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# Stability and uniqueness of traveling waves of a nonlocal dispersal SIR epidemic model

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Abstract. This paper is mainly concerned with the exponential stability and uniqueness of traveling waves of a delayed nonlocal dispersal SIR epidemic model. We first prove the stability of traveling waves by using the weighted energy method, where the traveling waves are allowed to be non-monotone. Next we establish the exact asymptotic behavior of traveling waves at $-\infty$ by using Ikehara's theorem. Then the uniqueness of traveling waves is obtained by the stability result. Finally, we discuss how the nonlocal dispersal affects the stability of traveling waves. The conclusion shows that the nonlocal dispersal slows down the convergence rate of the solution to the traveling waves.

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## 1. Introduction and Main Results

Nowadays, lots of emerging infectious diseases spread more and more quickly worldwide since globalization has made travels more convenient. It is not enough any more to use reaction diffusion equations to model the long range transmission of these diseases, and nonlocal dispersal is better to describe the transmission process,

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see [**42, 43**]. In this paper, we consider the following delayed nonlocal dispersal SIR epidemic model [**21**]

$$(1.1)\quad \begin{cases} \frac{\partial u_1(t,x)}{\partial t} = d_1[((J*u_1)-u_1)(t,x)] + B - \sigma u_1(t,x) - \frac{\beta u_1(t,x)u_2(t-\tau,x)}{1+\alpha u_2(t-\tau,x)}, \\ \frac{\partial u_2(t,x)}{\partial t} = d_2[((J*u_2)-u_2)(t,x)] + \frac{\beta u_1(t,x)u_2(t-\tau,x)}{1+\alpha u_2(t-\tau,x)} - (\mu+\gamma)u_2(t,x), \\ \frac{\partial u_3(t,x)}{\partial t} = d_3[((J*u_3)-u_3)(t,x)] + \gamma u_2(t,x) - \mu_1 u_3(t,x), x\in\mathbb{R}, t>0, \end{cases}$$

with the initial data

$$(1.2)\quad \begin{cases} u_1(0,x) = u_{10}(x),\ u_3(0,x) = u_{30}(x),\ x\in\mathbb{R}, \\ u_2(s,x) = u_{20}(s,x),\ s\in[-\tau,0],\ x\in\mathbb{R}. \end{cases}$$

Here, $u_1$, $u_2$, $u_3$ denote the susceptible class, infective class and removed class, respectively. $J(r)$ denotes the probability distribution of rates of dispersal over distance $r$ and $J*u_i - u_i$, $i=1,2,3$ can be interpreted as the net rate of increase due to dispersal of class $u_i$ (see [**14**]), where $J*u_i$ is the standard convolution with space invariable. $B$, $\alpha$, $\beta$, $\gamma$, $\sigma$, $\mu$, $\mu_1$ are all positive parameters (see e.g. [**21**] for detailed interpretation).

Taking $\tilde{u}_1(x,t) = \frac{\sigma}{B}u_1(x,\frac{t}{d_2})$, $\tilde{u}_2(x,t) = \frac{\sigma}{B}u_2(x,\frac{t}{d_2})$, $\tilde{u}_3(x,t) = \frac{\sigma}{B}u_3(x,\frac{t}{d_2})$, and letting $\tilde{d} = \frac{d_1}{d_2}$, $\tilde{\sigma} = \frac{\sigma}{d_2}$, $\tilde{\beta} = \frac{\beta B}{\sigma}$, $\tilde{\alpha} = \frac{\alpha B}{\sigma}$, $\tilde{\mu} = \frac{\mu}{d_2}$, $\tilde{\gamma} = \frac{\gamma}{d_2}$, $\tilde{d_3} = \frac{d_3}{d_2}$, $\tilde{\mu}_1 = \frac{\mu_1}{d_2}$, and dropping the tilde for convenience, system (1.1) reduces to

$$(1.3)\quad \begin{cases} \frac{\partial u_1(t,x)}{\partial t} = d[((J*u_1)-u_1)(t,x)] + \sigma - \sigma u_1(t,x) - \frac{\beta u_1(t,x)u_2(t-\tau,x)}{1+\alpha u_2(t-\tau,x)}, \\ \frac{\partial u_2(t,x)}{\partial t} = ((J*u_2)-u_2)(t,x) + \frac{\beta u_1(t,x)u_2(t-\tau,x)}{1+\alpha u_2(t-\tau,x)} - (\mu+\gamma)u_2(t,x), \\ \frac{\partial u_3(t,x)}{\partial t} = d_3[((J*u_3)-u_3)(t,x)] + \gamma u_2(t,x) - \mu_1 u_3(t,x). \end{cases}$$

Note that the corresponding reaction system of (1.3)

$$(1.4)\quad \begin{cases} \frac{du_1(t)}{dt} = \sigma - \sigma u_1(t) - \frac{\beta u_1(t)u_2(t-\tau)}{1+\alpha u_2(t-\tau)}, \\ \frac{du_2(t)}{dt} = \frac{\beta u_1(t)u_2(t-\tau)}{1+\alpha u_2(t-\tau)} - (\mu+\gamma)u_2(t), \\ \frac{du_3(t)}{dt} = \gamma u_2(t) - \mu_1 u_3(t) \end{cases}$$

always has a disease-free equilibrium $\mathbf{F} = (1,0,0)$. Furthermore, if $\beta > \mu+\gamma$, then (1.4) admits a unique endemic equilibrium $\mathbf{E} = (S^*, I^*, R^*)$, where

$$S^* = \frac{\alpha\sigma + (\mu+\gamma)}{\alpha\sigma+\beta}, \quad I^* = \frac{\sigma[\beta-(\mu+\gamma)]}{(\mu+\gamma)(\alpha\sigma+\beta)}, \quad R^* = \frac{\gamma}{\mu_1}I^*.$$

It is not difficult to see from (1.4) that $\mathbf{F}$ is unstable and $\mathbf{E}$ is stable.

The traveling wave solution of (1.3) connecting $\mathbf{F}$ and $\mathbf{E}$ is a special solution in the form of $(u_1,u_2,u_3)(x,t) = (\phi_1,\phi_2,\phi_3)(\xi)$ satisfying

$$(1.5)\quad \begin{cases} c\phi_1'(\xi) = d[(J*\phi_1)(\xi) - \phi_1(\xi)] + \sigma - \sigma\phi_1(\xi) - \frac{\beta\phi_1(\xi)\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}, \\ c\phi_2'(\xi) = (J*\phi_2)(\xi) - \phi_2(\xi) + \frac{\beta\phi_1(\xi)\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)} - (\mu+\gamma)\phi_2(\xi), \\ c\phi_3'(\xi) = d_3[(J*\phi_3)(\xi) - \phi_3(\xi)] + \gamma\phi_2(\xi) - \mu_1\phi_3(\xi), \\ (\phi_1,\phi_2,\phi_3)(-\infty) = (1,0,0), \quad (\phi_1,\phi_2,\phi_3)(+\infty) = (S^*, I^*, R^*), \end{cases}$$

where $\xi := x + ct$ and $c > 0$ is the wave speed.

Recently, under the following assumption **(J)**:

**(J):** $J \in C^1(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(x)\,dx = 1$ and $J$ is compactly supported.

Li et al. [**21**] obtained the existence and nonexistence of traveling waves of the subsystem

$$(1.6)\quad \begin{cases} \frac{\partial u_1(t,x)}{\partial t} = d[(J * u_1)(t,x) - u_1(t,x)] + \sigma - \sigma u_1(t,x) - \frac{\beta u_1(t,x)u_2(t-\tau,x)}{1+\alpha u_2(t-\tau,x)}, \\ \frac{\partial u_2(t,x)}{\partial t} = (J * u_2)(t,x) - u_2(t,x) + \frac{\beta u_1(t,x)u_2(t-\tau,x)}{1+\alpha u_2(t-\tau,x)} - (\mu+\gamma)u_2(t,x) \end{cases}$$

as follows:

PROPOSITION 1.1. (Existence). Suppose that $R_0 := \frac{\beta}{\mu+\gamma}$ and **(J)**. If the threshold value $R_0 > 1$, then there exists $c^* > 0$ such that for every $c > c^*$, the system (1.6) admits a traveling wave solution $(\phi_1(\xi), \phi_2(\xi))$ satisfying

$$\phi_1(-\infty) = 1, \quad \phi_1(+\infty) = S^*, \quad \phi_2(-\infty) = 0, \quad \phi_2(+\infty) = I^*,$$

and $\lim_{\xi\to-\infty} e^{-\lambda_1\xi}\phi_2(\xi) = 1$, where $\lambda_1$ is the smallest positive real root of the following equation

$$\int_{\mathbb{R}} J(y)\,[e^{-\lambda y} - 1]dy - c\lambda + \beta e^{-\lambda c\tau} - (\mu+\gamma) = 0.$$

If $R_0 < 1$ and $c \geq 0$; or $R_0 > 1$ and $c \in (0, c^*)$, then (1.6) admits no traveling wave solution. In particular, if $R_0 > \frac{\alpha\sigma+\beta}{\alpha\sigma}$, then (1.6) admits a traveling wave $(\phi_1(\xi), \phi_2(\xi))$ with wave speed $c = c^*$.

Since $u_3$ in (1.3) can be determined completely by $u_2$, Proposition 1.1 implies that the system (1.3) admits traveling waves connecting $\mathbf{F}$ and $\mathbf{E}$ for $R_0 > 1$ and $c > c^*$, while it has no traveling waves if $R_0 < 1$ and $c \geq 0$; or $R_0 > 1$ and $c \in (0, c^*)$. In particular, if $R_0 > \frac{\alpha\sigma+\beta}{\alpha\sigma}$, then (1.3) admits a traveling wave $(\phi_1, \phi_2, \phi_3)$ with wave speed $c = c^*$.

As is known, the stability of traveling waves for epidemic models is an interesting and important issue in the study of disease invasion since it has a strong influence on the dynamical behavior of the epidemic models. In the past decades, there have been extensively investigations on the stability of wavefronts for various diffusive equations, see e.g. [**5, 6, 15, 19, 25–32, 34–36, 40**] and references therein. One of the most effective methods to study the stability of monostable waves is the weighted energy method used by Mei and coauthors (see [**25–30**]) for the Nicholson's blowflies equations. Recently, Huang et al. [**19**] obtained the global stability of wavefronts of monotone monostable nonlocal dispersal equations by combining the weighted energy method with the comparison principle and the Fourier transform. Note that these stabilities in [**19, 25–30**] depend on the monotonicity of both the equations and the waves. However, under some conditions, the equations in those papers may not be monotone. Due to the lack of monotonicity, the equations do not possess the comparison principle. Hence, the weighted energy method together with the comparison principle fail in obtaining the stability of the traveling waves. Fortunately, the technical weighted energy method does not require the monotonicity of the equations and works for any non-monotone equations. More recently, by using the technical weighted energy method, Chern et al. [**8**], Lin et al. [**23**] and Wu et al. [**39**] respectively established asymptotic stability of non-monotone traveling waves of the Nicholson's blowflies equation with local diffusion. Zhang and Ma [**47**] considered a nonlocal dispersal equation and proved that the monotone/non-monotone waves with sufficiently large $c \gg 1$ are locally stable. Huang et al. [**20**] further studied the nonlocal dispersion equation, and established the local stability of non-monotone traveling waves with speed greater than the minimum speed. We should point out that the works mentioned above are mainly devoted to scalar equations. For delayed diffusive systems, little has been done for stability of traveling waves up to now (see [**41**, **44**, **45**]). Particularly, to the best of our knowledge, there has been no results on stability of traveling waves for nonlocal dispersal systems, even in quasi-monotone cases. As a result, this paper is concerned with the stability of traveling waves of nonlocal dispersal system (1.3), where the traveling waves are allowed to be non-monotone. It is much more challenging since the difficulties raise from the nonlocal diffusion term and the lack of monotonicity.

Regarding uniqueness of traveling waves, much work has been done for various scalar equations (see e.g. [**1**,**2**,**4**,**5**,**7**,**9**,**10**,**12**,**13**,**33**,**38**]). More recently, Aguerrea et al. [**1**] established the uniqueness of semi-wavefronts of nonlinear convolution equation and achieved an important extension of the uniqueness results in [**2**, **4**, **9**, **10**, **12**, **33**]. For systems, the results are rather limited. When the system is quasi-monotone, the uniqueness of traveling wavefronts can be obtained by sliding method, see e.g. [**16**]. For systems without quasi-monotonicity, the topic becomes more difficult since the traveling waves lack of monotonicity and the work is much less, see e.g. [**3**, **11**], where they studied some specific diffusive epidemic models and obtained the uniqueness of traveling waves by showing that the stable manifold at the endemic equilibrium is one-dimensional.

Based on the above facts, we are mainly concerned with the exponential stability and uniqueness of traveling waves of system (1.3) in this paper. To overcome the difficulties raised from the nonlocal term and the lack of monotonicity, we adopt the technical weighted energy method and construct the weight function $\omega(\xi)$ satisfying $\omega(+\infty) = 0$, which is motivated by [**23**]. Note that [**23**] uses the nonlinear Halanay's inequality to get the desired exponential decay estimate when $\phi(x + ct)$ is near $v_+$ (or, say, $x \to +\infty$). However, it is difficult, if not possible, to extend the nonlinear Halanay's inequality for scalar equation to system. Here we establish the desired decay estimate of $(\phi_1(x + ct), \phi_2(x + ct), \phi_3(x + ct))$ near $(S^*, I^*, R^*)$ with a new development (see Lemma 3.7). In addition, it is necessary to point out that, in [**23**], the uniform convergence $\lim_{\xi\to\infty} u(t, \xi) = 0$ for all $t \in [0, T]$ can not imply the uniformly boundedness $u(t, \xi) \le \epsilon$ for all $t \in [0, \infty)$ when $\xi >> 1$, see also [**8**, Observation 3]. To avoid such a trouble, inspired by [**8**], we first establish the global existence and uniqueness of solutions of the perturbed system and then the a priori estimates by the weighted energy method. Next, we get the exact asymptotic behavior of traveling waves at $-\infty$ by using Ikehara's theorem, and then establish the uniqueness of traveling waves of system (1.3) based on the stability result. Finally, we claim that the nonlocal dispersal affects the stability of traveling waves a lot. More precisely, we conclude that the nonlocal dispersal slows down the convergence rate for the solution to the traveling waves.

**Notations.** Throughout the paper, $C > 0$ always denotes a generic constant, while $C_i > 0 (i = 1, 2, ...)$ represents a specific constant. Let $I$ be an interval. $L^2(I)$ is the space of the square integrable functions defined on $I$, and $H^k(I)(k \ge 0)$ is the Sobolev space of the $L^2$-functions $f(x)$ defined on the interval $I$ whose derivatives $\frac{d^i}{dx^i} f (i = 1, 2, ..., k)$ also belong to $L^2(I)$. $L^2_\omega(I)$ denotes the weighted $L^2$-space with a weight function $\omega(x) > 0$ and its norm is defined by

$$||f||_{L^2_\omega} = \left(\int_I \omega(x)|f(x)|^2 dx\right)^{\frac{1}{2}}.$$

$H^k_\omega(I)$ is the weighted Sobolev space with the norm

$$||f||_{H^k_\omega} = \left(\sum_{i=0}^{k} \int_I \omega(x)\left|\frac{d^i}{dx^i}f(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$. $L^2([0,T];\mathcal{B})$ is the space of the $\mathcal{B}$-valued $L^2$-functions on $[0,T]$. The corresponding space of $\mathcal{B}$-valued functions on $[0,\infty)$ can be defined similarly.

Define

$$\triangle(\lambda, c) := \int_{\mathbb{R}} J(y)\,[e^{-\lambda y} - 1]dy - c\lambda + \beta e^{-\lambda c\tau} - (\mu + \gamma).$$

Then for $R_0 := \frac{\beta}{\mu+\gamma} > 1$, there exist $c^* > 0$ and $\lambda_* > 0$ such that

$$\frac{\partial\Delta(\lambda, c)}{\partial\lambda}\Big|_{(\lambda_*,c^*)} = 0, \quad \Delta(\lambda_*, c^*) = 0.$$

Moreover, if $c > c^*$, then $\Delta(\lambda, c) = 0$ has two positive solutions $\lambda_1(c) < \lambda_2(c)$ such that $\Delta(\lambda, c) < 0$ for $\lambda \in (\lambda_1(c), \lambda_2(c))$, while $\Delta(\lambda, c) = 0$ has no real roots if $c < c^*$ (see [**21**, Lemma 2.1]). In the rest of this paper, we always assume $R_0 > 1$.

Now define a weight function as

$$\omega(\xi) = e^{-\lambda_*(\xi-\xi_0)}, \quad \text{with a sufficient large number } \xi_0 \gg 1. \tag{1.7}$$

We are in position to state our main results.

