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Faedo-Galerkin approximations to fractional integro-differential equation of order $\alpha \in (1,\ 2]$ with deviated argument

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ABSTRACT. In this paper, we consider a fractional integro-differential equation of order $\alpha \in (1, 2]$ with deviated argument in a separable Hilbert space X. We used the α-order cosine family of linear operators and Banach fixed point theorem to study the existence and uniqueness of approximate solutions. We define the fractional power of the closed linear operator and used it to prove the convergence of the approximate solutions. Also, we prove the existence and convergence of the Faedo-Galerkin approximate solutions. Finally, an example is provided to illustrate the application of these abstract results.

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1. Introduction

We consider a fractional integro-differential equation of order $\alpha \in (1,\ 2]$ with deviated argument in a separable Hilber space X

$$\begin{aligned} {}^cD_t^\alpha x(t) + Ax(t) &= f(t, x(a(t)), x[h(x(t), t)]) \\ &\quad + \int_0^t k(t-s)g(s, x(s))ds, \quad t \in (0, T], \\ x(0) = x_0, &\quad x'(0) = y_0, \end{aligned} \tag{1.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative, $-A$ is the infinitesimal generator of a α-order cosine family $(C_\alpha(t))_{t\geq 0}$ on a separable Hilbert space X. $x : J(= [0, T]) \to X$ is the state function and $k : \mathbb{R}_+ \to \mathbb{R}$ is the kernal function. $f : J \times X \times X \to X$, $h : X \times [0, T] \to \mathbb{R}^+$, $a : [0, T] \to [0, T]$ and $g : J \times X \to X$ are the functions satisfying some suitable conditions to be specified later.

The theory of fractional calculus started with a correspondence between L'Hospital and Leibniz in 1695. Lots of literature available on theoretical as well as numerical work on this topic. It has application in numerous fields, for example, control theory, signal and image processing, aerodynamics and biophysics etc. Few years back, many scientists and engineers have shown a great interest in fractional theory due to the memory character of fractional derivative, which is the generalization of integer-order derivative and can describe many phenomena of physics, biology and finance etc. that integer-order derivative can't explain.

For the details on the different kind of fractional differential equations, we refers to [**1**]- [**9**] and the references cited in these papers. Recently, Li Kexue et al. [**5**] studied the exact controllability of the fractional differential system of order $\alpha \in (1, 2]$ with non-local conditions in an infinite dimensional Banach space by using the Sadovskii fixed point theorem.

Initial studies concerning existence, uniqueness and finite-time blow-up of solutions for the following equation

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \end{aligned}$$

have been considered by Segal [**10**], Murakami [**11**] and Heinz and Von Wahl [**12**]. Bazley [**13, 14**] has considered the following semilinear wave equation

$$\begin{aligned} u''(t) \ + \ Au(t) \ &= \ g(u(t)), \quad t \geq 0, \\ u(0) \ = \ \phi, \quad u'(0) \ &= \ \psi, \end{aligned} \tag{1.2}$$

and has established the uniform convergence of approximations of solutions to (1.2) using the results of Heinz and von Wahl [**12**]. Goethel [**15**] has proved the convergence of approximations of solutions to equation (1.2) but assumed g to be defined on the whole of H.

To my knowlege, Gal [**16**] was the first person who has considered the nonlinear abstract differential equations of order one with deviated arguments and study the existence and uniqueness of solutions by using the semigroup of linear operators. After the Gal [**16**], some authors [**17**]-[**19**] have worked on different types of abstract differential equations with deviated arguments. Several authors [**17**]-[**24**] studies the existence and convergence of approximate solutions of abstract differential equations of order one by using the analytic semigroup of linear operators in a separable Hilbert space.

To the best of author's knowledge, there are no papers discussing the fractional differential equations of order $\alpha \in (1, 2]$ with deviated arguments in infinite dimensional spaces. Therefore, we consider a fractional integro-differential equation (1.1) with deviated argument of order $\alpha \in (1, 2]$ in a separable Hilbert space and studied the Faedo-Galerkin approximations . The results of this paper will also be true if $g(t, x(t)) = 0$. Also, we can extend these results to nonlocal problems with some additional suitable conditions.

The work of this manuscript is motivated by [**3, 5**] and [**13**]. We use the ideas of Bazley [**13**], Miletta [**21**] and Muslim [**22**] to establish the existence and convergence of finite dimensional approximate solution of system (1.1).

In the first and second section, we give the introduction and provide some of the notions and the results required for later sections. In the third section, we studies the existence of approximate solutions and seciton 4 deals with the convergence of the approximate solutions obtained in section 3. In section 5, we study the existence and convergence of Faedo-Galerkin approximate solutions and in the last section, we have given an example to illustrate the application of these results.

2. Preliminaries and Assumptions

In this section, we briefly review some basic definitions and notions which will be used in the subsequent sections. Let X be a separable Hilbert space with norm $||.||$ and the space of all bounded linear operators form X into X is denoted by $L(X)$. $L^p([0, T], X), 1 \leq p < \infty$ denote the space of X-valued Bochner integrable functions $\tilde{f} : [0, T] \to X$ with the norm

$$||\tilde{f}||_{L^p} = \Big(\int_0^T ||\tilde{f}(t)||^p dt \Big)^{\frac{1}{p}}.$$

$C([0, T], X)$, $C^1([0, T], X)$ denote the spaces of functions $\tilde{f} : [0, T] \to X$, which are continuous, continuously differentiable respectively and endowed with the norms

$$||\tilde{f}||_C = \sup_{t \in J} ||\tilde{f}(t)||, \quad ||\tilde{f}||_{C^1} = \sup_{t \in J} \sum_{k=0}^{1} ||\tilde{f}^k(t)||.$$

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$J_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

where $x(t) \in L^1([0, T], X)$ and $\Gamma(.)$ is the gamma function.

Definition 2.2. *If $x(t) \in L^1([0, T], X)$, then the Riemann-Liouville fractional derivative of order $\alpha \in (1, 2)$ is defined by*

$$D_t^\alpha x(t) = \frac{d^2}{dt^2} J_t^{2-\alpha} x(t),$$

where $D_t^\alpha x(t) \in L^1([0, T], X)$.

Definition 2.3. *The Caputo fractional derivative of order $\alpha \in (1, 2]$ is defined by*

$${}^c D_t^\alpha x(t) = J_t^{2-\alpha} \frac{d^2}{dt^2} x(t),$$

where $x(t) \in L^1([0, T], X) \cap C^1([0, T], X)$.

