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Logarithmic stability in determining a boundary coefficient in an IBVP for the wave equation

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Abstract. In [**2**] we introduced a method combining together an observability inequality and a spectral decomposition to get a logarithmic stability estimate for the inverse problem of determining both the potential and the damping coefficient in a dissipative wave equation from boundary measurements. The present work deals with an adaptation of that method to obtain a logarithmic stability estimate for the inverse problem of determining a boundary damping coefficient from boundary measurements. As in our preceding work, the different boundary measurements are generated by varying one of the initial conditions.

Contents

1. Introduction

We are concerned with an inverse problem for the wave equation when the spatial domain is the square $\Omega = (0,1) \times (0,1)$. To this end we consider the

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following initial-boundary value problem (abbreviated to IBVP in the sequel) :

$$
\begin{cases}
\partial_t^2 u - \Delta u = 0 & \text{in } Q = \Omega \times (0,\tau),\\
u = 0 & \text{on } \Sigma_0 = \Gamma_0 \times (0,\tau),\\
\partial_\nu u + a\partial_t u = 0 & \text{on } \Sigma_1 = \Gamma_1 \times (0,\tau),\\
u(\cdot,0) = u^0,\ \partial_t u(\cdot,0) = u^1.
\end{cases} \tag{1.1}
$$

Here

$$
\Gamma_0 = ((0,1)\times\{1\}) \cup (\{1\}\times(0,1)),
$$
$$
\Gamma_1 = ((0,1)\times\{0\}) \cup (\{0\}\times(0,1))
$$

and $\partial_\nu = \nu\cdot\nabla$ is the derivative along ν, the unit normal vector pointing outward of Ω. We note that ν is everywhere defined except at the vertices of Ω. The boundary coefficient a is usually called the boundary damping coefficient.

In the rest of this text we identify $a_{|(0,1)\times\{0\}}$ by $a_1 = a_1(x)$, $x \in (0,1)$ and $a_{|\{0\}\times(0,1)}$ by $a_2 = a_2(y)$, $y \in (0,1)$. In that case it is natural to identify a, defined on Γ_1, by the pair (a_1, a_2).

1.1. The IBVP. We fix $1/2 < \alpha \le 1$ and we assume that $a \in \mathscr{A}$, where

$$
\mathscr{A} = \{b = (b_1,b_2) \in C^\alpha([0,1])^2,\ b_1(0) = b_2(0),\ b_j \ge 0\}.
$$

This assumption guarantees that the multiplication operator by a_j, $j = 1,2$, defines a bounded operator on $H^{1/2}((0,1))$. The proof of this fact will be proved in Appendix A.

Let $V = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_0\}$ and we consider on $V \times L^2(\Omega)$ the linear unbounded operator A given by

$$
A_a = (w, \Delta v), \quad D(A_a) = \{(v,w) \in V \times V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.
$$

One can prove that A_a is a m-dissipative operator on the Hilbert space $V \times L^2(\Omega)$ (for the reader's convenience we detail the proof in Appendix B). Therefore, A_a is the generator of a strongly continuous group of contractions e^{tA_a}. Hence, for each (u^0,u^1), the IBVP (1.1) possesses a unique solution denoted by $u_a = u_a(u^0,u^1)$ so that

$$
(u_a, \partial_t u_a) \in C([0,\infty); D(A_a)) \cap C^1([0,\infty), V \times L^2(\Omega)).
$$

1.2. Main result. For $0 < m \le M$, we set

$$
\mathscr{A}_{m,M} = \{b = (b_1,b_2) \in \mathscr{A} \cap H^1(0,1)^2;\ m \le b_j,\ \|b_j\|^2_{H^1(0,1)} \le M\}.
$$

Let $\mathscr{U}_0$ given by

$$
\mathscr{U}_0 = \{v \in V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = 0 \text{ on } \Gamma_1\}.
$$

We observe that $\mathscr{U}_0 \times \{0\} \subset D(A_a)$, for any $a \in \mathscr{A}$.

Let $C_a \in \mathscr{B}(D(A_a); L^2(\Sigma_1))$ defined by

$$
C_a(u^0,u^1) = \partial_\nu u_a(u^0,u^1)_{|\Gamma_1}.
$$

We define the initial to boundary operator

$$
\Lambda_a : u^0 \in \mathscr{U}_0 \longrightarrow C_a(u^0,0) \in L^2(\Sigma_1).
$$

Clearly $C_a \in \mathscr{B}(D(A_a); L^2(\Sigma_1))$ implies that $\Lambda_a \in \mathscr{B}(\mathcal{U}_0; L^2(\Sigma_1))$, when $\mathcal{U}_0$ is identified to a subspace of $D(A_a)$ endowed with the graph norm of A_a. Precisely the norm in $\mathcal{U}_0$ is the following one

$$\|u^0\|_{\mathcal{U}_0} = \left(\|u^0\|_V^2 + \|\Delta u^0\|_{L^2(\Omega)}^2\right)^{1/2}.$$

Henceforth, for simplicity sake, the norm of $\Lambda_a - \Lambda_0$ in $\mathscr{B}(\mathcal{U}_0; L^2(\Sigma_1))$ will denoted by $\|\Lambda_a - \Lambda_0\|$.