Theorem 1.2. (Stability). Suppose that **(J)** and $\tau < \tau_0$ with $\tau_0$ a positive constant. For any given traveling wave solution $(\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct))$ of (1.3) with speed $c > \max\{c^*, \widetilde{c}\}$, where $\widetilde{c} > 0$ is a constant defined in (3.23), if the initial perturbation satisfies

$$\begin{cases} u_{i0}(x) - \phi_i(x) \in C(\mathbb{R}) \cap H^1_\omega(\mathbb{R}),\ i = 1, 3, \\ u_{20}(s, x) - \phi_2(x + cs) \in C([-\tau, 0]; C(\mathbb{R}) \cap H^1_\omega(\mathbb{R})) \cap L^2([-\tau, 0]; H^1_\omega(\mathbb{R})) \end{cases}$$

and $\lim_{x\to+\infty}(u_{i0}(x) - \phi_i(x)) := u_{i0,\infty},\ i = 1, 3,\ \lim_{x\to+\infty}(u_{20}(s, x) - \phi_2(x + cs)) := u_{20,\infty}(s) \in C[-\tau, 0]$ exists uniformly with respect to $s \in [-\tau, 0]$, then there exist some constants $\delta, k_0, \gamma_0 > 0$ and $0 < k < \min\{k_0, \gamma_0\}$, all independent of $x$, $t$, $U_i(t, x), i = 1, 2, 3$, when the initial perturbation is small:

$$\sum_{i=1,3} ||(u_{i0} - \phi_i)(0)||^2_C + \max_{s\in[-\tau,0]} ||(u_{20} - \phi_2)(s)||^2_C$$
$$+\sum_{i=1}^{3} ||(u_{i0} - \phi_i)(0)||^2_{H^1_\omega(\mathbb{R})} + \int_{-\tau}^{0} ||(u_{20} - \phi_2)(s)||^2_{H^1_\omega(\mathbb{R})} ds \le \delta,$$

the solution $(U_1(t, x), U_2(t, x), U_3(t, x))$ of the Cauchy problem (1.3) and (1.2) exists uniquely and globally in time, and satisfies

$$U_i(t, x) - \phi_i(x + ct) \in C([-\tau_i, +\infty); C(\mathbb{R}) \cap H^1_\omega(\mathbb{R}))$$

$$\cap L^2([-\tau_i,+\infty);H^1_\omega(\mathbb{R}))\cap \mathcal{C}_{unif}[-\tau_i,+\infty)$$

and

$$\sup_{x\in\mathbb{R}}|U_i(t,x)-\phi_i(x+ct)|\le Ce^{-kt},\ t>0,\ i=1,2,3,$$

where $\tau_1=\tau_3=0$, $\tau_2=\tau$, and $\mathcal{C}_{unif}[-r,T]$, for $r\ge 0$ and $0<T\le\infty$, is defined by

$$\begin{aligned}(1.8)\quad \mathcal{C}_{unif}[-r,T] \ :=\ &\{u(t,x)\in C([-r,T]\times\mathbb{R}) \text{ such that}\\ &\lim_{x\to+\infty} e^{\gamma_0 t}u(t,x) \text{ exists uniformly in } t\in[-r,T]\}.\end{aligned}$$

Remark 1.3. Note that $\omega(\xi)\ge 1$ only for $\xi\le\xi_0$, and $\lim_{\xi\to-\infty}\omega(\xi)=+\infty$, $\lim_{\xi\to+\infty}\omega(\xi)=0$. It differs from the previous works (see e.g. [**25, 26, 28, 29**]), where the weight functions are selected to be greater than 1 for all $\xi$ and the initial perturbation in $H^1_\omega(\mathbb{R})$ with $\omega(\xi)\ge 1$ implies that the initial perturbation at $\pm\infty$ must be 0. Here we restrict the initial perturbation in $C(\mathbb{R})\cap H^1_\omega(\mathbb{R})$ with $\omega(+\infty)=0$, which allows the initial perturbation at $+\infty$, namely, $u_{i0,\infty}$, $i=1,3$, and $u_{20,\infty}(s)$, to be uniformly bounded but may not be 0. See also [**23**, Remark 1].

Remark 1.4. Theorem 1.2 ensures that when the wave speed is not too close to $c^*$, the wave is asymptotically stable. However, the stability problem for $c$ close to the critical wave speed $c^*$, especially for $c=c^*$, is still open. And we will leave it for study.

By the stability result, we can further obtain the following uniqueness result.

Theorem 1.5. (Uniqueness). Suppose that **(J)** and $\tau<\tau_0$. Then any traveling wave solution $(\phi_1(x+ct),\phi_2(x+ct),\phi_3(x+ct))$ of (1.3) with speed $c>\max\{c^*,\widetilde{c}\}$ is unique up to translations.

The rest of this paper is organized as follows. In Section 2, we reformulate the original system to the perturbed system around the given traveling waves, and state the stability results for the perturbed system. Section 3 is devoted to the proof of the stability theorem. We first prove the global existence and uniqueness of solutions to the perturbed system (see Proposition 3.2), and then establish the a priori estimates (see Proposition 3.3), which immediately derive the stability theorem. To establish the a priori estimates, it is much crucial to get the desired decay estimate of $(\phi_1(x+ct),\phi_2(x+ct),\phi_3(x+ct))$ near $(S^*,I^*,R^*)$ (see Lemmas 3.7-3.9). In Section 4, we first obtain the exact asymptotic behavior of traveling waves at $-\infty$ and then prove the uniqueness result based on the stability theorem. Section 5 discusses the effect of the nonlocal dispersal. The conclusion implies that the nonlocal dispersal slows down the convergence of the solution to the traveling waves of (1.3).

## 2. Reformulation of the Problem

Let $(U_1(t,x),U_2(t,x),U_3(t,x))$ be the solution of the Cauchy problem (1.3) and (1.2), and $(\phi_1(x+ct),\phi_2(x+ct),\phi_3(x+ct))$ be a given traveling wave solution of

(1.3). Let $\xi := x + ct$ and

$$\begin{cases} V_i(t,\xi) = U_i(t,x) - \phi_i(\xi) = U_i(t,\xi - ct) - \phi_1(\xi), \ i = 1,2,3, \\ V_i(0,\xi) = u_{i0}(x) - \phi_i(x) := V_{i0}(\xi), \ i = 1,3, \ x \in \mathbb{R}, \\ V_2(s,\xi) = u_{20}(s,x) - \phi_2(x+cs) := V_{20}(s,\xi), \ s \in [-\tau, 0], \ x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Then $(V_1(t,x), V_2(t,x), V_3(t,x))$ satisfies

(2.2)

$$\begin{cases} \frac{\partial V_1}{\partial t} + c\frac{\partial V_1}{\partial \xi} - d\int_{\mathbb{R}} J(y)V_1(t,\xi - y)dy + (d+\sigma)V_1 = P_1(V_1, V_{2\tau}), \\ \frac{\partial V_2}{\partial t} + c\frac{\partial V_2}{\partial \xi} - \int_{\mathbb{R}} J(y)V_2(t,\xi - y)dy + (1+\mu+\gamma)V_2 = P_2(V_1, V_{2\tau}), \\ \frac{\partial V_3}{\partial t} + c\frac{\partial V_3}{\partial \xi} - d_3\int_{\mathbb{R}} J(y)V_3(t,\xi - y)dy + (d_3+\mu_1)V_3 - \gamma V_2 = 0, \\ V_1(0,\xi) = V_{10}(\xi), \ V_3(0,\xi) = V_{30}(\xi), \ V_2(s,\xi) = V_{20}(s,\xi), \ s \in [-\tau, 0], \ \xi \in \mathbb{R}, \end{cases}$$

where $P_1(V_1, V_2) = \frac{\beta\phi_1\phi_2}{1+\alpha\phi_2} - \frac{\beta(V_1+\phi_1)(V_2+\phi_2)}{1+\alpha(V_2+\phi_2)}$ and $P_2(V_1,V_2) = -P_1(V_1,V_2)$ with $V_{2\tau} = V_2(t-\tau, \xi - c\tau)$ and $V_i = V_i(t,\xi)$, $i = 1,2$.

Furthermore, linearize the delay term $P_i$, $i = 1,2$, then the original problem (1.3) and (1.2) can be reformulated as

$$\begin{aligned} \frac{\partial V_1}{\partial t} + c\frac{\partial V_1}{\partial \xi} - d\int_{\mathbb{R}} J(y)V_1(t,\xi - y)dy + \left(d + \sigma + \frac{\beta\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)}\right)V_1 \\ + \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi - c\tau))^2}V_2(t-\tau,\xi - c\tau) = -Q(t-\tau,\xi - c\tau), \end{aligned} \tag{2.3}$$

$$\begin{aligned} \frac{\partial V_2}{\partial t} + c\frac{\partial V_2}{\partial \xi} - \int_{\mathbb{R}} J(y)V_2(t,\xi - y)dy + (1+\mu+\gamma)V_2 - \frac{\beta\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)}V_1 \\ - \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi - c\tau))^2}V_2(t-\tau,\xi - c\tau) = Q(t-\tau,\xi - c\tau) \end{aligned} \tag{2.4}$$

and

$$\frac{\partial V_3}{\partial t} + c\frac{\partial V_3}{\partial \xi} - d_3\int_{\mathbb{R}} J(y)V_3(t,\xi - y)dy - \gamma V_2 + (d_3+\mu_1)V_3 = 0 \tag{2.5}$$

with the initial conditions

$$V_1(0,\xi) = V_{10}(\xi), \ V_3(0,\xi) = V_{30}(\xi), \ V_2(s,\xi) = V_{20}(s,\xi), \ s \in [-\tau, 0], \ \xi \in \mathbb{R}, \tag{2.6}$$

where

$$\begin{aligned} & Q(t-\tau,\xi - c\tau) \\ = \ & \frac{\beta(V_1(t,\xi)+\phi_1(\xi))(V_2(t-\tau,\xi - c\tau)+\phi_2(\xi - c\tau))}{1+\alpha(V_2(t-\tau,\xi - c\tau)+\phi_2(\xi - c\tau))} - \frac{\beta\phi_1(\xi)\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)} \\ & -\frac{\beta\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)}V_1(t,\xi) - \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi - c\tau))^2}V_2(t-\tau,\xi - c\tau). \end{aligned}$$

For $r \geq 0$, $T > 0$, define

$$\begin{aligned} & X(r-\tau, r+T) \\ = \ & \big\{(V_1,V_2,V_3) | V_i \in C([r-\tau_i, r+T]; C(\mathbb{R}) \cap H^1_\omega(\mathbb{R})) \\ & \cap L^2([r-\tau_i, r+T]; H^1_\omega(\mathbb{R})) \cap \mathcal{C}_{unif}[r-\tau_i, r+T], \\ & i = 1,2,3, \tau_1 = \tau_3 = 0, \tau_2 = \tau\big\} \end{aligned}$$

with

$$M^2_{r,\mathbf{V}}(T) = \sum_{i=1}^{3}\left(\sup_{t\in[r,r+T]}\left(||V_i(t)||^2_C + ||V_i(t)||^2_{H^1_\omega}\right) + \int_r^{r+T} ||V_i(s)||^2_{H^1_\omega} ds\right)$$

and

$$M^2_{r,\mathbf{V}}(0) = \sum_{i=1}^{3}\left(\sup_{t\in[r-\tau_i,r]}\left(||V_{ir}(t)||^2_C + ||V_{ir}(0)||^2_{H^1_\omega}\right) + \int_{r-\tau}^{r} ||V_{2r}(s)||^2_{H^1_\omega} ds\right),$$

where $r \geq 0$ and $T > 0$. When $r = 0$, we denote $M_{\mathbf{V}}(T) = M_{0,\mathbf{V}}(T)$.

Now we state the stability result for the perturbed system (2.3)-(2.6), which implies Theorem 1.2 immediately.

Theorem 2.1. (Stability). Suppose that $\tau < \tau_0$. For any given traveling wave solution $(\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct))$ of (1.3) with speed $c > \max\{c^*, \widetilde{c}\}$, if

$$\begin{cases} V_{10}(\xi),\ V_{30}(\xi) \in C(\mathbb{R}) \cap H^1_\omega(\mathbb{R}), \\ V_{20}(s,\xi) \in C([-\tau,0]; C(\mathbb{R}) \cap H^1_\omega(\mathbb{R})) \cap L^2([-\tau,0]; H^1_\omega(\mathbb{R})), \end{cases}$$

and $\lim_{\xi\to+\infty} V_{i0}(\xi) := V_{i0,\infty}$, $i = 1, 3$, and $\lim_{\xi\to+\infty} V_{20}(s,\xi) := V_{20,\infty}(s) \in C[-\tau, 0]$ exists uniformly in $s$, where $\omega(\xi) = e^{-\lambda_*(\xi-\xi_0)}$ with a large number $\xi_0 \gg 1$. Then there exist some positive constants $\delta_0$ and $k$ such that, when $M_{\mathbf{V}}(0) \leq \delta_0$, the solution $\mathbf{V}(t,\xi) := (V_1(t,\xi), V_2(t,\xi), V_3(t,\xi))$ of the Cauchy problem (2.3)-(2.6) uniquely and globally exists in $X(-\tau,\infty)$ and satisfies

$$\sum_{i=1}^{3}\left(||V_i(t)||^2_C + ||V_i(t)||^2_{H^1_\omega} + \int_0^t e^{-2k(t-s)}||V_i(s)||^2_{H^1_\omega} ds\right) \leq CM^2_{\mathbf{V}}(0)e^{-2kt} \tag{2.7}$$

for $t \in [0,\infty)$.

## 3. Stability of Traveling Waves

This section is devoted to prove Theorem 2.1, which mainly depends on the results about global existence and uniqueness of solutions for the perturbed system (2.3)-(2.6) (see Proposition 3.2) and a prior estimate (see Proposition 3.3). We first show the local existence result which will be used later.

Proposition 3.1. (Local existence). Consider the Cauchy problem with the initial time $r \geq 0$,

(3.1)

$$\begin{cases} \frac{\partial V_1(t,\xi)}{\partial t} + c\frac{\partial V_1(t,\xi)}{\partial \xi} - d\int_{\mathbb{R}} J(y)V_1(t,\xi-y)dy + \left(d+\sigma+\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\right)V_1(t,\xi) \\ \qquad + \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}V_2(t-\tau,\xi-c\tau) = -Q(t-\tau,\xi-c\tau), \\ \frac{\partial V_2(t,\xi)}{\partial t} + c\frac{\partial V_2(t,\xi)}{\partial \xi} - \int_{\mathbb{R}} J(y)V_2(t,\xi-y)dy + (1+\mu+\gamma)V_2(t,\xi) - \frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)} \\ \qquad \times V_1(t,\xi) - \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}V_2(t-\tau,\xi-c\tau) = Q(t-\tau,\xi-c\tau), \\ \frac{\partial V_3(t,\xi)}{\partial t} + c\frac{\partial V_3(t,\xi)}{\partial \xi} - d_3\int_{\mathbb{R}} J(y)V_3(t,\xi-y)dy - \gamma V_2(t,\xi) + (d_3+\mu_1)V_3(t,\xi) \\ \qquad = 0,\ (t,\xi) \in [r,+\infty) \times \mathbb{R}, \\ V_i(r,\xi) = u_{ir}(\xi) - \phi_i(\xi) := V_{ir}(r,\xi), \quad i = 1,3,\ \xi \in \mathbb{R}, \\ V_2(s,\xi) = u_{2r}(s,\xi-cs) - \phi_2(\xi) := V_{2r}(s,\xi), \quad (s,\xi) \in [r-\tau, r] \times \mathbb{R}. \end{cases}$$

If $(V_{1r}(r,\xi), V_{2r}(s,\xi), V_{3r}(r,\xi)) \in X(r-\tau, r)$ and $M_{r,\mathbf{V}}(0) \leq \delta_1$ for a given constant $\delta_1 > 0$, then there exists $t_0 = t_0(\delta_1) > 1$ such that the local solution $\mathbf{V}(t,\xi)$ of (3.1) uniquely exists for $t \in [r-\tau, r+t_0]$ and satisfies $\mathbf{V}(t,\xi) := (V_1(t,\xi), V_2(t,\xi), V_3(t,\xi)) \in X(r-\tau, r+t_0)$ and $M_{r,\mathbf{V}}(t_0) \leq \bar{c}_0 M_{r,\mathbf{V}}(0)$ for some constant $\bar{c}_0 > 1$.

PROOF. The local existence of the solution can be proved by th standard iteration technique (see e.g. [**24**]). In contrast to previous works, here we need to show that the local solution $\mathbf{V} \in \mathcal{C}_{unif}[r, r+t_0] \times \mathcal{C}_{unif}[r-\tau, r+t_0] \times \mathcal{C}_{unif}[r, r+t_0]$ for some small $t_0 > 0$, which will be determined later. The proof is motivated by that of [**23**, Proposition 2.2] and we sketch the proof as follows.

