$ *and* $\gamma > 0$ *which are independent of* $t > 0$.

REMARK 1.2. *Theorem 1.2 indicates that the bacteria population is 'rearranged' by gravity to rapidly lose dependence on the horizontal variable and stratifies in the vertical direction, as time evolves, and a uniform linear gradient of the bacteria density is quickly established, due to diffusion and the naturally stabilizing boundary condition. The result shows consistency with the numerical observations reported in* [5]*, and roughly explains the sinking and piling of bacteria in a chemotactically inactive region.*

In the last part of this paper, we study the global stability of the naturally stabilizing equilibrium for the following initial-boundary value problem:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = -n\mathbf{e}_2, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t n + \mathbf{u} \cdot \nabla n = D_n \Delta n; \\ (\mathbf{u}, n)(\mathbf{x}, 0) = (\mathbf{u}_0, n_0)(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}; \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ n|_{\partial\Omega} = \bar{n} - \alpha y, \end{cases} \tag{1.6}$$

$\bar{n}, \alpha > 0$, which can be obtained from (1.5) by letting $\nu = 0$ and replacing the no-slip boundary condition by the slip boundary condition. Comparing with (1.5), the major technical difficulty for establishing certain time-asymptotic behavior for (1.6) comes from the vanished viscosity coefficient, resulting in a partially dissipative system, and the non-homogeneous boundary condition. Hence, it is natural to ask, in the absence of viscous dissipation, whether a uniform linear gradient of the bacteria density is still established as time evolves. By switching to a different approach, we can show the following:

THEOREM 1.3. *Consider the initial-boundary value problem* (1.6). *Suppose that the initial data* $(\mathbf{u}_0, n_0) \in H^3(\Omega)$ *satisfy* $\nabla \cdot \mathbf{u}_0 = 0$ *and are compatible with the boundary conditions. Then there exists a unique solution* $\mathbf{u} \in L^\infty(0, T; H^3(\Omega))$ *and* $n \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega))$ *for any* $0 < T < \infty$, *such that*

$$\lim_{t \to \infty} \|(n - \bar{n} + \alpha y)(t)\|_{L^2} = 0.$$

REMARK 1.3. *The last line recorded in Theorem 1.3 shows that the naturally stabilizing profile for the bacteria density is globally asymptotically stable even in the absence of viscous dissipation. This appears to be the first such result describing the long-time behavior of large-amplitude classical solutions to the inviscid 2D Boussinesq system with thermal diffusivity and subject to a non-homogeneous boundary condition.*

REMARK 1.4. *We would like to mention that the global well-posedness of classical solutions to* (1.6) *can be established by slightly modifying the proofs constructed in* [**18, 29**] *for the inviscid 2D Boussinesq system with linear or nonlinear thermal diffusivity. However, when the long-time asymptotic behavior of* (1.6) *is concerned, the arguments in* [**18, 29**] *are not sufficient to generate uniform-in-time energy estimates leading to the decay of the perturbation, mainly due to the non-homogeneous boundary condition. Roughly speaking, the idea we use here is to show that as a function of* t, *the square of* $\|(n - \bar{n} + \alpha y)(t)\|_{L^2}^2$ *belongs to* $W^{1,1}(0, \infty)$, *which yields the desired decay property.*

The rest of the paper contains the proofs of Theorems 1.1–1.3, which are given in Sections 2–4, respectively. The paper is finished with some concluding remarks in Section 5.

2. Instantaneous Smoothing Effect

To prove Theorem 1.1, we will view (1.1) as an infinite system of ODEs over a sequence space. Indeed, we will eventually identify the sequence as the corresponding sequence of Fourier coefficients of a periodic function over $\Omega = [-\pi, \pi]^d, d \geq 2$. We make the necessary preparations in the following section. In what follows, we may assume that f, χ are smooth real-valued functions in $\mathbb{R}$ and ϕ is a smooth real-valued function in Ω with Fourier coefficients given by $(\phi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$.

2.1. Preliminaries. Let $d \geq 2$ and $\Omega = [-\pi, \pi]^d$. Consider the following sequence spaces:

$$
\begin{aligned}
&\boldsymbol{\mathcal{K}}_1 := \{(\hat{\mathbf{u}}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in (\mathbb{C}^d)^{\mathbb{Z}^d} : \hat{\mathbf{u}}_{\mathbf{0}} = 0, \quad \hat{\mathbf{u}}_{\mathbf{k}} = \hat{\mathbf{u}}^*_{-\mathbf{k}}, \quad \mathbf{k}\cdot \hat{\mathbf{u}}_{\mathbf{k}} = 0\},\\
&\mathcal{K}_2 := \{(\hat{v}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d} : \hat{v}_{\mathbf{0}} = 0, \quad \hat{v}_{\mathbf{k}} = \hat{v}^*_{-\mathbf{k}}, \quad \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \hat{v}_{\boldsymbol{\ell}}\boldsymbol{\ell}\hat{\phi}_{-\boldsymbol{\ell}} = 0\},\\
&\mathcal{K}_3 := \{(\hat{w}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d} : \hat{w}_{\mathbf{k}} = \hat{w}^*_{-\mathbf{k}}\},
\end{aligned}
\tag{2.7}
$$

each whose topologies are induced by coordinate-wise convergence, where $a^* = \bar{a}$, denotes complex conjugation (cf. [3]). Note that $\hat{\mathbf{u}}_{\mathbf{0}} = \mathbf{0}$ corresponds to the zero spatial mean condition, $\hat{\mathbf{u}}_{\mathbf{k}} = \hat{\mathbf{u}}^*_{-\mathbf{k}}$ to the reality condition, $\mathbf{k}\cdot \hat{\mathbf{u}}_{\mathbf{k}} = 0$ the divergence-free condition, and $\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \hat{v}_{\boldsymbol{\ell}}\boldsymbol{\ell}\hat{\phi}_{-\boldsymbol{\ell}} = \mathbf{0}$ enforces the mean zero condition for the velocity equation. For convenience, we will often use the shorthand $\hat{\mathbf{u}} = (\hat{\mathbf{u}}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ and $\hat{u} = (\hat{u}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$, and let $\mathbf{u}$ denote the function whose corresponding Fourier coefficients are given by $\hat{\mathbf{u}}$.

Our ambient space will be defined as the following direct sum:

$$
\mathcal{K} := \boldsymbol{\mathcal{K}}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3. \tag{2.8}
$$

Given $\sigma \in \mathbb{R}$, we define

$$
\boldsymbol{\mathcal{V}}_\sigma := \{(\mathbf{u}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in (\mathbb{C}^d)^{\mathbb{Z}^d} : \|\mathbf{u}\|_\sigma < \infty\} \quad \text{and} \quad \mathcal{V}_\sigma := \{(u_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d} : \|u\|_\sigma < \infty\}, \tag{2.9}
$$

where we define

$$
\|u\|_\sigma := \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + |\mathbf{k}|^2)^{\sigma/2} |\hat{u}_{\mathbf{k}}|, \tag{2.10}
$$

and similarly for vector-valued sequences $\hat{\mathbf{u}}$. Note that when $\sigma = 0$, $\mathcal{V}_\sigma \subset \mathcal{W}$, where $\mathcal{W}$ is the Wiener algebra. Then for $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$, we define the subspace

$$
V_{\vec{\sigma}} := (\boldsymbol{\mathcal{V}}_{\sigma_1} \cap \boldsymbol{\mathcal{K}}_1) \oplus (\mathcal{V}_{\sigma_2} \cap \mathcal{K}_2) \oplus (\mathcal{V}_{\sigma_3} \cap \mathcal{K}_3) \subset \mathcal{K}. \tag{2.11}
$$

Now let $A = -\Delta$ with periodic boundary conditions. Let us define the *Gevrey norm* for scalar-valued sequences by

$$
\|u\|_{\lambda,\sigma} := \|e^{\lambda A^{1/2}} u\|_\sigma = \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{\lambda|\mathbf{k}|} \left(1 + |\mathbf{k}|^2\right)^{\sigma/2} |\hat{u}_{\mathbf{k}}|. \tag{2.12}
$$

For time-dependent sequences, $u = u(t)$, with $\lambda = \lambda(t)$ we define

$$
\|u(\cdot)\|_{\lambda(t),\sigma} := \|u(t)\|_{\lambda(t),\sigma},
$$

and similarly for vector-valued sequences, $\mathbf{u} = \mathbf{u}(t)$. Finally, we define for $\vec{\lambda} \in (\mathbb{R}_+)^3 := \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_j \geq 0, j = 1, 2, 3\}$, $\vec{\sigma} \in \mathbb{R}^3$, and $U = ((\mathbf{u}_{\mathbf{k}})_{\mathbf{k}}, (v_{\mathbf{k}})_{\mathbf{k}}, (w_{\mathbf{k}})_{\mathbf{k}}) \in V_{\vec{\sigma}}$ the following norm:

$$
\|U\|_{\vec{\lambda},\vec{\sigma}} := \|\mathbf{u}\|_{\lambda_1,\sigma_1} + \|v\|_{\lambda_2,\sigma_2} + \|w\|_{\lambda_3,\sigma_3},
$$

and we may similarly define the Gevrey norm for time-dependent elements, $U = U(t) \in V_{\vec{\sigma}}$, $t \geq 0$. We define the *analytic Gevrey class* by

$$
\mathcal{G}_{\lambda,\sigma} := \{(\hat{u}_{\mathbf{k}})_{\mathbf{k}} \in \mathbb{C}^{\mathbb{Z}^d} : \|u\|_{\lambda,\sigma} < \infty\}, \tag{2.13}
$$

and similarly for vector-valued sequences $\hat{\mathbf{u}}$. We recall the relation between real analytic functions and the analytic Gevrey classes in the following proposition (cf. [13, 16]):

PROPOSITION 2.1. *Let $\sigma \in \mathbb{R}$ and $\lambda > 0$. Let u be a function with Fourier coefficients $(\hat{u}_{\mathbf{k}})_{\mathbf{k}} \in \mathcal{V}_\sigma$.*

(1) *If* $u \in \mathcal{G}_{\lambda,\sigma}$*, then* u *admits an analytic extension on* $\{\mathbf{x}+\mathbf{iy} : |\mathbf{y}| < \lambda\}$.
(2) *If* u *admits an analytic extension on* $\{\mathbf{x}+\mathbf{iy} : |\mathbf{y}| < \lambda\}$*, then* $u \in \mathcal{G}_{\lambda',\sigma}$*, for all* $0 < \lambda' < \lambda$.

We will view (1.1) as a system of ODEs over a sequence space:

$$\begin{aligned}
&\frac{d}{dt}\hat{\mathbf{u}}_{\mathbf{k}} + \nu\,|\mathbf{k}|^2\,\hat{\mathbf{u}}_{\mathbf{k}} = -\mathbf{B}(\hat{\mathbf{u}},\hat{\mathbf{u}})(\mathbf{k}) + \mathbf{F}_{\nabla\phi}(\hat{n})(\mathbf{k}), \quad \mathbf{k}\cdot\hat{\mathbf{u}}_{\mathbf{k}} = 0, \quad \hat{\mathbf{u}}(\mathbf{k},0) = \hat{\mathbf{u}}_0(\mathbf{k}),\\
&\frac{d}{dt}\hat{n}_{\mathbf{k}} + D_n\,|\mathbf{k}|^2\,\hat{n}_{\mathbf{k}} = -B(\hat{\mathbf{u}},\hat{n})(\mathbf{k}) + T(\hat{n},\widehat{\chi(c)},\hat{c})(\mathbf{k}), \quad \hat{n}_{\mathbf{k}}(0) = \hat{n}_0(\mathbf{k}),\\
&\frac{d}{dt}\hat{c}_{\mathbf{k}} + D_c\,|\mathbf{k}|^2\,\hat{c}_{\mathbf{k}} = -B(\hat{\mathbf{u}},\hat{c})(\mathbf{k}) - C(\hat{n},\widehat{f(c)})(\mathbf{k}), \quad \hat{c}_{\mathbf{k}}(0) = \hat{c}_0(\mathbf{k}),
\end{aligned} \tag{2.14}$$

where $\hat{\mathbf{u}}_0(\mathbf{k}), \hat{n}_0(\mathbf{k}), \hat{c}_0(\mathbf{k})$ are the Fourier coefficients of $\mathbf{u}_0, n_0, c_0$, respectively, and

$$\mathbf{B}(\hat{\mathbf{v}},\hat{\mathbf{w}})(\mathbf{k}) := i\mathcal{P}\sum_{\boldsymbol{\ell}}(\hat{\mathbf{v}}_{\boldsymbol{\ell}}\cdot\mathbf{k})\hat{\mathbf{w}}_{\mathbf{k}-\boldsymbol{\ell}}, \tag{2.15}$$

$$\mathbf{F}_{\mathbf{w}}(\hat{v})(\mathbf{k}) := \mathcal{P}\sum_{\boldsymbol{\ell}}\hat{v}_{\boldsymbol{\ell}}\hat{\mathbf{w}}_{\mathbf{k}-\boldsymbol{\ell}}, \tag{2.16}$$

$$B(\hat{\mathbf{v}},\hat{w})(\mathbf{k}) := i\sum_{\boldsymbol{\ell}}(\hat{\mathbf{v}}_{\boldsymbol{\ell}}\cdot\mathbf{k})\hat{w}_{\mathbf{k}-\boldsymbol{\ell}}, \tag{2.17}$$

$$T(\hat{u},\hat{v},\hat{w})(\mathbf{k}) := \mathbf{k}\cdot\sum_{\boldsymbol{\ell}}\sum_{\mathbf{m}}\hat{u}_{\boldsymbol{\ell}}\widehat{v}_{\mathbf{m}-\boldsymbol{\ell}}(\mathbf{k}-\mathbf{m})\hat{w}_{\mathbf{k}-\mathbf{m}}, \tag{2.18}$$

$$C(\hat{v},\hat{w})(\mathbf{k}) := -\sum_{\boldsymbol{\ell}}\hat{v}_{\boldsymbol{\ell}}\widehat{w}_{\mathbf{k}-\boldsymbol{\ell}}, \tag{2.19}$$

where $\mathcal{P}$ denotes the Leray projection onto divergence-free sequences, i.e.,

$$(\mathcal{P}\hat{\mathbf{v}})_{\mathbf{k}} := \hat{\mathbf{v}}_{\mathbf{k}} - \langle\frac{\mathbf{k}}{|\mathbf{k}|},\hat{\mathbf{v}}_{\mathbf{k}}\rangle\frac{\mathbf{k}}{|\mathbf{k}|}, \quad |(\mathcal{P}\hat{\mathbf{v}})_{\mathbf{k}}| \le 2\,|\hat{\mathbf{v}}_{\mathbf{k}}|\,. \tag{2.20}$$

Observe that for all $\hat{\mathbf{u}},\hat{\mathbf{v}} \in \mathcal{K}_1$, we also have

$$\mathbf{B}(\hat{\mathbf{v}},\hat{\mathbf{w}})(\mathbf{k}) = i\mathcal{P}\sum_{\boldsymbol{\ell}}\hat{\mathbf{v}}_{\boldsymbol{\ell}}\cdot(\mathbf{k}-\boldsymbol{\ell})\hat{\mathbf{w}}_{\mathbf{k}-\boldsymbol{\ell}}, \quad B(\hat{\mathbf{v}},\hat{w})(\mathbf{k}) = i\sum_{\boldsymbol{\ell}}\hat{\mathbf{v}}_{\boldsymbol{\ell}}\cdot(\mathbf{k}-\boldsymbol{\ell})\hat{w}_{\mathbf{k}-\boldsymbol{\ell}}, \tag{2.21}$$

and also that

$$\mathbf{B}(\hat{\mathbf{v}},\hat{\mathbf{w}})(\mathbf{0}) = \mathbf{0}, \quad B(\hat{\mathbf{v}},w)(\mathbf{0}) = 0. \tag{2.22}$$

Next, we define a weak solution to (1.1) as follows.

Definition 2.1. *Let* $\tau \in (0,\infty]$*,* $\chi(\cdot), f(\cdot), \phi(\cdot)$ *be smooth functions, and*

$$(\hat{\mathbf{u}}_0,\hat{n}_0,\hat{c}_0) \in \mathcal{K}.$$

A weak solution to (1.1) is any element $\hat{U}(\cdot) = ((\hat{\mathbf{u}}_{\mathbf{k}}(\cdot)),(\hat{n}_{\mathbf{k}}(\cdot)),(\hat{c}_{\mathbf{k}}(\cdot))) \in \mathscr{C}([0,\tau];\mathcal{K})$ *such that*

$$|\mathbf{B}(\hat{\mathbf{u}},\hat{\mathbf{u}})(\mathbf{k},t)|\,, \quad \int_0^t |\mathbf{F}_{\nabla\phi}(\hat{n})(\mathbf{k},s)|\;ds < \infty, \tag{2.23}$$

and

$$|B(\hat{\mathbf{u}},\hat{n})(\mathbf{k},t)|,, \quad |T(\hat{n},\widehat{\chi(c)},\hat{c})(\mathbf{k},t)| < \infty, \tag{2.24}$$

and

$$|B(\hat{\mathbf{u}},\hat{c})(\mathbf{k},t)|, \quad |C(\hat{n},\widehat{f(c)})(\mathbf{k},t)| < \infty, \tag{2.25}$$

hold for all $k \in \mathbb{Z}^d$ *and a.e.* $t \in [0,\tau]$*, and* $U(\cdot)$ *satisfies (2.14) for all* $\mathbf{k} \in \mathbb{Z}^d$ *and a.e.* $t \in [0,\tau]$.