THEOREM 1.1. *There exists $\tau_0 > 0$ so that for any $\tau > \tau_0$, we find a constant $c > 0$ depending only on τ such that*

$$\|a - 0\|_{L^2((0,1))^2} \le cM\left(\left|\ln\left(m^{-1}\|\Lambda_a - \Lambda_0\|\right)\right|^{-1/2} + m^{-1}\|\Lambda_a - \Lambda_0\|\right), \tag{1.2}$$

for each $a \in \mathcal{A}_{m,M}$.

We point out that our choice of the domain Ω is motivated by the fact the spectral analysis of the laplacian under mixed boundary condition is very simple in that case. However this choice has the inconvenient that the square domain Ω is no longer smooth. So we need to prove an observability inequality associated to this non smooth domain. This is done by adapting the existing results. We note that the key point in establishing this observability inequality relies on a Rellich type identity for the domain Ω.

The inverse problem we discuss in the present paper remains largely open for an arbitrary (smooth) domain as well as for the stability around a non zero damping coefficient. Uniqueness and directional Lipschitz stability, around the origin, was established by the authors in [**3**].

The determination of a potential and/or the sound speed coefficient in a wave equation from the so-called Dirichlet-to-Neumann map was extensively studied these last decades. We refer to the comments in [**2**] for more details.

2. Preliminaries

2.1. Extension lemma. We decompose Γ_1 as follows $\Gamma_1 = \Gamma_{1,1} \cup \Gamma_{1,2}$, where $\Gamma_{1,1} = (0,1) \times \{0\}$ and $\Gamma_{1,2} = \{0\} \times (0,1)$. Similarly, we write $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}$, with $\Gamma_{0,1} = \{1\} \times (0,1)$ and $\Gamma_{0,2} = (0,1) \times \{1\}$.

Let $(g_1, g_2) \in L^2((0,1))^2$. We say that the pair (g_1, g_2) obeys the compatibility condition of the first order at the vertex $(0,0)$ if

$$\int_0^1 |g_1(t) - g_2(t)|^2 \frac{dt}{t} < \infty. \tag{2.1}$$

We can define in a similar manner the compatibility condition of the first order at the other vertices of Ω.

We need also to introduce compatibility conditions of the second order. Let $(f_j, g_j) \in H^1((0,1)) \times L^2((0,1))$, $j = 1, 2$. We say that the pair $[(f_1, g_1), (f_2, g_2)]$ satisfies the compatibility conditions of second order at the vertex $(0,0)$ when

$$f_1(0) = f_2(0), \quad \int_0^1 |f_1'(t) - g_2(t)|^2 \frac{dt}{t} < \infty \tag{2.2}$$

$$\text{and} \quad \int_0^1 |g_1(t) - f_2'(t)|^2 \frac{dt}{t} < \infty.$$

The compatibility conditions of the second order at the other vertices of Ω are defined in the same manner.

The following theorem is a special case of [**4**, Theorem 1.5.2.8, page 50].

THEOREM 2.1. *(1) The mapping*

$$w \longrightarrow (w_{|\Gamma_{0,1}}, w_{|\Gamma_{0,2}}, w_{|\Gamma_{1,1}}, w_{|\Gamma_{1,2}}) = (g_1, \ldots, g_4),$$

defined on $\mathscr{D}(\overline{\Omega})$ *is extended from* $H^1(\Omega)$ *onto the subspace of* $H^{1/2}((0,1))^4$ *consisting in functions* $(g_1, \ldots, g_4)$ *so that the compatibility condition of the first order is satisfied at each vertex of* Ω *in a natural way with the pairs* (g_j, g_k).
(2) The mapping

$$w \to (w_{|\Gamma_{0,1}}, \partial_x w_{|\Gamma_{0,1}}, w_{|\Gamma_{0,2}}, \partial_y w_{|\Gamma_{0,2}} w_{|\Gamma_{1,1}}, -\partial_y w_{|\Gamma_{1,1}}, w_{|\Gamma_{1,2}}, -\, \partial_x w_{|\Gamma_{1,2}}) = ((f_1, g_1), \ldots (f_4, g_4))$$

defined on $\mathscr{D}(\overline{\Omega})$ *is extended from* $H^2(\Omega)$ *onto the subspace of* $[H^{3/2}((0,1)) \times H^{1/2}((0,1))]^4$ *of functions* $((f_1, g_1), \ldots (f_4, g_4))$ *so that the compatibility conditions of the second order are satisfied at each vertex of* Ω *in a natural way with the pairs* $[(f_j, g_j), (f_k, g_k)]$.

LEMMA 2.1. *(Extension lemma) Let* $g_j \in H^{1/2}((0,1))$, $j = 1, 2$, *so that* (g_1, g_2), $(g_1, 0)$ *and* $(g_2, 0)$ *satisfy the first order compatibility condition respectively at the vertices* $(0,0)$, $(1,0)$ *and* $(0,1)$. *Then there exists* $u \in H^2(\Omega)$ *so that* $u = 0$ *on* Γ_0 *and* $\partial_\nu u = g_j$ *on* $\Gamma_{1,j}$, $j = 1, 2$.

PROOF. (i) We define $f_1(t) = \int_0^t g_2(s)ds$ and $f_2(t) = \int_0^t g_1(s)ds$. Then (f_1, g_1) and (f_2, g_2) satisfy the compatibility conditions of the second order at the vertex $(0,0)$.