Let $(V_1^{(0)}(t,\xi), V_2^{(0)}(t,\xi), V_3^{(0)}(t,\xi)) \in X(r-\tau, r+t_0)$. Let $\eta > \gamma_0$ be a constant, where $\gamma_0$ is a positive constant being specified in Lemma 3.8. Let $a_1 = \eta - d - \sigma$, $a_2 = \eta - 1 - (\mu+\gamma)$, $a_3 = \eta - d_3 - \mu_1$. For $i = 1,2,3$, we define the iteration $V_i^{(n+1)} = \mathcal{P}_i(V_i^{(n)})$ for $n \geq 0$ by

$$\begin{cases} \frac{\partial V_1^{(n+1)}}{\partial t} + c\frac{\partial V_1^{(n+1)}}{\partial \xi} + \eta V_1^{(n+1)} = d\int_{\mathbb{R}} J(y) V_1^{(n)}(t,\xi-y)dy \\ \qquad +h_1(V_1^{(n)}, V_2^{(n)}(t-\tau, \xi-c\tau)), \\ V_1^{(n+1)}(r,\xi) = V_{1r}(r,\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{3.2}$$

$$\begin{cases} \frac{\partial V_2^{(n+1)}}{\partial t} + c\frac{\partial V_2^{(n+1)}}{\partial \xi} + \eta V_2^{(n+1)} = \int_{\mathbb{R}} J(y) V_2^{(n)}(t,\xi-y)dy \\ \qquad +h_2(V_1^{(n)}, V_2^{(n)}(t-\tau, \xi-c\tau)), \\ V_2^{(n+1)}(s,\xi) = V_{2r}(s,\xi), \quad s \in [r-\tau, r],\ \xi \in \mathbb{R} \end{cases} \tag{3.3}$$

and

$$\begin{cases} \frac{\partial V_3^{(n+1)}}{\partial t} + c\frac{\partial V_3^{(n+1)}}{\partial \xi} + \eta V_3^{(n+1)} = d_3\int_{\mathbb{R}} J(y) V_3^{(n)}(t,\xi-y)dy + h_3(V_2^{(n)}, V_3^{(n)}), \\ V_3^{(n+1)}(r,\xi) = V_{3r}(r,\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} h_1(V_1^{(n)}(t,\xi), V_2^{(n)}(t-\tau,\xi-c\tau)) &= a_1 V_1^{(n)}(t,\xi) + \frac{\beta\phi_1(\xi)\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)} \\ &\quad - \frac{\beta\left(V_1^{(n)}(t,\xi)+\phi_1(\xi)\right)\left(V_2^{(n)}(t-\tau,\xi-c\tau)+\phi_2(\xi-c\tau)\right)}{1+\alpha\left(V_2^{(n)}(t-\tau,\xi-c\tau)+\phi_2(\xi-c\tau)\right)}, \end{aligned}$$

$$\begin{aligned} h_2(V_1^{(n)}(t,\xi), V_2^{(n)}(t-\tau,\xi-c\tau)) &= a_2 V_2^{(n)}(t,\xi) - \frac{\beta\phi_1(\xi)\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)} \\ &\quad + \frac{\beta\left(V_1^{(n)}(t,\xi)+\phi_1(\xi)\right)\left(V_2^{(n)}(t-\tau,\xi-c\tau)+\phi_2(\xi-c\tau)\right)}{1+\alpha\left(V_2^{(n)}(t-\tau,\xi-c\tau)+\phi_2(\xi-c\tau)\right)} \end{aligned}$$

and

$$h_3(V_2^{(n)}(t,\xi), V_3^{(n)}(t,\xi)) = a_3 V_3^{(n)}(t,\xi) + \gamma V_2^{(n)}(t,\xi).$$

Then (3.2)-(3.4) can be written in the integral form

(3.5) $\quad V_i^{(n+1)}(t,\xi)$

$$
\begin{aligned}
=&e^{-\eta t}V_{ir}(r,\xi-ct)+\int_r^t e^{-\eta(t-s)}\Big[d_i\int_{\mathbb{R}}J(y)V_i^{(n)}(s,\xi-y+c(s-t))dy\\
&+g_i\Big(V_1^{(n)}(s,\xi+c(s-t)),V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)),\\
&V_3^{(n)}(s,\xi+c(s-t))\Big)\Big]ds
\end{aligned}
$$

for $i=1,2,3$, where $d_1=d,\ d_2=1$, and

$$
g_{1,2}(V_1^{(n)},V_2^{(n)},V_3^{(n)})=h_{1,2}(V_1^{(n)},V_2^{(n)}),\quad g_3(V_1^{(n)},V_2^{(n)},V_3^{(n)})=h_3(V_2^{(n)},V_3^{(n)}).
$$

Furthermore, we have

$$
\begin{aligned}
e^{\gamma_0 t}V_i^{(n+1)}(t,\xi)\ =\ &e^{-(\eta-\gamma_0)t}V_{ir}(r,\xi-ct)+\int_r^t e^{-\eta(t-s)}e^{\gamma_0 t}\Big[d_i\int_{\mathbb{R}}J(y)\\
&\times V_i^{(n)}(s,\xi-y+c(s-t))dy+g_i\Big(V_1^{(n)}(s,\xi+c(s-t)),\\
&V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)),V_3^{(n)}(s,\xi+c(s-t))\Big)\Big]ds.
\end{aligned}
$$

Note that $V_i^{(n)}\in\mathcal{C}_{unif}[r-\tau_i,r+t_0]$ with $\tau_1=\tau_3=0$ and $\tau_2=\tau$, i.e., $\lim_{\xi\to+\infty}e^{\gamma_0 t}V_i^{(n)}(t,\xi)=e^{\gamma_0 t}V_{i,\infty}^{(n)}(t)\in C[r-\tau_i,r+t_0]$ exists uniformly in $t$ for $i=1,2,3$. We now prove $V_i^{(n+1)}\in\mathcal{C}_{unif}[r-\tau_i,r+t_0]$. It is easy to see that

$$
\begin{aligned}
&\lim_{\xi\to+\infty}V_i^{(n+1)}(t,\xi)\\
=&e^{-\eta t}\lim_{\xi\to+\infty}V_{ir}(r,\xi-ct)+\int_r^t e^{-\eta(t-s)}\Big[d_i\int_{\mathbb{R}}J(y)\\
&\times\lim_{\xi\to+\infty}V_i^{(n)}(s,\xi-y+c(s-t))dy+\lim_{\xi\to+\infty}g_i\Big(V_1^{(n)}(s,\xi+c(s-t)),\\
&V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)),V_3^{(n)}(s,\xi+c(s-t))\Big)\Big]ds\\
=&V_{ir,\infty}(r)e^{-\eta t}+\int_r^t e^{-\eta(t-s)}\Big[d_i\lim_{\xi\to+\infty}V_i^{(n)}(s,\xi+c(s-t))\\
&+\lim_{\xi\to+\infty}g_i\Big(V_1^{(n)}(s,\xi+c(s-t)),V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)),\\
&V_3^{(n)}(s,\xi+c(s-t))\Big)\Big]ds\\
=&V_{ir,\infty}(r)e^{-\eta t}+\int_r^t e^{-\eta(t-s)}\Big[d_iV_{i,\infty}^{(n)}(s)+g_i\Big(V_{1,\infty}^{(n)}(s),V_{2,\infty}^{(n)}(s-\tau),V_{3,\infty}^{(n)}(s)\Big)\Big]ds\\
=:&V_{i,\infty}^{(n+1)}(t),\quad\text{uniformly with respect to } t\in[r-\tau_i,r+t_0].
\end{aligned}
$$

We claim that $e^{\gamma_0 t}V_i^{(n+1)}(t,\xi)$ is uniformly convergent as $\xi\to+\infty$. In fact, since

$$
\lim_{\xi\to+\infty}\sup_{t\in[r,r+t_0]}\left|e^{\gamma_0 t}V_i^{(n)}(t,\xi)-e^{\gamma_0 t}V_{i,\infty}^{(n)}(t)\right|=0
$$

and

$$
\Big|g_i(V_1^{(n)}(s,\xi+c(s-t)),V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)),V_3^{(n)}(s,\xi+c(s-t)))
$$

$$
\begin{aligned}
&-g_i(V_{1,\infty}^{(n)}(s), V_{2,\infty}^{(n)}(s-\tau), V_{3,\infty}^{(n)}(s))\Big| \\
\leq & C \sup_{s\in[r,r+t_0]} \left( \sum_{i=1}^{3} \left| V_i^{(n)}(s,\xi+c(s-t)) - V_{i,\infty}^{(n)}(s) \right| \right. \\
& \left. + \left| V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)) - V_{2,\infty}^{(n)}(s-\tau) \right| \right),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& \lim_{\xi\to+\infty} \sup_{t\in[r,r+t_0]} \left| e^{\gamma_0 t} V_i^{(n+1)}(t,\xi) - e^{\gamma_0 t} V_{i,\infty}^{(n+1)}(t) \right| \\
= & \lim_{\xi\to+\infty} \sup_{t\in[r,r+t_0]} \int_r^t e^{-\eta(t-s)} e^{\gamma_0(t-s+\tau)} \left[ d_i \int_{\mathbb{R}} J(y) e^{\gamma_0(s-\tau)} \right. \\
& \times \left| V_i^{(n)}(s,\xi-y+c(s-t)) - V_{i,\infty}^{(n)}(s) \right| dy + e^{\gamma_0(s-\tau)} \\
& \times \Big| g_i(V_1^{(n)}(s,\xi+c(s-t)), V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)), V_3^{(n)}(s,\xi+c(s-t))) \\
& \left. -g_i(V_{1,\infty}^{(n)}(s), V_{2,\infty}^{(n)}(s-\tau), V_{3,\infty}^{(n)}(s)) \Big| \right] ds \\
\leq & \sup_{t\in[r,r+t_0]} \int_r^t e^{-\eta(t-s)} e^{\gamma_0(t-s+\tau)} \left[ d_i \int_{\mathbb{R}} J(y) \lim_{\xi\to+\infty} e^{\gamma_0(s-\tau)} \right. \\
& \left| V_i^{(n)}(s,\xi-y+c(s-t)) - V_{i,\infty}^{(n)}(s) \right| dy \\
& + C \lim_{\xi\to+\infty} \sup_{t\in[r,r+t_0]} \left( \sum_{i=1}^{3} \left| V_i^{(n)}(s,\xi+c(s-t)) - V_{i,\infty}^{(n)}(s) \right| \right. \\
& \left. \left. + \left| V_2^{(n)}(s-\tau,\xi+c(s-t-\tau)) - V_{2,\infty}^{(n)}(s-\tau) \right| \right) \right] e^{\gamma_0(s-\tau)} ds \\
= & 0.
\end{aligned}
$$

Here we also used the uniform boundedness of

$$
\int_r^t e^{-\eta(t-s)} e^{\gamma_0(t-s+\tau)} ds = \frac{e^{\gamma_0\tau}}{\eta-\gamma_0} \left( 1 - e^{(\eta-\gamma_0)(r-t)} \right) \leq \frac{e^{\gamma_0\tau}}{\eta-\gamma_0}.
$$

Furthermore, by taking

$$
\begin{aligned}
& \int_r^t \int_{\mathbb{R}} \sum_{k=0}^{1} \omega(\xi) \cdot \Big( \partial_\xi^k(3.2) \cdot \partial_\xi^k V_1^{(n+1)} + \partial_\xi^k(3.3) \cdot \partial_\xi^k V_2^{(n+1)} \\
& + \partial_\xi^k(3.4) \cdot \partial_\xi^k V_3^{(n+1)} \Big) d\xi ds,
\end{aligned}
$$

then for all $t \in [r-\tau_i, r+t_0]$, we can get

$$
\begin{aligned}
(3.6) \qquad & \sum_{i=1}^{3} \left( ||V_i^{(n+1)}(t)||_{H_\omega^1}^2 + \int_r^t e^{-2k(t-s)} ||V_i^{(n+1)}(s)||_{H_\omega^1}^2 ds \right) \\
\leq & \; C \Big( \sum_{i=1}^{3} ||V_{ir}(r)||_{H_\omega^1}^2 + \int_{r-\tau}^{r} ||V_{2r}(s)||_{H_\omega^1}^2 ds + \sum_{i=1}^{3} \int_r^t ||V_i^n(s)||_{H_\omega^1}^2 ds \Big)
\end{aligned}
$$

for some constant $C > 0$. From (3.5), we have

$$\sum_{i=1}^{3}||V_i^{(n+1)}(t,\xi)||_C \le \sum_{i=1}^{3}\left(C||V_{ir}(r)||_C + Ct_0 \sup_{t\in[r-\tau_i,r+t_0]}||V_i^{(n)}(t,\xi)||_C\right) \tag{3.7}$$

for $t \in [r-\tau_i, r+t_0]$. Combining (3.6) and (3.7), we get

$$\begin{aligned} M_{r,\mathbf{V^{(n+1)}}}(t_0) \le & C\Bigg(\sum_{i=1,3}||V_{ir}(r)||_C^2 + \max_{s\in[r-\tau,r]}||V_{2r}(s)||_C^2 + \sum_{i=1}^{3}||V_{ir}(r)||^2_{H^1_\omega} \\ & + \int_{r-\tau}^{r}||V_{2r}(s)||^2_{H^1_\omega}ds\Bigg)^{\frac{1}{2}} + Ct_0 M_{r,\mathbf{V^{(n)}}}(t_0). \end{aligned}$$

Then, we can prove that $\mathbf{V}^{(n+1)} = (\mathcal{P}_1(V_1^{(n+1)}), \mathcal{P}_2(V_2^{(n+1)}), \mathcal{P}_3(V_3^{(n+1)}))$ defined by (3.2) and (3.3) maps $X(r-\tau, r+t_0)$ into $X(r-\tau, r+t_0)$ and is a contraction mapping in $X(r-\tau, r+t_0)$ providing $0 < t_0 \ll 1$. Hence, by applying the Banach fixed point theorem, we can obtain the local existence of the solution in $X(r-\tau, r+t_0)$. Since the convergence $\lim_{\xi\to+\infty} V_i^{(n)}(t,\xi) = V_i(t,\xi)$ is uniform for $(t,\xi) \in [r, r+t_0]\times\mathbb{R}$ and $V_i^{(n)} \in \mathcal{C}_{unif}[r-\tau_i, r+t_0]$, we get that $V_i \in \mathcal{C}_{unif}[r-\tau_i, r+t_0]$, $i = 1,2,3$, $\tau_1 = \tau_3 = 0$ and $\tau_2 = \tau$. This completes the proof. □

Now we show the global existence and uniqueness of solutions of system (2.3)-(2.6).

Proposition 3.2. (Global existence and uniqueness). Let $(\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct))$ be a given traveling wave solution, and the initial perturbation $(V_{10}(\xi), V_{20}(s,\xi), V_{30}(\xi)) \in X(-\tau, 0)$ be arbitrary. Then the solution $\mathbf{V}(t,\xi) := (V_1(t,\xi), V_2(t,\xi), V_3(t,\xi))$ of (2.3)-(2.6) globally and uniquely exists in $X_{loc}(-\tau,\infty)$ with

$$\begin{aligned} X_{loc}(-\tau,\infty) \quad = \quad & \big\{(V_1,V_2,V_3)|V_i \in C_{loc}([-\tau_i,\infty); C(\mathbb{R})\cap H^1_\omega(\mathbb{R}))\cap L^2_{loc}([-\tau_i, \\ & \infty); H^1_\omega(\mathbb{R}))\cap\mathcal{C}_{unif}[-\tau_i,\infty), i = 1,2,3, \tau_1 = \tau_3 = 0, \tau_2 = \tau\big\}, \end{aligned}$$

where $L^2_{loc}([-\tau_i,\infty); H^1_\omega(\mathbb{R}))$ denotes the space whose $H^1_\omega$-valued functions are $L^2$-integrable on $[-\tau_i, T]$ for any $0 < T < \infty$, and $C_{loc}([-\tau_i,\infty); C(\mathbb{R})\cap H^1_\omega(\mathbb{R}))$ are similarly defined.

Proof. The proof is mainly motivated by that of [**8**, Proposition 2.2], see also [**22**, Proposition 3.2]. When $t \in [0,\tau]$, motivated by [**37**], system (2.2) with the initial value $(V_{10}(\xi), V_{20}(s,\xi), V_{30}(s,\xi)) \in X(-\tau,0)$ can be uniquely solved as

$$\begin{aligned} & V_i(t,\xi) \\ = \quad & e^{-a_i t}V_{i0}(0,\xi-ct) + \int_0^t e^{-a_i(t-s)}\Big[d_i\int_{\mathbb{R}}J(y)V_i(s,\xi-y+c(s-t))dy \\ & + P_i(V_1(s,\xi+c(s-t)), V_{20}(s-\tau,\xi+c(s-t-\tau))\Big]ds. \end{aligned} \tag{3.8}$$

We are going to prove $(V_1(t,\xi), V_2(t,\xi), V_3(t,\xi)) \in X_{loc}(0,\tau)$.

Multiplying $V_1$-equation in (2.2) by $\omega(\xi)V_1$ and using Cauchy-Schwarz inequality

$$de^{2kt}\left|\int_{\mathbb{R}}J(y)\omega(\xi)V_1(t,\xi)V_1(t,\xi-y)dy\right|$$

$$\begin{aligned} &\leq \frac{d}{2}e^{2kt}\int_{\mathbb{R}} J(y)\omega(\xi)\left(V_1^2(t,\xi)+V_1^2(t,\xi-y)\right)dy \\ &= \frac{d}{2}e^{2kt}\omega(\xi)V_1^2(t,\xi)+\frac{d}{2}e^{2kt}\int_{\mathbb{R}} J(y)\omega(\xi)V_1^2(t,\xi-y)dy. \end{aligned}$$

and integrating it with respect to $\xi$ over $\mathbb{R}$, we have

$$\frac{1}{2}\frac{d}{dt}||V_1||^2_{L^2_\omega}+m_1||V_1||^2_{L^2_\omega}\leq \int_{\mathbb{R}}\omega(\xi)V_1P_1(V_1(t,\xi),V_{20}(t-\tau,\xi-c\tau))d\xi, \tag{3.9}$$

where $m_1 := \frac{1}{2}c\lambda_* - \frac{1}{2}\int_{\mathbb{R}}[e^{-\lambda_* y}-1]dy+\sigma>0$ by the definition of $c$ (see (3.23)).

