

Small $\dot{B}_{\infty,\infty}^{-1}$ implies regularity

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ABSTRACT. We show that smallness of $\dot{B}_{\infty,\infty}^{-1}$ norm of solution to d -dimensional ($d \geq 3$) incompressible Navier-Stokes prevents blowups.

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1. Introduction

In recent [9], Farhat, Grujić and Leitmeyer proved that any unique L^∞ mild solution to 3D Navier-Stokes equation cannot develop finite-time blowups if the $B_{\infty,\infty}^{-1}$ norm is sufficiently small (near the first possible blowup time). This result is perhaps a bit surprising in view of the illposedness result of Bourgain-Pavlović [3]. The proof in [9] has a strong geometric flavor, and in particular relies on a geometric regularity criteria and characterization of the super-level sets developed in the series of works [6, 11, 10]. We refer the readers to the introduction in [9] and the references therein (see also [1]–[13]) for more details on these techniques and also related developments. The purpose of this note is to revisit this problem from the point of view of Littlewood-Paley calculus. In particular we will give a streamlined proof for all dimensions $d \geq 3$.

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Consider d -dimensional Navier-Stokes Equation (NSE):

$$(1.1) \quad \begin{cases} \partial_t v + (v \cdot \nabla)v = \Delta v - \nabla p, & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ \nabla \cdot v = 0, \\ v|_{t=0} = v_0. \end{cases}$$

THEOREM 1.1. *Let $d \geq 3$. Suppose v is a smooth solution to (1.1) and let $T > 0$ be the first possible blow-up time. There exists a positive constant m_0 depending only on the dimension d , such that if the solution v satisfies*

$$\sup_{t \in (T-\epsilon, T)} \|v(t)\|_{\dot{B}_{\infty, \infty}^{-1}} \leq m_0,$$

for some $0 < \epsilon < T$, then T is not a blow-up time, and the solution can be continued past T .

REMARK 1.2. Here to allow some generality we do not specify the particular class of smooth solution. As an example one can consider as in [9] the unique mild solution emanating from L^∞ initial data. By smoothing (cf. [7]) the solution is immediately in $W^{k, \infty}$ for all k . Other classes of solutions can also be considered and we will not dwell on this issue here.

We gather below some notation used in this note.

Notation. For any two quantities X and Y , we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence.

We will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let $\phi_0 \in C_c^\infty(\mathbb{R}^n)$ and satisfy

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \geq 7/6.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ which is supported in $\frac{1}{2} \leq |\xi| \leq \frac{7}{6}$. For any $f \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define

$$\begin{aligned} \widehat{P_{\leq j} f}(\xi) &= \phi_0(2^{-j}\xi)\hat{f}(\xi), \\ \widehat{P_j f}(\xi) &= \phi(2^{-j}\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}^n. \end{aligned}$$

Sometimes for simplicity we write $f_j = P_j f$, $f_{\leq j} = P_{\leq j} f$. Note that by using the support property of ϕ , we have $P_j P_{j'} = 0$ whenever $|j - j'| > 1$. The Bony paraproduct for a pair of functions f, g take the form

$$fg = \sum_{i \in \mathbb{Z}} f_i \tilde{g}_i + \sum_{i \in \mathbb{Z}} f_i g_{\leq i-2} + \sum_{i \in \mathbb{Z}} g_i f_{\leq i-2},$$

where $\tilde{g}_i = g_{i-1} + g_i + g_{i+1}$. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, the homogeneous Besov $\dot{B}_{\infty, \infty}^s$ norm is given by

$$\|f\|_{\dot{B}_{\infty, \infty}^s} = \sup_{j \in \mathbb{Z}} (2^{js} \|P_j f\|_\infty).$$

We will use without explicit mentioning the following simple estimate:

$$\|e^{t\Delta} P_j f\|_{L^\infty(\mathbb{R}^d)} \lesssim e^{-c2^{2j}t} \|P_j f\|_{L^\infty(\mathbb{R}^d)}, \quad \forall t > 0,$$

where $c > 0$ is a constant depending only on d .

