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Evolution of the two dimensional Boussinesq system

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Abstract. The smooth evolutions along the trajectories of the main physical quantities of the two dimensional Boussinesq system with viscousity and thermal diffusivity not both non-zero are studied. Specifically, for a spatially H^m solution with $m > 4$ (only $m > 3$ is needed for some result), quantities including the speed, vorticity, temperature gradient and their stretching rates are shown to evolve smoothly along the trajectories. Conclusions on their evolutions are obtained. Results on some of the stretching rates give information on the evolutions of the relative sizes of some basic quantities. When the viscousity and thermal diffusivity are zero, it is not known if smooth solutions exist globally and we study the dichotomy between finite time singularity and the long time behaviors of the main quantities. If either the viscousity or thermal diffusivity is non-zero, it is known that smooth solutions are global and this investigation provides some information about them by describing the dynamics of the main quantities.

Contents

1. Introduction, results and intepretations

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1.1. Introduction. We consider the Boussinesq system in $\mathbb{R}^2$

$$\begin{cases} \text{div } v = 0, \\ v_t + (v \cdot \nabla)v = \nu \Delta v - \nabla p + \theta e_2, \\ \theta_t + v \cdot \nabla \theta = \kappa \Delta \theta, \end{cases} \tag{1.1}$$

where $v = (v^1(x,t), v^2(x,t))$ is the velocity with $x = (x_1, x_2) \in \mathbb{R}^2$ and $t > 0$, $\theta = \theta(x,t)$ is the temperature, $p = p(x,t)$ is the pressure, $e_2 = (0,1)^\top$, $\nu \geq 0$ and $\kappa \geq 0$ are the viscousity and thermal diffusion coefficients respectively. We denote the i-th equation of (1.1) by $(1.1)_i$. It describes the motion of a slightly compressible fluid subject to convective heat transfer with the presence of viscousity or thermal diffusion, and models geophysical flows like ocean circulations. See for instance [**17, 19, 22**]. Though a lot has been written on the equations in two or three spatial dimensions, with or without viscousity and thermal diffusivity, isotropic or anisotropic, they are usually on well-posedness and regularity issues. A very incomplete list is [**1, 3, 4, 6, 11, 12, 13, 16, 21, 23**]. Little has been written on the evolution of the system. We take up this topic in this paper, and investigate the smooth evolutions along the trajectories of the main physical quantities of the 2D Boussinesq system with ν and κ not both non-zero. There are two reasons for this interest. First, when $\nu = \kappa = 0$, it is not known if local smooth solutions can be extended to global ones, and the evolutions of the quantities along the trajectories may give information on whether this is possible. Specifically, if contradictions in the dynamics derived under the assumption that a global smooth solution exists, then it does not. See the discussions in [**7, 8**] in studying the finite time blow-up problem for the three dimensional incompressible Euler system. Second, in case ν and κ are not all zero, it is known that smooth solutions are global [**6, 16**]. The dynamics of the main quantities then give some of their global and asymptotic properties, on which little has been written.

The following existence results provide the foundations for our discussions. In [**11**], it is proved that for initial values $(v_0, \theta_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ with $m > 2$ (same notation for vector- and scalar-valued Sobolev functions), (1.1) with $\kappa = \nu = 0$ has a local solution $(v, \theta) \in C([0,T]; H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$. In fact, it also belongs to $C^1([0,T]; H^{m-2}(\mathbb{R}^2) \times H^{m-2}(\mathbb{R}^2))$ (see Proposition 3.1). It blows up at $T_* > 0$ if and only if $\int_0^{T_*} \|\nabla \theta(t)\|_{L^\infty(\mathbb{R}^2)} dt = \infty$. Hence $\nabla \theta$ is a quantity determining the finite time blow-up of smooth solutions, like the vorticity or deformation tensor in the three dimensional incompressible Euler equations [**2, 20, 18**]. If either ν or κ is non-zero, global $C([0,\infty); H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$ solution exists [**6**] (and [**16**] for the case $\nu > 0$ and $\kappa = 0$ only). When ν and κ are both positive, the existence of global smooth solution is well known (see the discussions in [**3, 6, 16**]), and for instance can be proved by modifying the argument in [**6**]. Though the method employed here also yields results for this case, we choose not to discuss them as the results and their interpretations are clumsy. Quantities like $\nabla \Delta \theta$, $\Delta \omega$ (where ω is the vorticity) and Δv would appear in the theorems and obscure the clear results we now have.