Now we claim that, for $t\in[0,\tau]$,

$$P_1(V_1(t,\xi),V_{20}(t-\tau,\xi-c\tau))\leq C\left(|V_{10}(\xi)|+|V_{20}(t-\tau,\xi-c\tau)|\right). \tag{3.10}$$

By Taylor's formula, it is not difficult to verify that, (3.10) holds in $[0,t_0]$ for some small positive constant $t_0 << \tau$. For $t\in[t_0,2t_0]$, again by Taylor's formula, it holds

$$P_1(V_1(t,\xi),V_{20}(t-\tau,\xi-c\tau))\leq C\left(|V_1(t_0,\xi)|+|V_{20}(t-\tau,\xi-c\tau)|\right).$$

By (3.8),

$$\begin{aligned} |V_1(t_0,\xi)| &\leq C_1(|V_{10}(\xi-ct)|+|P_1(V_1(t,\xi+c(s-t)), \\ &\qquad V_{20}(t-\tau,\xi+c(s-t-\tau)))|) \\ &\leq C_2(|V_{10}(\xi)|+|V_{20}(t-\tau,\xi-c\tau)|) \text{ with } 0\leq t\leq t_0, \end{aligned}$$

for some constants $C_1$ and $C_2$, where the last inequality is true due to the initial condition $(V_{10}(\xi),V_{20}(s,\xi),V_{30}(s,\xi))\in X(-\tau,0)$. Then (3.10) holds for $t\in[t_0,2t_0]$. Doing this procedure in each of the intervals $[nt_0,(n+1)t_0]$, $n=1,2,3,...$, one by one, we obtain that (3.10) holds for all $t\in[0,\tau]$.

Using Cauchy-Schwarz inequality again, we have

$$\begin{aligned} &\int_{\mathbb{R}}\omega(\xi)V_1P_1(V_1(t,\xi),V_{20}(t-\tau,\xi-c\tau))d\xi \\ \leq\ & C\int_{\mathbb{R}}\omega(\xi)V_1(|V_{10}(\xi)|+|V_{20}(t-\tau,\xi-c\tau)|)d\xi \\ \leq\ & 2\epsilon||V_1(t)||^2_{L^2_\omega}+\frac{C}{4\epsilon}\left(||V_{10}(0)||^2_{L^2_\omega}+||V_{20}(t-\tau)||^2_{L^2_\omega}\right) \end{aligned}$$

for some small constant $\epsilon>0$. Substituting this into (3.9), we have

$$\frac{1}{2}\frac{d}{dt}||V_1||^2_{L^2_\omega}+(m_1-2\epsilon)||V_1||^2_{L^2_\omega}\leq C(||V_{10}(0)||^2_{L^2_\omega}+||V_{20}(t-\tau)||^2_{L^2_\omega}). \tag{3.11}$$

Integrating (3.11) over $[0,t]$ for $t\in[0,\tau]$, and taking $\epsilon<\frac{m_1}{2}$, we have

$$\begin{aligned} &||V_1||^2_{L^2_\omega}+\int_0^t ||V_1(s)||^2_{L^2_\omega}ds \\ \leq\ & C(||V_{10}(0)||^2_{L^2_\omega}+\int_0^t ||V_{10}(0)||^2_{L^2_\omega}ds+\int_0^t ||V_{20}(s-\tau)||^2_{L^2_\omega}ds) \\ \leq\ & C(||V_{10}(0)||^2_{L^2_\omega}+\int_{-\tau}^0 ||V_{20}(s)||^2_{L^2_\omega}ds)<\infty \text{ for } t\in[0,\tau]. \end{aligned}$$

Similarly, we can prove

$$||V_2||^2_{L^2_\omega}+\int_0^t ||V_2(s)||^2_{L^2_\omega}ds$$

$$\leq \quad C(\sum_{i=1}^{2}||V_{i0}(0)||^2_{L^2_\omega} + \int_{-\tau}^{0}||V_{20}(s)||^2_{L^2_\omega}ds) < \infty \text{ for } t \in [0,\tau]$$

and

$$\begin{aligned} & ||V_3||^2_{L^2_\omega} + \int_0^t ||V_3(s)||^2_{L^2_\omega}ds \\ \leq \quad & C(||V_{20}(0)||^2_{L^2_\omega} + ||V_{30}(s)||^2_{L^2_\omega}) < \infty \text{ for } t \in [0,\tau]. \end{aligned}$$

Thus,

$$\begin{aligned} & \sum_{i=1}^{3}\left(||V_i||^2_{L^2_\omega} + \int_0^t ||V_i(s)||^2_{L^2_\omega}ds\right) \\ \leq \quad & C\left(\sum_{i=1}^{3}||V_{i0}(0)||^2_{L^2_\omega} + \int_{-\tau}^{0}||V_{20}(s)||^2_{L^2_\omega}ds\right), \quad t \in [0,\tau]. \end{aligned} \tag{3.12}$$

Furthermore, by differentiating (2.2) with respect to $\xi$ and multiplying the resultant equations by $\omega(\xi)V_{i\xi}(t,\xi)$, $i = 1,2,3$, respectively, and by the same arguments as above, we obtain

$$\begin{aligned} & \sum_{i=1}^{3}\left(||V_{i\xi}||^2_{L^2_\omega} + \int_0^t ||V_{i\xi}(s)||^2_{L^2_\omega}ds\right) \\ \leq \quad & C\left(\sum_{i=1}^{3}||V_{i0}(0)||^2_{H^1_\omega} + \int_{-\tau}^{0}||V_{20}(s)||^2_{H^1_\omega}ds\right) < \infty \text{ for } t \in [0,\tau]. \end{aligned} \tag{3.13}$$

Now by (3.8) and (3.10), we have

$$\begin{aligned} ||V_i(t)||_C \quad \leq \quad & e^{-a_i t}||V_{i0}(0)||_C + d_i||V_i(t)||_C\int_0^t e^{-a_i(t-s)}ds \\ & +C\left(||V_{10}(0)||_C + \sup_{s-\tau\in[-\tau,0]}||V_{20}(s-\tau)||_C\right)\int_0^t e^{-a_i(t-s)}ds. \end{aligned}$$

Since $d_i\int_0^t e^{-a_i(t-s)}ds < 1$, we can further get

$$\begin{aligned} ||V_i(t)||_C \quad \leq \quad & e^{-a_i t}||V_{i0}(0)||_C + C\left(||V_{10}(0)||_C + \sup_{s\in[-\tau,0]}||V_{20}(s)||_C\right) \\ < \quad & \infty \text{ for } t \in [0,\tau]. \end{aligned} \tag{3.14}$$

On the other hand, for $V_{i0} \in \mathcal{C}_{unif}[-\tau_i,0]$, by Proposition 3.1, we have $V_i \in \mathcal{C}_{unif}[0,t_0]$ for some small $t_0 > 0$. Again using Proposition 3.1 in each of the interval $[nt_0,(n+1)t_0]$, we can prove $V_i \in \mathcal{C}_{unif}[nt_0,(n+1)t_0]$, $n = 1,2,...,$ and hence, $V_i \in \mathcal{C}_{unif}[0,\tau]$. Then it follows from the above arguments that $(V_1,V_2,V_3) \in X_{loc}(0,\tau)$ and

$$\begin{aligned} & \sum_{i=1}^{3}\left(||V_i||^2_C + ||V_i||^2_{H^1_\omega} + \int_0^t ||V_i(s)||^2_{H^1_\omega}ds\right) \\ \leq \quad & C\left(\sum_{i=1}^{3}\sup_{s\in[-\tau_i,0]}||V_{i0}(s)||^2_C + \sum_{i=1}^{3}||V_{i0}(0)||^2_{H^1_\omega} + \int_{-\tau}^{0}||V_{20}(s)||^2_{H^1_\omega}ds\right) \end{aligned} \tag{3.15}$$

for $t \in [0,\tau]$.

When $t \in [\tau, 2\tau]$, System (2.2) with the initial data $(V_1(s,\xi), V_2(s,\xi), V_2(s,\xi)) \in X_{loc}(0,\tau)$ can be uniquely solved as

$$\begin{aligned} V_i(t,\xi) &= e^{-a_i t} V_i(\tau,\xi) + \int_\tau^t e^{-a_i(t-s)} \Big[ d_i \int_{\mathbb{R}} J(y) V_i(s, \xi - y) dy \\ &\quad + P_i(V_1(s,\xi), V_2(s-\tau, \xi - c\tau) \Big] ds. \end{aligned}$$

By the same arguments as (3.12)-(3.15), we can prove that $(V_1, V_2, V_3) \in X_{loc}(\tau, 2\tau)$ and

$$\begin{aligned} &\sum_{i=1}^3 \left( ||V_i||_C^2 + ||V_i||_{H^1_\omega}^2 + \int_0^t ||V_i(s)||_{H^1_\omega}^2 ds \right) \\ \le\ & C \left( \sum_{i=1}^3 \sup_{s \in [0,\tau_i]} ||V_i(s)||_C^2 + \sum_{i=1}^3 ||V_i(\tau)||_{H^1_\omega}^2 + \int_0^\tau ||V_2(s)||_{H^1_\omega}^2 ds \right) \\ \le\ & C^2 \left( \sum_{i=1}^3 \sup_{s \in [-\tau_i, 0]} ||V_{i0}(s)||_C^2 + \sum_{i=1}^3 ||V_{i0}(0)||_{H^1_\omega}^2 + \int_{-\tau}^0 ||V_{20}(s)||_{H^1_\omega}^2 ds \right) \end{aligned}$$

for $t \in [\tau, 2\tau]$, where the last inequality is by (3.15).

Repeating the above procedure, step by step, we can get that $(V_1, V_2, V_3) \in X_{loc}((n-1)\tau, n\tau)$ uniquely exists, and satisfies

$$\begin{aligned} &\sum_{i=1}^3 \left( ||V_i||_C^2 + ||V_i||_{H^1_\omega}^2 + \int_0^t ||V_i(s)||_{H^1_\omega}^2 ds \right) \\ \le\ & C^n \left( \sum_{i=1}^3 \sup_{s \in [-\tau_i, 0]} ||V_{i0}(s)||_C^2 + \sum_{i=1}^3 ||V_{i0}(0)||_{H^1_\omega}^2 + \int_{-\tau}^0 ||V_{20}(s)||_{H^1_\omega}^2 ds \right) \end{aligned}$$

for $t \in [(n-1)\tau, n\tau]$, and finally we proved that $(V_1, V_2, V_3)$ is unique and $(V_1, V_2, V_3) \in X_{loc}(-\tau, \infty)$ with, for any $T > 0$,

$$\begin{aligned} &\sum_{i=1}^3 \left( ||V_i||_C^2 + ||V_i||_{H^1_\omega}^2 + \int_0^t ||V_i(s)||_{H^1_\omega}^2 ds \right) \\ \le\ & C_T \left( \sum_{i=1}^3 \sup_{s \in [-\tau_i, 0]} ||V_{i0}(s)||_C^2 + \sum_{i=1}^3 ||V_{i0}(0)||_{H^1_\omega}^2 + \int_{-\tau}^0 ||V_{20}(s)||_{H^1_\omega}^2 ds \right) \end{aligned}$$

with $t \in [0, T]$. This completes the proof. □

Next we establish the a priori estimates.

PROPOSITION 3.3. (A prior estimate). Let $\mathbf{V}(t,\xi) \in X_{loc}(-\tau, \infty)$ be a local solution of the Cauchy problem (2.3)-(2.6). Then there exist some positive constants $k$, $\delta_2$ and $C_0$ such that when $M_{\mathbf{V}}(0) \le \delta_2$,

$$\sum_{i=1}^3 \left( ||V_i(t)||_C^2 + ||V_i(t)||_{H^1_\omega}^2 + \int_0^t e^{-2k(t-s)} ||V_i(s)||_{H^1_\omega}^2 ds \right) \le C_0 M_{\mathbf{V}}^2(0) e^{-2kt} \tag{3.16}$$

for $t > 0$, where $\mathbf{V}(t,\xi) := (V_1, V_2, V_3)(t,\xi)$.

Then by Propositions 3.2 and 3.3, Theorem 2.1 is immediately followed.

In the following we prove Proposition 3.3 by a series of lemmas. We first prove three key inequalities. Define

$$B_{1,k,\omega}(\xi) := -c\frac{\omega'}{\omega} + d + 2\sigma + \frac{\beta\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi - c\tau))^2} - d\int_{\mathbb{R}} J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy - 2k, \tag{3.17}$$

$$B_{2,k,\omega}(\xi) := -c\frac{\omega'}{\omega} - \frac{\beta\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi - c\tau))^2} + 2\mu + 1 + \gamma - \int_{\mathbb{R}} J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy - 2e^{2k\tau}\frac{\beta\phi_1(\xi + c\tau)}{(1+\alpha\phi_2(\xi))^2} - 2k, \tag{3.18}$$

$$B_{3,k,\omega}(\xi) := -c\frac{\omega'}{\omega} + d_3 + 2\mu_1 - \gamma - d_3\int_{\mathbb{R}} J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy - 2k \tag{3.19}$$

and

$$C_1(k) := c\lambda_* - d\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\sigma - \tilde{L} - 2k, \tag{3.20}$$

$$C_2(k) := c\lambda_* - \int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu + \gamma - L - 3\tilde{L} - 2k - 2\tilde{L}\left(e^{2k\tau} - 1\right), \tag{3.21}$$

$$C_3(k) := c\lambda_* - d_3\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu_1 - \gamma - 2k, \tag{3.22}$$

where

$$L := \sup_{\xi\in\mathbb{R}} \frac{\beta\phi_2(\xi - c\tau)}{1+\alpha\phi_2(\xi - c\tau)}, \quad \tilde{L} := \sup_{\xi\in\mathbb{R}} \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi - c\tau))^2}.$$

Furthermore, let

$$c_1 := \frac{d\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + \tilde{L} - 2\sigma}{\lambda_*}, \quad c_3 := \frac{d_3\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + \gamma - 2\mu_1}{\lambda_*},$$

$$c_2 := \frac{\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + L + 3\tilde{L} - (2\mu + \gamma)}{\lambda_*},$$

and

$$\widetilde{c} := \max\{c_1, c_2, c_3\}. \tag{3.23}$$

Now we give the following lemma.

Lemma 3.4. (Key inequalities). Let $\omega(\xi)$ be given in (1.7) and $c > \max\{c^*, \widetilde{c}\}$. Then

$$B_{i,k,\omega}(\xi) > C_i(k) > 0$$

for all $\xi \in \mathbb{R}$, and $0 < k < k_0 := \min\{k_1, k_2, k_3\}$, where $k_i > 0$ are respectively the unique positive solution of equations $C_i(k) = 0$, $i = 1, 2, 3$.

PROOF. Since $c > \max\{c^*, \widetilde{c}\}$, we have

$$c\lambda_* - d\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\sigma - \tilde{L} > 0,$$

$$c\lambda_* - \int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu + \gamma - L - 3\tilde{L} > 0,$$

$$c\lambda_* - d_3\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu_1 - \gamma > 0.$$

Then $C_i(k) = 0$ exists a unique positive solution, say, $k_i$, $i = 1, 2, 3$.

Since $\omega(\xi) = e^{-\lambda_*(\xi-\xi_0)}$, $\frac{\omega'(\xi)}{\omega(\xi)} = -\lambda_*$, we get

$$\begin{aligned} B_{1,k,\omega}(\xi) &= c\lambda_* - d\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\sigma + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} \\ &\quad - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - 2k \\ &> c\lambda_* - d\int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\sigma - \tilde{L} - 2k \\ &= C_1(k) > 0 \text{ for } 0 < k < k_1. \end{aligned}$$

Similarly,

$$\begin{aligned} B_{2,k,\omega}(\xi) &= c\lambda_* - \int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu + \gamma - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} \\ &\quad - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - 2k - 2e^{2k\tau}\frac{\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2} \\ &> c\lambda_* - \int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu + \gamma - L - \tilde{L} - 2k - 2\tilde{L}e^{2k\tau} \\ &= c\lambda_* - \int_{\mathbb{R}} J(y)[e^{-\lambda_* y} - 1]dy + 2\mu + \gamma - L - 3\tilde{L} \\ &\quad -2k - 2\tilde{L}\left(e^{2k\tau} - 1\right) \\ &= C_2(k) > 0 \text{ for } 0 < k < k_2, \end{aligned}$$

and $B_{3,k,\omega}(\xi) = C_3(k) > 0$ for $0 < k < k_3$. Thus, $B_{i,k,\omega}(\xi) > C_i(k) > 0$ for all $\xi \in \mathbb{R}$ and $0 < k < k_0 := \min\{k_1, k_2, k_3\}$, $i = 1, 2, 3$. This completes the proof. □

LEMMA 3.5. *Let $\mathbf{V}(t, \xi) \in X_{loc}(-\tau, \infty)$. Then for $0 < k < k_0$, it holds that*

$$\begin{aligned} &\sum_{i=1}^{3}\left(||V_i(t)||^2_{H^1_\omega} + \int_0^t e^{-2k(t-s)}||V_i(s)||^2_{H^1_\omega}ds\right) \\ &\quad \leq Ce^{-2kt}\left(\sum_{i=1}^{3}||V_{i0}(0)||^2_{H^1_\omega} + \int_{-\tau}^0 ||V_{20}(s)||^2_{H^1_\omega}ds\right) \end{aligned} \tag{3.24}$$