We also define a mild solution to (1.1) as follows.

Definition 2.2. *Let* $\tau \in (0,\infty]$ *and* $\chi(\cdot), f(\cdot), \phi(\cdot)$ *be smooth functions, and* $(\hat{\mathbf{u}}_0, \hat{n}_0, \hat{c}_0) \in \mathcal{K}$. *A mild solution to (1.1) is any element* $\hat{U}(\cdot) = ((\hat{\mathbf{u}}_{\mathbf{k}}), (\hat{n}_{\mathbf{k}}), (\hat{c}_{\mathbf{k}})) \in \mathscr{C}([0,\tau];\mathcal{K})$ *such that*

$$\int_0^t e^{-(t-s)\nu|\mathbf{k}|^2} |\mathbf{B}(\hat{\mathbf{u}},\hat{\mathbf{u}})(\mathbf{k},s)| \, ds, \quad \int_0^t e^{-(t-s)\nu|\mathbf{k}|^2} |\mathbf{F}_{\nabla\phi}(\hat{n})(\mathbf{k},s)| \, ds < \infty, \tag{2.26}$$

and

$$\int_0^t e^{-(t-s)D_n|\mathbf{k}|^2}|B(\hat{\mathbf{u}},\hat{n})(\mathbf{k},s)| \, ds, \quad \int_0^t e^{-(t-s)D_n|\mathbf{k}|^2}|T(\hat{n},\widehat{\chi(c)},\hat{c})(\mathbf{k},s)| \, ds < \infty, \tag{2.27}$$

and

$$\int_0^t e^{-(t-s)D_c|\mathbf{k}|^2}|B(\hat{\mathbf{u}},\hat{c})(\mathbf{k},s)| \, ds, \quad \int_0^t e^{-(t-s)D_c|\mathbf{k}|^2}|C(\hat{n},\widehat{f(c)})(\mathbf{k},s)| \, ds < \infty, \tag{2.28}$$

hold for all $\mathbf{k} \in \mathbb{Z}^d$, *and* $U(\cdot)$ *satisfies*

$$\begin{aligned}
\hat{\mathbf{u}}_{\mathbf{k}}(t) &= e^{-\nu t|\mathbf{k}|^2}\hat{\mathbf{u}}_0(\mathbf{k}) - \int_0^t e^{-(t-s)\nu|\mathbf{k}|^2}\mathbf{B}(\hat{\mathbf{u}},\hat{\mathbf{u}})(\mathbf{k},s) \, ds \\
&\quad - \int_0^t e^{-(t-s)\nu|\mathbf{k}|^2}\mathbf{F}_{\nabla\phi}(\hat{n})(\mathbf{k},s) \, ds, \\
\hat{n}_{\mathbf{k}}(t) &= e^{-D_n t|\mathbf{k}|^2}\hat{n}_0(\mathbf{k}) - \int_0^t e^{-(t-s)D_n|\mathbf{k}|^2}B(\hat{\mathbf{u}},\hat{n})(\mathbf{k},s) \, ds \\
&\quad + \int_0^t e^{-(t-s)D_n|\mathbf{k}|^2}T(\hat{n},\widehat{\chi(c)},\hat{c})(\mathbf{k},s) \, ds, \\
\hat{c}_{\mathbf{k}}(t) &= e^{-D_c t|\mathbf{k}|^2}\hat{c}_0(\mathbf{k}) - \int_0^t e^{-(t-s)D_c|\mathbf{k}|^2}B(\hat{\mathbf{u}},\hat{c})(\mathbf{k},s) \, ds \\
&\quad + \int_0^t e^{-(t-s)D_c|\mathbf{k}|^2}C(\hat{n},\widehat{f(c)})(\mathbf{k},s) \, ds,
\end{aligned} \tag{2.29}$$

for all $0 \le t \le \tau$.

Finally, we define *Gevrey regular* functions as follows:

Definition 2.3. *A mild or weak solution* $U(\cdot) \in \mathscr{C}([0,\tau];\mathcal{K})$ *of (1.1) is Gevrey regular if there exist* $\vec{\sigma} \in \mathbb{R}^3$ *and* $\vec{\lambda} : \mathbb{R}_+ \to (\mathbb{R}_+)^3$ *with* $\vec{\lambda}(0) = \vec{0}$ *such that*

$$\sup_{0\le t\le\tau} \|U(t)\|_{\vec{\lambda}(t),\vec{\sigma}} < \infty. \tag{2.30}$$

For convenience, will use the following notation in the analysis below.

Remark 2.4. *We will denote by* $A \lesssim B$ *the relation that* $A \le cB$, *for some absolute constant* $c > 0$, *and use subscripts on* $\lesssim$ *to emphasize dependencies of this constant, c, on certain parameters. We will also use*

$$A \wedge B := \min\{A, B\} \quad \text{and} \quad A \vee B := \max\{A, B\}. \tag{2.31}$$

Let us point out here that in the analysis we perform below, we may often abuse notation and use c, C *to denote generic constants which may change line-to-line in the estimates.*

2.2. Convolution inequalities and Heat Semigroup estimates in Gevrey Spaces. To prove the required heat semigroup estimates, we will make use of the following elementary convolution inequalities. Note that we will state the most of the following inequalities for scalar-valued sequences, but they hold as well for vector-valued sequences. We will omit most of the details in the proof of some of these statements as some of them can either be found in [3, 4], or are otherwise readily adaptable. Otherwise, we prove slightly modified versions of them here. We recall that we denote $A = -\Delta$ with periodic boundary conditions, i.e.,

$$\widehat{A^{\beta/2}v}(\mathbf{k}) := |\mathbf{k}|^{\beta}\hat{v}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^d. \tag{2.32}$$

PROPOSITION 2.2. *Let $\lambda > 0, \gamma \geq 0$. Then*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{\lambda|\mathbf{k}|}|\mathbf{k}|^{\gamma}|(\hat{v}*\hat{w})(\mathbf{k})| \lesssim \left(\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{\lambda|\mathbf{k}|}|\mathbf{k}|^{\gamma}|\hat{v}(\mathbf{k})|\right)\left(\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{\lambda|\mathbf{k}|}|\mathbf{k}|^{\gamma}|\hat{w}(\mathbf{k})|\right). \tag{2.33}$$

Moreover, we have

$$\|\hat{v}*\hat{w}\|_{\lambda,\gamma} \lesssim \|v\|_{\lambda,\gamma}\|w\|_{\lambda,\gamma}. \tag{2.34}$$

PROOF. The inequality (2.33) is precisely Proposition 5.12 in [3]. To prove (2.34), observe that for $\gamma \geq 0$,

$$\begin{aligned}\|\hat{v}*\hat{w}\|_{\lambda,\gamma} &\leq \sum_{\boldsymbol{\ell}} |\hat{v}(\boldsymbol{\ell})||\hat{w}(-\boldsymbol{\ell})| + C\sum_{\mathbf{k}\neq\mathbf{0}} e^{\lambda|\mathbf{k}|}|\mathbf{k}|^{\gamma}|(\hat{v}*\hat{w})(\mathbf{k})| \\ &\leq \left(\sum_{\mathbf{k}}|\hat{v}(\mathbf{k})|\right)\left(\sum_{\mathbf{k}}|\hat{w}(\mathbf{k})|\right) + C\|v\|_{\lambda,\gamma}\|w\|_{\lambda,\gamma} \\ &\leq C\|v\|_{\lambda,\gamma}\|w\|_{\lambda,\gamma},\end{aligned}$$

where we have applied (2.33) and the facts that $1 \leq e^{\lambda|\mathbf{k}|}$ and $\max\{1, |\mathbf{k}|^{\gamma}\} \leq (1+|\mathbf{k}|^2)^{\gamma/2}$ to obtain the final and penultimate inequalities. □

We will assume from now on that χ, f are real analytic functions on $\mathbb{R}$ with majorants g_χ, g_f, respectively, and that ϕ is a real analytic function on $\mathbb{T}^2$. In particular, we may assume that the majorants take the following form (see for instance [12]):

$$g_h(x) = \sum_{n\geq 0}|c_n(h)|x^n, \quad h(x) = \sum_{n\geq 0} c_n(h)x^n, \quad \text{for } h = \chi, f. \tag{2.35}$$

We have following fact regarding majorants and our norms.

PROPOSITION 2.3. *Let $h : \mathbb{R} \to \mathbb{R}$, be a real analytic function with majorant g_h and $u \in \mathcal{G}_{\lambda,\gamma}$, for some $\lambda > 0, \gamma \geq 0$. Then*

$$\|h(v)\|_{\lambda,\gamma} \lesssim g_h(\|v\|_{\lambda,\gamma}). \tag{2.36}$$

PROOF. Since $u \in \mathcal{G}_{\lambda,\sigma}$, the function defined by

$$v(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{v}_{\mathbf{k}}$$

admits an analytic extension to the set $\{\mathbf{z} = \mathbf{x} + i\mathbf{y} : |\mathbf{y}| < \lambda\}$. Since

$$h(v(x)) = c_0 + \sum_{n\geq 1} c_n(v(x))^n,$$

it follows that

$$\widehat{h(v)}(\mathbf{k}) = c_0\delta_{\mathbf{0}}(\mathbf{k}) + \sum_{n\geq 1} c_n(\hat{v}^{*n})_{\mathbf{k}}, \tag{2.37}$$

where $\delta_{\mathbf{0}}$ denotes the Dirac delta function supported on $\{\mathbf{k}=\mathbf{0}\}$ and $\hat{v}^{*n}$ denotes the n-fold convolution product. It follows from repeated application of Proposition 2.2 that

$$\|h(v)\|_{\lambda,\gamma} \leq |c_0| + \sum_{n\geq 1} |c_n|\,\|\hat{v}^{*n}\|_{\lambda,\gamma} \lesssim g_h(\|v\|_{\lambda,\gamma}),$$

as desired. □

We will also make crucial use of the following Gevrey-norm adapted version of the mean value theorem.

PROPOSITION 2.4. *Let $\lambda > 0, \gamma \geq 0$ and $M > 0$. Suppose h is real analytic on $\mathbb{R}$ with a majorant g_h, and $u, v \in \mathcal{G}_{\lambda,\gamma}$ such that $\|u\|_{\lambda,\gamma}, \|v\|_{\lambda,\gamma} \leq M$. Then*

$$\|h(u) - h(v)\|_{\lambda,\gamma} \lesssim g_h'(M)\|u-v\|_{\lambda,\gamma}.$$

PROOF. Since $u, v \in \mathcal{G}_{\lambda,\sigma}$, the functions defined by

$$u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{u}_{\mathbf{k}}, \quad v(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{v}_{\mathbf{k}},$$

admit analytic extensions to the set $\{\mathbf{z} = \mathbf{x} + i\mathbf{y} : |\mathbf{y}| < \lambda\}$, where $\hat{u}$ denotes the Fourier transform of u. We know

$$h(u(x)) = c_0 + \sum_{n\geq 1} c_n (u(x))^n, \quad \widehat{h(u)}(\mathbf{k}) = c_0\delta_{\mathbf{0}}(\mathbf{k}) + \sum_{n\geq 1} c_n (\hat{u}^{*n})_{\mathbf{k}}. \tag{2.38}$$

Let $\hat{w} = \hat{u} - \hat{v}$ and let

$$C_{u^{*\ell}}^{v^{*k}}\hat{w} = \hat{v}^{*k} * \hat{w} * \hat{u}^{*\ell}. \tag{2.39}$$

Observe that

$$\begin{aligned}
(\hat{u}^{*n})_{\mathbf{k}} &= (\hat{w} * \hat{u}^{*(n-1)})_{\mathbf{k}} + (\hat{v} * \hat{u}^{*(n-1)})_{\mathbf{k}} \\
&= (\hat{w} * \hat{u}^{*(n-1)})_{\mathbf{k}} + (\hat{v} * \hat{w} * \hat{u}^{*(n-2)})_{\mathbf{k}} + (\hat{v}^{*2} * \hat{u}^{*(n-2)})_{\mathbf{k}} \\
&\;\;\vdots \\
&= (\hat{w} * \hat{u}^{*(n-1)})_{\mathbf{k}} + (\hat{v} * \hat{w} * \hat{u}^{*(n-2)})_{\mathbf{k}} + \cdots + (\hat{v}^{*(n-1)} * \hat{w})_{\mathbf{k}} + (v^{*n})_{\mathbf{k}} \\
&= \sum_{0\leq j\leq n-1} (C_{u^{*(n-1-j)}}^{v^{*j}}\hat{w})_{\mathbf{k}} + (v^{*n})_{\mathbf{k}}. \qquad (2.40)
\end{aligned}$$

Now consider

$$h(u) - h(v) = \sum_{n\geq 1} c_n(u^n - v^n).$$

Then from (2.38) and (2.40) we have

$$\widehat{h(u)}_{\mathbf{k}} - \widehat{h(v)}_{\mathbf{k}} = \sum_{n\geq 1} c_n \sum_{0\leq j\leq n-1} (C_{u^{*(n-1-j)}}^{v^{*j}}\hat{w})_{\mathbf{k}}. \tag{2.41}$$

Hence, from Proposition 2.2, 2.3, and (2.41) we deduce that

$$\begin{aligned}
\|h(u) - h(v)\|_{\lambda,\gamma} &\leq \sum_{n\geq 1} |c_n|\|\hat{u}^{*n} - \hat{v}^{*n}\|_{\lambda,\gamma} \\
&\lesssim \|w\|_{\lambda,\gamma} \sum_{n\geq 1} |c_n| \sum_{0\leq j\leq n-1} \|v\|_{\lambda,\sigma}^j \|u\|_{\lambda,\sigma}^{n-1-j} \\
&\lesssim \|w\|_{\lambda,\gamma} \sum_{n\geq 1} n|c_n|M^{n-1} \\
&\lesssim \|w\|_{\lambda,\gamma} g_h'(M), \qquad (2.42)
\end{aligned}$$

as desired. □

PROPOSITION 2.5. *Let $\rho > 0$ and $\beta, \lambda \geq 0$. Then for any $\sigma \in \mathbb{R}$ and $\hat{v}$ with $\hat{v}_{\mathbf{0}} \neq 0$*

$$(1 \wedge \rho)^{\beta/2} \|e^{-\rho A} v\|_{\lambda,\sigma+\beta} \lesssim_{\sigma,\beta} \|v\|_{\lambda,\sigma}.$$

If $\hat{v}_{\mathbf{0}} = 0$, then

$$\rho^{\beta/2} \|e^{-\rho A} v\|_{\lambda,\sigma+\beta} \lesssim_{\beta} \|v\|_{\lambda,\sigma}.$$

PROOF. Suppose that $\hat{v}_{\mathbf{0}} \neq 0$; the case $\hat{v}_{\mathbf{0}} = 0$ will be clear from the proof. Then observe that

$$\begin{aligned} \|e^{-\rho A} v\|_{\lambda,\sigma+\beta} &= \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{\lambda|\mathbf{k}|} (1 + |\mathbf{k}|^2)^{(\sigma+\beta)/2} e^{-\rho|\mathbf{k}|^2} |\hat{v}_{\mathbf{k}}| \\ &= |\hat{v}_{\mathbf{0}}| + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} e^{\lambda|\mathbf{k}|} (1 + |\mathbf{k}|^2)^{(\sigma+\beta)/2} e^{-\rho|\mathbf{k}|^2} |\hat{v}_{\mathbf{k}}| \,. \end{aligned}$$

Thus

$$\begin{aligned} \|e^{-\rho A} v\|_{\lambda,\sigma+\beta} &\lesssim_{\sigma,\beta} \|v\|_{\lambda,\sigma} + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} e^{\lambda|\mathbf{k}|} \, |\mathbf{k}|^{\sigma+\beta} \, e^{-\rho|\mathbf{k}|^2} \, |\hat{v}_{\mathbf{k}}| \\ &\lesssim_{\sigma,\beta} \|v\|_{\lambda,\sigma} + \left(\sup_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} |\mathbf{k}|^{\beta} \, e^{-(\rho/2)|\mathbf{k}|^2} \right) \|v\|_{\lambda,\sigma} \\ &\lesssim_{\sigma,\beta} (1 \vee \rho^{-\beta/2}) \|v\|_{\lambda,\sigma}. \end{aligned}$$

Observe that

$$(1 \vee \rho^{-\beta/2})(1 \wedge \rho)^{\beta/2} \leq 1. \tag{2.43}$$

Thus

$$(1 \wedge \rho)^{\beta/2} \|e^{-\rho A} v\|_{\lambda,\sigma+\beta} \lesssim_{\sigma,\beta} \|v\|_{\lambda,\sigma},$$

as claimed. □

PROPOSITION 2.6 ([4]). *Let $\alpha > 0$ and $\sigma \in \mathbb{R}$. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Then*

$$\|e^{-\rho(t-s)A} v\|_{\lambda(t),\sigma} \lesssim C(s,t) \|e^{-(\rho/2)(t-s)A} v\|_{\lambda(s),\sigma},$$

for all $0 \leq s < t$, where

$$C(s,t) := \exp\left(\frac{1}{2\rho} \frac{\lambda(t-s)^2}{(t-s)} \right).$$

PROPOSITION 2.7 ([3, 4]). *Let $\lambda, \gamma \geq 0$. Suppose that $\hat{\mathbf{v}}, \in \mathcal{K}_{\mathbf{1}}$. Then for any $\delta \in \mathbb{R}$*

$$\|e^{-\rho A} B(\hat{\mathbf{v}}, \hat{\mathbf{w}})\|_{\lambda,\delta} \lesssim_{\gamma,\delta} \rho^{-\max\{0,(1/2)(1+\delta-\gamma)\}} \|\mathbf{v}\|_{\lambda,\gamma} \|\mathbf{w}\|_{\lambda,\gamma}.$$

The adaptation of Proposition 2.7 to our definition of the Gevrey norm can be gleaned from the proof of following variant of it.