Let $a \in \mathbb{R}^2$. Let $X(a,t)$ be the particle trajectory starting from a at $t = 0$, the solution to

$$\frac{dX(a,t)}{dt} = v(X(a,t),t), \quad X(a,0) = a. \tag{1.2}$$

For a smooth function f on $\mathbb{R}^2 \times (0,T)$, the material derivative of f is $\frac{Df}{Dt} = \partial_t f + v \cdot \nabla f$. We will write $f(X(a,t),t)'$ or simply f' for $\frac{d}{dt} f(X(a,t),t) = \left(\frac{Df}{Dt}\right)(X(a,t),t)$. We study the main quantities of the system including speed, vorticity, temperature gradient and their stretching rates which in turn are composed of more basic quantities like the first three mentioned quantities together with pressure gradient, buoyancy force and deformation tensor. From (1.1), we derive some ordinary differential equations that govern their evolutions along a trajectory, which yield results on their evolutions. The evolutions of the stretching rates in turn give results on the evolution of the relative sizes of the basic quantities. These are the contents of Theorems 1.1 and 1.2, phrased in terms of f, g, α, Φ and Ψ to be defined in Section 1.2

This paper is inspired by Chae [**8**] and uses some of the methods in there (see also [**9, 10**]). The purpose of [**8**] is to investigate the finite time blow up problem of the three dimensional incompressible Euler equations. The approach in there is to study the evolutions of the vorticity and quantities related to its growth like the vorticity stretching rate and the eigenvalues of the deformation tensor (see [**5**]) with an eye on possible contradictions arising from the assumption of global existence of smooth solutions. The followings are the main differences between [**8**] and this paper. First, we investigate the 2D Boussinesq system with ν and κ not both nonzero instead of the 3D incompressible Euler system. Second, when either κ or ν is positive, there is no finite-time blow-up. In that case, our interest is on the global behaviors of global smooth solutions. Third, the focus of [**8**] is on the vorticity and related quantities which are relevant to the blow up of local smooth solutions. In contrast, we investigate all the main variables like velocity, temperature, pressure and vorticity of the Boussinesq equations, the evolutions of which give a good picture of the system. Consequently, we work with the momentum equation, the heat convection equation as well as the vorticity equation. Finally, the scenario in which the magnitude of vorticity decreases along the trajectories is not discussed in [**8**], possibly because finite-time blow-up is the main concern. In contrast, we also discuss the scenarios in which $|v|$, $|\omega|$ and $|\nabla\theta|$ are decreasing. They make up scenario 2 in Theorem 1.1. As a result, we can describe scenario changes in addition to single-scenario long time behaviors and give a more complete description of the evolution of the system. The method in [**8**] is generally applicable and has been used to discuss the dynamics of the Camassa-Holm equation in [**15**].

To facilitate the description of our result, we formally define the concept of vague monotonicity which has already appeared in [**8**].

Definition 1. A continuous function $\Phi : [0,\infty) \to \mathbb{R}$ is *vaguely increasing (decreasing)* if there is an infinite sequence $\{t_j\}_{j=0}^{\infty}$ with $0 = t_0 < t_1 < \cdots < t_j < t_{j+1} \to \infty$ such that for all $j = 0, 1, \ldots$, $\Phi(t_j) < \Phi(t_{j+1})$ ($\Phi(t_j) > \Phi(t_{j+1})$) and $\Phi(t_j) = \max_{t\in[0,t_j]} \Phi(t)$ ($\Phi(t_j) = \min_{t\in[0,t_j]} \Phi(t)$). Φ is said to be *vaguely monotonic* if it is vaguely increasing or decreasing.

This is a weak sense of monotonicity. In particular, that Φ is vaguely increasing implies that there exists a sequence $\{t_j\}$ increasing to infinity along which Φ is strictly increasing and that Φ is definitely not monotonic decreasing. Similar statements hold if Φ is vaguely decreasing. However, Φ can be simultaneously vaguely increasing and decreasing when it fluctuates increasingly wildly. We will use the fact that if $\Phi : [0,\infty) \to \mathbb{R}$ is continuous, positive (negative) and $\liminf_{t\to\infty} \Phi(t) = 0$

($\limsup_{t\to\infty} \Phi(t) = 0$), then Φ is vaguely decreasing (increasing). See Proposition 2.1 for the proof.