2. Proof of Theorem 1.1

LEMMA 2.1. *Let $\gamma > 1$. Then for any $j \in \mathbb{Z}$, we have*

$$\|P_j((v \cdot \nabla)v)\|_\infty \lesssim 2^{j(2-\gamma)} \|v\|_{\dot{B}_{\infty,\infty}^{-1}} \|v\|_{\dot{B}_{\infty,\infty}^\gamma}.$$

PROOF OF LEMMA 2.1. Although this is utterly standard we give a proof for completeness. By paraproduct decomposition, we have

$$\begin{aligned} (v \cdot \nabla)v &= \sum_{l \in \mathbb{Z}} (v_{\leq l-2} \cdot \nabla)v_l + \sum_{l \in \mathbb{Z}} (v_l \cdot \nabla)v_{\leq l-2} + \sum_{l \in \mathbb{Z}} (v_l \cdot \nabla)\tilde{v}_l \\ &=: A + B + C, \end{aligned}$$

where $\tilde{v}_l = v_{l-1} + v_l + v_{l+1}$. Then by frequency localization, we have

$$\|P_j(A)\|_\infty \lesssim \sum_{|l-j| \leq 2} \|v_{\leq l-2} \cdot \nabla v_l\|_\infty \lesssim 2^j \|v\|_{\dot{B}_{\infty,\infty}^{-1}} \cdot 2^{j(1-\gamma)} \|v\|_{\dot{B}_{\infty,\infty}^\gamma}.$$

Similar estimate hold for B . Now for the estimate of C , note that by using the divergence-free property we can write $(v_l \cdot \nabla)\tilde{v}_l = \nabla \cdot (v_l \otimes \tilde{v}_l)$ and this gives

$$\|P_j(C)\|_\infty \lesssim 2^j \sum_{l \geq j-2} 2^{-l} \|v_l\|_\infty \cdot \|\tilde{v}_l\|_\infty \cdot 2^{\gamma l} \cdot 2^{-l(\gamma-1)} \lesssim 2^{j(2-\gamma)} \|v\|_{\dot{B}_{\infty,\infty}^{-1}} \|v\|_{\dot{B}_{\infty,\infty}^\gamma}.$$

Here we used the assumption $\gamma > 1$. □

LEMMA 2.2. *Suppose $v = v(t)$ is a smooth solution to (1.1) on some time interval $[0, T]$ with smooth initial data v_0 . Let $\gamma > 1$. There exists constants $C_1 > 0$, $\delta_1 > 0$ which depend only on (γ, d) , such that if*

$$\sup_{0 \leq t \leq T} \|v(t)\|_{\dot{B}_{\infty,\infty}^{-1}} \leq \delta_1,$$

then

$$\max_{0 \leq t \leq T} \|v(t)\|_{\dot{B}_{\infty,\infty}^\gamma} \leq C_1 \|v_0\|_{\dot{B}_{\infty,\infty}^\gamma}.$$

PROOF OF LEMMA 2.2. Write $v_j = P_j v$. Then

$$\partial_t v_j - \Delta v_j = -P_j(\Pi((v \cdot \nabla)v)),$$

where Π is the usual Leray projection operator. Then for any $t > 0$, by using Lemma 2.1, we have

$$\begin{aligned} \|v_j(t)\|_\infty &\lesssim e^{-c \cdot 2^{2j} t} \|v_j(0)\|_\infty + \int_0^t e^{-c \cdot 2^{2j}(t-s)} 2^{j(2-\gamma)} \|v(s)\|_{\dot{B}_{\infty,\infty}^{-1}} \|v(s)\|_{\dot{B}_{\infty,\infty}^\gamma} ds \\ &\lesssim e^{-c \cdot 2^{2j} t} \|v_j(0)\|_\infty + (1 - e^{-c \cdot 2^{2j} t}) \cdot 2^{-j\gamma} \cdot \sup_{0 \leq s \leq t} \|v(s)\|_{\dot{B}_{\infty,\infty}^{-1}} \cdot \max_{0 \leq s \leq t} \|v(s)\|_{\dot{B}_{\infty,\infty}^\gamma}. \end{aligned}$$

This implies that for some constants $\tilde{C}_1 > 0$, $\tilde{C}_2 > 0$ depending only on (γ, d) ,

$$\max_{0 \leq t \leq T} \|v(t)\|_{\dot{B}_{\infty,\infty}^\gamma} \leq \tilde{C}_1 \|v_0\|_{\dot{B}_{\infty,\infty}^\gamma} + \tilde{C}_2 \cdot \sup_{0 \leq t \leq T} \|v(t)\|_{\dot{B}_{\infty,\infty}^{-1}} \cdot \max_{0 \leq t \leq T} \|v(t)\|_{\dot{B}_{\infty,\infty}^\gamma}.$$

The result obviously follows. □

PROOF OF THEOREM 1.1. Choose $\gamma = 3/2$ and $m_0 = \delta_1$ as specified in Lemma 2.2. Consider the solution $v = v(t)$ on the time interval $[T - \epsilon, T - \eta]$, where $\eta > 0$ will tend to zero. By Lemma 2.2 (regarding $v(T - \epsilon)$ as initial data), we then obtain uniform estimate on $\|v\|_{\dot{B}_{\infty,\infty}^\gamma}$ independent of η . A standard argument then implies that v must be regular beyond T . □

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