PROPOSITION 2.8. *Let $\lambda, \gamma \geq 0$ and $\rho > 0$. Suppose that $\hat{\mathbf{v}} \in \mathcal{K}_{\mathbf{1}}$. Then for any $\delta \in \mathbb{R}$*

$$\|e^{-\rho A} B(\hat{\mathbf{v}}, \hat{w})\|_{\lambda,\delta} \lesssim_{\gamma,\delta} \rho^{-\max\{0,(1/2)(1+\delta-\gamma)\}} \|\mathbf{v}\|_{\lambda,\gamma} \|w\|_{\lambda,\gamma}$$

as well as

$$\|e^{-\rho A} B(\hat{\mathbf{v}}, \hat{w})\|_{\lambda,\delta} \lesssim_{\gamma,\delta} \rho^{-\max\{0,(1/2)(\delta-\gamma)\}} \|\mathbf{v}\|_{\lambda,\gamma} \|w\|_{\lambda,\gamma+1}$$

Proof. Since $\hat{\mathbf{v}}_{\mathbf{0}} = \mathbf{0}$ and $\hat{\mathbf{v}}_{\mathbf{k}} \cdot \mathbf{k} = 0$, it follows from (2.21), (2.22), Proposition 2.2, and (2.32) that

$$\begin{aligned}\|e^{-\rho A}B(\hat{\mathbf{v}},\hat{w})\|_{\lambda,\delta} &\lesssim \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{\lambda|\mathbf{k}|}(1+|\mathbf{k}|^2)^{\delta/2}e^{-\rho|\mathbf{k}|^2}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}}|\hat{\mathbf{v}}_{\boldsymbol{\ell}}|\,|\mathbf{k}-\boldsymbol{\ell}|\,|\hat{w}_{\mathbf{k}-\boldsymbol{\ell}}|\\ &\lesssim \sup_{\mathbf{k}\in\mathbb{Z}^d}\left(|\mathbf{k}|^{\delta-\gamma}\,e^{-\rho|\mathbf{k}|^2}\right)\|\mathbf{v}\|_{\lambda,\gamma}\|A^{1/2}w\|_{\lambda,\gamma}\\ &\lesssim \rho^{-\max\{0,(\delta-\gamma)/2\}}\|\mathbf{v}\|_{\lambda,\gamma}\|w\|_{\lambda,\gamma+1},\end{aligned}$$

as desired. □

For the other nonlinear terms, T and C, we have the following.

Proposition 2.9. *Let $\lambda,\gamma\geq 0,\rho>0$, and $\delta\in\mathbb{R}$. Suppose that χ is real analytic on $\mathbb{R}$ with majorant g_χ. Then*

$$\|e^{-\rho A}T(\hat{v},\widehat{\chi(w)},\hat{w})\|_{\lambda,\delta}\lesssim_{\gamma,\delta}\rho^{-\max\{0,(1/2)(1+\delta-\gamma)\}}\|v\|_{\lambda,\gamma}g_\chi(\|w\|_{\lambda,\gamma})\|w\|_{\lambda,1+\gamma}.$$

Proof. Suppose that $1+\delta-\gamma>0$. Observe that since χ is majorized by g_χ, by repeated application of Proposition 2.2, and then Proposition 2.3 and (2.32), we have

$$\begin{aligned}&\|e^{-\rho A}T(\hat{v},\widehat{\chi(w)},\hat{w})\|_{\lambda,\delta} &\text{(2.44)}\\ &\lesssim \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{-\rho|\mathbf{k}|^2}e^{\lambda|\mathbf{k}|}|\mathbf{k}|^{1+\delta}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\sum_{\mathbf{m}\in\mathbb{Z}^d}|\hat{v}_{\boldsymbol{\ell}}||\widehat{\chi(w)}_{\mathbf{m}-\boldsymbol{\ell}}||\mathbf{k}-\mathbf{m}||\hat{w}_{\mathbf{k}-\mathbf{m}}|\\ &\lesssim \left(\sup_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{-\rho|\mathbf{k}|^2}|\mathbf{k}|^{1+\delta-\gamma}\right)\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}}e^{\lambda|\mathbf{k}|}|\mathbf{k}|^\gamma|(\hat{v}*\widehat{\chi(w)}*\widehat{A^{1/2}w})(\mathbf{k})|.\\ &\lesssim \rho^{-(1+\delta-\gamma)/2}\|\hat{v}*\widehat{\chi(w)}*(A^{1/2}w)\|_{\lambda,\gamma}\\ &\lesssim \rho^{-(1+\delta-\gamma)/2}\|v\|_{\lambda,\gamma}g_\chi(\|w\|_{\lambda,\gamma})\|w\|_{\lambda,\gamma+1}. &\text{(2.45)}\end{aligned}$$

Otherwise, $1+\delta-\gamma\leq 0$, and we have

$$\|e^{-\rho A}T(\hat{v},\widehat{\chi(w)},\hat{w})\|_{\lambda,\delta}\lesssim\|v\|_{\lambda,\gamma}g_\chi(\|w\|_{\lambda,\gamma})\|w\|_{\lambda,\gamma+1}, \tag{2.46}$$

as desired. □

Proposition 2.10. *Let $\lambda,\gamma\geq 0,\rho>0$, $\delta\in\mathbb{R}$, and $M>0$. Suppose that χ is real analytic on $\mathbb{R}$ with majorant g_χ. Suppose that $\|w_1\|_{\lambda,\gamma},\|w_2\|_{\lambda,\gamma}\leq M$. Then*

$$\begin{aligned}&\|e^{-\rho A}T(\hat{v},\widehat{\chi(w_1)}-\widehat{\chi(w_2)},\hat{w})\|_{\lambda,\delta}\\ &\lesssim_{\gamma,\delta} g_\chi'(M)\rho^{-\max\{0,(1/2)(1+\delta-\gamma)\}}\|v\|_{\lambda,\gamma}\|w_1-w_2\|_{\lambda,\gamma}\|w\|_{\lambda,1+\gamma}. &\text{(2.47)}\end{aligned}$$

Proof. The proof proceeds as in Proposition 2.9, except that we apply Proposition 2.4 (instead of Proposition 2.3) in conjunction with Proposition 2.2. In particular, we have

$$\begin{aligned}\|e^{-\rho A}T(\hat{v},\widehat{\chi(w_1)}-\widehat{\chi(w_2)},\hat{w})\|_{\lambda,\delta}&\lesssim\rho^{-(1+\delta-\gamma)/2}\|\hat{v}*(\widehat{\chi(w_1)}-\widehat{\chi(w_2)})*(A^{1/2}w)\|_{\lambda,\gamma}\\ &\lesssim g_\chi'(M)\rho^{-(1+\delta-\gamma)/2}\|v\|_{\lambda,\gamma}\|w_1-w_2\|_{\lambda,\gamma}\|w\|_{\lambda,\gamma+1}, &\text{(2.48)}\end{aligned}$$

as desired. □

Proposition 2.11. *Let $\lambda,\gamma\geq 0,\rho>0$, and $\delta\in\mathbb{R}$. Suppose that f is real analytic on $\mathbb{R}$ with majorant g_f. Then*

$$\|e^{-\rho A}C(\hat{v},\widehat{f(w)})\|_{\lambda,\delta}\lesssim_{\gamma,\delta}(1\wedge\rho)^{-\max\{0,(1/2)(\delta-\gamma)\}}\|v\|_{\lambda,\gamma}g_f(\|w\|_{\lambda,\gamma}).$$

PROOF. Using the fact that f is majorized by g_f and Proposition 2.2, we estimate

$$\begin{aligned}
&\|e^{-\rho A}C(\hat{v},\widehat{f(w)})\|_{\lambda,\delta}\\
&\lesssim \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}|\hat{v}_{\boldsymbol{\ell}}|\left|\widehat{f(w)}_{-\boldsymbol{\ell}}\right| + \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} e^{\lambda|\mathbf{k}|}(1+|\mathbf{k}|^2)^{\delta/2}e^{-\rho|\mathbf{k}|^2}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}|\hat{v}_{\boldsymbol{\ell}}||\widehat{f(w)}_{\mathbf{k}-\boldsymbol{\ell}}|\\
&\lesssim_{\delta,\gamma} \|v\|_{\lambda,\gamma}\|f(w)\|_{\lambda,\gamma} + \rho^{-\max\{0,(1/2)(\delta-\gamma)\}}\|v\|_{\lambda,\gamma}\|f(w)\|_{\lambda,\gamma}\\
&\lesssim_{\delta,\gamma} (1\vee\rho^{-\max\{0,(1/2)(\delta-\gamma)\}})\|v\|_{\lambda,\gamma}g_f(\|w\|_{\lambda,\gamma}).
\end{aligned}$$

The result then follows from (2.43). □

PROPOSITION 2.12. *Let $\lambda,\gamma\geq 0,\rho>0,M>0$, and $\delta\in\mathbb{R}$. Suppose that f is real analytic on $\mathbb{R}$ with majorant g_f. Suppose that $\|w_1\|_{\lambda,\gamma},\|w_2\|_{\lambda,\gamma}\leq M$. Then*

$$\|e^{-\rho A}C(\hat{v},\widehat{f(w_1)}-\widehat{f(w_2)})\|_{\lambda,\delta}\lesssim_{\gamma,\delta} g_f'(M)(1\wedge\rho)^{-\max\{0,(1/2)(\delta-\gamma)\}}\|v\|_{\lambda,\gamma}\|w_1-w_2\|_{\lambda,\gamma}.$$

PROOF. The proof proceeds as in Proposition 2.11, except that we apply Proposition 2.4 (instead of Proposition 2.3) in conjunction with Proposition 2.2. In particular, we have

$$\begin{aligned}
&\|e^{-\rho A}C(\hat{v},\widehat{f(w_1)}-\widehat{f(w_2)})\|_{\lambda,\delta}\\
&\lesssim_{\delta,\gamma} \|v\|_{\lambda,\gamma}\|f(w_1)-f(w_2)\|_{\lambda,\gamma}+\rho^{-\max\{0,(1/2)(\delta-\gamma)\}}\|v\|_{\lambda,\gamma}\|f(w_1)-f(w_2)\|_{\lambda,\gamma}\\
&\lesssim_{\delta,\gamma} g_f'(M)(1\vee\rho^{-\max\{0,(1/2)(\delta-\gamma)\}})\|v\|_{\lambda,\gamma}\|w_1-w_2\|_{\lambda,\gamma}).
\end{aligned}$$

The result then follows from (2.43). □

PROPOSITION 2.13. *Let $\lambda,\gamma\geq 0$ and $\rho>0$. Suppose $\hat{v}\in\mathcal{K}_2$. Then for any $\delta\in\mathbb{R}$*

$$\|e^{-\rho A}\mathbf{F}_{\nabla\phi}(\hat{v})\|_{\lambda,\delta}\lesssim_{\gamma,\delta}\rho^{-\max\{0,(1/2)(\delta-\gamma)\}}\|v\|_{\lambda,\gamma}\|\nabla\phi\|_{\lambda,\gamma}.$$

PROOF. We need only observe that $\mathbf{F}_{\nabla\phi}(\hat{v})(\mathbf{0})=\mathbf{0}$ since $\hat{v}\in\mathcal{K}_2$. The proof then follows as in Proposition 2.8. □

Next, we prepare the abstract setting in which we prove Theorem 1.1.

2.3. Abstract framework for Theorem 1.1. Let $\tau>0$. Suppose $\sigma>-1$, $\beta\geq 0$, and $\epsilon\in(0,1]$. We define the Banach spaces

$$\begin{aligned}
&\boldsymbol{\mathcal{X}}_{1,\tau}:=\{\mathbf{u}\in C([0,\tau];\boldsymbol{\mathcal{V}}_\sigma):\|\mathbf{u}\|_{X_{1,\tau}}<\infty\},\quad \|\mathbf{u}\|_{\boldsymbol{\mathcal{X}}_{1,\tau}}:=\sup_{0\leq t\leq\tau}\|\mathbf{u}(t)\|_{\lambda\sqrt{t},\sigma},\\
&\mathcal{X}_{2,\tau}:=\{v\in C([0,\tau];\mathcal{V}_\sigma):\|v\|_{X_{2,\tau}}<\infty\},\quad \|v\|_{\mathcal{X}_{2,\tau}}:=\sup_{0\leq t\leq\tau}\|v(t)\|_{\lambda\sqrt{t},\sigma},\\
&\mathcal{X}_{3,\tau}:=\{w\in C([0,\tau];\mathcal{V}_\epsilon):\|u\|_{X_{3,\tau}}<\infty\},\quad \|w\|_{\mathcal{X}_{3,\tau}}:=\sup_{0\leq t\leq\tau}\|w(t)\|_{\lambda\sqrt{t},\epsilon},
\end{aligned}\tag{2.49}$$

and

(2.50)

$$\begin{aligned}
&\boldsymbol{\mathcal{Y}}_{1,\tau}:=\{\mathbf{u}\in C((0,\tau];\boldsymbol{\mathcal{V}}_{\sigma+\beta}):\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}<\infty\},\quad \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}:=\sup_{0<t\leq\tau}(\nu t)^{\beta/2}\|\mathbf{u}(t)\|_{\lambda\sqrt{t},\sigma+\beta},\\
&\mathcal{Y}_{2,\tau}:=\{v\in C((0,\tau];\mathcal{V}_{\sigma+\beta}):\|v\|_{\mathcal{Y}_{2,\tau}}<\infty\},\quad \|v\|_{\mathcal{Y}_{2,\tau}}:=\sup_{0<t\leq\tau}(D_nt)^{\beta/2}\|v(t)\|_{\lambda\sqrt{t},\sigma+\beta},\\
&\mathcal{Y}_{3,\tau}:=\{w\in C((0,\tau];\mathcal{V}_1):\|w\|_{\mathcal{Y}_{3,\tau}}<\infty\},\|u\|_{\mathcal{Y}_{3,\tau}}:=\sup_{0<t\leq\tau}(1\wedge(D_ct))^{(1-\epsilon)/2}\|w(t)\|_{\lambda\sqrt{t},1}.
\end{aligned}$$