Throughout this paper, we will stick to the wordings and meanings in the following definitions of blowing up. Let (v, θ) be a time-continuous $H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2)$-valued solution of (1.1). We say that (v, θ) blows up at $T_* < \infty$ if $(v(t), \theta(t))$ is continuous on $[0, T_*)$ but cannot be extended continuously to T_*. Next, let T_* be the blow-up time of a solution or ∞ if it never blows up in finite time. For $\tilde{T} \le T_*$, we say that a quantity β blows up along a trajectory $(X(a, t), t)$ at $\tilde{T}$ if $\beta(X(a, \cdot), \cdot)$ is continuous on $[0, \tilde{T})$ and $\lim_{T\nearrow\tilde{T}} \sup_{t\in[0,T]} |\beta(X(a, t), t)| = \infty$.

We make two remarks. First, we have only defined the blowing up of a quantity along a trajectory before or at the solution blow-up time (i.e. $\tilde{T} \le T_*$) and do not worry about $\tilde{T} > T_*$. Notice that even if a solution blows up at $T_* < \infty$ and can only be continued beyond T_* as a weaker solution, it is possible that some or even most of the trajectories and the solution along them can be continued smoothly. Then the smooth evolution and blowing-up of a quantity along a trajectory can still be discussed after T_*. However in this paper, in Theorem 1.1 and 1.2, our discussion will stop at any finite blow-up time of the solution. Hence we need not consider this issue, saving the trouble of deciding if a trajectory and a quantity defined along it can be smoothly extended beyond T_*. The second remark is that sometimes the blow-up of a quantity along a trajectory at $\tilde{T}$ implies that the solution blows up at the same time. This is the case if for instance the quantity is v, ω or $\nabla\theta$ and $m > 2$. However, if a quantity like θ_{x_1}/ω blows up along a trajectory at $\tilde{T}$, it could be that ω goes to 0 as t increases to $\tilde{T}$ and the solution need not blow up at that time.

1.2. Evolutions of the quantities. The evolutions of the main quantities of the system along the trajectories are governed by several ordinary differential equations derived from (1.1). We group the quantities into three groups and organize our discussions under three cases. Each one of them deals with one of $|v|$, $|\omega|$, $|\nabla\theta|$ and its stretching rate. In case I, we assume $\nu = 0$, $\kappa \ge 0$ and discuss the evolution of speed $|v|$ and its stretching rate, which is the ratio of the trajectorial component of force and speed. Taking the dot product of $(1.1)_2$ with v, we get $|v|\frac{D|v|}{Dt} = v\frac{Dv}{Dt} = (-v \cdot \nabla p + v^2\theta)$. For $v \ne 0$, let $\xi = (\xi^1, \xi^2) := v/|v|$. Then $D_\xi p := \xi \cdot \nabla p$ is the directional derivative of the pressure and $\xi^2\theta$ is the component of the buoyancy force along the flow direction. Hence $F := -D_\xi p + \xi^2\theta$ is the trajectorial component of the force. Therefore when $|v| \ne 0$,

$$\frac{D|v|}{Dt} = \alpha_1|v|, \quad \text{where} \quad \alpha_1 := \frac{F}{|v|}. \tag{1.3}$$

α_1 is the $|v|$-stretching rate. We will also draw conclusion on the relative sizes of $|v|^2$ and F, the quantity denoted

$$\Phi_1(a, t) := \frac{\alpha_1}{|v|}(X(a, t), t) = \frac{F}{|v|^2}(X(a, t), t). \tag{1.4}$$