*provided $M_{\mathbf{V}}(\infty) \ll 1$, where $k_0$ is defined in Lemma 3.4.*

PROOF. We first estimate $\mathbf{V}(t, x)$ in $L^2_\omega(\mathbb{R})$. Multiplying (2.3) by $e^{2kt}\omega V_1$, where $k$ is a constant and will be specified later, we obtain

$$\left\{\frac{1}{2}e^{2kt}\omega {V_1}^2\right\}_t + e^{2kt}\left\{\frac{1}{2}c\omega {V_1}^2\right\}_\xi - de^{2kt}\int_{\mathbb{R}} J(y)\omega(\xi)V_1(t, \xi)V_1(t, \xi - y)dy \tag{3.25}$$

$$+e^{2kt}\left\{-\frac{c}{2}\frac{\omega'}{\omega}+d+\sigma+\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}-k\right\}\omega {V_1}^2$$
$$+\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2kt}\omega V_1V_2(t-\tau,\xi-c\tau)=-e^{2kt}\omega V_1Q(t-\tau,\xi-c\tau).$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&de^{2kt}\left|\int_{\mathbb{R}}J(y)\omega(\xi)V_1(t,\xi)V_1(t,\xi-y)dy\right|\\
\le\ &\frac{d}{2}e^{2kt}\int_{\mathbb{R}}J(y)\omega(\xi)\left(V_1^2(t,\xi)+V_1^2(t,\xi-y)\right)dy\\
=\ &\frac{d}{2}e^{2kt}\omega(\xi)V_1^2(t,\xi)+\frac{d}{2}e^{2kt}\int_{\mathbb{R}}J(y)\omega(\xi)V_1^2(t,\xi-y)dy.
\end{aligned}$$

Integrating (3.25) over $\mathbb{R}\times[0,t]$ with respect to $\xi$ and $t$, and noting that $\left\{\frac{1}{2}c\omega {V_1}^2\right\}\big|_{-\infty}^{\infty}=0$ since $\sqrt{\omega}V_1\in H^1(\mathbb{R})$ which implies that $(\sqrt{\omega}V_1)\big|_{\pm\infty}=0$, then

$$\begin{aligned}
(3.26)\qquad &e^{2kt}||V_1||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}}e^{2ks}\left\{-c\frac{\omega'}{\omega}+d+2\sigma+\frac{2\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}-2k\right\}\\
&\times\omega V_1^2(s,\xi)d\xi ds-d\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}e^{2ks}J(y)\omega(\xi)V_1^2(s,\xi-y)dyd\xi ds\\
&+2\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_1(s,\xi)V_2(s-\tau,\xi-c\tau)d\xi ds\\
\le\ &||V_{10}(0)||^2_{L^2_\omega}-2\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_1(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|\frac{2\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_1(s,\xi)V_2(s-\tau,\xi-c\tau)\right|\\
\le\ &\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)\left[V_1^2(s,\xi)+V_2^2(s-\tau,\xi-c\tau)\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\left|2\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_1(s,\xi)V_2(s-\tau,\xi-c\tau)d\xi ds\right|\\
\le\ &\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_1^2(s,\xi)d\xi ds\\
&+\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_2^2(s-\tau,\xi-c\tau)d\xi ds\\
=\ &\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_1^2(s,\xi)d\xi ds\\
&+\beta e^{2k\tau}\int_{-\tau}^{t-\tau}\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_2^2(s,\xi)d\xi ds\\
\le\ &\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_1^2(s,\xi)d\xi ds
\end{aligned}$$

$$+\beta e^{2k\tau}\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_2^2(s,\xi)d\xi ds$$
$$+\beta e^{2k\tau}\int_{-\tau}^0\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_{20}^2(s,\xi)d\xi ds.$$

Furthermore, by changing variable $\xi-y\to\xi$, one has

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}e^{2ks}J(y)\omega(\xi)V_1^2(s,\xi-y)dyd\xi ds\\ =\;&\int_0^t\int_{\mathbb{R}}e^{2ks}\left[\int_{\mathbb{R}}J(y)\omega(\xi+y)dy\right]V_1^2(s,\xi)d\xi ds\\ =\;&\int_0^t\int_{\mathbb{R}}e^{2ks}\left[\int_{\mathbb{R}}J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy\right]\omega(\xi)V_1^2(s,\xi)d\xi ds.\end{aligned}$$

Therefore, (3.26) reduces to

$$\begin{aligned}&e^{2kt}||V_1||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}}e^{2ks}\Bigg\{-c\frac{\omega'}{\omega}+d+2\sigma+\frac{2\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}-2k\\ &-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}-d\int_{\mathbb{R}}J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy\Bigg\}\omega V_1^2(s,\xi)d\xi ds\\ &-\beta e^{2k\tau}\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}\frac{\omega(\xi+c\tau)}{\omega(\xi)}e^{2ks}\omega(\xi)V_2^2(s,\xi)d\xi ds\\ &\le||V_{10}(0)||^2_{L^2_\omega}+\beta e^{2k\tau}\int_{-\tau}^0\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_{20}^2(s,\xi)d\xi ds\\ &-2\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_1(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds.\end{aligned}\tag{3.27}$$

Similarly, multiplying (2.4) by $e^{2kt}\omega(\xi)V_2(t,\xi)$, we have

$$\begin{aligned}&\left\{\frac{1}{2}e^{2kt}\omega {V_2}^2\right\}_t+e^{2kt}\left\{\frac{1}{2}c\omega {V_2}^2\right\}_\xi-e^{2kt}\int_{\mathbb{R}}J(y)\omega(\xi)V_2(t,\xi)V_2(t,\xi-y)dy\\ &+\left\{-\frac{c}{2}\frac{\omega'}{\omega}-k+1+(\mu+\gamma)\right\}e^{2kt}\omega {V_2}^2-\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2kt}\omega V_1V_2\\ &-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2kt}\omega V_2V_2(t-\tau,\xi-c\tau)=e^{2kt}\omega V_2Q(t-\tau,\xi-c\tau).\end{aligned}\tag{3.28}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&e^{2kt}\left|\int_{\mathbb{R}}J(y)\omega(\xi)V_2(t,\xi)V_2(t,\xi-y)dy\right|\\ \le\;&\frac{1}{2}e^{2kt}\int_{\mathbb{R}}J(y)\omega(\xi)\left(V_2^2(t,\xi)+V_2^2(t,\xi-y)\right)dy\\ =\;&\frac{1}{2}e^{2kt}\omega(\xi)V_2^2(t,\xi)+\frac{1}{2}e^{2kt}\int_{\mathbb{R}}J(y)\omega(\xi)V_2^2(t,\xi-y)dy.\end{aligned}$$

Integrating (3.28) over $\mathbb{R}\times[0,t]$ with respect to $\xi$ and $t$, it follows that

$$e^{2kt}||V_2||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}}e^{2ks}\left\{-c\frac{\omega'}{\omega}+1+2(\mu+\gamma)-2k\right\}\omega V_2^2(s,\xi)d\xi ds\tag{3.29}$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2ks}J(y)\omega(\xi)V_2^2(s,\xi-y)dyd\xi ds\\
&-2\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2ks}\omega(\xi)V_1(s,\xi)V_2(s,\xi)d\xi ds\\
&-2\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_2(s,\xi)V_2(s-\tau,\xi-c\tau)d\xi ds\\
= \quad & ||V_{20}(0)||^2_{L^2_\omega}+2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds.
\end{aligned}
$$

By Cauchy-Schwarz inequality, one has

$$
\begin{aligned}
&\left|\frac{2\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2ks}\omega(\xi)V_1(s,\xi)V_2(s,\xi)\right|\\
\leq \quad & \frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2ks}\omega(\xi)\left[V_1^2(s,\xi)+V_2^2(s,\xi)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|2\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_2(s,\xi)V_2(s-\tau,\xi-c\tau)d\xi ds\right|\\
\leq \quad & \beta\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2ks}\omega(\xi)V_2^2(s,\xi)d\xi ds\\
&+\beta e^{2k\tau}\int_0^t\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_2^2(s,\xi)d\xi ds\\
&+\beta e^{2k\tau}\int_{-\tau}^0\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_{20}^2(s,\xi)d\xi ds.
\end{aligned}
$$

Furthermore, by changing variable $\xi-y\to\xi$, we get

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2ks}J(y)\omega(\xi)V_2^2(s,\xi-y)dyd\xi ds\\
= \quad & \int_0^t\int_{\mathbb{R}} e^{2ks}\left[\int_{\mathbb{R}} J(y)\omega(\xi+y)dy\right]V_2^2(s,\xi)d\xi ds\\
= \quad & \int_0^t\int_{\mathbb{R}} e^{2ks}\left[\int_{\mathbb{R}} J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy\right]\omega(\xi)V_2^2(s,\xi)d\xi ds.
\end{aligned}
$$

Then (3.29) reduces to

$$
\begin{aligned}
(3.30)\qquad & e^{2kt}||V_2||^2_{L^2_\omega}-\beta\int_0^t\int_{\mathbb{R}}\frac{\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2ks}\omega(\xi)V_1^2(s,\xi)d\xi ds\\
&+\int_0^t\int_{\mathbb{R}} e^{2ks}\Bigg\{-c\frac{\omega'}{\omega}+1+2(\mu+\gamma)-2k-\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\\
&-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}-e^{2k\tau}\frac{\omega(\xi+c\tau)}{\omega(\xi)}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}\\
&-\int_{\mathbb{R}} J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy\Bigg\}\omega(\xi)V_2^2(s,\xi)d\xi ds\\
\leq \quad & ||V_{20}(0)||^2_{L^2_\omega}+\beta e^{2k\tau}\int_{-\tau}^0\int_{\mathbb{R}}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}e^{2ks}\omega(\xi+c\tau)V_{20}^2(s,\xi)d\xi ds
\end{aligned}
$$

$$+2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds.$$

By the same arguments as above for (2.5), we can obtain

$$\begin{aligned}(3.31)\qquad & e^{2kt}||V_3||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}} e^{2ks}\Big\{-c\frac{\omega'}{\omega}+d_3+2\mu_1-\gamma-2k\\ & -d_3\int_{\mathbb{R}} J(y)\frac{\omega(\xi+y)}{\omega(\xi)}dy\Big\}\omega(\xi)V_3^2(s,\xi)d\xi ds\\ & -\gamma\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_2^2(s,\xi)d\xi ds\le ||V_{30}(0)||^2_{L^2_\omega}.\end{aligned}$$

Combining (3.30), (3.31) with (3.27), we get

$$\begin{aligned}(3.32)\qquad & \sum_{i=1}^3\left(e^{2kt}||V_i||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}} B_{i,k,\omega}(\xi)e^{2ks}\omega(\xi)V_i^2(s,\xi)d\xi ds\right)\\ \le\ & \sum_{i=1}^3||V_{i0}(0)||^2_{L^2_\omega}+2\beta\int_{-\tau}^0\int_{\mathbb{R}} e^{2k\tau}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}\frac{\omega(\xi+c\tau)}{\omega(\xi)}\\ & \times e^{2ks}\omega(\xi)V_{20}^2(s,\xi)d\xi ds-2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_1(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\\ & +2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\\ \le\ & \sum_{i=1}^3||V_{i0}(0)||^2_{L^2_\omega}+2\tilde{L}e^{2k\tau}\int_{-\tau}^0\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_{20}^2(s,\xi)d\xi ds\\ & -2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_1(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\\ & +2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds,\end{aligned}$$

due to $\frac{\omega(\xi+c\tau)}{\omega(\xi)}=e^{-c\lambda_*\tau}<1$, where $B_{i,k,\omega}(\xi),\ i=1,2,3$, are respectively defined in (3.17)- (3.19).

By Lemma 3.4, (3.32) reduces to

$$\begin{aligned}(3.33)\qquad & \sum_{i=1}^3\left(e^{2kt}||V_i||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}} C_i(k)e^{2ks}\omega(\xi)V_i^2(s,\xi)d\xi ds\right)\\ \le\ & \sum_{i=1}^3||V_{i0}(0)||^2_{L^2_\omega}+2\tilde{L}e^{2k\tau}\int_{-\tau}^0\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_{20}^2(s,\xi)d\xi ds\\ & -2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_1(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\\ & +2\int_0^t\int_{\mathbb{R}} e^{2ks}\omega(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds.\end{aligned}$$

Now we estimate the nonlinear term on the right-hand side of (3.33). By Taylor's formula, we have

$$\begin{aligned}|Q(t-\tau,\xi-c\tau)| &= \Bigg|\frac{\beta}{(1+\alpha\overline{V}_2(t-\tau,\xi-c\tau))^2}V_1(t,\xi)V_2(t-\tau,\xi-c\tau)\\ &\quad+\frac{\alpha\beta\overline{V}_1(t,\xi)}{(1+\alpha\overline{V}_2(t-\tau,\xi-c\tau))^3}V_2^2(t-\tau,\xi-c\tau)\Bigg|\\ &\le L_1|V_1(t,\xi)V_2(t-\tau,\xi-c\tau)|+L_2|V_2(t-\tau,\xi-c\tau)|^2,\end{aligned}$$

where $\overline{V}_1=\phi_1+\theta_1V_1$, $\overline{V}_2(t-\tau,\xi-c\tau)=\phi_2+\theta_2V_2(t-\tau,\xi-c\tau)$, $\theta_1,\theta_2\in(0,1)$, $L_1:=\sup\limits_{\xi\in\mathbb{R}}\frac{\beta}{(1+\alpha\overline{V}_2(t-\tau,\xi-c\tau))^2}$ and $L_2:=\sup\limits_{\xi\in\mathbb{R}}\frac{2\alpha\beta\overline{V}_1(t,\xi)}{(1+\alpha\overline{V}_2(t-\tau,\xi-c\tau))^3}$. On the other hand, by the definition of $M_{\mathbf{V}}(\infty)$,

$$|V_i(t,\xi)|\le CM_{\mathbf{V}}(\infty),\ t\ge0,\ \xi\in\mathbb{R},\ i=1,2.$$

Then

$$\begin{aligned}&2\left|\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\right| \qquad (3.34)\\ \le& CM_{\mathbf{V}}(\infty)\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)\Big(|V_2(s-\tau,\xi-c\tau)|^2\\ &+|V_2(s,\xi)V_2(s-\tau,\xi-c\tau)|\Big)d\xi ds\\ \le& CM_{\mathbf{V}}(\infty)\int_{-\tau}^{t-\tau}\int_{\mathbb{R}}e^{2k(s+\tau)}\omega(\xi+c\tau)\left|V_2(s,\xi)\right|^2d\xi ds\\ &+CM_{\mathbf{V}}(\infty)\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)\left|V_2(s,\xi)\right|^2d\xi ds\\ \le& CM_{\mathbf{V}}(\infty)\Bigg(\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)\left|V_2(s,\xi)\right|^2d\xi ds\\ &+\int_{-\tau}^0\int_{\mathbb{R}}e^{2ks}\omega(\xi)|V_{20}(s,\xi)|^2d\xi ds\Bigg).\end{aligned}$$

Similarly, we can get that

$$\begin{aligned}&2\left|\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_1(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\right| \qquad (3.35)\\ \le& CM_{\mathbf{V}}(\infty)\Bigg\{\int_{-\tau}^0\int_{\mathbb{R}}e^{2ks}\omega(\xi)|V_{20}(s,\xi)|^2d\xi ds\\ &+\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)\left|V_2(s,\xi)\right|^2d\xi ds+\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)\left|V_1(s,\xi)\right|^2d\xi ds\Bigg\}.\end{aligned}$$

Substituting (3.34) and (3.35) into (3.33), we obtain

$$\begin{aligned}&\sum_{i=1}^3e^{2kt}||V_i||^2_{L^2_\omega}+\sum_{i=1}^2\int_0^t\int_{\mathbb{R}}\left(C_i(k)-C_iM_{\mathbf{V}}(\infty)\right)e^{2ks}\omega(\xi)V_i^2(s,\xi)d\xi ds\\ &+\int_0^t\int_{\mathbb{R}}C_3(k)e^{2ks}\omega(\xi)V_3^2(s,\xi)d\xi ds\\ \le& C\Bigg(\sum_{i=1}^3||V_{i0}(0)||^2_{L^2_\omega}+\int_{-\tau}^0\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_{20}^2(s,\xi)d\xi ds\Bigg)\end{aligned}$$

for some positive constants $C_i$, $i = 1, 2, 3$.

Since $C_i(k) > 0$ for $0 < k < k_0$, $i = 1, 2, 3$, now letting $M_{\mathbf{V}}(\infty) \ll 1$, for example, let

$$0 < M_{\mathbf{V}}(\infty) \le \delta_2 := \min_{i=1,2,3} \left\{ \frac{C_i(k)}{2C_i} \right\}, \tag{3.36}$$

then

$$\begin{aligned} &\sum_{i=1}^{3} e^{2kt} \left( ||V_i||^2_{L^2_\omega} + \int_0^t e^{-2k(t-s)} ||V_i(s)||^2_{L^2_\omega} ds \right) \\ \le\ & C\bigg( \sum_{i=1}^{3} ||V_{i0}(0)||^2_{L^2_\omega} + \int_{-\tau}^{0} ||V_{20}(s)||^2_{L^2_\omega} ds \bigg). \end{aligned} \tag{3.37}$$

Now we estimate $\mathbf{V}_\xi(t, \xi)$ in $L^2_\omega(\mathbb{R})$.