Then we define the Banach spaces

$$\mathcal{X}_\tau:=\boldsymbol{\mathcal{X}}_{1,\tau}\times\mathcal{X}_{2,\tau}\times\mathcal{X}_{3,\tau},\quad \mathcal{Y}_\tau:=\boldsymbol{\mathcal{Y}}_{1,\tau}\times\mathcal{Y}_{2,\tau}\times\mathcal{Y}_{3,\tau},\tag{2.51}$$

and

$$\boldsymbol{\mathcal{Z}}_{1,\tau} := \boldsymbol{\mathcal{X}}_{1,\tau} \cap \boldsymbol{\mathcal{Y}}_{1,\tau}, \quad \mathcal{Z}_{j,\tau} := \mathcal{X}_{j,\tau} \cap \mathcal{Y}_{j,\tau}, \quad j = 2,3, \quad \mathcal{Z}_{\tau} := \boldsymbol{\mathcal{Z}}_{1,\tau} \oplus \mathcal{Z}_{2,\tau} \oplus \mathcal{Z}_{3,\tau}, \tag{2.52}$$

equipped with the usual induced norms, i.e.,

$$\|\mathbf{u}\|_{\mathcal{X}_\tau} := \|\mathbf{u}\|_{\mathcal{X}_{1,\tau}} \vee \|\mathbf{u}\|_{\mathcal{X}_{2,\tau}} \vee \|\mathbf{u}\|_{\mathcal{X}_{3,\tau}}, \quad \|v\|_{\mathcal{Y}_\tau} := \|v\|_{\mathcal{Y}_{1,\tau}} \vee \|v\|_{\mathcal{Y}_{2,\tau}} \vee \|v\|_{\mathcal{Y}_{3,\tau}}. \tag{2.53}$$

and

$$\|\mathbf{u}\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} := \|\mathbf{u}\|_{\boldsymbol{\mathcal{X}}_{1,\tau}} \vee \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}, \quad \|v\|_{\boldsymbol{\mathcal{Z}}_{2,\tau}} := \|v\|_{\boldsymbol{\mathcal{X}}_{2,\tau}} \vee \|v\|_{\boldsymbol{\mathcal{Y}}_{2,\tau}}, \tag{2.54}$$

$$\|w\|_{\boldsymbol{\mathcal{Z}}_{3,\tau}} := \|w\|_{\boldsymbol{\mathcal{X}}_{3,\tau}} \vee \|w\|_{\boldsymbol{\mathcal{Y}}_{3,\tau}}, \tag{2.55}$$

and

$$\|U\|_{\mathcal{Z}_\tau} := \|\mathbf{u}\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} + \|v\|_{\mathcal{Z}_{2,\tau}} + \|w\|_{\mathcal{Z}_{3,\tau}}, \quad U = (\mathbf{u}, v, w). \tag{2.56}$$

Also, we will denote the zero element of $\mathcal{X}_\tau, \mathcal{Y}_\tau$ and $\mathcal{Z}_\tau$ as

$$\vec{0} = (\mathbf{0}, 0, 0). \tag{2.57}$$

Consider $(\hat{\mathbf{u}}_0, \hat{n}_0, \hat{c}_0) \in V_{(\sigma,\sigma,\epsilon)}$. We define $\mathbf{\Phi}_1, \Phi_2, \Phi_3$ as follows:

$$\mathbf{\Phi}_1(\hat{n})(t) = e^{-\nu t A}\hat{\mathbf{u}}_0(\mathbf{k}), \quad \Phi_2(t) = e^{-D_n t A}\hat{n}_0(\mathbf{k}), \quad \Phi_3(t) = e^{-D_c t A}\hat{c}_0(\mathbf{k}). \tag{2.58}$$

Let

$$\begin{aligned} \mathbf{W}_1(\hat{\mathbf{u}}, \hat{n}, \hat{c})(t) &= \mathbf{G}(\hat{n})(t) + \mathbf{W}_1^{(2)}(\hat{\mathbf{u}}, \hat{\mathbf{u}})(t), \\ W_2(\hat{\mathbf{u}}, \hat{n}, \hat{c})(t) &= W_2^{(1)}(\hat{\mathbf{u}}, \hat{n})(t) + W_2^{(2)}(\hat{n}, \widehat{\chi(c)}, \hat{c})(t), \\ W_3(\hat{\mathbf{u}}, \hat{n}, \hat{c})(t) &= W_3^{(1)}(\hat{\mathbf{u}}, \hat{c})(t) + W_3^{(2)}(\hat{n}, \widehat{f(c)})(t), \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} \mathbf{G}(\hat{n})(t) &:= -\int_0^t e^{-(t-s)\nu A} F_{\nabla\phi}(\hat{n})(\mathbf{k}, s)\, ds, \\ \mathbf{W}_1^{(2)}(\hat{\mathbf{u}}, \hat{\mathbf{u}})(t) &:= \int_0^t e^{(t-s)\nu A}\mathbf{B}(\hat{\mathbf{u}}, \hat{\mathbf{u}})(s)\, ds, \\ W_2^{(1)}(\hat{\mathbf{u}}, \hat{n})(t) &:= -\int_0^t e^{-(t-s)D_n A} B(\hat{\mathbf{u}}, \hat{n})(s)\, ds, \\ W_2^{(2)}(\hat{n}, \widehat{\chi(c)}, \hat{c})(t) &:= \int_0^t e^{(t-s)D_n A} T(\hat{n}, \widehat{\chi(c)}, \hat{c})(s)\, ds, \\ W_3^{(1)}(\hat{\mathbf{u}}, \hat{c})(t) &:= -\int_0^t e^{-(t-s)D_c A} B(\hat{\mathbf{u}}, \hat{c})(s)\, ds, \\ W_3^{(2)}(\hat{n}, \hat{c})(t) &:= \int_0^t e^{(t-s)D_c A} C(\hat{n}, \widehat{f(c)})(s)\, ds. \end{aligned} \tag{2.60}$$

Now define the following maps over $C([0,\tau]; V_{(\sigma,\sigma,\epsilon)})$:

$$\boldsymbol{\mathcal{S}}_1(\mathbf{u}, n, c)(t) := \mathbf{\Phi}_1(t) + \mathbf{W}_1(t), \quad \mathcal{S}_j(\mathbf{u}, n, c)(t) := \Phi_j(t) + W_j(t), \quad j = 2,3. \tag{2.61}$$

Then define

$$\mathcal{S}(\mathbf{u}, n, c) := (\boldsymbol{\mathcal{S}}_1(\mathbf{u}, n, c), \mathcal{S}_2(\mathbf{u}, n, c), \mathcal{S}_3(\mathbf{u}, n, c)). \tag{2.62}$$

Observe that for $\Phi = (\mathbf{\Phi}_1, \Phi_2, \Phi_3)$ and $W = (\mathbf{W}_1, W_2, W_3)$, we have

$$\mathcal{S} = \Phi + W.$$

Let $R > 0$ and define

$$B_{\mathcal{Z}_\tau}(\Phi, R) := \{U = (\mathbf{u}, v, w) : \|U - \Phi\|_{\mathcal{Z}_\tau} \leq R\}. \tag{2.63}$$

Then our main task is to ensure that

(1) $\mathcal{S} : B_{\mathcal{Z}_\tau}(\Phi, R) \to B_{\mathcal{Z}_\tau}(\Phi, R)$, and
(2) $\mathcal{S}$ is a contraction.

We establish these claims in Section 2.5 below. Before we do so, we make some preliminary estimates for Φ and W.

2.4. Estimates of Φ and W. We will establish the basic estimates needed for Φ and W in order for $\mathcal{S}$ to be a self-map on $B_{\mathcal{Z}_\tau}(\Phi, R)$, and ultimately a contraction map.

2.4.1. *Estimate of* Φ.

LEMMA 2.14. *Let $\sigma > -1$, $\epsilon \in (0, 1]$, and $\beta \geq 0$ such that $\sigma + \beta \geq 0$. Then for $\gamma = \sigma + \beta$ we have*

$$\|\mathbf{\Phi}_1\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} \lesssim \|\mathbf{u}_0\|_\sigma, \quad \|\Phi_2\|_{\mathcal{Z}_{2,\tau}} \lesssim \|n_0\|_\sigma, \quad \|\Phi_3\|_{\mathcal{Z}_{3,\tau}} \lesssim \|c_0\|_\epsilon,$$

and

$$\begin{aligned}\lim_{t\to 0^+} (\nu t)^{\beta/2} \|\mathbf{\Phi}_1(t)\|_{\lambda\sqrt{t},\gamma} &= \lim_{t\to 0^+} (D_n t)^{\beta/2} \|\Phi_2(t)\|_{\lambda\sqrt{t},\gamma} \\ &= \lim_{t\to 0^+} (1 \wedge (D_c t)^{(1-\epsilon)/2}) \|\Phi_3(t)\|_{\lambda\sqrt{t},1} = 0. \end{aligned} \tag{2.64}$$

PROOF. It suffices to consider $\mathbf{\Phi}_1, \Phi_3$; Φ_2 can be treated similarly to $\mathbf{\Phi}_1$. Let us treat $\mathbf{\Phi}_1$ first. By Proposition 2.6, we have

$$\|\mathbf{\Phi}_1(t)\|_{\lambda\sqrt{t},\sigma} \lesssim \|e^{-\nu t A}\mathbf{u}_0\|_{\lambda\sqrt{t},\sigma} \lesssim \|e^{-(\nu/2)tA}\mathbf{u}_0\|_\sigma \lesssim \|\mathbf{u}_0\|_\sigma. \tag{2.65}$$

On the other hand, by Proposition 2.5 we have

$$\|\mathbf{\Phi}_1(t)\|_{\lambda\sqrt{t},\gamma} \lesssim \|e^{-\nu t A}\mathbf{u}_0\|_{\lambda\sqrt{t},\gamma} \lesssim (\nu t)^{-\beta/2} \|e^{-(\nu/2)tA}\mathbf{u}_0\|_{\lambda\sqrt{t},\sigma} \lesssim (\nu t)^{-\beta/2}\|\mathbf{u}_0\|_\sigma. \tag{2.66}$$

Upon taking supremum over $[0, \tau]$ (resp. $(0, \tau]$) of (2.65) (resp. (2.66)), we obtain

$$\|\Phi_1\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} \lesssim \|\mathbf{u}_0\|_\sigma. \tag{2.67}$$

Now observe that for all $N > 0$

$$\begin{aligned}(\nu t)^{\beta/2}\|\Phi_1(t)\|_{\lambda\sqrt{t},\sigma+\beta} &\lesssim (\nu t)^{\beta/2}\|e^{-\nu t A}(I - P_N)\mathbf{u}_0\|_{\lambda\sqrt{t},\sigma+\beta} + (\nu t)^{\beta/2}\|P_N\mathbf{u}_0\|_{\lambda\sqrt{t},\sigma+\beta}, \\ &\lesssim \|(I - P_N)\mathbf{u}_0\|_\sigma + (\nu t)^{\beta/2}N^\beta\|\mathbf{u}_0\|_\sigma.\end{aligned}$$

where P_N denotes the operator that truncates sequences up to the terms $|\mathbf{k}| \leq N$. Since N is arbitrary and $\mathbf{u}_0 \in \boldsymbol{\mathcal{V}}_\sigma$, it follows that

$$\lim_{t\to 0^+} (\nu t)^{\beta/2}\|\mathbf{\Phi}_1(t)\|_{\lambda\sqrt{t},\sigma+\beta} \lesssim \|(I - P_N)\mathbf{u}_0\|_\sigma \to 0 \quad \text{as } N \to \infty,$$

as desired.

Let us treat Φ_3 now. Similar to before, by Proposition 2.6 we have

$$\|\Phi_3(t)\|_{\lambda\sqrt{t},\epsilon} \lesssim \|e^{-D_c t A}c_0\|_{\lambda\sqrt{t},\epsilon} \lesssim \|e^{-(D_c/2)tA}c_0\|_\epsilon \lesssim \|c_0\|_\epsilon, \tag{2.68}$$

while Proposition 2.5 implies

$$\|\Phi_3(t)\|_{\lambda\sqrt{t},1} \lesssim \|e^{-D_c t A}c_0\|_{\lambda\sqrt{t},(1-\epsilon)+\epsilon} \lesssim (1 \wedge (D_c t))^{-(1-\epsilon)/2}\|c_0\|_\epsilon. \tag{2.69}$$

Therefore, taking the supremum over $[0, \tau]$ (resp. $(0, \tau]$) of (2.68) (resp. (2.69)), we obtain

$$\|\Phi_3\|_{\mathcal{Z}_{3,\tau}} \lesssim \|c_0\|_\epsilon.$$

Similarly, we can show that for all $N > 0$

$$(1 \wedge (D_c t))^{(1-\epsilon)/2}\|\Phi_3(t)\|_{\lambda\sqrt{t},1} \lesssim \|(I - P_N)c_0\|_\epsilon + (1 \wedge (D_c t))^{(1-\epsilon)/2}N^{1-\epsilon}\|P_N c_0\|_1.$$

This implies

$$\lim_{t\to 0^+} (1 \wedge (D_c t))^{(1-\epsilon)/2}\|\Phi_3(t)\|_{\lambda\sqrt{t},1} \lesssim \|(I - P_N)c_0\|_\epsilon \to 0 \quad \text{as } N \to \infty.$$

$\square$

2.4.2. *Estimate of W.* To estimate W in $\mathcal{Z}_\tau$, we estimate $\mathbf{W}_1, W_2, W_3$ in

$$\boldsymbol{\mathcal{Z}}_{1,T}, \mathcal{Z}_{2,T}, \mathcal{Z}_{3,T},$$

respectively. We do this in Lemmas 2.15, 2.18, and 2.21 below.

LEMMA 2.15. *Let $-1 < \sigma \leq 0$ and $\beta \in [0,1)$ such that $\sigma + \beta \geq 0$. Then*

$$\|\mathbf{W}_1\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} \lesssim \nu^{-1}((\nu\tau)^{1-\beta} \vee (\nu\tau)^{(1-\beta)/2} \vee (\nu\tau)) \left(C_\phi(\tau)\|n\|_{\mathcal{Y}_{2,\tau}} + \|\mathbf{u}\|^2_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\right),$$

where $C_\phi(\tau) := \|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}$. In particular, we have

$$\begin{aligned} \|\mathbf{G}\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} &\lesssim C_\phi \nu^{-1}((\nu\tau)^{1-\beta} \vee (\nu\tau))\|n\|_{\mathcal{Y}_{2,\tau}}, \\ \|\mathbf{W}_1^{(2)}\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} &\lesssim C_\phi \nu^{-1}((\nu\tau) \vee (\nu\tau)^{(1-\beta)/2})\|\mathbf{u}\|^2_{\boldsymbol{\mathcal{Y}}_{1,\tau}}. \end{aligned} \tag{2.70}$$

We immediately obtain the following corollary.

COROLLARY 2.16. *Let $\rho > 0$ and $U = (\mathbf{u}, n, c)$. Suppose that $U \in B_{\mathcal{Z}_\tau}(\vec{0}, \rho)$. Then*

$$\|\mathbf{W}_1\|_{\mathcal{Z}_{1,\tau}} \leq \nu^{-1} C_{2.15}(\tau)(C_\phi(\tau) + \rho)\|U\|_{\mathcal{Y}_\tau},$$

where

$$C_{2.15}(\tau) := C_0((\nu\tau)^{1-\beta} \vee (\nu\tau)^{(1-\beta)/2} \vee (\nu\tau)), \tag{2.71}$$

for some absolute constant $C_0 > 0$. In particular, $C_{2.15}(\tau)$ is an increasing function of $\tau \geq 0$.