Notice that the positivity or negativity of F, α_1 and Φ_1 means that the force is pushing or working against the fluid along its trajectory. Looking further into the evolution of $|v|$ (by considering $D^2|v|/Dt^2$) will involve the derivative of α_1, given by $\alpha_1' = \frac{F'}{|v|} - \alpha_1^2 = \left(\frac{F'|v|}{F^2} - 1\right)\alpha_1^2$. When $F(X(a, t), t)$ is defined and non-zero, if we define

$$\Psi_1(a,t) := \frac{F'|v|}{F^2}(X(a,t),t) = \left[-\frac{DF^{-1}}{Dt}|v|\right](X(a,t),t), \tag{1.5}$$

then

$$\alpha_1(X(a,t),t)' = [\Psi_1(a,t)-1]\alpha_1(X(a,t),t)^2. \tag{1.6}$$

In case II, we still suppose $\nu = 0$ and $\kappa \geq 0$. We consider the evolution of the vorticity $\omega = v^2_{x_1} - v^1_{x_2}$ and its stretching rate. Take the curl of $(1.1)_2$ to get the vorticity equation

$$\frac{D\omega}{Dt} = \theta_{x_1}. \tag{1.7}$$

When $\omega \neq 0$, multiply (1.7) by ω to get

$$\frac{D|\omega|}{Dt} = \alpha_2|\omega| \quad \text{with} \quad \alpha_2 = \frac{\operatorname{sgn}(\omega)\theta_{x_1}}{|\omega|} = \frac{\theta_{x_1}}{\omega}, \tag{1.8}$$

where $\operatorname{sgn}(\omega)$ is 1, -1 or undefined when ω is positive, negative or 0. α_2 is the vorticity stretching rate. Define

$$\Phi_2(a,t) := \frac{\alpha_2}{|\omega|}(X(a,t),t) = \frac{(\operatorname{sgn}\,\omega)\theta_{x_1}}{\omega^2}(X(a,t),t). \tag{1.9}$$

Hence α_2 and Φ_2 are positive if and only if θ_{x_1} and ω are of the same sign. Their evolutions give information on that of ω relative to θ_{x_1}. Again for a refinement (consider $D^2|\omega|/Dt^2$) we need to look at α_2'. When $[\operatorname{sgn}(\omega)\theta_{x_1}](X(a,t),t)$ is defined and non-zero, if we define

$$\begin{aligned}\Psi_2(a,t) \quad &:= \quad \frac{\theta'_{x_1}\omega}{\theta^2_{x_1}}(X(a,t),t) = \left[-\frac{D\theta^{-1}_{x_1}}{Dt}\omega\right](X(a,t),t) \\ &= \quad \left[-\frac{D[\operatorname{sgn}(\omega)\theta_{x_1}]^{-1}}{Dt}|\omega|\right](X(a,t),t),\end{aligned} \tag{1.10}$$

then

$$\alpha_2(X(a,t),t)' = [\Psi_2(a,t)-1]\alpha_2(X(a,t),t)^2. \tag{1.11}$$

In case III, let $\nu \geq 0$, $\kappa = 0$ and consider the evolution of $|\nabla\theta|$ and its stretching rate. Take the gradient of $(1.1)_3$ to get

$$\frac{D\nabla\theta}{Dt} = -V\nabla\theta, \tag{1.12}$$

where V is a 2×2 matrix with $V_{ij} = \partial_{x_i}v^j$. Take the dot product with $\nabla\theta$ to get $|\nabla\theta|\frac{D|\nabla\theta|}{Dt} = \nabla\theta\cdot\frac{D\nabla\theta}{Dt} = -\nabla\theta\cdot V\nabla\theta$. If $\nabla\theta \neq 0$, let $\eta := \nabla\theta/|\nabla\theta|$. Let $S = (V+V^T)/2$ be the deformation tensor. Then

$$\frac{D|\nabla\theta|}{Dt} = \alpha_3|\nabla\theta|, \quad \alpha_3 = -\eta\cdot V\eta = -\eta\cdot S\eta. \tag{1.13}$$

Thus α_3 is the temperature gradient stretching rate. Define

$$\Phi_3(a,t) := \frac{\alpha_3}{|\nabla\theta|}(X(a,t),t) = \frac{-\nabla\theta\cdot S\nabla\theta}{|\nabla\theta|^3}(X(a,t),t). \tag{1.14}$$