By differentiating (2.3)-(2.5) with respect to $\xi$, and multiplying the resultant equations by $e^{2kt}\omega(\xi)V_{i\xi}(t, \xi)$, $i = 1, 2, 3$, respectively, we obtain

$$\begin{aligned} &\left\{ \frac{1}{2} e^{2kt} \omega V_{1\xi}^2 \right\}_t + e^{2kt} \left\{ \frac{1}{2} c\omega V_{1\xi}^2 \right\}_\xi + de^{2kt} \int_{\mathbb{R}} J(y)\omega(\xi)V_{1\xi}(t,\xi)V_{1\xi}(t,\xi-y)dy \\ &+e^{2kt} \left\{ -\frac{c}{2}\frac{\omega'}{\omega} + d + \sigma + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - k \right\} \omega V_{1\xi}^2 \\ &+\frac{\beta e^{2kt}\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2} \omega V_{1\xi} V_{2\xi}(t-\tau, \xi - c\tau) = -e^{2kt}\omega V_{1\xi} Q_1(t-\tau, \xi - c\tau), \end{aligned}$$

$$\begin{aligned} &\left\{ \frac{1}{2} e^{2kt} \omega V_{2\xi}^2 \right\}_t + e^{2kt} \left\{ \frac{1}{2} c\omega V_{2\xi}^2 \right\}_\xi + e^{2kt} \int_{\mathbb{R}} J(y)\omega(\xi)V_{2\xi}(t,\xi)V_{2\xi}(t,\xi-y)dy \\ &+e^{2kt} \left\{ -\frac{c}{2}\frac{\omega'}{\omega} + 1 + (\mu + \gamma) - k \right\} \omega V_{2\xi}^2 - \frac{\beta e^{2kt}\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} \omega V_{1\xi} V_{2\xi} \\ &-\frac{\beta e^{2kt}\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2} \omega V_{2\xi} V_{2\xi}(t-\tau, \xi - c\tau) = e^{2kt}\omega V_{2\xi} Q_1(t-\tau, \xi - c\tau) \end{aligned}$$

and

$$\begin{aligned} &\left\{ \frac{1}{2} e^{2kt} \omega V_{3\xi}^2 \right\}_t + e^{2kt} \left\{ \frac{1}{2} c\omega V_{3\xi}^2 \right\}_\xi + d_3 e^{2kt} \int_{\mathbb{R}} J(y)\omega(\xi)V_{3\xi}(t,\xi)V_{3\xi}(t,\xi-y)dy \\ &+e^{2kt} \left\{ -\frac{c}{2}\frac{\omega'}{\omega} + d_3 + \mu_1 - k \right\} \omega V_{3\xi}^2 - \gamma e^{2kt}\omega V_{2\xi} V_{3\xi} = 0, \end{aligned}$$

where

$$\begin{aligned} &Q_1(t-\tau, \xi - c\tau) \\ =\ & \left( \frac{\beta(V_2(t-\tau,\xi-c\tau) + \phi_2(\xi - c\tau))}{1 + \alpha(V_2(t-\tau,\xi-c\tau) + \phi_2(\xi - c\tau))} - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} \right) \\ &\times (V_{1\xi}(t,\xi) + \phi_1'(\xi)) + \left( \frac{\beta(V_1(t,\xi) + \phi_1(\xi))}{1 + \alpha(V_2(t-\tau,\xi-c\tau) + \phi_2(\xi - c\tau))^2} \right. \\ &\left. -\frac{\beta\phi_1(\xi)}{1 + \alpha\phi_2(\xi - c\tau)} \right) (V_{2\xi}(t-\tau,\xi-c\tau) + \phi_2'(\xi - c\tau)). \end{aligned}$$

Integrating them over $\mathbb{R}\times[0,t]$ with respect to $\xi$ and $t$, and then by similar arguments as above, we can obtain

$$\begin{aligned}(3.38)\qquad &\sum_{i=1}^{3}\left(e^{2kt}||V_{i\xi}||^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}}B_{i,k,\omega}(\xi)e^{2ks}\omega V^2_{i\xi}(s,\xi)d\xi ds\right)\\ \le\ &\sum_{i=1}^{3}||V_{i0\xi}(0)||^2_{L^2_\omega}-2\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_{1\xi}(s,\xi)Q_1(s-\tau,\xi-c\tau)d\xi ds\\ &+2\int_0^t\int_{\mathbb{R}}e^{2ks}\omega(\xi)V_{2\xi}(s,\xi)Q_1(s-\tau,\xi-c\tau)d\xi ds.\end{aligned}$$

Now using the same arguments as above for the last two terms of (3.38) and combining the basic energy estimate (3.37), we can get that

$$\begin{aligned}(3.39)\qquad &\sum_{i=1}^{3}e^{2kt}\left(||V_{i\xi}||^2_{L^2_\omega}+\int_0^t e^{-2k(t-s)}||V_{i\xi}(s)||^2_{L^2_\omega}ds\right)\\ \le\ & C\left(\sum_{i=1}^{3}||V_{i0}(0)||^2_{H^1_\omega}+\int_{-\tau}^0||V_{20}(s)||^2_{H^1_\omega}ds\right).\end{aligned}$$

Combining (3.37) and (3.39), (3.24) is proved. The proof is complete. □

Next, we establish the following Sobolev inequality.

LEMMA 3.6. *Let* $V_i\in H^1_\omega(\mathbb{R})$, $i=1,2,3$. *Then* $\sqrt{\omega}V_i\in H^1(\mathbb{R})$ *and*

$$(3.40)\qquad ||\sqrt{\omega}V_i||_C\le C||V_i||_{H^1_\omega}$$

*and*

$$(3.41)\qquad \sup_{\xi\in(-\infty,\xi_0]}||V_i(t,\xi)||\le Ce^{-kt},\quad t>0,\ i=1,2,3.$$

PROOF. Since $V_i\in H^1_\omega(\mathbb{R})$, i.e. $\sqrt{\omega}V_i,\sqrt{\omega}V_{i\xi}\in L^2(\mathbb{R})$, it yields that

$$\partial_\xi(\sqrt{\omega}V_i)=\sqrt{\omega}V_{i\xi}-\frac{\lambda_*}{2}\sqrt{\omega}V_i\in L^2(\mathbb{R}),$$

namely, $\sqrt{\omega}V_i\in H^1(\mathbb{R})$. Then by using the Sobolev inequality $H^1(\mathbb{R})\hookrightarrow C(\mathbb{R})$, (3.40) follows. Furthermore, by (3.24) and note that $\omega(\xi)=e^{-\lambda_*(\xi-\xi_0)}\ge 1$ for $\xi\in(-\infty,\xi_0]$, (3.41) holds provided $M_{\mathbf{V}}(\infty)\ll 1$. This completes the proof. □

Finally, to complete the proof of Proposition 3.3, it is necessary to establish the uniform decay estimate of $\mathbf{V}(t,\xi)$ at $\xi=+\infty$ for all $t>0$.

LEMMA 3.7. *If* $\tau<\tau_0$. *Then there exists a large number* $\xi_0\gg 1$ *(independent of* $t$*) such that*

$$||V_i(t)||_{L^\infty[\xi_0,+\infty)}\le CM_{\mathbf{V}}(0)e^{-\gamma_0 t},\quad t>0,\ i=1,2,3.$$

PROOF. Since $\mathbf{V}(t,\xi)\in X_{loc}(-\tau,\infty)$, we have $\lim_{\xi\to+\infty}\mathbf{V}(t,\xi)$ exists uniformly in $t\in[0,\infty)$. Now we go back to the original system (1.3) and (1.2) and denote

$$\begin{cases}\widetilde{V}_i(t,x)=U_i(t,x)-\phi_i(x+ct),\ i=1,2,3,\\ \widetilde{V}_1(0,x)=u_{10}(x)-\phi_1(x):=\widetilde{V}_{10}(x),\ \widetilde{V}_3(0,x)=u_{30}(x)-\phi_3(x):=\widetilde{V}_{30}(x),\\ \widetilde{V}_2(s,x)=u_{20}(s,x)-\phi_2(x+cs):=\widetilde{V}_{20}(s,x),\ s\in[-\tau,0],\ x\in\mathbb{R}.\end{cases}$$

Then $\widetilde{\mathbf{V}}(t,x) = \mathbf{V}(t,\xi)$ and satisfies

$$\begin{cases} \frac{\partial \widetilde{V}_1(t,x)}{\partial t} - d\int_{\mathbb{R}} J(y)\widetilde{V}_1(t,x-y)dy + \left(d+\sigma+\frac{\beta\phi_2(x+ct-c\tau)}{1+\alpha\phi_2(x+ct-c\tau)}\right)\widetilde{V}_1(t,x) \\ \qquad +\frac{\beta\phi_1(x+ct)}{(1+\alpha\phi_2(x+ct-c\tau))^2}\widetilde{V}_2(t-\tau,x) = -Q(t-\tau,x), \\ \frac{\partial \widetilde{V}_2(t,x)}{\partial t} - \int_{\mathbb{R}} J(y)\widetilde{V}_2(t,x-y)dy + (1+\mu+\gamma)\widetilde{V}_2(t,x) - \frac{\beta\phi_2(x+ct-c\tau)}{1+\alpha\phi_2(x+ct-c\tau)}\widetilde{V}_1(t,x) \\ \qquad -\frac{\beta\phi_1(x+ct)}{(1+\alpha\phi_2(x+ct-c\tau))^2}\widetilde{V}_2(t-\tau,x) = Q(t-\tau,x),\ t>0,\ x\in\mathbb{R}, \\ \frac{\partial \widetilde{V}_3(t,x)}{\partial t} - d_3\int_{\mathbb{R}} J(y)\widetilde{V}_3(t,x-y)dy - \gamma\widetilde{V}_2(t,x) + (d_3+\mu_1)\widetilde{V}_3(t,x) = 0, \\ \widetilde{V}_1(0,x) = \widetilde{V}_{10}(x),\ \widetilde{V}_3(0,x) = \widetilde{V}_{30}(x),\ \widetilde{V}_2(s,x) = \widetilde{V}_{20}(s,x),\ s\in[-\tau,0],\ x\in\mathbb{R}. \end{cases}$$

By Proposition 3.2, $\mathbf{V}(t,\xi) \in \mathcal{C}_{unif}[0,\infty)$. Then $\lim_{\xi\to+\infty}\mathbf{V}(t,\xi) = \lim_{x\to+\infty}\widetilde{\mathbf{V}}(t,x) = \widetilde{\mathbf{V}}(t,\infty)$ exists uniformly in $t\in[0,\infty)$. Letting $x\to+\infty$ in above system, we get (3.42)

$$\begin{cases} \widetilde{V}_{1t}(t,\infty) + \left(\sigma+\frac{\beta I^*}{1+\alpha I^*}\right)\widetilde{V}_1(t,\infty) + \frac{\beta S^*}{(1+\alpha I^*)^2}\widetilde{V}_2(t-\tau,\infty) = -Q(t-\tau,\infty), \\ \widetilde{V}_{2t}(t,\infty) - \frac{\beta I^*}{1+\alpha I^*}\widetilde{V}_1(t,\infty) - \frac{\beta S^*}{(1+\alpha I^*)^2}\widetilde{V}_2(t-\tau,\infty) \\ \qquad +(\mu+\gamma)\widetilde{V}_2(t,\infty) = Q(t-\tau,\infty), \\ \widetilde{V}_{3t}(t,\infty) - \gamma\widetilde{V}_2(t,\infty) + \mu_1\widetilde{V}_3(t,\infty) = 0, \\ \widetilde{V}_i(0,\infty) = \widetilde{V}_{i0}(\infty), \quad \widetilde{V}_2(s,\infty) = \widetilde{V}_{20}(s,\infty), \quad i=1,3,\ s\in[-\tau,0]. \end{cases}$$

In order to estimate nonlinear delayed system (3.42), we first consider the corresponding linear system

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}' = A\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} + B\begin{pmatrix} u(t-\tau) \\ v(t-\tau) \\ w(t-\tau) \end{pmatrix} \tag{3.43}$$

with

$$A = \begin{pmatrix} -\sigma-\frac{\beta I^*}{1+\alpha I^*} & 0 & 0 \\ \frac{\beta I^*}{1+\alpha I^*} & -(\mu+\gamma) & 0 \\ 0 & \gamma & -\mu_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{\beta S^*}{(1+\alpha I^*)^2} & 0 \\ 0 & \frac{\beta S^*}{(1+\alpha I^*)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the initial value $\mathbf{\Psi}(s) = \begin{pmatrix} \widetilde{V}_{10}(\infty) \\ \widetilde{V}_{20}(s,\infty) \\ \widetilde{V}_{30}(\infty) \end{pmatrix}$ for $s\in[-\tau,0]$, and the system of ordinary differential equations

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}' = P\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \tag{3.44}$$

with the initial value $\mathbf{\Psi}(s)$ and $P = A+B$.

Note that $\frac{\beta S^*}{(1+\alpha I^*)^2} = \frac{\mu+\gamma}{1+\alpha I^*}$. Then $|\lambda I - P| = 0$ if and only if

$$(\lambda+\mu_1)\Big\{\lambda^2 + \Big(\sigma+\frac{\beta I^*}{1+\alpha I^*}+\frac{\alpha(\mu+\gamma)I^*}{1+\alpha I^*}\Big)\lambda + \Big(\sigma+\frac{\beta I^*}{1+\alpha I^*}\Big)\frac{\alpha(\mu+\gamma)I^*}{1+\alpha I^*} + \frac{\beta(\mu+\gamma)I^*}{(1+\alpha I^*)^2}\Big\} = 0,$$

and hence, it is easy to see that all the eigenvalues of the matrix $P$ have negative real parts.

Motivated by [**17**, Section 4.4.5], we can prove the following lemma.

LEMMA 3.8. *Let* $\mathbf{X}(t)$ *be the solution of* (3.43). *Then there exist positive constants* $\tau_0, \gamma_0$ *such that, when* $\tau < \tau_0$,

$$|\mathbf{X}(t)| \leq Ce^{-\gamma_0 t}, \quad t > 0.$$

PROOF. Assume $\mathbf{U}(t,s)$ is a fundamental matrix of solutions of the system (3.44), then by [**46**, Page 432, Theorem 8.18], there exists positive constants $K$ and $h$ such that

$$|\mathbf{U}(t,s)| \leq Ke^{-h(t-s)} \text{ for any } t > s.$$

Let $\mathbf{X}(t)$ be any solution of system (3.43). Then

$$\mathbf{X}(t) = \mathbf{U}(t,0)\mathbf{X}(0) + \int_0^t B\mathbf{U}(t,s)(\mathbf{X}(s-\tau) - \mathbf{X}(s))ds.$$

For $t > 2\tau$, $\mathbf{X}(t)$ can be written as

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{U}(t,0)\mathbf{X}(0) + \int_0^{2\tau} B\mathbf{U}(t,s)(\mathbf{X}(s-\tau) - \mathbf{X}(s))ds \\ &\quad + \int_{2\tau}^t B\mathbf{U}(t,s)(\mathbf{X}(s-\tau) - \mathbf{X}(s))ds. \end{aligned}$$

Now we estimate $|\mathbf{X}(s-\tau) - \mathbf{X}(s)|$. By the fundamental theory of ODE, it is not difficult to get that, for $0 \leq s \leq 2\tau$,

$$|\mathbf{X}(s)| \leq e^{2(|A|+|B|)\tau}||\mathbf{\Psi}||,$$

where $|A| = \sup_{|\mathbf{x}|\leq 1} |A\mathbf{x}|$ and $|\mathbf{x}|$ is the Euclidean norm (see e.g. [**17**, Page 39], then $|A|$ is given by $\sqrt{\Lambda}$, where $\Lambda$ is the largest eigenvalue of the matrix $A'A$, $'$ denotes the transpose). Then for $0 \leq s \leq 2\tau$, it follows that

$$|\mathbf{X}(s-\tau) - \mathbf{X}(s)| \leq 2e^{2(|A|+|B|)\tau}||\mathbf{\Psi}||.$$

For $s \geq 2\tau$, we have

$$\mathbf{X}(s-\tau) - \mathbf{X}(s) = \int_s^{s-\tau} \mathbf{X}'(\eta)d\eta = \int_s^{s-\tau} [A\mathbf{X}(\eta) - B\mathbf{X}(\eta-\tau)]d\eta,$$

hence

$$|\mathbf{X}(s-\tau) - \mathbf{X}(s)| \leq (|A|+|B|)\tau \sup_{s-2\tau\leq\eta\leq s} |\mathbf{X}(\eta)|.$$

It follows that

$$\begin{aligned} |\mathbf{X}(t)| &\leq Ke^{-ht}||\mathbf{\Psi}|| + 2\int_0^{2\tau} |B|Ke^{-h(t-s)}e^{2(|A|+|B|)\tau}||\mathbf{\Psi}||ds \\ &\quad + \int_{2\tau}^t |B|Ke^{-h(t-s)}(|A|+|B|)\tau \sup_{s-2\tau\leq\eta\leq s} |\mathbf{X}(\eta)|ds \\ &= Ke^{-ht}||\mathbf{\Psi}|| + \frac{2}{h}(e^{2h\tau}-1)|B|Ke^{-ht}e^{2(|A|+|B|)\tau}||\mathbf{\Psi}|| \\ &\quad + K|B|(|A|+|B|)\tau \int_{2\tau}^t e^{-h(t-s)} \sup_{s-2\tau\leq\eta\leq s} |\mathbf{X}(\eta)|ds. \end{aligned}$$

Let

$$L_0 = K||\mathbf{\Psi}|| \left(1 + \frac{2}{h}(e^{2h\tau} - 1)|B|e^{2(|A|+|B|)\tau}\right), \quad M = K|B|(|A| + |B|)\tau.$$

Then

$$|\mathbf{X}(t)| \leq L_0 e^{-ht} + M \int_{2\tau}^{t} e^{-h(t-s)} \sup_{s-2\tau \leq \eta \leq s} |\mathbf{X}(\eta)| ds.$$

Now let

$$Y(t) = e^{-ht} \left( L_0 + M \int_{2\tau}^{t} e^{hs} \sup_{s-2\tau \leq \eta \leq s} |\mathbf{X}(\eta)| ds \right).$$

We have

$$\begin{aligned} Y'(t) &= -he^{-ht} \left( L_0 + M \int_{2\tau}^{t} e^{hs} \sup_{s-2\tau \leq \eta \leq s} |\mathbf{X}(\eta)| ds \right) \\ &\quad + e^{-ht} M e^{ht} \sup_{t-2\tau \leq \eta \leq t} |\mathbf{X}(\eta)| \\ &= -hY(t) + M \sup_{t-2\tau \leq \eta \leq t} |\mathbf{X}(\eta)|. \end{aligned}$$

Since $|\mathbf{X}(t)| \leq Y(t)$, $\sup_{t-2\tau \leq \eta \leq t} |\mathbf{X}(\eta)| \leq \sup_{t-2\tau \leq \eta \leq t} Y(\eta)$. Thus,

$$Y'(t) \leq -hY(t) + M \sup_{t-2\tau \leq \eta \leq t} Y(\eta).$$

If $M < h$, namely,

$$\tau < \frac{h}{K} |B|(|A| + |B|) := \tau_0,$$

then by [**17**, Lemma in Page 378], there exist constants $C$ and $\gamma_0$ such that

$$Y(t) \leq Ce^{-\gamma_0 t}, \quad t > 0.$$

and hence a similar inequality holds for $|\mathbf{X}(t)|$. This completes the proof. $\square$

Thus, as a nonlinear perturbation to the linear delay system (3.43), by Lemma 3.8 and [**18**, Corollary 9.2.2], (3.42) satisfies the following nonlinear stability.