Consider the following fractional order differential problem

$$ {}^cD_t^\alpha x(t) = Ax(t), \quad x(0) = \eta,\ x'(0) = 0, \tag{2.3} $$

where $\alpha \in (1, 2]$, $A : D(A) \subset X \to X$ is a closed densely defined linear operator in separable Hilbert space X. By applying the Riemann-Liouville fractional integral of order $\alpha \in (1, 2]$ on both sides of (2.3), we have

$$ x(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)ds. \tag{2.4} $$

Definition 2.4. *([**4**]). A family $(C_\alpha(t))_{t\geq 0} \subset L(X)$, $\alpha \in (1, 2]$ is called the solution operator (or a strongly continuous α-order fractional cosine family) for (2.3) and A is called the infinitesimal generator of $C_\alpha(t)$, if the following conditions are hold:*
(i) $C_\alpha(t)$ is strongly continuous for $t \geq 0$ and $C_\alpha(0) = I$, where I is identity operator;
(ii) $C_\alpha(t)D(A) \subset D(A)$ and $AC_\alpha(t)\eta = C_\alpha(t)A\eta$ for all $\eta \in D(A)$, $t \geq 0$;
(iii) $C_\alpha(t)\eta$ is solution for (2.3) for all $\eta \in D(A)$.

Definition 2.5. *The fractional sine family $S_\alpha : [0, \infty) \to L(X)$ associated with C_α is defined by*

$$ S_\alpha(t) = \int_0^t C_\alpha(s)ds,\ t \geq 0. \tag{2.5} $$

Definition 2.6. *The fractional Riemann- Liouville family $P_\alpha : [0, \infty) \to L(X)$ associated with C_α is defined by*

$$ P_\alpha(t) = J^{\alpha-1}C_\alpha(t). \tag{2.6} $$

Definition 2.7. *The α-order cosine family $C_\alpha(t)$ is called exponentially bounded if there are constants $M_1 \geq 1$ and $\omega \geq 0$ such that*

$$ ||C_\alpha(t)|| \leq M_1 e^{\omega t},\ t \geq 0. \tag{2.7} $$

An operator A is said to belong to $C^\alpha(X; M, \omega)$, if the problem (1.1) has an solution operator $C_\alpha(t)$ satisfying (2.7). Throughout this paper, we assume that $A \in C^\alpha(X; M, \omega)$ for $\alpha \in (1,\ 2]$, hence from Theorem (3.3) in [**4**], A generates an analytic semigroup and hence the fractional power A^β, $0 \leq \beta \leq 1$ is defined. For the details on the fractional power of operators please see Pazy [**25**].

In order to prove the existence and convergence of approximate solution of the problem (1.1), we need the following assumptions.

(A1). Operator A is a closed, positive definite, linear, self-adjoint with domain $D(A)$ dense in X, A has the pure poitn spectrum,

$$ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots $$

with $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions ϕ_i, *i.e*

$$ A\phi_i = \lambda_i\phi_i, \text{ and } \langle \phi_i, \phi_j \rangle = \delta_{ij}, $$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If the condition $(A1)$ is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ in X (cf., [Pazy [**25**], pp. 69-75]). Therefore, the fractional powers A^{β} of A are well defined from domain $D(A^{\beta})$ into X. $D(A^{\beta})$ is a Banach space endowed with the norm

$$\|x\|_{\beta} = \|A^{\beta}x\|.$$

We denote this space by X_{β}. Also, for each $\beta > 0$, we define $X_{-\beta} = (X_{\beta})^{*}$, the dual space of X_{β} is a Banach space endowed with the norm $\|x\|_{-\beta} = \|A^{-\beta}x\|$.

It can be seen easily that $\mathcal{C}_t^{\beta} = C([0,t]; X_{\beta})$, for all $t \in [0,T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\beta} := \sup_{0\le\eta\le t} \|\psi(\eta)\|_{\beta}, \quad \psi \in \mathcal{C}_t^{\beta}.$$

We set, $\mathcal{C}_T^{\beta-1} = C([0,T]; X_{\beta-1}) = \{y \in \mathcal{C}_T^{\beta} : \|y(t) - y(s)\|_{\beta-1} \le L|t-s|, \forall\ \ t,s \in [0,T]\}$, where L is a suitable positive constant to be specified later and $0 \le \beta < 1$.

(A2). $f : J \times X_{\beta} \times X_{\beta-1} \to X$ is a continuous function and there exists positive constants K_1 and K_2 such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le L_f(\|x_1 - x_2\|_{\beta} + ||y_1 - y_2\|_{\beta-1})$$

for every $x_1, x_2 \in X_{\beta}$ and $y_1, y_2 \in X_{\beta-1}$ and

$$\max_{t\in J} ||f(t, x(t), x[h(x(t),t)])|| = K_f.$$

(A3). $h : X_{\beta} \times J \to \mathbb{R}^{+}$ is a uniformly continuous and there exists a positive constant $L_h = L_h(\alpha)$ such that

$$|h(x_1,s) - h(x_2,s)| \le L_h||x_1 - x_2||_{\beta}, \forall\, x_1, x_2 \in X_{\beta} \quad 0 \le s \le T_0$$

and satisfies $h(.,0) = 0$.

(A4). **(i)** $g : J \times X_{\beta} \to X$ is a continuous function and there exists positive constants L_g and K_g such that

$$||g(t,x_1) - g(t,x_2)|| \le L_g||x_1 - x_2||_{\beta}$$

for every $x_1, x_2 \in X$ and $\max_{t\in J} ||g(t,x)|| = K_g$ for all $t \in [0,T]$, $x \in X_{\beta}$.
(ii) $K_T = \int_0^t |k(t-s)|ds$.
(iii) Delay function $a : [0,T] \to [0,T]$ is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$|a(t) - a(s)| \le L_a|t-s|, \ \forall\, s,t \in [0,T].$$

(A5). A is the infinitesimal generator of a α-order cosine family $C_{\alpha}(t)$ on X and there exists a constant $M \ge 1$ such that

$$||C_{\alpha}(t)|| \le M.$$

DEFINITION 2.8. *A continuous function $x \in C_T^{\beta-1} \cap C_T^{\beta}$ is said to be a mild solution of equation (1.1) if x is the solution of the following integral equation*

$$x(t) = C_{\alpha}(t)x_0 + S_{\alpha}(t)y_0 + \int_0^t P_{\alpha}(t-s)\Big[f(s, x(a(s)), x[h(x(s),s)]) + \int_0^s k(s-\eta)g(\eta, x(\eta))d\eta\Big]ds. \tag{2.8}$$

3. Existence of Approximate Solutions

In this section, we will study the existence of approximate solution of the problem (1.1). Let X_n denote the finite dimensional subspace of X spanned by $\{\phi_1, \phi_2, \cdots, \phi_n\}$ and $P^n : X \to X_n$ be the corresponding orthogonal projection operator for $n = 1, 2, 3, \cdots$.