Using (1.13),

$$\begin{aligned}\alpha_3' &= \left(\frac{-\eta\cdot S\nabla\theta}{|\nabla\theta|}\right)' = \frac{(-\eta\cdot S\nabla\theta)'}{|\nabla\theta|} - \frac{(-\eta\cdot S\nabla\theta)|\nabla\theta|'}{|\nabla\theta|^2}\\ &= \frac{(-\eta\cdot S\nabla\theta)'}{|\nabla\theta|} - \alpha_3^2 = \left[\frac{(-\eta\cdot S\nabla\theta)'|\nabla\theta|}{(-\eta\cdot S\nabla\theta)^2} - 1\right]\alpha_3^2.\end{aligned}$$

When $(\eta\cdot S\nabla\theta)(X(a,t),t)$ is defined and non-zero, if we define

$$\Psi_3(a,t) := \frac{(-\eta\cdot S\nabla\theta)'|\nabla\theta|}{(-\eta\cdot S\nabla\theta)^2}(X(a,t),t), \tag{1.15}$$

then

$$\alpha_3(X(a,t),t)' = [\Psi_3(a,t)-1]\alpha_3(X(a,t),t)^2. \tag{1.16}$$

One can calculate S', η' etc. to get an expression for Ψ_3 involving no derivatives, but the resulting expression would be complicate. Moreover the present form is convenient for us to give a uniform description of the dynamics in all the three cases.

The dynamics of $|v|$, $|\omega|$ and $|\nabla\theta|$ follow similar patterns and can be described in a uniform manner. Let $f:\mathbb{R}^2\times[0,T]\to\mathbb{R}$ be either $|v|$, $|\omega|$ or $|\nabla\theta|$. Correspondingly, let g be F, $\operatorname{sgn}(\omega)\theta_{x_1}$ or $-\eta\cdot S\nabla\theta$, defined on $\mathbb{R}^2\times[0,T]\setminus Z_f$ with $Z_f := \{(x,t)\in\mathbb{R}^2\times[0,T] : f(x,t)=0\}$ the zero set of f. Let

$$\alpha := \frac{g}{f} \qquad \text{on} \quad \mathbb{R}^2\times[0,T]\setminus Z_f. \tag{1.17}$$

It is the f-stretching rate, and is either α_1, α_2 or α_3 corresponding to the three choices of f and g. We call attention to the fact that $\alpha\neq 0$ implies that both f and g are non-zero. Indeed, for α to be defined, g has to be too and hence $f\neq 0$. Then $\alpha = g/f \neq 0$ implies that $g\neq 0$. Then (1.3), (1.8) and (1.13) can be written as

$$\frac{Df}{Dt} = \alpha f, \quad \alpha = \frac{g}{f} \quad \text{on } \mathbb{R}^2\times[0,T]\setminus Z_f.$$

In particular,

$$f(X(a,t),t)' = (\alpha f)(X(a,t),t) \quad \text{when } f(X(a,t),t)\neq 0. \tag{1.18}$$

Define

$$\Phi(a,t) := \frac{\alpha}{f}(X(a,t),t) = \frac{g}{f^2}(X(a,t),t) \quad \text{when } (X(a,t),t)\notin Z_f \tag{1.19}$$

to be one of the Φ_i's, $i=1,2,3$. If one attempt a refinement of the dynamics and looks into the second derivative of $f(X(a,t),t)$, one encounters α'. From (1.18), $\alpha' = (g'/f) - \alpha^2 = [(g'f/g^2)-1]\alpha^2$. The functions Ψ_i $(i=1,2,3)$ defined in (1.5), (1.10) and (1.15) are summarily given by

$$\begin{aligned}\Psi(a,t) &:= \frac{g'f}{g^2}(X(a,t),t)\\ &= \left[-\frac{Dg^{-1}}{Dt}f\right](X(a,t),t) \quad \text{when } g(X(a,t),t)\neq 0.\end{aligned} \tag{1.20}$$

Then the evolutions of the α_i's given by (1.6), (1.11) and (1.16) can be written as

$$\alpha(X(a,t),t)' = [\Psi(a,t)-1]\alpha(X(a,t),t)^2. \tag{1.21}$$

Information on Ψ and its evolution translates into those on g and f. For instance, $\Psi > 1$ means that $(\operatorname{sgn} g)(\log g)' > |g|/f$ or $g'/g^2 > f^{-1}$, or any form sensible to the reader. For easy reference, we summarize the quantities in the three cases in Table 1.