(ii) Let $\widetilde{g}_1 \in H^{1/2}((0,1))$ be such that $\int_0^1 \frac{|\widetilde{g}_1(t)|^2}{t} dt < \infty$. Let $\widetilde{f}_1(t) = \int_0^t g_2(s)ds$. Hence, it is straightforward to check that $(\widetilde{f}_1, \widetilde{g}_1)$ and $(0, g_2)$ satisfy the compatibility conditions of the second order at $(0,0)$.

(iii) From (i) and (ii) we derive that the pairs $[(f_1, g_1), (f_2, g_2)]$, $[(f_1, g_1), (0, g_2)]$ and $[(0, g_1), (f_2, g_2)]$ satisfy the second order compatibility conditions respectively at the vertices $(0,0)$, $(1,0)$ and $(0,1)$. We see that unfortunately the pair $[(0, g_1), (0, g_2)]$ doesn't satisfy necessarily the compatibility conditions of the second order at the vertex $(1,1)$. We pick $\chi \in C^\infty(\mathbb{R})$ so that $\chi = 1$ in a neighborhood of 0 and $\chi = 0$ in a neighborhood of 1. Then $[(0, \chi g_1), (0, \chi g_2)]$ satisfies the compatibility condition of the second order at the vertex $(1,1)$. Since this construction is of local character at each vertex, the cutoff function at the vertex $(1,1)$ doesn't modify the construction at the other vertices. In other words, the compatibility conditions of the second order are preserved at the other vertices. We complete the proof by applying Theorem 2.1. □

COROLLARY 2.1. *Let* $a = (a_1, a_2) \in \mathscr{A}$ *and* $g_j \in H^{1/2}((0,1))$, $j = 1, 2$, *so that* (g_1, g_2), $(g_1, 0)$ *and* $(g_2, 0)$ *satisfy the first order compatibility condition respectively at the vertices* $(0,0)$, $(1,0)$ *and* $(0,1)$. *Then there exists* $u \in H^2(\Omega)$ *so that* $u = 0$ *on* Γ_0 *and* $\partial_\nu u = a_j g_j$ *on* $\Gamma_{1,j}$, $j = 1, 2$.

PROOF. It is sufficient to prove that $(a_1 g_1, a_2 g_2)$ and $(a_j g_j, 0)$, $j = 1, 2$, satisfy the first order compatibility condition at $(0,0)$ with $a_1(0) = a_2(0)$ for the first pair and without any condition on a_j for the second pair.

Using $a_1(0) = a_2(0)$, we get

$$\begin{aligned} t^{-1}|a_1(t) - a_2(t)|^2 &\le 2t^{-1}|a_1(t) - a_1(0)|^2 + 2t^{-1}|a_2(t) - a_2(0)|^2 \\ &\le 2t^{-1+2\alpha}([a_1]_\alpha^2 + [a_2]_\alpha^2) \\ &\le 2([a_1]_\alpha^2 + [a_2]_\alpha^2). \end{aligned}$$

Here

$$[a_i]_\alpha = \sup\{|a_i(x) - a_i(y)||x-y|^{-\alpha};\ x, y \in [0,1],\ x \neq y\},\quad i = 1, 2.$$

The last estimate together with the following one

$$|a_1(t)g_1(t) - a_2(t)g_2(t)|^2 \le 2|a_1(t) - a_2(t)|^2|g_1(t)|^2 + 2|a_2(t)|^2|g_1(t) - g_2(t)|^2$$

yield

$$\begin{aligned} \int_0^1 |a_1(t)g_1(t) - a_2(t)g_2(t)|^2 \frac{dt}{t} &\le 4([a_1]_\alpha^2 + [a_2]_\alpha^2)\|f\|_{L^2((0,1))} \\ &\quad + 2\|a_2\|_{L^\infty((0,1))} \int_0^1 |g_1(t) - g_2(t)|^2 \frac{dt}{t}. \end{aligned}$$

Hence

$$\int_0^1 |g_1(t) - g_2(t)|^2 \frac{dt}{t} < \infty \Longrightarrow \int_0^1 |a_1(t)g_1(t) - a_2(t)g_2(t)|^2 \frac{dt}{t} < \infty.$$

If $(g_j, 0)$ satisfies the first compatibility at the vertex $(0,0)$. Then

$$\int_0^1 |g_j(t)|^2 \frac{dt}{t} < \infty.$$

Therefore

$$\int_0^1 |a_j g_j(t)|^2 \frac{dt}{t} \le \|a_j\|_{L^\infty((0,1))}^2 \int_0^1 |g_j(t)|^2 \frac{dt}{t} < \infty.$$

Thus $(a_j g_j, 0)$ satisfies also the first compatibility at the vertex $(0,0)$. □

2.2. Observability inequality. We discuss briefly how we can adapt the existing results to get an observability inequality corresponding to our IBVP. We first note that

$$\begin{aligned} \Gamma_0 &\subset \{x \in \Gamma;\ m(x)\cdot\nu(x) < 0\}, \\ \Gamma_1 &\subset \{x \in \Gamma;\ m(x)\cdot\nu(x) > 0\}, \end{aligned}$$

where $m(x) = x - x_0$, $x \in \mathbb{R}^2$, and $x_0 = (\alpha, \alpha)$ with $\alpha > 1$.