We define

$$h_n : D(A^\beta) \times J \longrightarrow \mathbb{R}_+ \text{ as } h_n(x(t), t) = h(P^n x(t), t)$$

and

$$g_n : \mathbb{R}_+ \times D(A^\beta) \longrightarrow X \text{ as } g_n(t, x(t)) = g(t, P^n x(t))$$

Similarly, we define

$$f_n : J \times D(A^\beta) \times D(A^{\beta-1}) \to X$$

such that

$$f_n(s, x(a(s)), x[h(x(s), s)]) = f(s, P^n x(a(s)), P^n x[h(P^n x(s), s)]).$$

We set

$$\mathcal{W} = \{x \in C^\beta_{T_0} \cap C^{\beta-1}_{T_0} : x(0) = x_0, \;\; x'(0) = y_0, \;\; \|x\|_{T_0,\beta} \le R\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of $C^{\beta-1}_T$.

For $n = 1, 2, 3, \cdots$, we define a map $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$ given by

$$\begin{aligned}(\mathcal{F}_n x)(t) = C_\alpha(t)x_0 + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)\Big[f_n(s, x(a(s)), x[h(x(s), s)])\\ + \int_0^s k(s-\eta) g_n(\eta, x(\eta)) d\eta\Big] ds. \end{aligned} \tag{3.9}$$

Theorem 3.1. *If $x_0, \;\; y_0 \in D(A)$ and all the assumptions (A1)-(A5) are satisfied. Then, there exist an unique $x_n \in \mathcal{W}$ such that $\mathcal{F}_n x_n = x_n$ for each $n = 1, 2, 3, \cdots$. i.e. x_n satisfies the approximate integral equation*

$$\begin{aligned} x_n(t) &= C_\alpha(t)x_0 + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)\Big[f_n(s, x(a(s)), x[h(x(s), s)])\\ &\quad + \int_0^s k(s-\eta) g_n(\eta, x(\eta)) d\eta\Big] ds, \quad t \in [0, T]. \end{aligned} \tag{3.10}$$

Proof: We denote

$$\sup_{0 \le t \le T_0} \|P_\alpha(t)\| = \rho_1 \quad \text{and} \quad \sup_{0 \le t \le T_0} \|AP_\alpha(t)\| = \rho_2,$$

where $\rho_1, \;\; \rho_2 > 0$ and we choose a suitable R such that

$$M\|x_0\|_\beta + M\|y_0\|_\beta T_0 + \|A^{\beta-1}\|\rho_2[K_f + K_T K_g]T_0 = R.$$

First, we need to show that $\mathcal{F}_n x \in \mathcal{C}_{T_0}^{\beta-1}$ for any $x \in \mathcal{C}_{T_0}^{\beta-1}$. If $x \in \mathcal{C}_{T_0}^{\beta-1}$ and $T_0 > t_2 > t_1 > 0$, then, we get

$$\begin{aligned}
\|(\mathcal{F}_n x)(t_2) - (\mathcal{F}_n x)(t_1)\|_{\beta-1} &\leq \|A^{\beta-1}\|\|(C_\alpha(t_2) - C_\alpha(t_1))x_0\| \\
&+\|A^{\beta-1}\|\|(S_\alpha(t_2) - S_\alpha(t_1))y_0\| \\
&+\int_0^{t_1} \|A^{\beta-1}\|\|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|\Big[\|f_n(s, x(s), x[h(x(s), s)])\| \\
&+\int_0^s |k(s-\tau)|\|g_n(\tau, x(\tau))\|d\tau\Big]ds \\
&+\int_{t_1}^{t_2} \|A^{\beta-1}\|\|P_\alpha(t_2 - s)\|\Big[\|f_n(s, x(s), x[h(x(s), s)])\| \\
&+\int_0^s |k(s-\tau)|\|g_n(\tau, x(\tau))\|d\tau\Big]ds \\
&\leq I_1 + I_2 + I_3 + I_4. \qquad (3.11)
\end{aligned}$$

We have,

$$\begin{aligned}
I_1 = \|A^{\beta-1}\|\|(C_\alpha(t_2) - C_\alpha(t_1))x_0\| &= \|A^{\beta-1}\|\|\int_{t_1}^{t_2} AP_\alpha(\tau)x_0 d\tau\| \\
&\leq C_1(t_2 - t_1), \qquad (3.12)
\end{aligned}$$

where $C_1 = \rho_2\|x_0\|\|A^{\beta-1}\|$.

Similarlly,

$$\begin{aligned}
I_2 = \|A^{\beta-1}\|\|(S(t_2) - S(t_1))y_0\| &= \|A^{\beta-1}\|\|\int_{t_1}^{t_2} C_\alpha(\tau)d\tau\|\|y_0\| \\
&\leq C_2(t_2 - t_1), \qquad (3.13)
\end{aligned}$$

where $C_2 = \|A^{\beta-1}\|M\|y_0\|$.

Third part of inequality (3.11) is calculated as follows

$$\begin{aligned}
I_3 = \|A^{\beta-1}\|\int_0^{t_1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|\Big[\|f_n(s, x(s), x[h(x(s), s)])\| \\
+\int_0^s |k(s-\tau)|\|g_n(\tau, x(\tau))\|d\tau\Big]ds.
\end{aligned}$$

We have,

$$\begin{aligned}
&\|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\| \\
&\leq \int_0^{t_1 - s} \Big[\frac{(t_2 - s - \tau)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_1 - s - \tau)^{\alpha-2}}{\Gamma(\alpha-1)}\Big]\|C_\alpha(\tau)\|d\tau \\
&\qquad + \int_{t_1 - s}^{t_2 - s} \frac{(t_2 - s - \tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|C_\alpha(\tau)\|d\tau \\
&\qquad \leq \frac{M}{(\alpha-1)\Gamma(\alpha-1)}\Big[(t_2 - s)^{\alpha-1} + (t_1 - s)^{\alpha-1}\Big].
\end{aligned}$$

We use the above inequality in I_3 and get the following

$$\begin{aligned}
I_3 &\leq \|A^{\beta-1}\|\frac{M[K_f + K_{T_0}K_g]}{(\alpha-1)\Gamma(\alpha-1)}\int_0^{t_1} \Big[(t_2 - s)^{\alpha-1} + (t_1 - s)^{\alpha-1}\Big]ds \\
&\leq C_3(t_2 - t_1), \qquad (3.14)
\end{aligned}$$

where

$$C_3 = \|A^{\beta-1}\| \frac{M[K_f + K_{T_0}K_g]}{(\alpha-1)\Gamma(\alpha-1)} [\frac{1}{\alpha}[-(t_2-t_1)^{\alpha-1} + t_2^{\alpha}(t_2-t_1)^{-1} + t_1^{\alpha}(t_2-t_1)^{-1}]].$$

Fourth part of the inequality (3.11) is calculated as

$$I_4 = \|A^{\beta-1}\| \int_{t_1}^{t_2} \|P_\alpha(t_2-s)\| \Big[\|f_n(s, x(s), x[h(x(s), s)])\| + \int_0^s |k(s-\tau)| \|g_n(\tau, x(\tau))\| d\tau\Big] ds$$

$$\leq C_4(t_2 - t_1), \tag{3.15}$$

where $C_4 = \rho_1 \|A^{\beta-1}\|[K_f + K_{T_0}K_g]$.