General	Case I	Case II	Case III
f	$\|v\|$	$\|\omega\|$	$\|\nabla\theta\|$
g	F	$\operatorname{sgn}(\omega)\theta_{x_1}$	$-\eta\cdot S\nabla\theta$
$\alpha=\frac{g}{f}$	$\alpha_1=\frac{F}{\|v\|}$	$\alpha_2=\frac{\operatorname{sgn}(\omega)\theta_{x_1}}{\|\omega\|}=\frac{\theta_{x_1}}{\omega}$	$\alpha_3=-\eta\cdot S\eta$
$\Phi=\frac{\alpha}{f}=\frac{g}{f^2}$	$\Phi_1=\frac{\alpha_1}{\|v\|}=\frac{F}{\|v\|^2}$	$\Phi_2=\frac{\alpha_2}{\|\omega\|}=\frac{\operatorname{sgn}(\omega)\theta_{x_1}}{\|\omega\|^2}$	$\Phi_3=\frac{\alpha_3}{\|\nabla\theta\|}=-\frac{\eta\cdot S\eta}{\|\nabla\theta\|}$
$\Psi=\frac{g'f}{g^2}$	$\Psi_1=\frac{F'\|v\|}{F^2}$	$\Psi_2=\frac{\theta'_{x_1}\omega}{\theta_{x_1}^2}$	$\Psi_3=\frac{(-\eta\cdot S\nabla\theta)'\|\nabla\theta\|}{(-\eta\cdot S\nabla\theta)^2}$

TABLE 1. A summary of f, g, α, Φ and Ψ in the three cases. The entries in the last two rows should read $\Phi(a,t)=(g/f^2)(X(a,t),t)$ and $\Psi(a,t)=(g'f/g^2)(X(a,t),t)$ etc. In the first three rows, the arguments are (x,t) or $(X(a,t),t)$.

1.3. The main theorems and a description of the dynamics. Throughout the rest of this paper, $\alpha_0(a)$ and $f_0(a)$ stand for $\alpha(a,0)$ and $f(a,0)$ respectively. The following theorem describes the evolutions of f, g and Φ in the three cases. Recall that in Case I and Case II, $\nu=0$ and $\kappa\geq 0$, while in Case III, $\nu\geq 0$ and $\kappa=0$.

THEOREM 1.1. *Let $m>4$ ($m>3$ is sufficient for Case I). Let (θ,v) be a $C([0,T];H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2))\cap C^1([0,T];H^{m-2}(\mathbb{R}^2)\times H^{m-2}(\mathbb{R}^2))$ solution of (1.1) (notice that T is not the maximal time of existence). Define the sets*

$$\Sigma^+(t):=\{a\in\mathbb{R}^2|\alpha(X(a,t),t)>0\},\quad \Sigma^-(t):=\{a\in\mathbb{R}^2|\alpha(X(a,t),t)<0\}$$

associated with (v,θ). In particular, both $g(X(a,t),t)$ and $f(X(a,t),t)$ are non-zero for a to belong to any one of them (see the remark after the definition of α (1.17)). We say that the system is in scenario 1 (scenario 2) along $X(a,\cdot)$ at time t, or simply that the system is in scenario 1 (or 2), if $a\in\Sigma^+(t)$ ($a\in\Sigma^-(t)$). Then the following conclusions hold (for a continuation of (v,θ)) for any initial time $t_0\geq 0$, though only stated and proved for $t_0=0$ for clarity.

(1) *If $a\in\Sigma^+(0)$, then at least one of the following is true.*
    (a) *(finite time singularity, possible only if $\nu=\kappa=0$) The solution blows up in finite time.*
    (b) *(possible scenario change from 1 to 2) (finite time extinction of α) There is a $\tilde{t}<\infty$ such that $\alpha(X(a,\tilde{t}),\tilde{t})=0$.*
    (c) *(single-scenario long time behavior)(Φ vaguely decreasing) $\Phi(a,t)>0$ for all $t\geq 0$ and $\liminf_{t\to\infty}\Phi(a,t)=0$.*
(2) *If $a\in\Sigma^-(0)$, then at least one of the following is true.*
    (a) *(finite time singularity, possible only if $\nu=\kappa=0$) The solution blows up in finite time.*