The following Rellich identity is a particular case of identity [**5**, (3.5), page 227]: for each $3/2 < s < 2$ and $\varphi \in H^s(\Omega)$ satisfying $\Delta\varphi \in L^2(\Omega)$,

$$2\int_\Omega \Delta\varphi(m\cdot\nabla\varphi)dx = 2\int_\Gamma \partial_\nu\varphi(m\cdot\nabla\varphi)d\sigma - \int_\Gamma (m\cdot\nu)|\nabla\varphi|^2 d\sigma. \tag{2.3}$$

Lemma 2.2. *Let $(v, w) \in D(A_a)$. Then*

$$2\int_\Omega \Delta v(m\cdot\nabla v)dx = 2\int_\Gamma \partial_\nu v(m\cdot\nabla v)d\sigma - \int_\Gamma (m\cdot\nu)|\nabla v|^2 d\sigma.$$

Proof. Let $(v, w) \in D(A_a)$. By Corollary 2.1, there exists $\widetilde{v} \in H^2(\Omega)$ so that $\widetilde{v} = 0$ on Γ_0 and $\partial_\nu \widetilde{v} = -aw$ on Γ_1. In light of the fact that $z = v - \widetilde{v}$ is such that $\Delta z \in L^2(\Omega)$, $z = 0$ on Γ_0 and $\partial_\nu z = 0$ on Γ_1, we get $z \in H^s(\Omega)$ for some $3/2 < s < 2$ by [**5**, Theorem 5.2, page 237]. Therefore $v \in H^s(\Omega)$. We complete the proof by applying Rellich identity (2.3). □

Lemma 2.2 at hand, we can mimic the proof of [**7**, Theorem 7.6.1, page 252] in order to obtain the following theorem:

Theorem 2.2. *We assume that $a \geq \delta$ on Γ_1, for some $\delta > 0$. There exist $M \geq 1$ and $\omega > 0$, depending only on δ, so that*

$$\|e^{tA_a}(v, w)\|_{V \times L^2(\Omega)} \leq Me^{-\omega t}\|(v, w)\|_{V \times L^2(\Omega)}, \;\; (v, w) \in D(A_a),\; t \geq 0.$$

An immediate consequence of Theorem 2.2 is the following observability inequality.

Corollary 2.2. *We fix $0 < \delta_0 < \delta_1$. Then there exist $\tau_0 > 0$ and κ, depending only on δ_0 and δ_1 so that for any $\tau \geq \tau_0$ and $a \in \mathscr{A}$ satisfying $\delta_0 \leq a \leq \delta_1$ on Γ_1,*

$$\|(u^0, u^1)\|_{V \times L^2(\Omega)} \leq \kappa \|C_a(u^0, u^1)\|_{L^2(\Sigma_1)}.$$

Moreover, C_a is admissible for e^{-tA_a} and (C_a, A_a) is exactly observable.

We omit the proof of this corollary. It is quite similar to that of [**7**, Corollary 7.6.5, page 256].

Remark 2.1. It is worth mentioning that in Corollary 2.2 the observability region Γ_1 can be substituted by Γ_0 or $\Gamma_2 = (\{0\} \times (0, 1)) \cup ((0, 1) \times \{1\})$ or $\Gamma_3 = ((0, 1) \times \{0\}) \cup (\{1\} \times (0, 1))$. In that case, the statement of Theorem 1.1 still holds if Γ_1 is substituted by Γ_j, $j = 0$ or $j = 2$ or $j = 3$.

3. The inverse problem

3.1. An abstract framework for the inverse source problem. In the present subsection we consider an inverse source problem for an abstract evolution equation. The result of this subsection is the main ingredient in the proof of Theorem 1.1.

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be the generator of continuous semigroup $(T(t))$. An operator $C \in \mathscr{B}(D(A), Y)$, Y is a Hilbert space which is identified with its dual space, is called an admissible observation for $(T(t))$ if for some (and hence for all) $\tau > 0$, the operator $\Psi \in \mathscr{B}(D(A), L^2((0, \tau), Y))$ given by

$$(\Psi x)(t) = CT(t)x, \;\; t \in [0, \tau], \;\; x \in D(A),$$

has a bounded extension to H.

We introduce the notion of exact observability for the system

$$z'(t) = Az(t), \;\; z(0) = x, \tag{3.1}$$

$$y(t) = Cz(t), \tag{3.2}$$

where C is an admissible observation for $T(t)$. Following the usual definition, the pair (A, C) is said exactly observable at time $\tau > 0$ if there is a constant κ such that the solution (z, y) of (3.1) and (3.2) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 dt \geq \kappa^2 \|x\|_X^2, \;\; x \in D(A).$$

Or equivalently

$$\int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geq \kappa^2 \|x\|_X^2, \;\; x \in D(A). \tag{3.3}$$

Let $\lambda \in H^1((0,\tau))$ such that $\lambda(0) \neq 0$. We consider the Cauchy problem

$$z'(t) = Az(t) + \lambda(t)x, \;\; z(0) = 0 \tag{3.4}$$

and we set

$$y(t) = Cz(t), \;\; t \in [0,\tau]. \tag{3.5}$$

We fix β in the resolvent set of A. Let H_1 be the space $D(A)$ equipped with the norm $\|x\|_1 = \|(\beta - A)x\|$ and denote by H_{-1} the completion of H with respect to the norm $\|x\|_{-1} = \|(\beta - A)^{-1}x\|$. As it is observed in [**1**, Proposition 4.2, page 1644] and its proof, when $x \in H_{-1}$ (which is the dual space of H_1 with respect to the pivot space H) and $\lambda \in H^1((0,T))$, then, according to the classical extrapolation theory of semigroups, the Cauchy problem (3.4) has a unique solution $z \in C([0,\tau];H)$. Additionally y given in (3.5) belongs to $L^2((0,\tau),Y)$.