PROOF OF LEMMA 2.15. Let $\gamma = \sigma + \beta$, so that $\gamma = 0$. Observe that for $0 \leq s \leq t \leq \tau$ by Proposition 2.6 and 2.13 we have

$$\begin{aligned} \|e^{(t-s)\nu A} F_{\nabla\phi}(\hat{n})(s)\|_{\lambda\sqrt{t},\sigma} &\lesssim \|e^{-(\nu/2)(t-s)A} F_{\nabla\phi}(\hat{n})(s)\|_{\lambda\sqrt{s},\sigma} \\ &\lesssim (\nu(t-s))^{-\beta/2}\|F_{\nabla\phi}(\hat{n})(s)\|_{\lambda\sqrt{s},\gamma} \\ &\lesssim \|\nabla\phi\|_{\lambda\sqrt{s},\gamma}\|n(s)\|_{\lambda\sqrt{s},\gamma} \\ &\lesssim \|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}}(\nu(t-s))^{-\beta/2}(\nu s)^{-\beta/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \|\mathbf{G}(t)\|_{\lambda\sqrt{t},\sigma} &\lesssim \|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}} \int_0^t (\nu(t-s))^{-\beta/2}(\nu s)^{-\beta/2}\, ds \\ &\lesssim \|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}}\nu^{-1}(\nu t)^{1-\beta}. \end{aligned} \tag{2.72}$$

On the other hand, observe that by Proposition 2.7 we have

$$\begin{aligned} \|\mathbf{W}_1^{(2)}(t)\|_{\lambda\sqrt{t},\sigma} &\lesssim \int_0^t \|e^{-(t-s)\nu A}\mathbf{B}(\hat{\mathbf{u}}, \hat{\mathbf{u}})(s)\|_{\lambda\sqrt{s},\sigma}\, ds \\ &\lesssim \int_0^t (\nu(t-s))^{-(1/2)(1-\beta)}\|\mathbf{u}(s)\|^2_{\lambda\sqrt{s},\gamma}\, ds \\ &\lesssim \|\mathbf{u}\|^2_{\boldsymbol{\mathcal{Y}}_{1,\tau}} \int_0^t (\nu(t-s))^{-(1/2)(1-\beta)}(\nu s)^{-\beta}\, ds \\ &\lesssim \|\mathbf{u}\|^2_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\nu^{-1}(\nu t)^{(1-\beta)/2}. \end{aligned} \tag{2.73}$$

Thus, upon adding (2.72) and (2.73), and taking supremum over $[0,\tau]$, we arrive at

$$\sup_{0\leq t\leq\tau}\|\mathbf{W}_1(t)\|_{\lambda\sqrt{t},\sigma} \lesssim \nu^{-1}((\nu\tau)^{(1-\beta)/2} \vee (\nu\tau)^{1-\beta})(\|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}} + \|\mathbf{u}\|^2_{\boldsymbol{\mathcal{Y}}_{1,\tau}}). \tag{2.74}$$

On the other hand, Propositions 2.6 and 2.13 imply

$$\begin{aligned}\|e^{(t-s)\nu A}F_{\nabla\phi}(\hat{n})(s)\|_{\lambda\sqrt{t},\gamma} &\lesssim \|e^{-(\nu/2)(t-s)A}F_{\nabla\phi}(\hat{n})(s)\|_{\lambda\sqrt{s},\gamma}\\ &\lesssim \|\nabla\phi\|_{\lambda\sqrt{s},\gamma}\|n(s)\|_{\lambda\sqrt{s},\gamma}\\ &\lesssim \|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}}(\nu s)^{-\beta/2}.\end{aligned} \tag{2.75}$$

Upon integrating (2.75) over $(0,t)$, then taking supremum over $(0,\tau]$, we obtain

$$\sup_{0<t\leq\tau}(\nu t)^{\beta/2}\|\mathbf{F}(t)\|_{\lambda\sqrt{t},\gamma} \lesssim \|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}}\nu^{-1}(\nu\tau). \tag{2.76}$$

For $\mathbf{W}_1^{(2)}$, we apply Propositions 2.6 and 2.7 to estimate

$$\begin{aligned}\|\mathbf{W}_1^{(2)}(t)\|_{\lambda\sqrt{t},\gamma} &\lesssim \int_0^t \|e^{-(t-s)\nu A}\mathbf{B}(\hat{\mathbf{u}},\hat{\mathbf{u}})(s)\|_{\lambda\sqrt{s},\gamma}\, ds\\ &\lesssim \int_0^t (\nu(t-s))^{-1/2}\|\hat{\mathbf{u}}(s)\|^2_{\lambda\sqrt{s},\gamma}\, ds\\ &\lesssim \|\mathbf{u}\|^2_{\mathbfcal{Y}_{1,\tau}}\int_0^t (\nu(t-s))^{-1/2}(\nu s)^{-\beta}\, ds.\end{aligned}$$

It follows that

$$\sup_{0<t\leq\tau}(\nu t)^{\beta/2}\|\mathbf{W}_1^{(2)}(t)\|_{\lambda\sqrt{t},\gamma} \lesssim \nu^{-1}(\nu\tau)^{(1-\beta)/2}\|\mathbf{u}\|^2_{\mathbfcal{Y}_{1,\tau}}. \tag{2.77}$$

Upon adding (2.76) and (2.77), taking supremum over $(0,\tau]$, we arrive at

$$\sup_{0<t\leq\tau}\|\mathbf{W}_1(t)\|_{\lambda\sqrt{t},\gamma} \lesssim \nu^{-1}((\nu\tau)\vee(\nu\tau)^{(1-\beta)/2})(\|\nabla\phi\|_{\lambda\sqrt{\tau},\gamma}\|n\|_{\mathcal{Y}_{2,\tau}} + \|\mathbf{u}\|^2_{\mathbfcal{Y}_{1,\tau}}). \tag{2.78}$$

Combining (2.74) and (2.78) completes the proof. □

To obtain the estimates for W_2 and W_3, we will make use of the following elementary fact. A variation of this was used in [3, 4]. We provide the proof here for the sake of completion.

PROPOSITION 2.17. *Let $a,b,c>0$ such that $a,b<1$ and $a+b\leq 1$. Then*

$$\int_0^t (1\wedge(c(t-s)))^{-a}s^{-b}\, ds \lesssim_{a,b} \left((ct)^{-a}\vee 1\right)t^{1-b}, \tag{2.79}$$

and

$$\int_0^t (t-s)^{-a}(1\wedge(cs))^{-b}\, ds \lesssim_{a,b} \left((ct)^{-b}\vee 1\right)t^{1-a}. \tag{2.80}$$

PROOF. Let $\mathcal{B}(a,b) := \int_0^1(1-s)^{-a}s^{-b}\, ds$ denote the beta integral. By symmetry, it suffices to consider (2.79). Suppose that $t\leq c^{-1}$. Then for $0\leq s\leq t$, we have $c(t-s)\leq 1$. This implies that

$$\int_0^t (1\wedge(c(t-s)))^{-a}s^{-b}\, ds = \mathcal{B}(a,b)c^{-a}t^{1-a-b}.$$

On the other hand, if $t>c^{-1}$, then

$$\begin{aligned}\int_0^t (1\wedge(c(t-s)))^{-a}s^{-b}\, ds &= \mathcal{B}(a,b)c^{-1+b} + \int_{c^{-1}}^t s^{-b}\, ds\\ &= \mathcal{B}(a,b)c^{-(1-b)} + \frac{1}{1-b}(t^{1-b}-c^{-(1-b)})\\ &\leq \left(\mathcal{B}(a,b)\vee\frac{1}{1-b}\right)t^{1-b}.\end{aligned}$$

Therefore

$$\int_0^t (1\wedge (c(t-s)))^{-a}s^{-b}\,ds \le \left(\mathcal{B}(a,b)\vee \frac{1}{1-b}\right)\left((ct)^{-a}\vee 1\right)t^{1-b},$$

as desired. □

LEMMA 2.18. *Let $-1<\sigma\le 0$ and $\beta\in[0,1)$ such that $\sigma+\beta=0$. Then*

$$\|W_2\|_{\mathcal{Z}_{2,\tau}} \lesssim D_n^{-1}\left[\left(\frac{D_n}{\nu}\right)^{\beta/2}\vee\left(\frac{D_n}{D_c}\right)^{1/2}\right]\left((D_n\tau)^{(1-\beta)/2}\vee (D_c\tau)^{\epsilon/2}\vee (D_n\tau)^{(1+\beta)/2}\right)$$
$$\times\left(\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}+g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}\right)\|n\|_{\mathcal{Y}_{2,\tau}}.$$

In particular, we have

$$\|W_2^{(1)}\|_{\mathcal{Z}_{2,\tau}} \le D_n^{-1}\left(\frac{D_n}{\nu}\right)^{\beta/2}(D_n\tau)^{(1-\beta)/2}\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}},$$
$$\|W_2^{(2)}\|_{\mathcal{Z}_{2,\tau}} \lesssim D_n^{-1}\left(\frac{D_n}{D_c}\right)^{1/2}\left((D_c\tau)^{\epsilon/2}\vee (D_c\tau)^{(1+\beta)/2}\right)g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}}.$$

As before, we obtain the following corollary.

COROLLARY 2.19. *Let $\rho>0$ and $U=(\mathbf{u},n,c)$. Suppose that $U\in B_{\mathcal{Z}_\tau}(\vec{0},\rho)$. Then*

$$\|\mathbf{W}_2\|_{\mathcal{Z}_{1,\tau}} \le D_n^{-1}C_{2.18}(\epsilon,\tau)\rho(1+g_\chi(\rho))\|U\|_{\mathcal{Y}_\tau}, \tag{2.81}$$

where

$$C_{2.18}(\epsilon,\tau) := C_0\left[\left(\frac{D_n}{\nu}\right)^{\beta/2}\vee\left(\frac{D_n}{D_c}\right)^{1/2}\right]\left((D_n\tau)^{(1-\beta)/2}\vee (D_c\tau)^{\epsilon/2}\vee (D_n\tau)^{(1+\beta)/2}\right), \tag{2.82}$$

for some absolute constant $C_0>0$. In particular, $C_{2.18}(\epsilon,\tau)$ is an increasing function of $\tau\ge 0$ for $\epsilon>0$, while $C_{2.18}(0,\tau)$ is constant in τ for τ sufficiently small.

PROOF OF LEMMA 2.18. Let $\gamma=\sigma+\beta$, so that $\gamma=0$. First, we have

$$\|W_2(t)\|_{\lambda\sqrt{t},\sigma} \le \|W_2^{(1)}(t)\|_{\lambda\sqrt{t},\sigma}+\|W_2^{(2)}(t)\|_{\lambda\sqrt{t},\sigma}.$$

It follows from Propositions 2.6 and 2.8 that

$$\|W_2^{(1)}(t)\|_{\lambda\sqrt{t},\sigma} \lesssim \int_0^t (D_n(t-s))^{-(1/2)(1-\beta)}\|\mathbf{u}(s)\|_{\lambda\sqrt{s},\gamma}\|n(s)\|_{\lambda\sqrt{s},\gamma}\,ds$$
$$\lesssim \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}}D_n^{-\beta/2}\nu^{-\beta/2}\int_0^t (t-s)^{-(1/2)(1-\beta)}s^{-\beta}\,ds$$
$$\lesssim \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}}D_n^{-1}\left(\frac{D_n}{\nu}\right)^{\beta/2}(D_n\tau)^{(1-\beta)/2}. \tag{2.83}$$

On the other hand, by Propositions 2.6, 2.9, 2.17, and the fact that $\epsilon\ge 0$, we have

$$\|W_2^{(2)}(t)\|_{\lambda\sqrt{t},\sigma}$$
$$\lesssim \int_0^t (D_n(t-s))^{-(1/2)(1-\beta)}\|n\|_{\lambda\sqrt{s},\gamma}g_\chi(\|c\|_{\lambda\sqrt{s},0})\|c\|_{\lambda\sqrt{s},1}\,ds$$
$$\lesssim \|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}D_n^{-1/2}D_c^{\beta/2}\left(\int_0^t (t-s)^{-(1/2)(1-\beta)}(1\wedge(D_cs))^{-(1-\epsilon+\beta)/2}\,ds\right)$$
$$\lesssim D_n^{-1}\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}\left(\frac{D_n}{D_c}\right)^{1/2}\left((D_c\tau)^{\epsilon/2}\vee(D_c\tau)^{(1+\beta)/2}\right). \tag{2.84}$$

Upon combining (2.83) and (2.84) and taking supremum over $[0,\tau]$ we arrive at

$$\sup_{0\le t\le\tau}\|W_2(t)\|_{\lambda\sqrt{t},\sigma}\lesssim \tag{2.85}$$

$$D_n^{-1}\left[\left(\frac{D_n}{\nu}\right)^{\beta/2}\vee\left(\frac{D_n}{D_c}\right)^{1/2}\right]((D_n\tau)^{(1-\beta)/2}\vee(D_c\tau)^{\epsilon/2}\vee(D_n\tau)^{(1+\beta)/2})$$

$$\times\left(\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}+g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}\right)\|n\|_{\mathcal{Y}_{2,\tau}}. \tag{2.86}$$

Now we estimate

$$\|W_2(t)\|_{\lambda\sqrt{t},\gamma}\le\|W_2^{(1)}(t)\|_{\lambda\sqrt{t},\gamma}+\|W_2^{(2)}(t)\|_{\lambda\sqrt{t},\gamma}.$$

By Propositions 2.6 and 2.8, we have

$$\begin{aligned}(D_nt)^{\beta/2}\|W_2^{(1)}(t)\|_{\lambda\sqrt{t},\gamma}&\lesssim(D_nt)^{\beta/2}\int_0^t(D_n(t-s))^{-1/2}\|\mathbf{u}(s)\|_{\lambda\sqrt{s},\gamma}\|n(s)\|_{\lambda\sqrt{s},\gamma}\,ds\\&\lesssim(D_nt)^{\beta/2}\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}}D_n^{-1/2}(D_n\nu)^{-\beta/2}\int_0^t(t-s)^{-1/2}s^{-\beta}\,ds\\&\lesssim\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}}D_n^{-1}\left(\frac{D_n}{\nu}\right)^{\beta/2}(D_n\tau)^{(1-\beta)/2}.\end{aligned}\tag{2.87}$$

On the other hand, by Propositions 2.6, 2.9, 2.17, and the fact that $\epsilon\ge0$, we have

$$\begin{aligned}&(D_nt)^{\beta/2}\|W_2^{(2)}(t)\|_{\lambda\sqrt{t},\gamma}\\&\lesssim(D_nt)^{\beta/2}\int_0^t(D_n(t-s))^{-1/2}\|n\|_{\lambda\sqrt{s},\gamma}g_\chi(\|c\|_{\lambda\sqrt{s},0})\|c\|_{\lambda\sqrt{s},1}\,ds\\&\lesssim\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}D_n^{-1/2}D_c^{\beta/2}t^{\beta/2}\left(\int_0^t(t-s)^{-1/2}(1\wedge(D_cs))^{-(1-\epsilon+\beta)/2}\,ds\right)\\&\lesssim D_n^{-1}\left(\frac{D_n}{D_c}\right)^{1/2}((D_c\tau)^{\epsilon/2}\vee(D_c\tau)^{(1+\beta)/2})\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}.\end{aligned}\tag{2.88}$$

Thus, upon taking supremum over $(0,\tau]$, (2.87) and (2.88) imply

$$\begin{aligned}&\sup_{0<t\le\tau}(D_nt)^{\beta/2}\|W_2(t)\|_{\lambda\sqrt{t},\gamma}\\&\lesssim D_n^{-1}\left[\left(\frac{D_n}{\nu}\right)^{\beta/2}\vee\left(\frac{D_n}{D_c}\right)^{1/2}\right]\left(\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}+g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}\right)\|n\|_{\mathcal{Y}_{2,\tau}}\\&\times((D_n\tau)^{(1-\beta)/2}\vee(D_c\tau)^{\epsilon/2}\vee(D_c\tau)^{(1+\beta)/2}).\end{aligned}\tag{2.89}$$

Combining (2.85) and (2.89) then completes the proof. □

Invoking Proposition 2.10 in place of Proposition 2.9 appropriately, similar to Lemma 2.18, we have that the following estimate holds as well.

LEMMA 2.20. *Let $-1<\sigma\le0$ and $\beta\in[0,1)$ such that $\sigma+\beta=0$. Then*

$$\|\mathbf{W}_2^{(2)}(\hat{n},\widehat{\chi(c_1)}-\widehat{\chi(c_2)},\hat{c})\|_{\mathcal{Z}_{1,\tau}}\le D_n^{-1}g'_\chi(M)\widetilde{C}_{2.18}(\tau)\|c_1-c_2\|_{\mathcal{X}_{3,\tau}}\|c\|_{\mathcal{Y}_{3,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}},$$

where

$$\widetilde{C}_{2.18}(\epsilon,\tau):=C_0\left(\frac{D_n}{D_c}\right)^{1/2}((D_c\tau)^{\epsilon/2}\vee(D_n\tau)^{(1+\beta)/2}),\tag{2.90}$$

for some absolute constant $C_0>0$. In particular, $\widetilde{C}_{2.18}(\epsilon,\tau)$ is an increasing function of $\tau\ge0$ for $\epsilon>0$, while $\widetilde{C}_{2.18}(0,\tau)$ is constant in τ for τ sufficiently small.