We use the inequalities (3.12), (3.13), (3.14) and (3.15)) in inequality (3.11) and get the following inequality

$$||(\mathcal{F}_n x)(t_2) - (\mathcal{F}_n x)(t_1)||_{\beta-1} \leq L|t_2 - t_1|, \tag{3.16}$$

where $L = C_1 + C_2 + C_3 + C_4$. Hence, $\mathcal{F}_n x \in \mathcal{C}_{T_0}^{\beta-1}$ for any $x \in \mathcal{C}_{T_0}^{\beta-1}$.
Our next task is to prove that $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$. For any $t \in (0, T_0]$ and $x \in \mathcal{W}$, we have

$$\begin{aligned} ||(\mathcal{F}_n x)(t)||_\beta &\leq ||C_\alpha(t)x_0||_\beta \\ &+ ||S_\alpha(t)y_0||_\beta + \|A^{\beta-1}\| \int_0^t ||AP_\alpha(t-s)|| \Big[\|f_n(s, x(s), x[h(x(s), s)])\| \\ &+ \int_0^s |k(s-\tau)| \|g_n(\tau, x(\tau))\| d\tau\Big] ds \\ &\leq M\|x_0\|_\beta + M\|y_0\|_\beta T_0 + \|A^{\beta-1}\| \rho_2 [K_f + K_T K_g] T_0. \end{aligned}$$

Thus, we get $||\mathcal{F}_n x||_{T_0,\beta} \leq R$.

Hence, $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$.

Now, we want to prove that the mapping $\mathcal{F}_n$ is a strict contraction mapping on $\mathcal{W}$.

For any $x, y \in \mathcal{W}$, we have

$$\begin{aligned} ||(\mathcal{F}_n x)(t) - (\mathcal{F}_n y)(t)||_\beta &\leq \|A^{\beta-1}\| \int_0^t \|AP_\alpha(t-s)\| \\ &\Big[\|f_n(s, x(a(s), x[h(x(s), s)]) - f_n(s, y(a(s), y[h(y(s), s)])\| \\ &+ \int_0^s |k(s-\tau)| \|g_n(\tau, x(\tau)) - g_n(\tau, y(\tau))\| d\tau\Big] ds \\ &\leq \lambda ||x - y||_{T_0,\beta}. \end{aligned}$$

Therefore, $||(\mathcal{F}_n x) - (\mathcal{F}_n y)||_{T_0,\beta} \leq \lambda ||x - y||_{T_0,\beta}$,
where $\lambda = \Big[\|A^{\beta-1}\| \rho_2 [L_f(1 + LL_h + \|A^{-1}\|) + K_{T_0} L_g]\Big] T_0$. We choose T_0 in such a way that $\lambda < 1$. Hence, $\mathcal{F}_n$ is a strict contraction mapping. Therefore, $\mathcal{F}_n$ has a unique fixed point $x_n(t)$ in $\mathcal{W}$ which is the approximate solution of the equation (1.1). □

LEMMA 3.2. *Let the conditions (A1)-(A5) are hold. If $x_0, y_0 \in D(A)$ then $u_n(t) \in D(A^\vartheta)$ for all $t \in (0, T]$, where $0 \leq \vartheta < 1$.*

Proof: If $x_0, y_0 \in D(A)$ then $C_\alpha(t)x_0 \in D(A)$ and $S_\alpha(t)y_0 \in D(A)$. From proposition (3.3) in [**3**], $\int_0^t P_\alpha(t-s)f_n(s, x_n(s), x_n[h(x_n(s), s)])ds \in D(A)$ for all $f_n(s, x_n(s), x_n[h(x_n(s), s)]) \in X$. Hence, the required result follows from these facts and the facts that $D(A) \subseteq D(A^\vartheta)$ for all $0 \le \vartheta \le 1$. □

LEMMA 3.3. *Let all the conditions (A1) - (A5) are hold. If $x_0, y_0 \in D(A)$, then*

$$\|x_n\|_{T_0,\vartheta} \le U_0, \quad t \in [0, T_0], \quad n = 1, 2, \cdots,$$

for some suitable constant U_0.

Proof: We have

$$\begin{aligned} x_n(t) &= C_\alpha(t)x_0 + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)\Big[f_n(s, x_n(a(s)), x_n[h(x_n(s), s)]) \\ &\quad + \int_0^s k(s-\eta)g_n(\eta, x_n(\eta))d\eta\Big]ds. \end{aligned} \tag{3.17}$$

Let $0 \le \vartheta < 1$. By Applying the A^ϑ on the both side of equation (3.17), we get the following

$$\begin{aligned} \|x_n(t)\|_\vartheta &\le \|C_\alpha(t)\|\|A^\vartheta x_0\| + \|S_\alpha(t)\|\|A^\vartheta y_0\| \\ &+ \int_0^t \|A^{\vartheta-1}\|\|AP_\alpha(t-s)\|\Big[\|f_n(s, x_n(s), x_n[h(x_n(s), s)])\| \\ &+ \int_0^s |k(s-\eta)|\|g_n(\eta, x_n(\eta))\|d\eta\Big]ds \\ &\le U_0, \end{aligned} \tag{3.18}$$

where $U_0 = M\|x_0\|_\vartheta + T_0M\|y_0\|_\vartheta + \rho_2\|A^{\vartheta-1}\|[K_f + K_{T_0}K_g]T_0$. □

4. Convergence of Approximate Solutions

In this section, we will establish the convergence of the approximate solution $x_n \in \mathcal{W}$ to a unique mild solution x of equation (1.1).

THEOREM 4.1. *Let all the conditions (A1)-(A5) are hold. If $x_0, y_0 \in D(A)$, then*

$$\lim_{m\to\infty} \sup_{\{n \ge m,\ 0 \le t \le T_0\}} \|x_n(t) - x_m(t)\|_\beta = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathcal{W}$ which converges to the solution x of equation (1.1).