When $x \in H$, we have by Duhamel's formula

$$y(t) = \int_0^t \lambda(t-s)CT(s)xds = \int_0^t \lambda(t-s)(\Psi x)(s)ds. \tag{3.6}$$

Let

$$H_\ell^1((0,\tau),Y) = \left\{u \in H^1((0,\tau),Y);\; u(0) = 0\right\}.$$

We define the operator $S : L^2((0,\tau),Y) \longrightarrow H_\ell^1((0,\tau),Y)$ by

$$(Sh)(t) = \int_0^t \lambda(t-s)h(s)ds. \tag{3.7}$$

If $E = S\Psi$, then (3.6) takes the form

$$y(t) = (Ex)(t).$$

Let $\mathcal{Z} = (\beta - A^*)^{-1}(X + C^*Y)$.

Theorem 3.1. *We assume that (A,C) is exactly observable at time τ. Then*
(i) E is one-to-one from H onto $H_\ell^1((0,\tau),Y)$.
(ii) E can be extended to an isomorphism, denoted by $\widetilde{E}$, from $\mathcal{Z}'$ onto $L^2((0,\tau);Y)$.
(iii) There exists a constant $\widetilde{\kappa}$, independent on λ, so that

$$\|x\|_{\mathcal{Z}'} \leq \widetilde{\kappa}|\lambda(0)|e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|\widetilde{E}x\|_{L^2((0,\tau),Y)}. \tag{3.8}$$

Proof. (i) and (ii) are contained in [**1**, Theorem 4.3, page 1645]. We need only to prove (iii). To do this, we start by observing that S^*, the adjoint of S, maps $L^2((0,\tau),Y)$ into $H_r^1((0,\tau);Y) = \left\{u \in H^1((0,\tau),Y);\; u(\tau) = 0\right\}$. Moreover

$$S^*h(t) = \int_t^\tau \lambda(s-t)h(s)ds, \;\; h \in L^2((0,\tau);Y).$$

We fix $h \in L^2((0,\tau);Y)$ and we set $k = S^*h$. Then

$$k'(t) = \lambda(0)h(t) - \int_t^\tau \lambda'(s-t)h(s)ds.$$

Hence

$$
\begin{aligned}
|\lambda(0)\|h(t)\|^2 &\le \left(\int_t^\tau \frac{|\lambda'(s-t)|}{|\lambda(0)|}[|\lambda(0)|\|h(s)\|]ds + \|k'(t)\|\right)^2 \\
&\le 2\left(\int_t^\tau \frac{|\lambda'(s-t)|}{|\lambda(0)|}[|\lambda(0)|\|h(s)\|]ds\right)^2 + 2\|k'(t)\|^2 \\
&\le 2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\int_0^t [|\lambda(0)|\|h(s)\|]^2 ds + 2\|k'(t)\|^2.
\end{aligned}
$$

The last estimate is obtained by applying Cauchy-Schwarz's inequality.

A simple application of Gronwall's lemma entails

$$[|\lambda(0)|\|h(t)\|]^2 \le 2e^{2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|k'(t)\|^2.$$

Therefore,

$$\|h\|_{L^2((0,\tau);Y)} \le \frac{\sqrt{2}}{|\lambda(0)|}e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|k'\|_{L^2((0,\tau);Y)}.$$

This inequality yields

$$\|h\|_{L^2((0,\tau);Y)} \le \frac{\sqrt{2}}{|\lambda(0)|}e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|S^*h\|_{H^1_r((0,\tau);Y)}. \tag{3.9}$$

The adjoint operator of S^*, acting as a bounded operator from $[H_r^1((0,\tau);Y)]'$ into $L^2((0,\tau);Y)$, gives an extension of S. We denote by $\widetilde{S}$ this operator. By [**1**, Proposition 4.1, page 1644] $\widetilde{S}$ defines an isomorphism from $[H_r((0,1);Y)]'$ onto $L^2((0,\tau);Y)$. In light of the fact that

$$\|\widetilde{S}\|_{\mathscr{B}([H^1_r((0,\tau);Y)]';L^2((0,\tau);Y))} = \|S^*\|_{\mathscr{B}(L^2((0,\tau);Y);H^1_r((0,\tau);Y))},$$

(3.9) implies

$$\frac{|\lambda(0)|}{\sqrt{2}}e^{-\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau} \le \|\widetilde{S}\|_{\mathscr{B}([H^1_r((0,\tau);Y)]';L^2((0,\tau);Y))}. \tag{3.10}$$

On the other hand, according to [**1**, Proposition 2.13, page 1641], Ψ possesses a unique bounded extension, denoted by $\widetilde{\Psi}$ from $\mathscr{Z}'$ into $[H_r^1((0,\tau);Y)]'$ and there exists a constant $c>0$ so that

$$\|\widetilde{\Psi}\|_{\mathcal{B}(\mathscr{Z}';[H^1_r((0,\tau);Y)]')} \ge c. \tag{3.11}$$

Consequently, $\widetilde{E} = \widetilde{S}\widetilde{\Psi}$ gives the unique extension of E to an isomorphism from $\mathscr{Z}'$ onto $L^2((0,\tau);Y)$.