REMARK 2.1. *Let $0<\alpha<1$. Suppose that the nonlinear term $\nabla\cdot(n\chi(c)\nabla c)$ is given instead by*

$$\nabla\cdot(n\chi(c)|\nabla|^{\alpha}c). \tag{2.91}$$

Redefine the norms for $\mathcal{X}_{3,\tau}$ and $\mathcal{Y}_{3,\tau}$ by

$$\|c\|_{\mathcal{X}_{3,\tau}}:=\sup_{0\le t\le\tau}\|c(t)\|_{\lambda\sqrt{t},0},\quad \|c\|_{\mathcal{Y}_{3,\tau}}:=\sup_{0<t\le\tau}(1\wedge(D_ct)^{\alpha/2})\|c(t)\|_{\lambda\sqrt{t},\alpha}, \tag{2.92}$$

and, consequently, the spaces $\mathcal{X}_\tau,\mathcal{Y}_\tau$. Then, in the proof of Lemma 2.18, we may instead estimate

$$\begin{aligned}
&\|W_2^{(2)}(t)\|_{\lambda\sqrt{t},\sigma}\\
&\lesssim\int_0^t(D_n(t-s))^{-(1/2)(1-\beta)}\|n\|_{\lambda\sqrt{s},\gamma}g_\chi(\|c\|_{\lambda\sqrt{s},0})\|c\|_{\lambda\sqrt{s},\alpha}\,ds\\
&\lesssim\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}D_n^{-1/2}D_c^{\beta/2}\left(\int_0^t(t-s)^{-(1-\beta)/2}(1\wedge(D_cs))^{-(\alpha+\beta)/2}\,ds\right)
\end{aligned}\tag{2.93}$$

$$\lesssim\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}D_n^{-1}\left(\frac{D_n}{D_c}\right)^{1/2}((D_ct)^{(1-\alpha)/2}\vee(D_ct)^{(1+\beta)/2}). \tag{2.94}$$

Similarly, we have

$$\begin{aligned}
&(D_nt)^{\beta/2}\|W_2^{(2)}(t)\|_{\lambda\sqrt{t},\gamma}\lesssim\\
&(D_ct)^{\beta/2}\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}D_n^{-1/2}D_c^{-\beta/2}\\
&\left(\int_0^t(t-s)^{-1/2}(1\wedge(D_cs))^{-(\alpha+\beta)/2}\,ds\right)
\end{aligned}\tag{2.95}$$

$$\lesssim\|n\|_{\mathcal{Y}_{2,\tau}}g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}D_n^{-1} \tag{2.96}$$

$$\left(\frac{D_n}{D_c}\right)^{1/2}((D_ct)^{(1-\alpha)/2}\vee(D_ct)^{(1+\beta)/2}) \tag{2.97}$$

This implies that

$$\|W_2\|_{\mathcal{Z}_{2,\tau}}\lesssim D_n^{-1}C_{2.18}(\epsilon,\tau)\left(\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}}+g_\chi(\|c\|_{\mathcal{X}_{3,\tau}})\|c\|_{\mathcal{Y}_{3,\tau}}\right)\|n\|_{\mathcal{Y}_{2,\tau}}, \tag{2.98}$$

where

$$C_{2.18}(\epsilon,\tau):=\left[\left(\frac{D_n}{\nu}\right)^{\beta/2}\vee\left(\frac{D_n}{D_c}\right)^{1/2}\right]((D_n\tau)^{(1-\beta)/2}\vee(D_c\tau)^{(1-\alpha)/2}\vee(D_n\tau)^{(1+\beta)/2}). \tag{2.99}$$

In an entirely similar vein, we also obtain the analog to Lemma 2.20:

$$\|\mathbf{W}_2^{(2)}(\hat{n},\widehat{\chi(c_1)}-\widehat{\chi(c_2)},\hat{c})\|_{\mathcal{Z}_{1,\tau}}\le D_n^{-1}g_\chi'(M)\widetilde{C}_{2.18}(\epsilon,\tau)\|c_1-c_2\|_{\mathcal{X}_{3,\tau}}\|c\|_{\mathcal{Y}_{3,\tau}}\|n\|_{\mathcal{Y}_{2,\tau}}, \tag{2.100}$$

where

$$\widetilde{C}_{2.18}(\epsilon,\tau):=C_0\left(\frac{D_n}{D_c}\right)^{1/2}((D_c\tau)^{(1-\alpha)/2}\vee(D_n\tau)^{(1+\beta)/2}), \tag{2.101}$$

The inequalities (2.98) and (2.100), replacing the corresponding ones provided by Lemmas 2.18 and 2.20, will allow us to establish the analog of Theorem 1.1 corresponding to the case where the "cubic" nonlinearity in (1.1) is given by (2.91), and the spaces $\mathcal{X}_\tau,\mathcal{Y}_\tau$ are redefined according to (2.92) (see Remark 2.2).

LEMMA 2.21. *Let $-1 < -\sigma \le 0$ and $\epsilon, \beta \in [0, 1)$ such that $\sigma + \beta = 0$. Then*

$$\|W_3\|_{\mathcal{Z}_{3,\tau}} \lesssim D_c^{-1} \left[\left(\frac{D_c}{\nu}\right) \vee \left(\frac{D_c}{D_n}\right) \right]^{\beta/2} \left(\|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}} \|c\|_{\mathcal{Y}_{3,\tau}} + \|n\|_{\mathcal{Y}_{2,\tau}} g_f(\|c\|_{\mathcal{X}_{3,\tau}}) \right)$$
$$\times \left((D_c\tau)^{(1-\beta)/2} \vee (D_c\tau)^{1-\epsilon/2} \vee (D_c\tau)^{1-\beta/2} \vee (D_c\tau)^{(3-(\epsilon+\beta))/2} \right). \tag{2.102}$$

In particular, we have

$$\|W_3^{(1)}\|_{\mathcal{Z}_{3,\tau}} \lesssim D_c^{-1} \left(\frac{D_c}{\nu}\right)^{\beta/2} \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}} \|c\|_{\mathcal{Y}_{3,\tau}} \left((D_c\tau)^{(1-\beta)/2} \vee (D_c\tau)^{1-\epsilon/2} \right)$$
$$\|W_3^{(2)}\|_{\mathcal{Z}_{3,\tau}} \lesssim D_c^{-1} \left(\frac{D_c}{D_n}\right)^{\beta/2} \|n\|_{\mathcal{Y}_{2,\tau}} g_f(\|c\|_{\mathcal{X}_{3,\tau}}) \left((D_c\tau)^{1-\beta/2} \vee (D_c\tau)^{(3-(\epsilon+\beta))/2} \right).$$

COROLLARY 2.22. *Let $\rho > 0$ and $U = (\mathbf{u}, n, c)$. Suppose that $U \in B_{\mathcal{Z}_\tau}(\vec{0}, \rho)$. Then*

$$\|\mathbf{W}_3\|_{\mathcal{Z}_{1,\tau}} \le D_c^{-1} C_{2.21}(\tau)(\rho + g_f(\rho)) \|U\|_{\mathcal{Z}_\tau},$$

where

$$C_{2.21}(\tau) = C_0 \left[\left(\frac{D_c}{\nu}\right) \vee \left(\frac{D_c}{D_n}\right) \right]^{\beta/2} \tag{2.103}$$
$$\left((D_c\tau)^{(1-\beta)/2} \vee (D_c\tau)^{1-\epsilon/2} \vee (D_c\tau)^{1-\beta/2} \vee (D_c\tau)^{(3-(\epsilon+\beta))/2} \right) \tag{2.104}$$

for some absolute constant $C_0 > 0$. In particular, $C_{2.21}(\tau)$ is an increasing function of $\tau \ge 0$.

PROOF OF LEMMA 2.21. Let $\gamma = \sigma + \beta$, so that $\gamma = 0$. Proceeding as in the proofs of Lemmas 2.15 and 2.18, we observe that

$$\|W_3(t)\|_{\lambda\sqrt{t},\delta} \le \|W_3^{(1)}(t)\|_{\lambda\sqrt{t},\delta} + \|W_3^{(2)}(t)\|_{\lambda\sqrt{t},\delta}$$

for any $\delta \in \mathbb{R}$. By Propositions 2.6 and 2.8, for $0 \le t \le \tau$ we have

$$\begin{aligned} \|W_3^{(1)}(t)\|_{\lambda\sqrt{t},\epsilon} &\lesssim \int_0^t \|e^{-(D_c/2)(t-s)A} B(\hat{\mathbf{u}}, \hat{c})(s)\|_{\lambda\sqrt{s},\epsilon}\, ds \\ &\lesssim \int_0^t (D_c(t-s))^{-\epsilon/2} \|\mathbf{u}\|_{\lambda\sqrt{s},\gamma} \|c\|_{\lambda\sqrt{s},\gamma+1}\, ds \\ &\lesssim \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}} \|c\|_{\mathcal{X}_{3,\tau}} \nu^{-\beta/2} \int_0^t (t-s)^{-\epsilon/2} s^{-(1-\epsilon+\beta)/2}\, ds \\ &\lesssim \|\mathbf{u}\|_{\boldsymbol{\mathcal{Y}}_{1,\tau}} \|c\|_{\mathcal{X}_{3,\tau}} D_c^{-1} \left(\frac{D_c}{\nu}\right)^{\beta/2} (D_c\tau)^{(1-\beta)/2}. \end{aligned} \tag{2.105}$$

By Propositions 2.6, 2.11, and 2.17, along with the fact that $\epsilon + \beta < 2$, we have

$$\begin{aligned} \|W_3^{(2)}(t)\|_{\lambda\sqrt{t},\epsilon} &\lesssim \int_0^t \|e^{-(D_c/2)(t-s)A} C(\hat{n}, \widehat{f(c)})(s)\|_{\lambda\sqrt{s},\epsilon}\, ds \\ &\lesssim \int_0^t (1 \wedge ((D_c/2)(t-s)))^{-\epsilon/2} \|n\|_{\lambda\sqrt{s},0} g_f(\|c\|_{\lambda\sqrt{s},0})\, ds \\ &\lesssim \|n\|_{\mathcal{Y}_{2,\tau}} g_f(\|c\|_{\mathcal{X}_{3,\tau}}) D_n^{-\beta/2} \int_0^t (1 \wedge ((D_c/2)(t-s)))^{-\epsilon/2} s^{-\beta/2}\, ds \\ &\lesssim D_c^{-1} \|n\|_{\mathcal{Y}_{2,\tau}} g_f(\|c\|_{\mathcal{X}_{3,\tau}}) \left(\frac{D_c}{D_n}\right)^{\beta/2} \left((D_c t)^{1-\beta/2} \vee (D_c t)^{1-(\epsilon+\beta)/2} \right). \end{aligned} \tag{2.106}$$

Taking supremum over $[0,\tau]$ (resp. $(0,\tau]$) in (2.105) and (2.106), then adding the results

$$\sup_{0\le t\le\tau}\|W_3(t)\|_{\lambda\sqrt{t},\epsilon}\lesssim \tag{2.107}$$

$$D_c^{-1}\left[\left(\frac{D_c}{\nu}\right)\vee\left(\frac{D_c}{D_n}\right)\right]^{\beta/2}\left((D_c\tau)^{(1-\beta)/2}\vee(D_c\tau)^{1-\beta/2}\vee(D_c\tau)^{1-(\epsilon+\beta)/2}\right)$$

$$\times\left[\|\mathbf{u}\|_{\mathcal{Y}_{1,\tau}}\|c\|_{\mathcal{X}_{3,\tau}}+\|n\|_{\mathcal{Y}_{2,\tau}}g_f(\|c\|_{\mathcal{X}_{3,\tau}})\right]. \tag{2.108}$$

On the other hand, for $0<t\le\tau$, we use Propositions 2.6, 2.8, and 2.17 to estimate

$$\begin{aligned}
&(1\wedge(D_ct))^{(1-\epsilon)/2}\|W_3^{(1)}(t)\|_{\lambda\sqrt{t},1}\\
&\lesssim(1\wedge(D_ct))^{(1-\epsilon)/2}\int_0^t\|e^{-(D_c/2)(t-s)A}B(\hat{\mathbf{u}},\hat{c})(s)\|_{\lambda\sqrt{s},1}\,ds\\
&\lesssim(D_ct)^{(1-\epsilon)/2}\int_0^t(D_c(t-s))^{-1/2}\|\mathbf{u}\|_{\lambda\sqrt{s},\gamma}\|c\|_{\lambda\sqrt{s},1}\,ds\\
&\lesssim\|\mathbf{u}\|_{\mathcal{Y}_{1,T}}\|c\|_{\mathcal{Y}_{3,T}}D_c^{-1/2}(D_ct)^{(1-\epsilon)/2}\left(\frac{D_c}{\nu}\right)^{\beta/2}\int_0^t(t-s)^{-1/2}(1\wedge(D_cs))^{-(1-\epsilon+\beta)/2}\,ds
\end{aligned}$$

$$\lesssim D_c^{-1}\left(\frac{D_c}{\nu}\right)^{\beta/2}\|\mathbf{u}\|_{\mathcal{Y}_{1,T}}\|c\|_{\mathcal{Y}_{3,T}}\left((D_c\tau)^{(1-\beta)/2}\vee(D_c\tau)^{1-\epsilon/2}\right). \tag{2.109}$$

Using Propositions 2.6, 2.11, 2.17, and the facts (2.43) and $\epsilon\ge 0$, we also have

$$\begin{aligned}
&(1\wedge(D_ct))^{(1-\epsilon)/2}\|W_3^{(2)}(t)\|_{\lambda\sqrt{t},1}\\
&\lesssim(1\wedge(D_ct))^{(1-\epsilon)/2}\int_0^t(1\wedge((D_c/2)(t-s)))^{-(1-\epsilon)/2}\|n(s)\|_{\lambda\sqrt{s},\gamma}g_f\left(\|c(s)\|_{\lambda\sqrt{s},0}\right)\,ds\\
&\lesssim(D_ct)^{(1-\epsilon)/2}\|n\|_{\mathcal{Y}_{2,\tau}}g_f(\|c\|_{\mathcal{X}_{3,T}})D_n^{-\beta/2}\int_0^t(1\wedge((D_c/2)(t-s)))^{-(1-\epsilon)/2}s^{-\beta/2}\,ds
\end{aligned}$$

$$\lesssim D_c^{-1}\|n\|_{\mathcal{Y}_{2,\tau}}g_f(\|c\|_{\mathcal{X}_{3,\tau}})\left(\frac{D_c}{D_n}\right)^{\beta/2}\left((D_c\tau)^{1-\beta/2}\vee(D_c\tau)^{(3-(\epsilon+\beta))/2}\right) \tag{2.110}$$

Upon taking supremum over $(0,T]$, then adding (2.109), (2.110), we have

$$\sup_{0<t\le\tau}(D_ct)^{(1-\epsilon)/2}\|W_3(t)\|_{\lambda\sqrt{t},1}\lesssim \tag{2.111}$$

$$D_c^{-1}\left[\left(\frac{D_c}{\nu}\right)\vee\left(\frac{D_c}{D_n}\right)\right]^{\beta/2}\left(\|\mathbf{u}\|_{\mathcal{Y}_{1,\tau}}\|c\|_{\mathcal{Y}_{3,\tau}}+\|n\|_{\mathcal{Y}_{2,\tau}}g_f(\|c\|_{\mathcal{X}_{3,\tau}})\right)$$

$$\times\left((D_c\tau)^{(1-\beta)/2}\vee(D_c\tau)^{1-\epsilon/2}\vee(D_c\tau)^{1-\beta/2}\vee(D_c\tau)^{(3-(\epsilon+\beta))/2}\right). \tag{2.112}$$

Combining (2.107) and (2.111) completes the proof. □

As before, with Lemma 2.20, we also have the following variant of Lemma 2.21, whose proof follows along the lines of Lemma 2.21, except that we appropriately invoke Proposition 2.12 in place of Proposition 2.11.

Lemma 2.23. *Let $-1<-\sigma\le 0$, $\epsilon\in[0,1)$, and $\beta\in[0,1)$ such that $\sigma+\beta=0$. Then*

$$\|W_3^{(2)}(\hat{n},\widehat{f(c_1)}-\widehat{f(c_2)})\|_{\mathcal{Z}_{3,\tau}}\le D_c^{-1}g_f'(M)\widetilde{C}_{2.21}(\tau)\|n\|_{\mathcal{Y}_{2,\tau}}\|c_1-c_2\|_{\mathcal{X}_{3,\tau}},$$

where

$$\widetilde{C}_{2.21}(\tau):=C_0\left(\frac{D_c}{D_n}\right)^{\beta/2}\left((D_c\tau)^{1-\beta/2}\vee(D_c\tau)^{(3-(\epsilon+\beta))/2}\right), \tag{2.113}$$

for some absolute constant $C_0 > 0$. In particular, $C_{2.21}(\tau)$ is an increasing function of $\tau \geq 0$.

2.5. Contraction Mapping Argument. Let $\sigma > -1$ and $\epsilon \in [0, 1)$. Let $\beta \geq 0$ be given such that $\beta = \min\{0, -\sigma\}$. Let $\gamma = \sigma + \beta$ and $(\mathbf{u}_0, n_0, c_0) \in \mathcal{V}_{(\sigma,\sigma,\epsilon)}$. From Lemma 2.14, there exists a constant $C_{2.14} > 0$ such that

$$\|\Phi\|_{\mathcal{Z}_T} \leq C_{2.14}\|U_0\|_{(\sigma,\sigma,\epsilon)} =: R, \tag{2.114}$$

where

$$U_0 = (\mathbf{u}_0, n_0, c_0) \quad \text{and} \quad U = (\mathbf{u}, n, c).$$

Let

$$B_{\mathcal{Z}_\tau}(\Phi, R) := \{U \in \mathcal{Z}_\tau : \|U - \Phi\|_{\mathcal{Z}_\tau} \leq R\}.$$

We show that $\mathcal{S}$ is a self-map on $B_{\mathcal{Z}_\tau}(\Phi, R)$ and that the map, $\mathcal{S}$, given by (2.62) is a contraction.