Proof Let $0 < \beta < \vartheta < 1$. We have the following inequality

$$\|(P^n - P^m)x_m(t)\|_\beta \le \|A^{\beta-\vartheta}(P^n - P^m)A^\vartheta x_m(t)\| \le \frac{1}{\lambda_m^{\vartheta-\beta}}\|A^\vartheta x_m(t)\| \le \frac{1}{\lambda_m^{\vartheta-\beta}}U_0.$$

For $n \ge m$, we have

$$\begin{aligned} &\|f_n(t, x_n(t), x_n[h(x_n(t), t)]) - f_m(t, x_m(t), x_m[h(x_m(t), t)])\| \\ &\quad \le \|f_n(t, x_n(t), x_n[h(x_n(t), t)]) - f_n(t, x_m(t), x_m[h(x_m(t), t)])\| \\ &\qquad + \|f_n(t, x_m(t), x_m[h(x_m(t), t)]) - f_m(t, x_m(t), x_m[h(x_m(t), t)])\| \\ &\quad \le J_1 + J_2. \end{aligned} \tag{4.19}$$

We calculate J_1 as follows:

$$\begin{aligned}
J_1 &= \|f_n(t, x_n(t), x_n[h(x_n(t), t)]) - f_n(t, x_m(t), x_m[h(x_m(t), t)])\| \\
&\le L_f[\|P^n x_n(t) - P^n x_m(t)\|_\beta \\
&\quad + \|P^n x_n[h(P^n x_n(t), t)] - P^n x_m[h(P^n x_m(t), t)]\|_{\beta-1}] \\
&\le L_f[\|x_n(t) - x_m(t)\|_\beta + \|x_n[h(P^n x_n(t), t)] - x_m[h(P^n x_m(t), t)]\|_{\beta-1}] \\
&\le L_f[\|x_n - x_m\|_{T,\beta} + \|A^{-1}\|\|x_n - x_m\|_{T,\beta}] \\
&\le L_f[1 + \|A^{-1}\|]\|x_n - x_m\|_{T,\beta}. \qquad (4.20)
\end{aligned}$$

Similarly, we calculate I_2 as follows:

$$\begin{aligned}
J_2 &= \|f_n(t, x_m(t), x_m[h(x_m(t), t)]) - f_m(t, x_m(t), x_m[h(x_m(t), t)])\| \\
&\le L_f[\|(P^n - P^m)x_m(t)\|_\beta \\
&\quad + \|P^n x_m[h(P^n x_m(t), t)] - P^m x_m[h(P^m x_m(t), t)]\|_{\beta-1}] \\
&\le L_f[\|(P^n - P^m)x_m(t)\|_\beta \\
&\quad + \|P^n x_m[h(P^n x_m(t), t)] - P^m x_m[h(P^n x_m(t), t)]\|_{\beta-1} \\
&\quad + \|P^m x_m[h(P^n x_m(t), t)] - P^m x_m[h(P^m x_m(t), t)]\|_{\beta-1}] \\
&\le L_f[\|(P^n - P^m)x_m(t)\|_\beta + \|(P^n - P^m)x_m[h(P^n x_m(t), t)]\|_{\beta-1} \\
&\qquad + \|x_m[h(P^n x_m(t), t)] - x_m[h(P^m x_m(t), t)]\|_{\beta-1}] \\
&\le L_f[\|(P^n - P^m)x_m(t)\|_\beta + \|(P^n - P^m)x_m[h(P^n x_m(t), t)]\|_{\beta-1} \\
&\qquad + L|h(P^n x_m(t), t) - h(P^m x_m(t), t)\|] \\
&\le L_f[\|(P^n - P^m)x_m(t)\|_\beta + \|A^{-1}\|\|(P^n - P^m)x_m[h(P^n x_m(t), t)]\|_\beta \\
&\qquad + LL_h\|(P^n - P^m)x_m(t)\|_\beta] \\
&\le L_f[(1 + LL_h)\|(P^n - P^m)x_m(t)\|_\beta \\
&+ \|A^{-1}\|\|(P^n - P^m)x_m[h(P^n x_m(t), t)]\|_\beta]
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\|f_n(t, x_n(t), x_n[h(x_n(t), t)]) - f_m(t, x_m(t), x_m[h(x_m(t), t)])\| \\
&\le L_f[1 + \|A^{-1}\|]\|x_n - x_m\|_{T,\beta} + L_f[(1 + LL_h)\|(P^n - P^m)x_m(t)\|_\beta \\
&\qquad + \|A^{-1}\|\|(P^n - P^m)x_m[h(P^n x_m(t), t)]\|_\beta] \\
&\le L_f[1 + \|A^{-1}\|]\|x_n - x_m\|_{T,\beta} + L_f(1 + LL_h + \|A^{-1}\|)\frac{1}{\lambda_m^{\vartheta-\beta}}U_0. \qquad (4.21)
\end{aligned}$$

Also, for $n \ge m$, we have

$$\begin{aligned}
&\|g_n(t, x_n(t)) - g_m(t, x_m(t))\| \\
&\quad \le \|g_n(t, x_n(t)) - g_n(t, x_m(t))\| \\
&\qquad + \|g_n(t, x_m(t)) - g_m(t, x_m(t))\| \\
&\quad \le L_g[\|x_n - x_m\|_{T_0,\beta} + \frac{1}{\lambda_m^{\vartheta-\beta}}U_0]. \qquad (4.22)
\end{aligned}$$

Hence,

$$\begin{aligned}&\|x_n(t)-x_m(t)\|_\beta\\ &\le \int_0^t \|A^{\beta-1}\|\|AP_\alpha(t-s)\|\Big[\|f_n(s,x_n(a(s)),x_n[h(x_n(s),s)])\\ &-f_m(s,x_m(a(s)),x_m[h(x_m(s),s)])\|\\ &\quad+\int_0^s |k(s-\eta\|\|g_n(\eta,x_n(\eta))-g_m(\eta,x_m(\eta))\|d\eta\Big]ds\\ &\le \rho_2\|A^{\beta-1}\|T_0\Big((L_f[1+\|A^{-1}\|]+K_{T_0}L_g)\|x_n-x_m\|_{T_0,\beta}\\ &\qquad\qquad +L_F(1+LL_h+\|A^{-1}\|+K_{T_0}L_g)\frac{1}{\lambda_m^{\vartheta-\beta}}U_0\Big).\end{aligned}$$

Therefore, we take the supremum and get

$$\begin{aligned}\|x_n-x_m\|_{T,\beta}\le \rho_2\|A^{\beta-1}\|T_0\Big(&(L_f[1+\|A^{-1}\|]+K_{T_0}L_g)\|x_n-x_m\|_{T_0,\beta}\\ &+L_f(1+LL_h+\|A^{-1}\|+K_{T_0}L_g)\frac{1}{\lambda_m^{\vartheta-\beta}}U_0\Big).\end{aligned}$$

Hence,

$$\|x_n-x_m\|_{T,\beta}\le \frac{\rho_2\|A^{\beta-1}\|T_0(L_f[1+\|A^{-1}\|]+K_{T_0}L_g)}{1-\rho_2\|A^{\beta-1}\|T_0(L_f(1+LL_h+\|A^{-1}\|)+K_{T_0}L_g)}\frac{1}{\lambda_m^{\vartheta-\beta}}U_0.$$

Therefore,

$$\lim_{m\to\infty}\ \sup_{\{n\ge m,\ 0\le t\le T_0\}}\|x_n(t)-x_m(t)\|_\beta=0$$

since $\frac{1}{\lambda_m^{\vartheta-\beta}}\to 0$ as $m\to\infty$. This completes the proof of the theorem. □

With the help of Theorem (3.1)and Theorem (4.1), we can state the following existence, uniqueness and convergence results.