LEMMA 3.9. *Let* $\widetilde{\mathbf{V}}_\infty(t) := (\widetilde{V}_1(t, \infty), \widetilde{V}_2(t, \infty), \widetilde{V}_3(t, \infty))$ *be the solution of* (3.42)*. If* $\tau < \tau_0$*, then*

$$|\widetilde{\mathbf{V}}_\infty(t)| \leq CM_{\widetilde{\mathbf{V}}}(0)e^{-\gamma_0 t}, \quad t > 0,$$

*provided* $M_{\widetilde{\mathbf{V}}}(0) \ll 1$*, where* $\tau_0, \gamma_0 > 0$ *are defined in Lemma 3.8.*

For $i = 1, 2, 3$, since

$$\lim_{x \to +\infty} \left| e^{\gamma_0 t} \widetilde{V}_i(t, x) - e^{\gamma_0 t} \widetilde{V}_i(t, \infty) \right| = 0 \text{ uniformly in } t \in [0, \infty),$$

namely, for any given positive number $\epsilon > 0$, there exists a positive number $\xi_0 = \xi_0(\epsilon)$ sufficient large and independent of $t$ such that when $x \geq \xi_0$,

$$\left| e^{\gamma_0 t} \widetilde{V}_i(t, x) - e^{\gamma_0 t} \widetilde{V}_i(t, \infty) \right| < \epsilon,$$

which implies that

$$\left| e^{\gamma_0 t} |\widetilde{V}_i(t, x)| - e^{\gamma_0 t} |\widetilde{V}_i(t, \infty)| \right| \leq \left| e^{\gamma_0 t} \widetilde{V}_i(t, x) - e^{\gamma_0 t} \widetilde{V}_i(t, \infty) \right| < \epsilon.$$

From Lemma 3.9, $e^{\gamma_0 t}|\widetilde{V}_i(t,\infty)| \le CM_{\widetilde{\mathbf{V}}}(0)$ is uniformly bounded with respect to $t$, then

$$e^{\gamma_0 t}|\widetilde{V}_i(t,x)| \le CM_{\widetilde{\mathbf{V}}}(0) + \epsilon \quad \text{for } x \ge \xi_0,\ t \in [0,\infty).$$

Let $\epsilon = M_{\widetilde{\mathbf{V}}}(0)$, then

$$\sup_{x\in[\xi_0,+\infty)} |\widetilde{\mathbf{V}}(t,x)| \le CM_{\widetilde{\mathbf{V}}}(0)e^{-\gamma_0 t} \quad \text{for } t \in [0,\infty).$$

Note that $\mathbf{V}(t,\xi) = \widetilde{\mathbf{V}}(t,x)$ and $\xi = x + ct \ge x \ge \xi_0$ for $x \ge \xi_0$ and $t \ge 0$. Thus we have

$$\sup_{\xi\in[\xi_0,+\infty)} |\mathbf{V}(t,\xi)| \le CM_{\mathbf{V}}(0)e^{-\gamma_0 t} \text{ for } t \in [0,\infty).$$

This completes the proof of lemma 3.7. □

Proof of Proposition 3.3. Now by Lemmas 3.5-3.7, we can get

$$M^2_{\mathbf{V}}(\infty) \le C_4 M^2_{\mathbf{V}}(0)e^{-2kt}, \quad t \ge 0, \tag{3.45}$$

with $0 < k \le \min\{k_0, \gamma_0\}$ and some positive constant $C_4$, and hence, Proposition 3.3 is followed.

In order to guarantee the estimates $M_{\mathbf{V}}(\infty) \le \delta_2$ (see (3.36)), we take $\delta_0 > 0$ as

$$\delta_0 := \frac{\delta_2}{\sqrt{C_4}}.$$

By (3.45), we have $M_{\mathbf{V}}(\infty) \le \sqrt{C_4}M_{\mathbf{V}}(0)e^{-kt}$ for $t \ge 0$. Then when $M_{\mathbf{V}}(0) \le \delta_0$, we can guarantee

$$M_{\mathbf{V}}(\infty) \le \sqrt{C_4}M_{\mathbf{V}}(0)e^{-kt} \le \sqrt{C_4}M_{\mathbf{V}}(0) = \delta_2.$$

This completes the proof. □

## 4. Uniqueness of Traveling Waves

The aim of this section is to show the uniqueness of traveling wave solutions of (1.3). It suffices to consider the subsystem (1.6). We first establish the exact asymptotic behavior of traveling waves as $\xi \to -\infty$ by using a modified version of Ikehara's theorem (see Lemma 4.1), and then prove the uniqueness by using the stability result in Theorem 1.2.

Lemma 4.1 ( [**16**]). *Let $F(\lambda) = \int_{-\infty}^0 u(\xi)e^{\lambda\xi}d\xi$, where $u(\xi)$ is a positive increasing function for $\xi \in \mathbb{R}$. Assume that $F(\lambda)$ can be written in the form*

$$F(\lambda) = \frac{H(\lambda)}{(\alpha-\lambda)^{k+1}},$$

*where $k > -1$, $\alpha > 0$ and $H(\lambda)$ is analytic in the strip $\alpha - \epsilon < Re\lambda \le \alpha$ for $\epsilon > 0$. Then*

$$\lim_{\xi\to-\infty} \frac{u(\xi)}{|\xi|^k e^{\lambda\xi}} = \frac{H(\lambda)}{\Gamma(\lambda+1)}.$$

To prove the results, it is more convenient to work on $(U, I)$, where $U = 1 - u_1$, $I = u_2$. Then system (1.6) is equivalent to

$$\begin{cases} \frac{\partial U(t,x)}{\partial t} = d[(J*U)(t,x) - U(t,x)] - \sigma U(t,x) + \frac{\beta(1-U(t,x))I(t-\tau,x)}{1+\alpha I(t-\tau,x)}, \\ \frac{\partial I(t,x)}{\partial t} = (J*I)(t,x) - I(t,x) + \frac{\beta(1-U(t,x))I(t-\tau,x)}{1+\alpha I(t-\tau,x)} - (\mu+\gamma)I(t,x). \end{cases} \tag{4.1}$$

The wave solution equation of system (4.1) is

$$(4.2)\qquad \begin{cases} cU'(\xi) = d[(J*U)(\xi) - U(\xi)] - \sigma U(\xi) + \frac{\beta(1-U(\xi))I(\xi-c\tau)}{1+\alpha I(\xi-c\tau)}, \\ cI'(\xi) = (J*I)(\xi) - I(\xi) + \frac{\beta(1-U(\xi))I(\xi-c\tau)}{1+\alpha I(\xi-c\tau)} - (\mu+\gamma)I(\xi), \\ (U,I)(-\infty) = (0,0), \quad (U,I)(+\infty) = (1-S^*, I^*). \end{cases}$$

We first give a lemma as follows.

LEMMA 4.2. *Any nonconstant positive solution* $(U, I)$ *of* (4.2) *satisfies*

$$U(\xi) = O(e^{\nu\xi}),\ I(\xi) = O(e^{\lambda_1\xi}) \text{ as } \xi \to -\infty,$$

*where* $\nu = \min\{\lambda_1, \lambda_4\}$*, and* $\lambda_4$ *is the unique positive root of the following equation*

$$d\int_{-\infty}^{+\infty} J(y)[e^{-\lambda y} - 1]dy - c\lambda - \sigma = 0.$$

PROOF. By Proposition 1.1, $I(\xi) = O(e^{\lambda_1\xi})$ as $\xi \to -\infty$. We further claim that there exists a number $\xi_0 < 0$ sufficient negative and a constant $\widehat{C} > 0$ large enough such that $U(\xi) \le \widehat{C}e^{\nu\xi}$ for $\xi \le \xi_0$, and hence the conclusion follows.

Indeed, by [**21**, Lemma 2.3], $I(\xi) \le e^{\lambda_1\xi} := I_+(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, by [**21**, Remark 2.9], it is easy to get that $U(\xi) \le \frac{\beta}{\alpha\sigma+\beta}$ for $\xi \in \mathbb{R}$. Letting

$$U_+(\xi) := \min\left\{\widehat{C}e^{\nu\xi}, \frac{\beta}{\alpha\sigma+\beta}\right\},$$

and taking $\widehat{C} > 1$, then there exists a number $\xi_1 < 0$ such that $\widehat{C}e^{\nu\xi_1} = \frac{\beta}{\alpha\sigma+\beta}$. We aim to prove that $U_+(\xi)$ is an upper solution of the U-equation of (4.2), namely, $U_+(\xi)$ satisfies

$$(4.3)\qquad cU_+'(\xi) \ge d[(J*U_+)(\xi) - U_+(\xi)] - \sigma U_+(\xi) + \frac{\beta(1-U_+(\xi))I_+(\xi-c\tau)}{1+\alpha I_+(\xi-c\tau)}.$$

For $\xi \ge \xi_1$, $U_+(\xi) = \frac{\beta}{\alpha\sigma+\beta}$. It is not difficult to verify that (4.3) holds.

For $\xi < \xi_1$, $U_+(\xi) = \widehat{C}e^{\nu\xi}$. Since

$$\begin{aligned}
(4.4)\qquad & cU_+'(\xi) - d[(J*U_+)(\xi) - U_+(\xi)] + \sigma U_+(\xi) - \frac{\beta(1-U_+(\xi))I_+(\xi-c\tau)}{1+\alpha I_+(\xi-c\tau)} \\
\ge\ & c\nu\cdot\widehat{C}e^{\nu\xi} - d\Bigg[\int_{-\infty}^{\xi_1} J(\xi-y)\cdot\widehat{C}e^{\nu y}dy + \int_{\xi_1}^{+\infty} J(\xi-y)\cdot\frac{\beta}{\alpha\sigma+\beta}dy \\
& -\widehat{C}e^{\nu\xi}\Bigg] + \sigma\cdot\widehat{C}e^{\nu\xi} - \beta e^{\lambda_1(\xi-c\tau)} \\
=\ & c\nu\cdot\widehat{C}e^{\nu\xi} - d\left[\int_{-\infty}^{+\infty} J(\xi-y)\cdot\widehat{C}e^{\nu y}dy - \widehat{C}e^{\nu\xi}\right] + \sigma\cdot\widehat{C}e^{\nu\xi} \\
& +d\int_{\xi_1}^{+\infty} J(\xi-y)\left[\widehat{C}e^{\nu y} - \frac{\beta}{\alpha\sigma+\beta}\right]dy - \beta e^{\lambda_1(\xi-c\tau)} \\
=\ & \left(c\nu - d\int_{\mathbb{R}} J(y)[e^{-\nu y} - 1]dy + \sigma\right)\cdot\widehat{C}e^{\nu\xi} - \beta e^{\lambda_1(\xi-c\tau)} \\
& +d\int_{\xi_1}^{+\infty} J(\xi-y)\left[\widehat{C}e^{\nu y} - \frac{\beta}{\alpha\sigma+\beta}\right]dy.
\end{aligned}$$

Note that if $\lambda_4 > \lambda_1$, then $\nu = \lambda_1$ and $c\lambda_1 - d\int_{\mathbb{R}} J(y)[e^{-\lambda_1 y} - 1]dy + \sigma > 0$, while if $\lambda_4 \leq \lambda_1$, then $\nu = \lambda_4$ and $c\lambda_4 - d\int_{\mathbb{R}} J(y)[e^{-\lambda_4 y} - 1]dy + \sigma = 0$. Now take

$$\widehat{C} > \max\left\{\frac{\beta + d}{d\int_0^\infty J(\xi - y)e^{\nu y}dy}, 1\right\}.$$

Since $\xi \leq \xi_1 < 0$, we have

$$\begin{aligned} (4.4) \quad &\geq d\int_{\xi_1}^{+\infty} J(\xi - y)\left[\widehat{C}e^{\nu y} - \frac{\beta}{\alpha\sigma + \beta}\right]dy - \beta e^{\lambda_1(\xi - c\tau)} \\ &> \widehat{C}d\int_0^\infty J(\xi - y)e^{\nu y}dy - (d + \beta) > 0. \end{aligned}$$

Hence (4.3) is true. Consequently, $U(\xi) \leq U_+(\xi)$ for all $\xi \in \mathbb{R}$.

Then for any $\xi_0 \leq \xi_1 < 0$, we can get that $U(\xi) \leq \widehat{C}e^{\nu\xi}$ for all $\xi \leq \xi_0$. The conclusion follows and the proof is completed. □

Now we have the following results on asymptotic behavior at $-\infty$.

THEOREM 4.3. (Asymptotic behavior at $-\infty$). Let $(U, I)$ be a nontrivial positive solution of (4.2). Then the following statements hold.

(i) there exist a constant $\theta_1 \in \mathbb{R}$ such that

$$\lim_{\xi \to -\infty} \frac{I(\xi + \theta_1)}{|\xi|^k e^{\lambda_1 \xi}} = 1,$$

where $k = 0$ for $c > c^*$, and $k = 1$ for $c = c^*$.

(ii) there exist constants $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{R}$ such that

$$\begin{aligned} &\lim_{\xi \to -\infty} \frac{U(\xi + \eta_1)}{|\xi|^k e^{\lambda_1 \xi}} = 1 \text{ if } \lambda_4 > \lambda_1, \\ &\lim_{\xi \to -\infty} \frac{U(\xi + \eta_2)}{|\xi|^{k+1} e^{\lambda_1 \xi}} = 1 \text{ if } \lambda_4 = \lambda_1, \\ &\lim_{\xi \to -\infty} \frac{U(\xi + \eta_3)}{e^{\lambda_4 \xi}} = 1 \text{ if } \lambda_4 < \lambda_1, \end{aligned}$$

where $k = 0$ for $c > c^*$, and $k = 1$ for $c = c^*$.

PROOF. (i) By Lemma 4.2, we can define the following two-sided Laplace transforms

$$\begin{aligned} \mathcal{L}(\lambda, U) &:= \int_{-\infty}^{+\infty} e^{-\lambda\xi} U(\xi)d\xi, \quad \lambda \in \mathbb{C},\ 0 < Re\lambda < \nu, \\ \mathcal{L}(\lambda, I) &:= \int_{-\infty}^{+\infty} e^{-\lambda\xi} I(\xi)d\xi, \quad \lambda \in \mathbb{C},\ 0 < Re\lambda < \lambda_1. \end{aligned}$$

Now we claim that $U'(\xi) = O(e^{\nu\xi})$, $I'(\xi) = O(e^{\lambda_1\xi})$ as $\xi \to -\infty$. In fact, by Lemma 4.2, we have

$$\frac{\beta(1 - U(\xi))I(\xi - c\tau)}{1 + \alpha I(\xi - c\tau)} = O(e^{\lambda_1\xi}) \text{ as } \xi \to -\infty,$$

and

$$J * U(\xi) = \int_{\mathbb{R}} J(y)U(\xi - y)dy = O(e^{\nu\xi}), \quad J * I(\xi) = \int_{\mathbb{R}} J(y)I(\xi - y)dy = O(e^{\lambda_1\xi})$$

as $\xi \to -\infty$. Then the conclusion follows by (4.2).