2.5.1. *$\mathcal{S}$ is a self-map, $\mathcal{S} : B_{\mathcal{Z}_\tau}(\Phi, R) \to B_{\mathcal{Z}_\tau}(\Phi, R)$.* Observe that from (2.114), for $U = (\mathbf{u}, n, c) \in B_{\mathcal{Z}_\tau}(\Phi, R)$ we have

$$\|U\|_{\mathcal{Z}_\tau} \leq \|U - \Phi\|_{\mathcal{Z}_\tau} + \|\Phi\|_{\mathcal{Z}_\tau} \leq 2R. \tag{2.115}$$

Suppose that $\tau, R > 0$ and ϕ satisfy

(2.116)
$$\max\left\{\nu^{-1}C_{2.15}(\tau)(C_\phi(\tau) + R), D_n^{-1}C_{2.18}(\epsilon, \tau)R(1 + g_\chi(2R)), D_c^{-1}C_{2.21}(\tau)(R + g_f(2R))\right\}$$
(2.117)
$$\leq \frac{1}{48},$$

where $C_{2.15}(\tau), C_{2.18}(\epsilon, \tau), C_{2.21}(\tau)$ are given by (2.71),(2.82),(2.103), respectively. Then by Corollaries 2.16, 2.19, and 2.22, we have

$$\|\mathbf{W}_1\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} \leq \frac{1}{24}\|U\|_{\mathcal{Y}_\tau}, \quad \|W_2\|_{\mathcal{Z}_{2,\tau}} \leq \frac{1}{24}\|U\|_{\mathcal{Y}_\tau}, \quad \|W_3\|_{\mathcal{Z}_{3,\tau}} \leq \frac{1}{24}\|U\|_{\mathcal{Y}_\tau}, \tag{2.118}$$

It follows from (2.115) and (2.118) that

$$\begin{aligned}
\|\mathcal{S}(U) - \Phi\|_{\mathcal{Z}_\tau} &= \|\boldsymbol{\mathcal{S}}_1(\mathbf{u}, n, c) - \boldsymbol{\Phi}_1\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} + \|\mathcal{S}_2(\mathbf{u}, n, c) - \Phi_2\|_{\mathcal{Z}_{2,\tau}} + \|\mathcal{S}_3(\mathbf{u}, n, c) - \Phi_3\|_{\mathcal{Z}_{3,\tau}} \\
&\leq \|\mathbf{W}_1\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} + \|W_2\|_{\mathcal{Z}_{2,\tau}} + \|W_3\|_{\mathcal{Z}_{3,\tau}} \\
&\leq \frac{1}{4}R, \qquad (2.119)
\end{aligned}$$

as claimed.

2.5.2. *$\mathcal{S}$ is a Contraction.* Let $U_1 = (\mathbf{u}_1, n_1, c_1), U_2 = (\mathbf{u}_2, n_2, c_2) \in B_{\mathcal{Z}_\tau}(\Phi, R)$. In addition to (2.116), suppose that ϕ and $\tau, R > 0$ satisfy

$$D_n^{-1}\widetilde{C}_{2.18}(\epsilon, \tau)R(1 + g'_\chi(2R)R) \leq \frac{1}{96}, \quad D_c^{-1}\widetilde{C}_{2.21}(\tau)(R + g'_f(2R)R) \leq \frac{1}{48}, \tag{2.120}$$

$\widetilde{C}_{2.18}(\tau), \widetilde{C}_{2.21}(\tau)$ are given by (2.90), (2.113), respectively. Observe that

$$\begin{aligned}
\|\mathcal{S}(U_1) - \mathcal{S}(U_2)\|_{\mathcal{Z}_\tau} = \|W_1(U_1) - W_1(U_2)\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} + \\
\|W_2(U_1) - W_2(U_2)\|_{\mathcal{Z}_{2,\tau}} + \|W_3(U_1) - W_3(U_2)\|_{\mathcal{Z}_{3,\tau}}.
\end{aligned} \tag{2.121}$$

Since (2.120) holds, we see from Lemma 2.15 and (2.118) that

$$\begin{aligned}
&\|W_1(U_1) - W_1(U_2)\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} \quad (2.122)\\
&\le \|\mathbf{G}(\hat{n}_1 - \hat{n}_2)\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} + \|W_1^{(2)}(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_1) - W_1^{(2)}(\hat{\mathbf{u}}_2, \hat{\mathbf{u}}_2)\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}}\\
&\le \frac{1}{24}\|n_1 - n_2\|_{\mathcal{Y}_{2,\tau}} + \|W_1^{(2)}(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_2)\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}} + \|W_1^{(2)}(\hat{\mathbf{u}}_2, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\|_{\boldsymbol{\mathcal{Z}}_{1,\tau}}\\
&\le \frac{1}{24}\|n_1 - n_2\|_{\mathcal{Y}_{2,\tau}} + \frac{1}{12}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{Y}_{1,\tau}}\\
&\le \frac{1}{8}\|U_1 - U_2\|_{\mathcal{Y}_\tau}. \quad (2.123)
\end{aligned}$$

Similarly, (2.120), Lemmas 2.18 and 2.20, and (2.118) imply

$$\begin{aligned}
&\|W_2(U_1) - W_2(U_2)\|_{\mathcal{Z}_{2,\tau}}\\
&\le \|W_2^{(1)}(\hat{\mathbf{u}}_1, \hat{n}_1) - W_2^{(1)}(\hat{\mathbf{u}}_2, \hat{n}_2)\|_{\mathcal{Z}_{2,\tau}} \quad (2.124)\\
&+ \|W_2^{(2)}(\hat{n}_1, \widehat{\chi(c_1)}, \hat{c}_1) - W_2^{(2)}(\hat{n}_2, \widehat{\chi(c_2)}, \hat{c}_2)\|_{\mathcal{Z}_{2,\tau}}\\
&\le \|W_2^{(1)}(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{n}_1)\|_{\mathcal{Z}_{2,\tau}} + \|W_2^{(1)}(\hat{\mathbf{u}}_2, \hat{n}_1 - \hat{n}_2)\|_{\mathcal{Z}_{2,\tau}}\\
&+ \|W_2^{(2)}(\hat{n}_1 - \hat{n}_2, \widehat{\chi(c_1)}, \hat{c}_1)\|_{\mathcal{Z}_{2,\tau}} + \|W_2^{(2)}(\hat{n}_2, \widehat{\chi(c_1)} - \widehat{\chi(c_2)}, \hat{c}_1)\|_{\mathcal{Z}_{2,\tau}} \quad (2.125)\\
&+ \|W_2^{(2)}(\hat{n}_2, \widehat{\chi(c_2)}, \hat{c}_1 - \hat{c}_2)\|_{\mathcal{Z}_{2,\tau}}\\
&\le \frac{1}{24}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{Y}_{1,\tau}} + \frac{1}{24}\|n_1 - n_2\|_{\mathcal{Y}_{2,\tau}} + \frac{1}{24}\|n_1 - n_2\|_{\mathcal{Y}_{2,\tau}} + \frac{1}{12}\|c_1 - c_2\|_{\mathcal{Y}_{3,\tau}}\\
&\le \frac{5}{24}\|U_1 - U_2\|_{\mathcal{Y}_\tau}. \quad (2.126)
\end{aligned}$$

Finally, (2.120), Lemmas 2.21 and 2.23, and (2.118) imply

$$\begin{aligned}
&\|W_3(U_1) - W_3(U_2)\|_{\mathcal{Z}_{3,\tau}} \quad (2.127)\\
&\le \|W_3^{(1)}(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{c}_1)\|_{\mathcal{Z}_{3,\tau}} + \|W_3^{(1)}(\hat{\mathbf{u}}_2, \hat{c}_1 - \hat{c}_2)\|_{\mathcal{Z}_{3,\tau}}\\
&\quad + \|W_3^{(2)}(\hat{n}_1 - \hat{n}_2, \widehat{f(c_1)})\|_{\mathcal{Z}_{3,\tau}} + \|W_3^{(2)}(\hat{n}_2, \widehat{f(c_1)} - \widehat{f(c_2)})\|_{\mathcal{Z}_{3,T}}\\
&\le \frac{1}{24}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{Y}_{1,\tau}} + \frac{1}{24}\|n_1 - n_2\|_{\mathcal{Y}_{2,\tau}} + \frac{1}{12}\|c_1 - c_2\|_{\mathcal{Y}_{3,\tau}}\\
&\le \frac{1}{6}\|U_1 - U_2\|_{\mathcal{Y}_\tau}. \quad (2.128)
\end{aligned}$$

Upon combining (2.122)-(2.127), we may therefore conclude that

$$\|\mathcal{S}(U_1) - \mathcal{S}(U_2)\|_{\mathcal{Z}_\tau} \le \frac{1}{2}\|U_1 - U_2\|_{\mathcal{Z}_T}, \quad (2.129)$$

so that $\mathcal{S}$ is a contraction, as desired.

This proves that there exists a unique $U \in B_{\mathcal{Z}_\tau}(\Phi, R_0)$, for some $R_0 := R_0(U_0)$, that satisfies (1.1) as a mild solution, provided that $U_0 \in \mathcal{V}_{(\sigma,\sigma,\epsilon)}$. Moreover, the solution is Gevrey regular by definition of $\mathcal{Z}_T$. The fact that it is weak solution also follows from Propositions 2.7, 2.9, and 2.11, which ensure that the mild solution formulation can be differentiated in time. To avoid repetition of argument, the reader is referred to [3] for additional details. This completes the proof of Theorem 1.1.

Remark 2.2. *We recall that when* $\epsilon = 0$, *then* $C_{2.18}(\epsilon, \tau) \lesssim 1$. *Thus, even when* τ *is sufficiently small, (2.116) and (2.120) can be satisfied only when* $R > 0$ *is chosen sufficiently small. In particular, when* $\epsilon = 0$, *we are unable to obtain a "small-data/large-time" type result.*

From another point of view, in light of Remark 2.1, by replacing the constants

$$C_{2.18}(\epsilon, \tau), \widetilde{C}_{2.18}(\epsilon, \tau)$$

with (2.99), (2.101), and changing (2.116) and (2.120) accordingly, then we may repeat the above argument to produce the analog to Theorem 1.1 in the case where the "cubic" nonlinearity in (1.1) is given by (2.91), and the spaces $\mathcal{X}_\tau, \mathcal{Y}_\tau$ are redefined according to (2.92). This theorem is stated below for convenience.

THEOREM 2.24. *Let $0 < \alpha < 1$. Suppose that χ, f are real analytic on $\mathbb{R}$ with majorants g_χ, g_f, respectively, and that ϕ is real analytic on Ω. Let $\sigma \in (-1, 0]$ and $(\mathbf{u}_0, n_0, c_0) \in \boldsymbol{\mathcal{V}}_\sigma \times \mathcal{V}_\sigma \times \mathcal{W}$. Then there exist $R, T > 0$ sufficiently small such that if*

$$\|(\mathbf{u}_0, n_0, c_0)\|_{\mathcal{Z}_T} \leq CR, \tag{2.130}$$

then there is a unique solution $(\mathbf{u}, n, c) \in \mathcal{Z}_T$ of $NSKS_\alpha$, corresponding to $(\mathbf{u}_0, n_0, c_0)$, where $NSKS_\alpha$ is given by

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} - n \nabla \phi, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t n + \mathbf{u} \cdot \nabla n = D_n \Delta n - \nabla \cdot (n \chi(c) |\nabla|^\alpha c), \\ \partial_t c + \mathbf{u} \cdot \nabla c = D_c \Delta c - n f(c). \end{cases} \tag{$NSKS_\alpha$}$$

□

3. Long-time Behavior of (1.5)

In this section, we prove Theorem 1.2. Let us consider the following initial-boundary value problem:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \tilde{P} = \nu \Delta \mathbf{u} - \tilde{n} \mathbf{e}_2, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \tilde{n} + \mathbf{u} \cdot \nabla \tilde{n} = \alpha \mathbf{u} \cdot \mathbf{e}_2 + D_n \Delta \tilde{n}; \\ (\mathbf{u}, \tilde{n})(\cdot, 0) = (\mathbf{u}_0, n_0 + \alpha y - \bar{n})(\cdot) \in H^2(\Omega), \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \tilde{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{3.131}$$

which is obtained from (1.5) by letting $n = \tilde{n} - \alpha y + \bar{n}$ and $P = \tilde{P} + \frac{\alpha}{2} y^2 - \bar{n} y$. We begin with the estimate of the zero frequency part of the solution. We point out that although the energy estimates performed below are formal, they can be made rigorous by standard arguments, e.g., studying Galerkin approximations of (3.131) to establish existence of strong solutions in the class $H^2(\Omega)$.

3.1. L^2-Estimate. By taking the L^2-inner products of the first equation of (3.131) with $\alpha \mathbf{u}$ and the third equation with $\tilde{n}$, and using the boundary conditions, we have

$$\frac{1}{2} \frac{d}{dt} \left(\alpha \|\mathbf{u}\|^2 + \|\tilde{n}\|^2 \right) + \alpha \nu \|\nabla \mathbf{u}\|^2 + D_n \|\nabla \tilde{n}\|^2 = 0, \tag{3.132}$$

which will later be coupled with different levels of energy estimates to generate the exponential decay of the solution. Next, we move on to the estimate of the first order derivatives of the solution.

3.2. H^1-Estimate. Taking the L^2-inner product of the third equation of (3.131) with $-\Delta \tilde{n}$ and using the boundary conditions, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{n}\|^2 + D_n \|\Delta \tilde{n}\|^2 = \int_\Omega (\mathbf{u} \cdot \nabla \tilde{n})(\Delta \tilde{n}) d\mathbf{x} - \alpha \int_\Omega (\mathbf{u} \cdot \mathbf{e}_2)(\Delta \tilde{n}) d\mathbf{x}. \tag{3.133}$$

For the first term on the RHS of (3.133), by using the Hölder, Gagliardo-Nirenberg and Young's inequalities, we have

$$\begin{aligned}\left|\int_\Omega (\mathbf{u}\cdot\nabla\tilde{n})(\Delta\tilde{n})d\mathbf{x}\right| &\le \|\mathbf{u}\|_{L^4}\|\nabla\tilde{n}\|_{L^4}\|\Delta\tilde{n}\| \\ &\le C\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\nabla\tilde{n}\|^{\frac{1}{2}}\|\Delta\tilde{n}\|^{\frac{3}{2}} \\ &\le C\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\nabla\tilde{n}\|^2 + \frac{D_n}{4}\|\Delta\tilde{n}\|^2.\end{aligned} \tag{3.134}$$

For the second term on the RHS of (3.133), we apply the Cauchy-Schwarz inequality to get

$$\left|-\alpha\int_\Omega (\mathbf{u}\cdot\mathbf{e}_2)(\Delta\tilde{n})dx\right| \le C\|\mathbf{u}\|^2 + \frac{D_n}{4}\|\Delta\tilde{n}\|^2. \tag{3.135}$$

Taking (3.134) and (3.135) back to (3.133), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{n}\|^2 + \frac{D_n}{2}\|\Delta\tilde{n}\|^2 \le C\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\nabla\tilde{n}\|^2 + C\|\mathbf{u}\|^2. \tag{3.136}$$

After applying the uniform estimate of $\|\mathbf{u}\|^2$ obtained from Section 3.1, and applying the Poincaré's inequality to the last term on the RHS of (3.136), we update (3.136) as

$$\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{n}\|^2 + \frac{D_n}{2}\|\Delta\tilde{n}\|^2 \le C\|\nabla\mathbf{u}\|^2\|\nabla\tilde{n}\|^2 + C\|\nabla\mathbf{u}\|^2. \tag{3.137}$$

Applying the Gronwall's inequality to (3.137), and using the uniform temporal integrability of $\|\nabla\mathbf{u}\|^2$ obtained from Section 3.1, we then have

$$\|\nabla\tilde{n}(t)\|^2 \le C \tag{3.138}$$

for some time-independent constant. Hence, we update (3.137) as

$$\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{n}\|^2 + \frac{D_n}{2}\|\Delta\tilde{n}\|^2 \le C\|\nabla\mathbf{u}\|^2 \tag{3.139}$$

for some time-independent constant.

Next, taking the L^2-inner product of the first equation of (3.131) with $\partial_t\mathbf{u}$ and using the boundary conditions, we find

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 + \|\partial_t\mathbf{u}\|^2 = -\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot(\partial_t\mathbf{u})d\mathbf{x} - \int_\Omega \tilde{n}\mathbf{e}_2\cdot(\partial_t\mathbf{u})dx, \tag{3.140}$$

where the first term on the RHS can be estimated as

$$\begin{aligned}\left|-\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot(\partial_t\mathbf{u})d\mathbf{x}\right| &\le \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\partial_t\mathbf{u}\| \\ &\le C\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|^{\frac{1}{2}}\|\partial_t\mathbf{u}\| \\ &\le \frac{1}{8}\|\partial_t\mathbf{u}\|^2 + C\|\mathbf{u}\|\|\nabla\mathbf{u}\|^2\|\Delta\mathbf{u}\| \\ &\le \frac{1}{8}\|\partial_t\mathbf{u}\|^2 + C\|\mathbf{u}\|\|\nabla\mathbf{u}\|^2\left(\|\partial_t\mathbf{u}\| + \|\tilde{n}\| + \|\mathbf{u}\|\|\nabla\mathbf{u}\|^2\right) \\ &\le \frac{1}{4}\|\partial_t\mathbf{u}\|^2 + C\left(\|\nabla\mathbf{u}\|^4 + \|\nabla\mathbf{u}\|^2\right),\end{aligned} \tag{3.141}$$

where we have invoked the classic results on the Stokes equation and the uniform estimates of $\|\mathbf{u}\|$ and $\|\tilde{n}\|$ obtained from Section 3.1. In addition, for the second term on the RHS of (3.140), we have

$$\begin{aligned}\left|\int_\Omega \tilde{n}\mathbf{e}_2\cdot(\partial_t\mathbf{u})d\mathbf{x}\right| &\le \frac{1}{4}\|\partial_t\mathbf{u}\|^2 + C\|\tilde{n}\|^2 \\ &\le \frac{1}{4}\|\partial_t\mathbf{u}\|^2 + C\|\nabla\tilde{n}\|^2.\end{aligned}$$

Hence, we update (3.140) as

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2+\frac{1}{2}\|\partial_t\mathbf{u}\|^2\le C\left(\|\nabla\mathbf{u}\|^4+\|\nabla\mathbf{u}\|^2\right)+C\|\nabla\tilde{n}\|^2. \tag{3.142}$$

Applying the Gronwall's inequality to (3.142) and applying the uniform-in-time integrability of both $\|\nabla\mathbf{u}\|^2$ and $\|\nabla\tilde{n}\|^2$ obtained from Section 3.1, we find

$$\|\nabla\mathbf{u}\|^2\le C \tag{3.143}$$

for some time-independent constant. So we update (3.142) as

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2+\frac{1}{2}\|\partial_t\mathbf{u}\|^2\le C\left(\|\nabla\mathbf{u}\|^2+\|\nabla\tilde{n}\|^2\right) \tag{3.144}$$

for some time-independent constant. Next, we estimate the second order spatial derivatives of the solutions.