We end up the proof by noting that (3.8) is a consequence of (3.10) and (3.11).

□

3.2. An inverse source problem for an IBVP for the wave equation. In the present subsection we are going to apply the result of the preceding subsection to $H = V \times L^2(\Omega)$, $H_1 = D(A_a)$ equipped with its graph norm and $Y = L^2(\Gamma_1)$.

We consider the IBVP

$$
\begin{cases}
\partial_t^2 u - \Delta u = \lambda(t)w & \text{in } Q,\\
u = 0 & \text{on } \Sigma_0,\\
\partial_\nu u + a\partial_t u = 0 & \text{on } \Sigma_1,\\
u(\cdot,0)=0,\ \partial_t u(\cdot,0)=0,
\end{cases}
\tag{3.12}
$$

Let $(0, w) \in H_{-1}$ and $\lambda \in H^1((0, \tau))$. From the comments in the preceding subsection, (3.12) has a unique solution u_w so that $(u_w, \partial_t u_w) \in C([0, \tau]; V \times L^2(\Omega))$ and $\partial_\nu u_{w|\Gamma_1} \in L^2(\Sigma_1)$.

We consider the inverse problem consisting in the determination of w, so that $(0, w) \in H_{-1}$, appearing in the IBVP (3.12) from the boundary measurement $\partial_\nu u_{w|\Sigma_1}$. Here the function λ is assumed to be known.

Taking into account that $\{0\} \times V' \subset H_{-1}$, where V' is the dual space of V, we obtain as a consequence of Corollary 2.1:

PROPOSITION 3.1. *There exists a constant $C > 0$ so that for any $\lambda \in H^1((0, \tau))$ and $w \in V'$,*

$$\|w\|_{V'} \le C|\lambda(0)|e^{\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\tau}\|\partial_\nu u_w\|_{L^2(\Sigma_1)}. \tag{3.13}$$

3.3. Proof of Theorem 1.1. We start by observing that u_a is also the unique solution of

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} au'(t)v, \quad v \in V. \\ u(0) = u^0, \quad u'(0) = u^1. \end{cases}$$

Let $u = u_a - u_0$. Then u is the solution of the following problem

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} au'(t)v - \int_{\Gamma_1} au_0'(t)v, \quad v \in V. \\ u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{3.14}$$

For k, $\ell \in \mathbb{Z}$, we set

$$\lambda_{k\ell} = [(k + 1/2)^2 + (\ell + 1/2)^2]\pi^2$$
$$\phi_{k\ell}(x, y) = 2\cos((k + 1/2)\pi x)\cos((\ell + 1/2)\pi y).$$

We check in a straightforward manner that $u_0 = \cos(\sqrt{\lambda_{k\ell}}t)\phi_{k\ell}$ when $(u^0, u^1) = (\phi_{k\ell}, 0)$.

In the sequel k, ℓ are arbitrarily fixed. We set $\lambda(t) = \cos(\sqrt{\lambda_{k\ell}}t)$ and we define $w_a \in V'$ by

$$w_a(v) = -\sqrt{\lambda_{k\ell}}\int_{\Gamma_1} a\phi_{k\ell}v.$$

In that case (3.14) becomes

$$\begin{cases} \displaystyle\int_\Omega u''(t)v dx = \int_\Omega \nabla u(t)\cdot\nabla v dx - \int_{\Gamma_1} au'(t)v + \lambda(t)w_a(v), \quad v \in V. \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

Consequently, u is the solution of (3.12) with $w = w_a$. Applying Proposition 3.1, we find

$$\|w_a\|_{V'} \le Ce^{\lambda_{k\ell}\tau^2}\|\partial_\nu u\|_{L^2(\Sigma_1)}. \tag{3.15}$$

But

$$a_1(0)\Big|\int_{\Gamma_1}(a\phi_{k\ell})^2 d\sigma\Big| = \frac{1}{\sqrt{\lambda_{k\ell}}}\big|w_a((a_1\otimes a_2)\phi_{k\ell})\big| \le \frac{1}{\sqrt{\lambda_{k\ell}}}\|w_a\|_{V'}\|(a_1\otimes a_2)\phi_{k\ell}\|_V, \tag{3.16}$$

where we used $a_1(0)=a_2(0)$, and

$$\|(a_1\otimes a_2)\phi_{k\ell}\|_V \le C_0\sqrt{\lambda_{kl}}\|a_1\otimes a_2\|_{H^1(\Omega)}, \tag{3.17}$$

Here C_0 is a constant independent on a and $\phi_{k\ell}$.

We note $(a_1\otimes a_2)\phi_{k\ell}\in V$ even if $a_1\otimes a_2\notin V$.