THEOREM 4.2. *If $x_0\in D(A)$, $y_0\in D(A)$ and all the assumptions (A1)-(A5) are satisfied. Then, there exist an unique $x_n\in\mathcal{W}$ for each $n=1,2,3,\cdots$ and $x\in\mathcal{W}$ satisfying*

$$\begin{aligned}x_n(t) &= C_\alpha(t)x_0+S_\alpha(t)y_0+\int_0^t P_\alpha(t-s)\Big[f_n(s,x_n(a(s)),x_n[h(x_n(s),s)])\\ &\quad+\int_0^s k(s-\eta)g_n(\eta,x_n(\eta))d\eta\Big]ds \qquad (4.23)\end{aligned}$$

and

$$\begin{aligned}x(t) &= C_\alpha(t)x_0+S_\alpha(t)y_0+\int_0^t P_\alpha(t-s)\Big[f(s,x(a(s)),x[h(x(s),s)])\\ &\quad+\int_0^s k(s-\eta)g(\eta,x(\eta))d\eta\Big]ds \qquad (4.24)\end{aligned}$$

such that $x_n\to x$ in $\mathcal{W}$ as $n\to\infty$, where f_n and g_n are defined as earlier.

Proof: Existence and convergence of x_n is already proved in Theorem (3.1) and Theorem (4.1). We only need to prove that the limit of x_n is given by equation (4.24). We have

$$\begin{aligned} &\|x_n(t) - x(t)\|_\beta \\ &\le d_1 \int_0^t \|f_n(s, x_n(a(s)), x_n[h(x_n(s), s)]) - f(s, x(s), x[h(x(s), s)])\| ds \\ &+ d_1 \int_0^t \int_0^s |k(s-\eta)| \|g_n(\eta, x_n(\eta)) - g(\eta, x(\eta))\| d\eta ds, \quad t \in [0, T_0], \end{aligned} \tag{4.25}$$

where $d_1 = \rho_2 \|A^{\beta-1}\|$. We have the following inequlities

$$\begin{aligned} &\|f_n(t, x_n(t), x_n[h(x_n(t), t)]) - f(t, x(t), x[h(x(t), t)])\|, t \in [0, T_0] \\ &\le K_1[\|P^n x_n(t) - x(t)\|_\beta + \|P^n x_n[h(x_n(t), t)] - x[h(x(t), t)]\|_{\beta-1}] \\ &\le K_1[\|P^n(x_n(t) - x(t))\|_\beta + \|(P^n - I)x(t)\|_\beta] \\ &\quad + K_1\|A^{-1}\| \|P^n x_n[h(x_n(t), t)] - P^n x[h(x(t), t)]\|_\beta \\ &\quad + K_1\|A^{-1}\| \|(P^n - I)x[h(x(t), t)]\|_\beta \\ &\le K_1[\|x_n - x\|_{T_0,\beta} + \|(P^n - I)x\|_{T_0,\beta}] \\ &\quad + K_1\|A^{-1}\| \|x_n - x\|_{T_0,\beta} + K_1\|A^{-1}\| \|(P^n - I)x\|_{T_0,\beta}. \end{aligned}$$

and

$$\begin{aligned} &\|g_n(\eta, x_n(\eta)) - g(\eta, x(\eta))\|, \quad t \in [0, T_0] \\ &\quad \le L_g \|P^n x_n(t) - x(t)\|_\beta \\ &\quad \le L_g[\|P^n(x_n(t) - x(t))\|_\beta + \|(P^n - I)x(t)\|_\beta] \\ &\quad \le L_g[\|x_n - x\|_{T_0,\beta} + \|(P^n - I)x\|_{T_0,\beta}]. \end{aligned}$$

Hence, $\|f_n(t, x_n(t), x_n[h(x_n(t), t)]) - f(t, x(t), x[h(x(t), t)])\| \to 0$ and $\|g_n(\eta, x_n(\eta)) - g(\eta, x(\eta))\| \to 0$ as $n \to \infty$ because $x_n \to x$ and $P^n x \to x$ as $n \to \infty$. This completes the proof of the theorem. □

5. Faedo-Galerkin Approximation

For any $0 < t < T_0$, we have a unique $x \in \mathcal{W}$ satisfying the integral equation

$$\begin{aligned} x(t) = C_\alpha(t)x_0 + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)\Big[f(s, x(a(s)), x[h(x(s), s)]) \\ + \int_0^s k(s-\eta) g(\eta, x(\eta)) d\eta\Big] ds. \end{aligned} \tag{5.26}$$

Also, we have a unique solution $x_n \in \mathcal{W}$ of the approximate integral equation

$$\begin{aligned} x_n(t) \quad = \quad C_\alpha(t)x_0 + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)\Big[f_n(s, x_n(a(s)), x_n[h(x_n(s), s)]) \\ + \int_0^s k(s-\eta) g_n(\eta, x_n(\eta)) d\eta\Big] ds. \end{aligned} \tag{5.27}$$

The Faedo-Galerkin approximation of solution to equation (1.1) is defined as $\hat{x}_n(t) = P^n x_n(t)$. Faedo-Galerkin Approximate solution $\hat{x}_n(t) = P^n x_n(t)$ satisfies the following equation

$$\begin{aligned}\hat{x}_n(t) &= C_\alpha(t)P^n x_0 + S_\alpha(t)P^n y_0 \\ &+ \int_0^t P_\alpha(t-s)\Big[P^n f_n(s, x_n(a(s)), x_n[h(x_n(s), s)]) \\ &+ \int_0^s k(s-\eta)P^n g_n(\eta, x_n(\eta))d\eta\Big] ds. \end{aligned} \tag{5.28}$$

Solutions x and $\hat{x}_n$ which are given by equation (5.26) and equation (5.28) respectively, have the following representation

$$x(t) = \sum_{i=1}^{\infty} \alpha_i(t)\phi_i, \quad \alpha_i(t) = \langle x(t), \phi_i\rangle, \quad i = 1, 2, \ldots; \tag{5.29}$$

$$\hat{x}_n(t) = \sum_{i=1}^{n} \alpha_i^n(t)\phi_i, \quad \alpha_i^n(t) = \langle \hat{x}_n(t), \phi_i\rangle, \quad i = 1, 2, \ldots, n. \tag{5.30}$$