3.3. H^2 **Estimate.** In order to build up higher order spatial regularity of the solution, we now turn to the estimate of the temporal derivatives, and then invoke the classic theory on elliptic regularity. Taking the temporal derivatives of the first and third equations of (3.131), we have

$$\begin{cases}\partial_{tt}\mathbf{u}+\nabla\cdot(\partial_t\mathbf{u}\otimes\mathbf{u})+\mathbf{u}\cdot\nabla\partial_t\mathbf{u}+\nabla\partial_t\tilde{P}=\nu\Delta\partial_t\mathbf{u}-\partial_t\tilde{n}\mathbf{e}_2,\\ \partial_{tt}\tilde{n}+\nabla\cdot(\tilde{n}\partial_t\mathbf{u})+\mathbf{u}\cdot\nabla\partial_t\tilde{n}=\alpha\partial_t\mathbf{u}\cdot\mathbf{e}_2+D_n\Delta\partial_t\tilde{n}.\end{cases} \tag{3.145}$$

Taking the L^2 inner products of the two equations of (3.145) with $\alpha\partial_t\mathbf{u}$ and $\partial_t\tilde{n}$, respectively, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\alpha\|\partial_t\mathbf{u}\|^2+\|\partial_t\tilde{n}\|^2\right)+\alpha\nu\|\nabla\partial_t\mathbf{u}\|^2+D_n\|\nabla\partial_t\tilde{n}\|^2\\ &=-\alpha\int_\Omega[\nabla\cdot(\partial_t\mathbf{u}\otimes\mathbf{u})]\cdot\partial_t\mathbf{u}\,d\mathbf{x}-\int_\Omega[\nabla\cdot(\tilde{n}\partial_t\mathbf{u})]\,\partial_t\tilde{n}\,d\mathbf{x}\\ &=\alpha\int_\Omega\mathbf{u}\cdot[\partial_t\mathbf{u}\cdot\nabla\partial_t\mathbf{u}]\,d\mathbf{x}+\int_\Omega\tilde{n}\left(\partial_t\mathbf{u}\cdot\nabla\partial_t\tilde{n}\right)d\mathbf{x}\\ &\le\frac{\alpha\nu}{4}\|\nabla\partial_t\mathbf{u}\|^2+\frac{D_n}{2}\|\nabla\partial_t\tilde{n}\|^2+C\left(\|\mathbf{u}\|_{L^4}^2+\|\tilde{n}\|_{L^4}^2\right)\|\partial_t\mathbf{u}\|_{L^4}^2.\end{aligned} \tag{3.146}$$

We note that, by the Gagliardo-Nirenberg interpolation inequality,

$$\begin{aligned}C\left(\|\mathbf{u}\|_{L^4}^2+\|\tilde{n}\|_{L^4}^2\right)\|\partial_t\mathbf{u}\|_{L^4}^2&\le C\left(\|\mathbf{u}\|\|\nabla\mathbf{u}\|+\|\tilde{n}\|\|\nabla\tilde{n}\|\right)\|\partial_t\mathbf{u}\|\|\nabla\partial_t\mathbf{u}\|\\ &\le C\|\partial_t\mathbf{u}\|\|\nabla\partial_t\mathbf{u}\|\\ &\le\frac{\alpha\nu}{4}\|\nabla\partial_t\mathbf{u}\|^2+C\|\partial_t\mathbf{u}\|^2.\end{aligned} \tag{3.147}$$

Using (3.147) we update (3.146) as

$$\frac{1}{2}\frac{d}{dt}\left(\alpha\|\partial_t\mathbf{u}\|^2+\|\partial_t\tilde{n}\|^2\right)+\frac{\alpha\nu}{2}\|\nabla\partial_t\mathbf{u}\|^2+\frac{D_n}{2}\|\nabla\partial_t\tilde{n}\|^2\le C\|\partial_t\mathbf{u}\|^2. \tag{3.148}$$

3.4. Coupling and Decay. So far, we have established the following energy estimates:

$$\begin{aligned}&(3.132):\quad\frac{1}{2}\frac{d}{dt}\left(\alpha\|\mathbf{u}\|^2+\|\tilde{n}\|^2\right)+\alpha\nu\|\nabla\mathbf{u}\|^2+D_n\|\nabla\tilde{n}\|^2=0,\\ &(3.139):\quad\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{n}\|^2+\frac{D_n}{2}\|\Delta\tilde{n}\|^2\le C\|\nabla\mathbf{u}\|^2,\\ &(3.144):\quad\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2+\frac{1}{2}\|\partial_t\mathbf{u}\|^2\le C\left(\|\nabla\mathbf{u}\|^2+\|\nabla\tilde{n}\|^2\right),\\ &(3.148):\quad\frac{1}{2}\frac{d}{dt}\left(\alpha\|\partial_t\mathbf{u}\|^2+\|\partial_t\tilde{n}\|^2\right)+\frac{\alpha\nu}{2}\|\nabla\partial_t\mathbf{u}\|^2+\frac{D_n}{2}\|\nabla\partial_t\tilde{n}\|^2\le C\|\partial_t\mathbf{u}\|^2.\end{aligned}$$

It is easy to see that the RHS of (3.148) can be absorbed by the LHS of a suitably large constant multiple of (3.144). Then the RHS of the resulting estimate, together with the RHS of (3.139) can be absorbed by the LHS of a suitable large constant multiple of (3.132). The final resulting estimate then reads

$$\frac{d}{dt}K_1(t) + D_1(t) \leq 0, \tag{3.149}$$

where

$$K_1(t) \cong \|\mathbf{u}\|^2 + \|\tilde{n}\|^2 + \|\nabla\mathbf{u}\|^2 + \|\nabla\tilde{n}\|^2 + \|\partial_t\mathbf{u}\|^2 + \|\partial_t\tilde{n}\|^2,$$
$$D_1(t) \cong \|\nabla\mathbf{u}\|^2 + \|\nabla\tilde{n}\|^2 + \|\partial_t\mathbf{u}\|^2 + \|\Delta\tilde{n}\|^2 + \|\nabla\partial_t\mathbf{u}\|^2 + \|\nabla\partial_t\tilde{n}\|^2.$$

By Poincaré's inequality, it is then easy to derive the exponential decay of $K_1(t)$, which yields the exponential decay of the H^2 norm of the perturbation, due to the classic Stokes estimates and elliptic regularity. This completes the proof of Theorem 1.2.

□

4. Long-time Behavior of (1.6)

In this section, we prove Theorem 1.3. We note that the perturbed system associated with (1.6) around the naturally stabilizing equilibrium reads

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\tilde{P} = -\tilde{n}\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\tilde{n} + \mathbf{u}\cdot\nabla\tilde{n} = \alpha\mathbf{u}\cdot\mathbf{e}_2 + D_n\Delta\tilde{n}; \\ (\mathbf{u},\tilde{n})(\mathbf{x},0) = (\mathbf{u}_0, n_0 - \bar{n} + \alpha y)(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega}; \\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \ \tilde{n}|_{\partial\Omega} = 0, \end{cases} \tag{4.150}$$

where $\tilde{P} = P - \frac{\alpha}{2}y^2 + \bar{n}y$ and $\tilde{n} = n - \bar{n} + \alpha y$. As in Section 3, the estimates we perform below are formal, but can be made rigorous by standard arguments.

By taking the L^2 inner products of the first equation of (4.150) with $\alpha\mathbf{u}$ and the third one with $\tilde{n}$, then adding the results, we have

$$\frac{1}{2}\frac{d}{dt}\left(\alpha\|\mathbf{u}\|^2 + \|\tilde{n}\|^2\right) + D_n\|\nabla\tilde{n}\|^2 = 0,$$

which yields

$$\alpha\|\mathbf{u}(t)\|^2 + \|\tilde{n}(t)\|^2 + 2D_n\int_0^t\|\nabla\tilde{n}(\tau)\|^2 = \alpha\|\mathbf{u}_0\|^2 + \|\tilde{n}_0\|^2, \qquad \forall\, t>0. \tag{4.151}$$

Since $\tilde{n}|_{\partial\Omega} = 0$, applying the Poincaré's inequality to (4.151), we have

$$\int_0^t \|\tilde{n}(\tau)\|_{H^1}^2 d\tau \leq C, \qquad \forall\, t>0, \tag{4.152}$$

for some constant independent of time. If we define $G_1(t) \equiv \|\tilde{n}(t)\|^2$ for $t>0$, then according to (4.151) and (4.152), we see that

$$\int_0^t [G_1(\tau)]^2\, d\tau \leq C, \qquad \forall\, t>0, \tag{4.153}$$

for some constant independent of time.

Note that, by taking the L^2 inner product of the third equation of (4.150) with $\tilde{n}$, we have

$$\frac{d}{dt}\|\tilde{n}\|^2 = 2\alpha\int_\Omega(\mathbf{u}\cdot\mathbf{e}_2)\tilde{n}\ d\mathbf{x} - 2D_n\|\nabla\tilde{n}\|^2. \tag{4.154}$$

We note that the first term on the RHS of (4.154) is not uniformly integrable with respect to time. Hence, it is not possible to conclude $\|\tilde{n}(t)\|^2 \in W^{1,1}(0,\infty)$. However, we observe that

(4.155)
$$\begin{aligned}\int_0^t \left|\frac{d}{d\tau}[G_1(\tau)]^2\right| d\tau &= 4\int_0^t \|\tilde{n}(\tau)\|^2 \left|\alpha\int_\Omega (\mathbf{u}\cdot\mathbf{e}_2)\tilde{n}\, d\mathbf{x} - D_n\|\nabla\tilde{n}\|^2\right| d\tau \\ &\le 4\alpha\int_0^t \|\tilde{n}(\tau)\|^2\|\mathbf{u}(\tau)\|\|\tilde{n}(\tau)\|d\tau + 4D_n\int_0^t \|\tilde{n}(\tau)\|^2\|\nabla\tilde{n}(\tau)\|^2 d\tau \\ &\le C\int_0^t \|\nabla\tilde{n}(\tau)\|^2 d\tau,\end{aligned}$$

where we have used the Poincaré's inequality and the uniform estimates of $\|\mathbf{u}(t)\|$ and $\|\tilde{n}(t)\|$ obtained from (4.151). Here, the constant C is independent of time. Then, according to (4.151) and (4.155), we see that

$$\int_0^t \left|\frac{d}{d\tau}[G_1(\tau)]^2\right| d\tau \le C, \qquad \forall\, t>0, \tag{4.156}$$

for some constant independent of time. From (4.153) and (4.156) we see that

$$\|\tilde{n}(t)\|^4 = [G_1(t)]^2 \in W^{1,1}(0,\infty).$$

This dictates that $\lim_{t\to\infty}\|\tilde{n}(t)\|^4 = 0$, and hence $\|n(t) - \bar{n} + \alpha y\| \to 0$ as $t\to\infty$. Therefore the naturally stabilizing profile, $\bar{n} - \alpha y$, for the bacteria density function is globally asymptotically stable, even in the absence of viscous dissipation. We further remark that the global stability result is indeed far from being trivial. This is because the third equation of (4.150) contains the second component of the velocity field, while there is no indication from the first equation that the velocity field is decaying. Finally, we note that higher order regularity of solutions to (4.150) can be established by slightly modifying the proofs constructed in [**18, 29**] for the 2D inviscid Boussinesq equations with linear or nonlinear thermal diffusivity. The technical details are omitted in this paper.

□

5. Conclusion

In this paper, we studied the qualitative behavior of the coupled chemotaxis-fluid model, (1.1), on bounded domains in $\mathbb{R}^d$ subject to various boundary conditions. First, by adopting approaches from [**3, 4**], we showed that in any space dimension, that starting with potentially rough initial data, solutions to (1.1) on periodic domains become instantaneously spatially analytic, and the spatial analyticity radius grows like $\sqrt{t}$ at least up to some time T. Since real analyticity can be characterized by exponential decay in the Fourier spectrum with rate proportional to the analyticity radius, this result can be used to identify the number of degrees of freedom in the system. Indeed, the reciprocal of the analyticity radius at time t indicates a wave-number beyond which the remaining ones experience exponential decay. Thus, from the point of view of numerical computations, wave-numbers beyond this cut-off are irrelevant. Theorem 1.1 appears to be the first such result in the field for (1.1). In addition, since the coupled system consists of the parabolic-parabolic type Keller-Segel model and the incompressible Navier-Stokes and Boussinesq equations, our result shows that this instantaneous smoothing effect is also enjoyed by these classic models in chemotaxis and fluid dynamics.

In the second part of the paper, we studied the long-time asymptotic behavior of large-amplitude classical solutions to the model on a bounded domain in two space dimensions with physical boundary. In particular, we considered a scenario in which the chemotactic movement is inactive. In this case, the model reduces to the standard 2D incompressible Boussinesq equations for buoyancy driven fluid flows. Inspired by the numerical studies conducted in [**5**], we prescribed a naturally stabilizing (non-homogeneous) boundary condition for the bacteria density function to corroborate the numerical evidence observed therein. Under such setting, it is shown that for the fully dissipative system (i.e., when $\nu > 0, \kappa > 0$), the naturally stabilizing profile is globally asymptotically stable regardless of the magnitude of initial data. The result is proved by using standard L^2-based energy method. On the other hand, by developing a novel approach, we showed that even in the absence of viscous dissipation, the naturally stabilizing profile is still globally asymptotically stable. Our results show consistency with the numerical results obtained in [5] and appear to be the first ones describing the long-time behavior of large-amplitude classical solutions to the 2D Boussinesq equations on bounded domains with physical boundaries subject to non-homogeneous boundary conditions. In particular, the second result concerning the inviscid model is highly trivial, due to the appearance of the second component of the velocity field in the perturbed equation for the bacteria density function, and the non-decaying of the velocity field. Moreover, the analytical approach developed for proving the last result can be of independent interests, and can be adopted to study the long-time behavior of other partially dissipative PDE systems.

However, we would like to mention that many fundamental questions, concerning the qualitative behavior of the coupled model, still remain open at the present time. Here we just mention a few:

- The global well-posedness (GWP) of classical solutions under biologically relevant structural conditions on the chemotactic sensitivity and oxygen consumption functions is still unknown even in two space dimensions (of course it is too ambitious to pursue the GWP in three space dimensions since the model contains the 3D incompressible Navier-Stokes equations whose GWP still remains open as one of the Millennium Prize Problems). We shall provide definite answers to such a question in a forthcoming paper.
- From Theorem 1.1 we see that the instantaneous smoothing effect holds for large time when the initial data is sufficiently small and for short time when the initial data is large, provided that the initial data belongs to a suitably strong regularity class. However, if the initial oxygen concentration belongs only to $\mathcal{W}$, i.e., the space of functions whose Fourier series converge absolutely, then our methods are unable to deduce a short-time/large-data result. Hence, it is interesting to ask whether the coupled model may still enjoy this type of result in that setting. Indeed, this apparent obstruction to the well-posedness theory of (1.1) presents itself as a type of critical phenomena. We leave investigation of this criticality for the future.
- In [5], the authors numerically produced plumes aligned with the vertical direction, which showed consistency with the experimental observation reported in [10]. Nevertheless, a rigorous mathematical explanation of such a phenomenon is still missing in the literature, which corresponds to the study of steady state solutions of the model subject to appropriate boundary conditions. We suspect that the underlying mechanisms for the emerging of plumes are gravity and the interaction between hydrodynamic effect and chemotactic movement, resulting in bifurcation or spectral instability. We also leave the detailed study for the future.

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