(b) *(possible scenario change from 2 to 1)*

(i) *(finite time extinction of α) There is a $\tilde{t} < \infty$ such that*

$$\alpha(X(a, \tilde{t}), \tilde{t}) = 0.$$

(ii) *(finite time extinction of f) There is a $\bar{t} < \infty$ such that*

$$f(X(a, \bar{t}), \bar{t}) = 0.$$

(c) *(single-scenario long time behavior)*

(i) *(f decaying at the rate $1/t$) There is an $\epsilon^* > 0$, $t^* > 0$ such that for $t \geq t^*$,*

$$f(X(a, t), t) \leq \frac{f(X(a, t^*), t^*)}{1 + \epsilon^*(t - t^*) f(X(a, t^*), t^*)}. \tag{1.22}$$

(ii) *(Φ vaguely increasing) $\Phi(a, t) < 0$ for all $t \geq 0$ and*

$$\limsup_{t \to \infty} \Phi(a, t) = 0.$$

We make three remarks about the options in the theorem. First, notice that in scenario 2, even though equation (1.18) implies that $f(X(a, t), t)$ (one of the $|v|$, $|\omega|$ or $|\nabla\theta|$ under discussion) decreases, the finite-time blow up option 2(a) is still a possibility (when $\nu = \kappa = 0$). Second, notice that no option in scenario 1 correspond to option 2(b)ii, the finite time extinction of f. In fact, we claim that in scenario 1, f cannot vanish unless some other option has already happened. The reason is as follows. Write $f(t)$ for $f(X(a, t), t)$ temporarily. Suppose that $f(t_1) = 0$ but $f > 0$ on $[0, t_1)$. Then Proposition 3.2 implies that α is continuous over $[0, t_1)$. If 1(b) has not happened before t_1, α remains positive over $[0, t_1)$. Then (1.18) implies that f increases over there, contradictory to $f(t_1) = 0$. Our claim is proved. In contrast, in scenario 2, if 2(a) and 2(b)i have not happened, $\alpha(X(a, t), t)$ remains negative. Then from (1.18), f decreases and the possibility that it vanishes at some finite $\bar{t}$ cannot be ruled out. Third, let us explain why the finite time extinction of f (option 2(b)ii) is a scenario change option. Consider Case I, when $f = |v|$ and $\alpha = \alpha_1 = F/|v|$. If $|v(X(a, t), t)|$ vanishes, the smoothly varying $v(X(a, t), t)$ can 'reverse' direction. Then ξ can change direction abruptly causing F and hence α to change sign. Then the system enters scenario 1 from scenario 2. Notice that the locus of the particle can have a cusp even though the trajectory $X(a, t)$ and $v(X(a, t), t)$ are smooth functions of t. Next consider Case II where $f = |\omega|$ and $\alpha = \alpha_2 = \mathrm{sgn}(\omega)\theta_{x_1}/\omega$. If $\omega(X(a, \bar{t}), \bar{t}) = 0$, then $\alpha(X(a, t), t)$ can go to $\pm\infty$ as $t \nearrow \bar{t}$ and can change sign after $\bar{t}$ resulting in a scenario change. Consider Case III where $f = |\nabla\theta|$ and $\alpha = \alpha_3 = -\eta \cdot S\eta$. When $|\nabla\theta|$ vanishes, η may change direction abruptly even though $\nabla\theta$ varies smoothly, possibly causing α to change sign and a scenario change. Of course, in all the three cases, options 1(b) and 2(b) do not necessarily result in scenario changes, as α can retain its original sign after these options happened.

From Theorem 1.1, the evolutions of f and g can be described as follows. From (1.18), $f(X(a, t), t)$ increases in scenario 1 and decreases in scenario 2 . If 1(b) happens, α may change sign after $\tilde{t}$ and the system may enter scenario 2 from scenario 1, and vice versa if any one of the options in 2(b) happens. Hence along a particle trajectory, the system can go back and forth among the two scenarios. If it stays in scenario 1 after a certain time, then $f(X(a, t), t)$ keeps on increasing. It may blow up in finite time (possible only if $\nu = \kappa = 0$), or $\liminf_{t \to \infty} \Phi(a, t) = 0$, which implies that Φ is vaguely decreasing (see Proposition 2.1) or equivalently g decreases relative to f^2 in a vague sense. If it stays in scenario 2, then f keeps on decreasing. The solution may blow up in finite time, or f decays at the rate $1/t$ or $\limsup_{t\to\infty}\Phi(a,t)=0$ which implies that Φ is vaguely increasing. As $g<0$ in this case, $|g|$ still decreases vaguely relative to f^2.