Now a combination of (3.15), (3.16) and (3.17) yields

$$a_1(0)\Big(\|a_1\phi_k\|^2_{L^2((0,1))}+\|a_2\phi_\ell\|^2_{L^2((0,1))}\Big) \le C\|a_1\|_{H^1(0,1)}\|a_2\|_{H^1(0,1)}e^{\lambda_{k\ell}\tau^2/2}\|\partial_\nu u\|_{L^2(\Sigma_1)},$$

where $\phi_k(s)=\sqrt{2}\cos((k+1/2)\pi s)$. This and the fact that $m\le a_j(0)$ and $\|a_j\|_{H^1((0,1))}\le M$ imply

$$\|a_1\phi_k\|^2_{L^2((0,1))}+\|a_2\phi_\ell\|^2_{L^2((0,1))}\le C\frac{M^2}{m}e^{\lambda_{k\ell}\tau^2/2}\|\partial_\nu u\|_{L^2(\Sigma_1)},$$

Hence, where $j=1$ or 2,

$$\|a_j\phi_k\|^2_{L^2((0,1))}\le C\frac{M^2}{m}e^{k^2\tau^2\pi^2}\|\partial_\nu u\|_{L^2(\Sigma_1)}.$$

Let

$$a_j^k=\int_0^1 a_j(x)\phi_k(x)dx,\quad j=1,2.$$

Since

$$|a_j^k|=\left|\int_0^1 a_j(x)\phi_k(x)dx\right|\le \|a_j\phi_k\|_{L^1((0,1))}\le\|a_j\phi_k\|_{L^2((0,1))},$$

we get

$$(a_j^k)^2\le C\frac{M^2}{m}e^{k^2\tau^2\pi^2}\|\partial_\nu u\|_{L^2(\Sigma_1)}.$$

On the other hand

$$\|\partial_\nu u\|_{L^2(\Sigma_1)}=\|\Lambda_a(\phi_{kl})-\Lambda_0(\phi_{kl})\|_{L^2(\Sigma)}\le Ck^2\|\Lambda_a-\Lambda_0\|.$$

Hence

$$(a_j^k)^2\le C\frac{M^2}{m}e^{k^2(\tau^2\pi^2+1)}\|\Lambda_a-\Lambda_0\|. \tag{3.18}$$

Let $q=\frac{M^2}{m}$ and $\alpha=\tau^2\pi^2+2$. We obtain in a straightforward manner from (3.18)

$$\sum_{|k|\le N}(a_j^k)^2\le Cqe^{\alpha N^2}\|\Lambda_a-\Lambda_0\|.$$

Consequently,

$$\begin{aligned}\|a_j\|^2_{L^2((0,1))} &\le \sum_{|k|\le N}(a_j^k)^2 + \frac{1}{N^2}\sum_{|k|>N} k^2 (a_j^k)^2 \\ &\le C\left(qe^{\alpha N^2}\|\Lambda_a-\Lambda_0\| + \frac{\|a_j\|^2_{H^1((0,1))}}{N^2}\right) \\ &\le C\left(qe^{\alpha N^2}\|\Lambda_a-\Lambda_0\| + \frac{M^2}{N^2}\right) \\ &\le CM^2\left(\frac{1}{m}e^{\alpha N^2}\|\Lambda_a-\Lambda_0\| + \frac{1}{N^2}\right).\end{aligned}$$

That is

$$\|a_j\|^2_{L^2((0,1))} \le CM^2\left(\frac{1}{m}e^{\alpha N^2}\|\Lambda_a-\Lambda_0\| + \frac{1}{N^2}\right). \tag{3.19}$$

Assume that $\|\Lambda_a-\Lambda_0\| \le \delta = me^{-\alpha}$. Let then $N_0 \ge 1$ be the greatest integer so that

$$\frac{C}{m}e^{\alpha N_0^2}\|\Lambda_a-\Lambda_0\| \le \frac{1}{N_0^2}.$$

Using

$$\frac{1}{m}e^{\alpha (N_0+1)^2}\|\Lambda_a-\Lambda_0\| \le \frac{1}{(N_0+1)^2},$$

we find

$$(2N_0)^2 \ge (N_0+1)^2 \ge \frac{1}{\alpha+1}\ln\left(\frac{m}{\|\Lambda_a-\Lambda_0\|}\right).$$

This estimate in (3.19) with $N = N_0$ gives

$$\|a_j\|_{L^2((0,1))} \le 2C\sqrt{\alpha+1}M\left|\ln\left(m^{-1}\|\Lambda_a-\Lambda_0\|\right)\right|^{-1/2}. \tag{3.20}$$

When $\|\Lambda_a-\Lambda_0\| \ge \delta$, we have

$$\|a_j\|_{L^2((0,1))} \le \frac{M}{\delta}\|\Lambda_a-\Lambda_0\|. \tag{3.21}$$

In light of (3.20) and (3.21), we find a constants $c > 0$, that can depend only on τ, so that

$$\|a_j\|_{L^2((0,1))} \le cM\left(\left|\ln\left(m^{-1}\|\Lambda_a-\Lambda_0\|\right)\right|^{-1/2} + m^{-1}\|\Lambda_a-\Lambda_0\|\right).$$

Appendix A

We prove the following lemma

Lemma A.1. *Let $1/2 < \alpha \le 1$ and $a \in C^\alpha([0,1])$. Then the mapping $f \mapsto af$ defines a bounded operator on $H^{1/2}((0,1))$.*