The Faedo-Galerkin method approximates equation (1.1) by

$$\begin{aligned}\frac{d^\alpha P^n x(t)}{dt^\alpha} &= P^n A P^n x(t) + P^n f(t, P^n x(t), P^n x[h(P^n x(t), t)]) \\ &+ \int_0^t k(t-\eta)P^n g_n(\eta, x_n(\eta))d\eta, \quad t \in (0, T_0], \\ P^n x(0) &= P^n x_0, \quad P^n x'(0) = P^n y_0. \end{aligned} \tag{5.31}$$

Equation (5.30) leads to the following system of fractional differential equations

$$\begin{aligned}&\frac{d^\alpha \alpha_i^n(t)}{dt^\alpha} = \sum_{j=1}^{n} \alpha_j^n(t)\langle A\phi_i, \phi_j\rangle + f_i^n(t, \alpha_1^n, \cdots, \alpha_n^n) + g_i^n(t, \alpha_1^n, \cdots, \alpha_n^n), \\ &\alpha_i^n(0) = \langle x_0, \phi_i\rangle, \ \ \dot{\alpha}_i^n(0) = \langle y_0, \phi_i\rangle \quad i = 1, 2, \cdots, n,\end{aligned}$$

where $t \in (0, T_0]$,

$$f_i^n(t, \alpha_1^n, \cdots, \alpha_n^n) = \langle f(t, \ \sum_{i=1}^{n} \alpha_i^n(t)\phi_i, \ \sum_{i=1}^{n} \alpha_i^n(h(\sum_{i=1}^{n} \alpha_i^n(t)\phi_i, t))\phi_i), \ \phi_i\rangle$$

and $g_i^n(t, \alpha_1^n, \cdots, \alpha_n^n) = \langle \int_0^t k(t-\tau)g(\tau, \quad \sum_{i=1}^{n} \alpha_i^n(\tau)\phi_i)d\tau, \ \phi_i\rangle$. Since $\phi_i, \ i = 1, 2, 3, \cdots$ are the eigenfunctions of A with corresponding eigenvalues λ_i, these above equation becomes

$$\begin{aligned}&\frac{d^\alpha \alpha_i^n(t)}{dt^\alpha} = \lambda_i \alpha_i^n(t) + f_i^n(t, \alpha_1^n, \cdots, \alpha_n^n) + g_i^n(t, \alpha_1^n, \cdots, \alpha_n^n), \quad t \in (0, T_0], \\ &\alpha_i^n(0) = \langle x_0, \phi_i\rangle, \ \ \dot{\alpha}_i^n(0) = \langle y_0, \phi_i\rangle \quad i = 1, 2, \cdots, n.\end{aligned}$$

THEOREM 5.1. *Let all the assumptions (A1)-(A5) are satisfying and $x_0, \ y_0 \in D(A)$. Then, we have the following*

$$\lim_{n\to\infty} \sup_{\{n \geq m, \ 0 \leq t \leq T_0\}} \|A^\beta[\hat{x}_n(t) - \hat{x}_m(t)]\| = 0.$$

Proof For $n \geq m$, we have

$$\begin{aligned}\|A^\beta[\hat{x}_n(t) - \hat{x}_m(t)]\| &= \|A^\beta[P^n x_n(t) - P^m x_m(t)]\| \\ &\leq \|P^n[x_n(t) - x_m(t)]\|_\beta + \|(P^n - P^m)x_m\|_\beta \\ &\leq \|x_n(t) - x_m(t)\|_\beta + \frac{1}{\lambda_m^{\vartheta-\beta}} U_0.\end{aligned}$$

We use the Theorem (4.1) and Lemma (3.3) to get the desired result. □

Now we can state a theorem which will ensure the existence and convergence of Faedo-Galerkin approximate solution of equation (1.1).

THEOREM 5.2. *If all the assumptions (A1)-(A5) are satisfying and $x_0,\ y_0 \in D(A)$. Then, there exists a unique function $\hat{x}_n \in \mathcal{W}$ given by*

$$\begin{aligned}\hat{x}_n(t) &= C_\alpha(t)P^n x_0 + S_\alpha(t)P^n y_0 \\ &+ \int_0^t P_\alpha(t-s)\Big[P^n f_n(s, x_n(a(s)), x_n[h(x_n(s), s)]) \\ &\quad + \int_0^s k(s-\eta)P^n g_n(\eta, x(\eta))d\eta\Big]ds\end{aligned}$$

and $x \in \mathcal{W}$ given by

$$\begin{aligned}x(t) &= C_\alpha(t)x_0 + S_\alpha(t)y_0 + \int_0^t P_\alpha(t-s)\Big[f(s, x(a(s)), x[h(x(s), s)]) \\ &\quad + \int_0^s k(s-\eta)g(\eta, x(\eta))d\eta\Big]ds\end{aligned}$$

such that $\hat{x}_n \to x$ as $n \to \infty$ in $\mathcal{W}$ on $[0, T_0]$.

Proof: Proof of this theorem is the consequence of Theorems (3.1) and Theorem (4.1). □

We have the following convergence theorem for $\{\alpha_i^n(t)\}$.

THEOREM 5.3. *Let all the assumptions (A1)-(A5) are satisfied and $x_0,\ y_0 \in D(A)$. Then, we have the following.*

$$\lim_{n\to\infty} \sup_{t_0 \leq t \leq T} \Big[\sum_{i=0}^{n} \lambda_i^{2\beta} |\alpha_i(t) - \alpha_i^n(t)|^2\Big] = 0.$$

Proof: We have

$$A^\beta[x(t) - \hat{x}_n(t)] = A^\beta\Big[\sum_{i=0}^{\infty} (\alpha_i(t) - \alpha_i^n(t))\phi_i\Big] = \sum_{i=0}^{\infty} \lambda_i^\beta (\alpha_i(t) - \alpha_i^n(t))\phi_i,$$

where $\alpha_i^n(t) = 0$ for all $i > n$.