Set $f(U,I) = \frac{\beta(1-U)I}{1+\alpha I}$. For any $0 < \kappa < \lambda_1 + \nu$, we can get

$$f(U(\xi), I(\xi)) - \partial_2 f(0,0)I(\xi) = O(|U(\xi)| \cdot |I(\xi)|) = O(e^{\kappa\xi}) \text{ as } \xi \to -\infty.$$

Let

$$Q(\lambda) := \beta \int_{-\infty}^{+\infty} \left( I(\eta - c\tau) - \frac{(1-U(\eta))I(\eta - c\tau)}{1+\alpha I(\eta - c\tau)} \right) e^{-\lambda\eta} d\eta.$$

Then $Q(\lambda)$ is analytic in the strip $Re\lambda \in (0, \lambda_1 + \nu)$. Moreover, it is not difficult to get that $Q(\lambda) > 0$ for $Re\lambda \in (0, \lambda_1 + \nu)$. Then equations (4.2) can be rewritten as

$$d[(J * U)(\xi) - U(\xi)] - cU'(\xi) - \sigma U(\xi) = -\frac{\beta(1-U(\xi))I(\xi - c\tau)}{1+\alpha I(\xi - c\tau)}, \tag{4.5}$$

$$(J*I)(\xi) - I(\xi) - cI'(\xi) + \beta I(\xi - c\tau) - (\mu+\gamma)I(\xi) = \beta I(\xi - c\tau) - \frac{\beta(1-U(\xi))I(\xi - c\tau)}{1+\alpha I(\xi - c\tau)}. \tag{4.6}$$

Note that

$$\begin{aligned} \int_{\mathbb{R}} e^{-\lambda\xi}(J * U)(\xi) d\xi &= \int_{\mathbb{R}} e^{-\lambda y} J(y) \int_{\mathbb{R}} U(\xi - y) e^{-\lambda(\xi - y)} d\xi dy \\ &= \mathcal{L}(\lambda, U) \int_{\mathbb{R}} e^{-\lambda y} J(y) dy \end{aligned}$$

and

$$\int_{\mathbb{R}} e^{-\lambda\xi}(J * I)(\xi) d\xi = \mathcal{L}(\lambda, I) \int_{\mathbb{R}} e^{-\lambda y} J(y) dy.$$

Multiplying (4.5) and (4.6) by $e^{-\lambda\xi}$ and integrating both sides of the equations from $-\infty$ to $+\infty$, we get

$$\left( d\int_{\mathbb{R}} J(y)[e^{-\lambda y} - 1]dy - c\lambda - \sigma \right) \mathcal{L}(\lambda, U) = Q(\lambda) - \beta e^{-\lambda c\tau} \mathcal{L}(\lambda, I), \ Re\lambda \in (0, \nu), \tag{4.7}$$

$$\left( \int_{\mathbb{R}} J(y)[e^{-\lambda y} - 1]dy - c\lambda + \beta e^{-\lambda c\tau} - (\mu+\gamma) \right) \mathcal{L}(\lambda, I) = Q(\lambda), \ Re\lambda \in (0, \lambda_1). \tag{4.8}$$

Let $\mathcal{J}(\lambda, u) = \int_{-\infty}^0 u(\xi) e^{-\lambda\xi} d\xi$. Then by (4.7) and (4.8),

$$\mathcal{J}(\lambda, I) = \frac{Q(\lambda)}{\Delta(\lambda, c)} - \int_0^{+\infty} I(\xi) e^{-\lambda\xi} d\xi, \ Re\lambda \in (0, \lambda_1),$$

$$\mathcal{J}(\lambda, U) = \frac{Q(\lambda)}{P_1(\lambda)} - \frac{\beta e^{-\lambda c\tau} Q(\lambda)}{P(\lambda)} - \int_0^{+\infty} U(\xi) e^{-\lambda\xi} d\xi, \ Re\lambda \in (0, \nu), \tag{4.9}$$

where $P_1(\lambda) = d\int_{\mathbb{R}} J(y)[e^{-\lambda y} - 1]dy - c\lambda - \sigma$ and $P(\lambda) = P_1(\lambda)\Delta(\lambda, c)$.

Denoting $\mathcal{H}(\lambda, v) = \mathcal{J}(\lambda, v)(\lambda_1 - \lambda)^{k+1}$, then

$$\mathcal{H}(\lambda, I) = \frac{Q(\lambda)}{\Delta(\lambda, c)/(\lambda_1 - \lambda)^{k+1}} - (\lambda_1 - \lambda)^{k+1} \int_0^{+\infty} I(\xi) e^{-\lambda\xi} d\xi,$$

where $k = 0$ for $c > c^*$ and $k = 1$ for $c = c^*$, since $\Delta(\lambda, c) = 0$ has a simple root $\lambda_1$ when $c > c^*$ and a double root $\lambda_1$ when $c = c^*$.

Now we prove that $\mathcal{H}$ is analytic for $0 < Re\lambda \le \lambda_1$. Since $I$ is bounded and $(\lambda_1 - \lambda)^{k+1} \int_0^{\infty} I(\xi) e^{-\lambda\xi} d\xi$ is analytic for $Re\lambda > 0$, it suffices to show that

$\frac{Q(\lambda)}{\Delta(\lambda,c)/(\lambda_1-\lambda)^{k+1}} := \mathcal{F}(\lambda)$ is analytic for $0 < Re\lambda \leq \lambda_1$. Since $\mathcal{F}(\lambda) = \mathcal{L}(\lambda, I)(\lambda_1 - \lambda)^{k+1}$ and $\mathcal{L}(\lambda, I)$ is well defined for $0 < Re\lambda < \lambda_1$, we get that $\mathcal{F}(\lambda)$ is analytic for $0 < Re\lambda < \lambda_1$. It remains to show that $\mathcal{F}(\lambda)$ is analytic for $Re\lambda = \lambda_1$. We claim that $\Delta(\lambda, c) = 0$ does not have any zeros with $Re\lambda = \lambda_1$ other than $\lambda = \lambda_1$. In fact, let $\lambda_0 = \lambda_1 + bi$, then by $\Delta(\lambda_0, c) = \Delta(\lambda_1, c) = 0$, we have

$$\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}\sin^2\frac{by}{2}dy + \beta e^{-\lambda_1 c\tau}\sin^2\frac{bc\tau}{2} = 0,$$

and

$$\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}\sin(by)dy + cb + \beta e^{-\lambda_1 c\tau}\sin(bc\tau) = 0,$$

which implies that $b = 0$. Thus $\mathcal{F}(\lambda)$ is analytic for $Re\lambda = \lambda_1$. Hence, $\mathcal{H}(\lambda, I)$ is analytic for $0 < Re\lambda \leq \lambda_1$. Furthermore, since $Q(\lambda_1) > 0$ and $(\lambda_1-\lambda)^{k+1}/\Delta(\lambda, c) > 0$ at $\lambda = \lambda_1$, we have $\mathcal{H}(\lambda_1, I) > 0$.

Note that $I(\xi)$ may not be monotone on $(-\infty, 0)$, we claim that there exists a $m > 0$ such that $\widetilde{I}(\xi) := I(\xi)e^{m\xi}$ is monotone increasing on $(-\infty, 0)$. Indeed, by (4.2), we have $cI' + (1 + \mu + \gamma)I \geq J * I$, then $\left(I(\xi)e^{\frac{1+\mu+\gamma}{c}\xi}\right)' \geq 0$ and hence $\widetilde{I}(\xi)$ is monotone provided $m \geq \frac{1+\mu+\gamma}{c}$. Thus we can apply Lemma 4.1 to $\tilde{I}(\xi)$. More precisely, let

$$\widetilde{\mathcal{J}}(\lambda, I) = \int_{-\infty}^{0} \widetilde{I}(\xi)e^{-\lambda\xi}d\xi = \mathcal{J}(\lambda - m, I).$$

It follows that

$$\widetilde{\mathcal{J}}(\lambda, I) = \widetilde{\mathcal{H}}(\lambda, I)/((\lambda_1 + m) - \lambda)^{k+1},$$

where $\widetilde{\mathcal{H}}(\lambda, I) = \mathcal{H}(\lambda - m, I)$ is analytic for $m < Re\lambda \leq \lambda_1 + m$.

By Lemma 4.1, we have

$$\frac{\mathcal{H}(\lambda_1, I)}{\Gamma(\lambda_1 + m + 1)} = \frac{\widetilde{\mathcal{H}}(\lambda_1 + m, I)}{\Gamma(\lambda_1 + m + 1)} = \lim_{\xi\to-\infty}\frac{\widetilde{I}(\xi)}{|\xi|^k e^{(\lambda_1+m)\xi}} = \lim_{\xi\to-\infty}\frac{I(\xi)}{|\xi|^k e^{\lambda_1\xi}}.$$

Then for any $\theta \in \mathbb{R}$,

$$\frac{\mathcal{H}(\lambda_1, I)}{\Gamma(\lambda_1 + m + 1)} = \lim_{\xi\to-\infty}\frac{I(\xi+\theta)}{|\xi+\theta|^k e^{\lambda_1(\xi+\theta)}} = e^{-\lambda_1\theta}\lim_{\xi\to-\infty}\frac{I(\xi+\theta)}{|\xi|^k e^{\lambda_1\xi}}.$$

Let $\theta_1$ be the constant such that $e^{\lambda_1\theta_1}\mathcal{H}(\lambda_1, I)/\Gamma(\lambda_1 + m + 1) = 1$, then

$$\lim_{\xi\to-\infty}\frac{I(\xi+\theta_1)}{|\xi|^k e^{\lambda_1\xi}} = 1.$$

(ii) For $c > c^*$, by (4.9) and note that

$$\begin{aligned}\mathcal{H}(\lambda, U) &= \mathcal{J}(\lambda, U)(\nu - \lambda)^{k+1}\\ &= \frac{Q(\lambda)}{P_1(\lambda)/(\nu-\lambda)^{k+1}} - \frac{\beta e^{-\lambda c\tau}Q(\lambda)}{P(\lambda)(\nu-\lambda)^{k+1}} - (\nu-\lambda)^{k+1}\int_0^{+\infty}U(\xi)e^{-\lambda\xi}d\xi\end{aligned}$$

for $0 < Re\lambda < \nu$, where $k = 0$ for $\lambda_4 \neq \lambda_1$ and $k = 1$ for $\lambda_4 = \lambda_1$, we know $\mathcal{H}(\lambda, U)$ is analytic for $0 < Re\lambda < \nu$. By similar arguments as (1), we can get that $\mathcal{H}(\lambda, U)$ is analytic for $Re\lambda = \nu$. Moreover, since $Q(\nu) > 0$ and $(\nu - \lambda)^{k+1}/\Delta(\lambda, c) > 0$ at $\lambda = \nu$, we have $\mathcal{H}(\lambda_1, I) > 0$. Then by Lemma 4.2 (if necessary we replace $U(\xi)$ by $e^{m\xi}U(\xi)$ for some $m >> 1$), we have

$$\lim_{\xi\to-\infty}\frac{U(\xi)}{|\xi|^k e^{\nu\xi}} = \frac{\mathcal{H}(\nu, U)}{\Gamma(\nu+1)},$$

where $k = 0$ for $\lambda_4 \neq \lambda_1$ and $k = 1$ for $\lambda_4 = \lambda_1$. Then there exist constants $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$ such that $\lim_{\xi \to -\infty} \frac{U(\xi+\eta_1)}{e^{\lambda_1 \xi}} = 1$ for $\lambda_4 > \lambda_1$, $\lim_{\xi \to -\infty} \frac{U(\xi+\eta_2)}{|\xi| e^{\lambda_1 \xi}} = 1$ for $\lambda_4 = \lambda_1$, and $\lim_{\xi \to -\infty} \frac{U(\xi+\eta_3)}{e^{\lambda_4 \xi}} = 1$ for $\lambda_4 < \lambda_1$. Thus, the case for $c > c^*$ is proved. The case for $c = c^*$ can be done similarly and we omit it. The proof is complete. □

Now it is in a position to prove the uniqueness result.

Proof of Theorem 1.5. It is sufficient to consider the subsystem (1.6) with the initial condition

$$u_1(0, x) = u_{10}(x), u_2(s, x) = u_{20}(s, x),\ s \in [-\tau, 0],\ x \in \mathbb{R}. \tag{4.10}$$

Let $(\phi_1(x + ct), \phi_2(x + ct))$ and $(\varphi_1(x + ct), \varphi_2(x + ct))$ be two different traveling waves with the same speed $c > c^*$ and the same exponential decay at $-\infty$. By Theorem 4.3, let

$$(1 - \phi_1(\xi), \phi_2(\xi)) = (A_1 e^{\nu\xi}, A_2 e^{\lambda_1 \xi}) \text{ as } \xi \to -\infty$$

and

$$(1 - \varphi_1(\xi), \varphi_2(\xi)) = (B_1 e^{\nu\xi}, B_2 e^{\lambda_1 \xi}) \text{ as } \xi \to -\infty$$

for some constants $A_1, A_2, B_1, B_2 > 0$. We now shift $(\varphi_1(x + ct), \varphi_2(x + ct))$ to $(\varphi_1(x+ct+x_1), \varphi_2(x+ct+x_2))$ with some constant shifts $x_1$ and $x_2$, $x_i = \frac{1}{\lambda_1} \ln \frac{A_i}{B_i}$, $i = 1, 2$. Then by a simple calculation, we get

$$(1 - \varphi_1(\xi + x_1), \varphi_2(\xi + x_2)) = (A_1 e^{\nu\xi}, A_2 e^{\lambda_1 \xi}) \text{ as } \xi \to -\infty.$$

Thus,

$$|\varphi_1(\xi + x_1) - \phi_1(\xi)| = O(1)e^{a\xi}, \quad |\varphi_2(\xi + x_2) - \phi_2(\xi)| = O(1)e^{b\xi}$$

for all $a > \nu$ and $b > \lambda_1$ as $\xi \to -\infty$, which implies that

$$\varphi_1(\xi + x_1) - \phi_1(\xi), \quad \varphi_2(\xi + x_2) - \phi_2(\xi) \in C(\mathbb{R}) \cap H^1_\omega(\mathbb{R}).$$

Now take the initial data in (4.10) as

$$u_{10}(x) = \varphi_1(x + x_1), \quad u_{20}(s, x) = \varphi_2(x + cs + x_2),\ s \in [-\tau, 0],\ x \in \mathbb{R}.$$

Then the corresponding solution to (1.6) is

$$(U_1(x, t), U_2(x, t)) = (\varphi_1(x + ct + x_1), \varphi_2(x + ct + x_2)).$$

By Theorem 1.2, we have

$$\limsup_{t\to+\infty, x\in\mathbb{R}} |\varphi_1(x + ct + x_1) - \phi_1(x + ct)| = 0$$

and

$$\limsup_{t\to+\infty, x\in\mathbb{R}} |\varphi_2(x + ct + x_2) - \phi_2(x + ct)| = 0,$$

namely, $\varphi_1(x + ct + x_1) = \phi_1(x+ct)$ and $\varphi_2(x + ct + x_2) = \phi_2(x + ct)$ for all $x \in \mathbb{R}$ as $t >> 1$. This proves the uniqueness of the traveling waves up to shift. □

## 5. Discussion

Up to now, we have established the exponential stability and uniqueness of traveling waves of system (1.3). Especially, Theorem 1.5 implies that, the global solution of Cauchy problem (1.3)-(1.2) is exponential decay tending to the traveling waves of system (1.3), where the decay rate $k_0$ is determined by Lemma 3.4.

In the following we shall discuss how the nonlocal dispersal affects the stability of traveling waves, namely, how the range of the nonlocal interaction affects the decay rate $k_0$. Define

$$J_\rho(x) = \frac{1}{\rho} J(\frac{x}{\rho}), \tag{5.1}$$

where $\rho$ quantifies the range of the nonlocal interaction. It is not difficult to verify that $J_\rho(x) * u \to u$ as $\rho \to 0$ and hence (1.3) reduces to the corresponding reaction-diffusion equations.

Using (5.1), $C_i(k) = 0$, $i = 1, 2, 3$ in (3.20)-(3.22) can be rewritten as

$$\begin{aligned} C_1(k) &:= c\lambda_* - d \int_{\mathbb{R}} J_\rho(y)[e^{-\lambda_* y} - 1]dy + 2\sigma - \tilde{L} - 2k \\ &= c\lambda_* - d \int_{\mathbb{R}} J(y)[e^{-\lambda_* \rho y} - 1]dy + 2\sigma - \tilde{L} - 2k, \end{aligned}$$

$$\begin{aligned} C_2(k) &:= c\lambda_* - \int_{\mathbb{R}} J_\rho(y)[e^{-\lambda_* y} - 1]dy + 2\mu + \gamma - L - 3\tilde{L} - 2k - 2\tilde{L}\left(e^{2k\tau} - 1\right) \\ &= c\lambda_* - \int_{\mathbb{R}} J(y)[e^{-\lambda_* \rho y} - 1]dy + 2\mu + \gamma - L - 3\tilde{L} - 2k - 2\tilde{L}\left(e^{2k\tau} - 1\right) \end{aligned}$$

and

$$C_3(k) := c\lambda_* - d_3 \int_{\mathbb{R}} J_\rho(y)[e^{-\lambda_* y} - 1]dy + 2\mu_1 - \gamma - 2k.$$

By Lemma 3.4, there exist unique positive numbers $k_i$ such that $C_i(k_i) = 0$, $i = 1, 2, 3$. Now by a simple computation, we can get

$$\frac{dk_i}{d\rho} = \frac{d_i \lambda_*}{2} \int_{\mathbb{R}} yJ(y)e^{-\lambda_* \rho y}dy, \ \frac{dk_2}{d\rho} = \frac{\lambda_* \int_{\mathbb{R}} yJ(y)e^{-\lambda_* \rho y}dy}{2 + 4\tilde{L}\tau e^{2k_2\tau}}, \ i = 1, 3, \ d_1 = d.$$

Let $h(\rho) = \lambda_* \int_{\mathbb{R}} yJ(y)e^{-\lambda_* \rho y}dy$. Since $h(0) = 0$ and

$$h'(\rho) = -{\lambda_*}^2 \int_{\mathbb{R}} y^2 J(y)e^{-\lambda_* \rho y}dy \le 0,$$

we have

$$\frac{dk_i}{d\rho} \le 0, \quad i = 1, 2, 3.$$

That is, $k_i (i = 1, 2, 3)$ and hence $k_0 := \min\{k_1, k_2, k_3\}$ are decreasing with respect to $\rho \ge 0$.

Thus we conclude that the range of the nonlocal interaction slows down the convergence rate of the global solution of Cauchy problem (1.3)-(1.2) to the traveling waves of (1.3).

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