Proof. We recall that $H^{1/2}((0,1))$ consists in functions $f \in L^2(0,1)$ with finite norm

$$\|f\|_{H^{1/2}((0,1))} = \left(\|f\|^2_{L^2((0,1))} + \int_0^1\int_0^1 \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy\right)^{1/2}.$$

Let $a \in C^\alpha([0,1])$. We have

$$\frac{|a(x)f(x)-a(y)f(y)|^2}{|x-y|^2} \leq \|a\|^2_{L^\infty(0,1)}\frac{|f(x)-f(y)|^2}{|x-y|^2}+|f(y)|^2\frac{[a]^2_\alpha}{|x-y|^{2(1-\alpha)}},$$

where

$$[a]_\alpha = \sup\{|a(x)-a(y)||x-y|^{-\alpha};\ x,y\in[0,1],\ x\neq y\}.$$

Using that $1/2 < \alpha \leq 1$, we find that $x \to |x-y|^{-2(1-\alpha)} \in L^1((0,1))$, $y \in [0,1]$, and

$$\int_0^1 \frac{dx}{|x-y|^{2(1-\alpha)}} \leq \frac{1}{2\alpha-1},\ \ y\in[0,1].$$

Hence $af \in H^{1/2}((0,1)$ with

$$\|af\|_{H^{1/2}((0,1))} \leq \frac{1}{2\alpha-1}\|a\|_{C^\alpha([0,1])}\|f\|_{H^{1/2}((0,1))}.$$

Here

$$\|a\|_{C^\alpha([0,1])} = \|a\|_{L^\infty((0,1))} + [a]_\alpha.$$

□

Appendix B

We give the proof of the following lemma

Lemma B.1. *Let $a \in \mathscr{A}$ and A_a be the unbounded operator defined on $V\times L^2(\Omega)$ by*

$$A_a = (w, \Delta v),\quad D(A_a) = \{(v,w)\in V\times V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.$$

Then A_a est m-dissipative.

Proof. Let $\langle\cdot,\cdot\rangle$ be scalar product in $V\times L^2(\Omega)$. That is

$$\langle (v_1,w_1),(v_2,w_2)\rangle = \int_\Omega \nabla v_1\cdot\nabla\overline{v_2}dx + \int_\Omega w_1\overline{w_2}dx,\ \ (v_j,w_j)\in V\times L^2(\Omega),\ j=1,2.$$

For $(v_j,w_j)\in D(A_a)$, $j=1,2$, we have

$$\begin{aligned}\langle A_a(v_1,w_1),(v_2,w_2)\rangle &= \langle (w_1,\Delta v_1),(v_2,w_2)\rangle \\ &= \int_\Omega \nabla w_1\cdot\nabla\overline{v_2}dx + \int_\Omega \Delta v_1\overline{w_2}dx\end{aligned} \tag{B.1}$$

Applying twice Green's formula, we get

$$\int_\Omega \nabla w_1\cdot\nabla\overline{v_2}dx = -\int_\Omega w_1\Delta\overline{v_2}dx + \int_{\Gamma_1} w_1\partial_\nu\overline{v_2}, \tag{B.2}$$

$$\int_\Omega \Delta v_1\overline{w_2}dx = -\int_\Omega \nabla v_1\cdot\nabla\overline{w_2}dx + \int_{\Gamma_1} aw_1\overline{w_2}. \tag{B.3}$$

We take the sum side by side of identities (B.2) and (B.3). Using that $\partial_\nu v_2 = -aw_2$ on Γ_1 we obtain

$$\begin{aligned}\int_\Omega \nabla w_1\cdot\nabla\overline{v_2}dx + \int_\Omega \Delta v_1\overline{w_2}dx &= -\int_\Omega w_1\Delta\overline{v_2}dx - \int_\Omega \nabla v_1\cdot\nabla\overline{w_2}dx \\ &= -\langle (v_1,w_1),A_a(v_2,w_2)\rangle - 2\Re\int_{\Gamma_1} a\,w_1\overline{w}_2.\end{aligned}$$

This and (B.1) yield

$$\langle A_a(v_1,w_1),(v_2,w_2)\rangle = -\langle (v_1,w_1), A_a(v_2,w_2)\rangle - 2\Re\int_{\Gamma_1} a\, w_1\overline{w}_2.$$

In other words, A_a is dissipative.

We complete the proof by showing that A_a is onto implying that A_a is m-dissipative according to [**7**, Proposition 3.7.3, page 99]. To this end we are going to show that for each $(f,g)\in V\times L^2(\Omega)$, the problem

$$w=f \quad -\Delta v = g.$$

has a unique solution $(v,w)\in D(A_a)$.

In light of the fact $\psi \rightarrow \left(\int_\Omega |\nabla\psi|^2dx\right)^{1/2}$ defines an equivalent norm on V, we can apply Lax Milgram's lemma. We get that there exists a unique $v\in V$ satisfying

$$\int_\Omega \nabla v\cdot\nabla\overline{\psi}dx = \int_\Omega g\overline{\psi}dx - \int_{\Gamma_1} aw\overline{\psi},\quad \psi\in V.$$

From this identity, we deduce in a standard way that $-\Delta v = g$ in Ω and $\partial_\nu v = -aw$ on Γ_1. The proof is then complete □

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