Therefore, we have

$$\|A^\beta[x(t) - \hat{x}_n(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\beta} (\alpha_i(t) - \alpha_i^n(t))^2.$$

Result follows from Theorem (5.2). □

6. Application

Let $X = L^2[0,\pi]$. We consider the following partial differential equations with deviated argument,

$$(6.32)\quad \begin{cases} {}^cD_t^\alpha Z(t,y) = \partial_{yy}Z(t,y) + f_2(y, Z(a(t),y)), + f_3(t,y,Z(t,y)) \\ \quad + \int_0^t k(t-\tau)g_1(t,y,Z(t,y))d\tau, \quad y \in (0,\pi),\ t > 0, \\ \quad Z(t,0) = Z(t,\pi) = 0,\ t \in [0,T],\ a(t) \le t,\ 0 < T < \infty, \\ \quad Z(0,y) = x_0,\ y \in (0,\pi), \\ \quad \partial_t Z(0,y) = y_0, y \in (0,\pi), \end{cases}$$

where

$$\alpha \in (1,2], \quad f_3(t,y,Z(t,y)) = \int_0^y K(y,s)Z(s, h(t)(a_1|Z(t,s)| + b_1|Z(t,s)|))ds.$$

We assume that $a_1, b_1 \ge 0, (a_1,b_1) \neq (0,0)$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $h(0) = 0$ and $K : [0,\pi] \times [0,\pi] \to \mathbb{R}$, $b \in X$. We define an operator A, as follows,

$$(6.33)\qquad Ax = -\frac{d^2x}{dy^2} \quad \text{with} \quad x \in D(A) = \{x \in H_0^1(0,\pi) \cap H^2(0,\pi) :\ x'' \in X\},$$

where $H^2(0,\pi)$ and $H_0^1(0,\pi)$ are the sobolev spaces.

Let m be a positive integer and let $1 \le p < \infty$, we define the Sobolev space $W^{m,p}(\Omega)$ as

$$(6.34)\qquad W^{m,p}(\Omega) = \{x \in L^p(\Omega) \mid D^\eta x \in L^p(\Omega) \text{ for all } |\eta| \le m\},$$

where $\|x\|_{m,p,\Omega} = (\sum_{|\eta|\le m} \|D^\eta x\|_{L^p}^p)^{\frac{1}{p}}$. Here, η is a multi-index. If $p = 2$, we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. If $m = 1$ and $p = 2$ then $W^{1,2}(\Omega) = H^1(\Omega)$. The closure of the space $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ is a proper closed subspace of $H^1(\Omega)$ and denoted by $H_0^1(\Omega)$. Here, $\mathcal{D}(\Omega)$ denote the space of test functions in Ω. For the more details on Sobolev spaces, we refer to [**26**].

We observe some properties of the operators A defined by equation (6.33). Let $x \in D(A)$ and $\lambda \in \mathbb{R}$ such that $Ax = -x''$ that is

$$(6.35)\qquad x'' + \lambda x = 0.$$

Also, $\langle Ax,\ x\rangle = \langle \lambda x, x\rangle$. Hence, $\langle -x'',\ x\rangle = |x'|_{L^2}^2 = \lambda |x|_{L^2}^2$. Therefore, $\lambda > 0$. Solutions (orthonormal eigenfunctions) of equation (6.35) are given by $x_n(s) = \sqrt{2/\pi}\sin ns, n = 1,2,3,\cdots$, and eigenvalues are given by $\lambda_n = n^2$. Since $D(A)$ is a seprable Hilbert space hence for any $x \in D(A)$, there exists a sequence of reals numbers (α_n) such that

$$x = \sum_{n=1}^\infty \alpha_n x_n$$

with

$$\sum_{n=1}^\infty (\alpha_n)^2 < \infty,\ \sum_{n=1}^\infty (\lambda_n)^2(\alpha_n)^2 < \infty.$$

Here, $\alpha_n = \langle x, x_n\rangle$. We apply the operator A on x and get the infinite series representation

$$Ax = \sum_{n=1}^\infty n^2 \langle x, x_n\rangle x_n.$$

Moreover, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)_{t\in\mathbb{R}}$ on X which is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos nt(x, x_n)x_n, \quad x \in X,$$

and the associated sine family $\{S(t)\}_{t\in\mathbb{R}}$ on X which is given by

$$S(t)x = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(x, x_n)x_n, \quad x \in X.$$

For more details on operator A and their representation please see [**21, 25, 27, 28, 29**].

For $\alpha = 2$, the equation (6.32) can be reformulated as the following abstract equation in $X = L^2[0, \pi]$:

$$x''(t) + Ax(t) = f(t, x(a(t)), x[h(x(t), t)]) + \int_0^t k(t-s)g(s, x(s))ds, \quad t > 0,$$

(6.36) $\quad x(0) = x_0, \ x'(0) = y_0, \ a(t) \le t,$

where $x(t) = Z(t, .)$ that is $x(t)(y) = Z(t, y)$, $y \in [0, \pi]$. The operator A is same as in equation (6.33). The function $g : \mathbb{R}_+ \times X \to X$, is given by

$$g(t, \varsigma)(y) = g_1(t, y, \varsigma), \tag{6.37}$$

where g_1 is given by

$$g_1(t, y, Z(t, y)) = \int_0^y K(y, s)Z(t, s)ds. \tag{6.38}$$

The function $f : \mathbb{R}_+ \times X \times X \to X$, is given by

$$f(t, \psi, \xi)(y) = f_2(y, \xi) + f_3(t, y, \psi), \tag{6.39}$$

where $f_2 : [0, \pi] \times X \to H_0^1(0, \pi)$ is given by

$$f_2(y, \xi) = \int_0^y K(y, x)\xi(x)dx, \tag{6.40}$$

and

$$\|f_3(t, y, \psi)\| \le V(y, t)(1 + \|\psi\|_{H^2(0,1)}) \tag{6.41}$$

with $V(., t) \in X$ and V is continuous in its second argument. For more details see [**16**]. Thus, the theorem (3.1) can be applied to the problem (6.32).
For $\alpha \in (1, 2)$, since A is the infinitesimal generator of a strongly continuous cosine family $C(t)_{t\in\mathbb{R}}$, form the subordinate principle (Theorem 3.1, [**4**]), it follows that A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $C_\alpha(t)$ such that $C_\alpha(0) = I$, and

$$C_\alpha(t) = \int_0^\infty \varphi_{t,\alpha/2}(s)C(s)ds, \quad t > 0,$$

where $\varphi_{t,\alpha/2}(s) = t^{-\alpha/2}\phi_{\alpha/2}(st^{-\alpha/2})$, and

$$\phi_\gamma(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.$$

Thus, the equation (6.32) can be reformulated as the following fractional differential equation in $X = L^2[0,\pi]$

$$^cD_t^\alpha x(t) + Ax(t) = f(t, x(a(t)), x[h(x(t),t)]) + \int_0^t k(t-s)g(s,x(s))ds, \quad t > 0,$$

$$x(0) = x_0, \ \ x'(0) = y_0, \ a(t) \leq t. \tag{6.42}$$

Therefore, Theorem (3.1), Theorem (4.1) and other abstract results of the manuscript can be obtained for the problem (6.32).

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