If one wants more information on the evolution of f and consider its second derivative, one encounters α'. The following theorem considers the evolutions of α and Ψ, which in turn gives further information on those of f and g. It works with (1.21) instead of (1.18).

THEOREM 1.2. *Let $m>4$ and (θ,v) be a $C([0,T];H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2))\cap C^1([0,T];H^{m-2}(\mathbb{R}^2)\times H^{m-2}(\mathbb{R}^2))$ solution of (1.1) (T not the maximal time of existence). Define the sets*

$$\Sigma^{++}(t):=\{a\in\mathbb{R}^2|\alpha(X(a,t),t)>0,\Psi(a,t)>1\},$$

$$\Sigma^{+-}(t):=\{a\in\mathbb{R}^2|\alpha(X(a,t),t)>0,\Psi(a,t)<1\},$$

$$\Sigma^{-+}(t):=\{a\in\mathbb{R}^2|\alpha(X(a,t),t)<0,\Psi(a,t)>1\},$$

$$\Sigma^{--}(t):=\{a\in\mathbb{R}^2|\alpha(X(a,t),t)<0,\Psi(a,t)<1\}$$

associated with (v,θ). In particular, $g(X(a,t),t)$, $f(X(a,t),t)$ and $\Psi(a,t)-1$ are non-zero for a to belong to any one of them (see the remark after (1.17)). We say that the system is in scenario 1 (scenario 2) along $X(a,\cdot)$ at time t, or simply that the system is in scenario 1 (or 2), if $a\in\Sigma^{++}(t)\cup\Sigma^{--}(t)$ ($a\in\Sigma^{+-}(t)\cup\Sigma^{-+}(t)$). Then the following conclusions hold (for a continuation of (v,θ)) for any initial time $t_0\geq 0$, though only stated and proved for $t_0=0$ for clarity

(1) *If $a\in\Sigma^{++}(0)\cup\Sigma^{--}(0)$, then at least one of the following is true.*
   (a) *(finite time singularity, possible only if $\nu=\kappa=0$) The solution blows up in finite time.*
   (b) *(possible scenario change from 1 to 2)*
      (i) *(finite time extinction of f, possible only if $a\in\Sigma^{--}(0)$) There exists a $\bar{t}<\infty$ such that $f(X(a,\bar{t}),\bar{t})=0$.*
      (ii) *(finite time extinction of $\Psi-1$) There exists a $\hat{t}<\infty$ such that $\Psi(a,\hat{t})=1$.*
   (c) *(single-scenario long time behavior)*
      (i) *(Ψ vaguely decreasing) For $a\in\Sigma^{++}(0)$, $\Psi(a,t)>1$ for all $t\geq 0$ and $\liminf_{t\to\infty}\Psi(a,t)=1$.*
      (ii) *(Ψ vaguely increasing) For $a\in\Sigma^{--}(0)$, $\Psi(a,t)<1$ for all $t\geq 0$ and $\limsup_{t\to\infty}\Psi(a,t)=1$.*

(2) *If $a\in\Sigma^{+-}(0)\cup\Sigma^{-+}(0)$, then at least one of the following is true.*
   (a) *(finite time singularity, possible only if $\nu=\kappa=0$) The solution blows up in finite time.*
   (b) *(possible scenario change from 2 to 1)*
      (i) *(finite time extinction of α) There is a $\tilde{t}<\infty$ such that*

$$\alpha(X(a,\tilde{t}),\tilde{t})=0.$$

      (ii) *(finite time extinction of $\Psi-1$) There is a $\hat{t}<\infty$ such that $\Psi(a,\hat{t})=1$.*
   (c) *(single-scenario long time behavior)*

(i) *(α decays at the rate $1/t$) There is an $\epsilon^* > 0$, $t^* > 0$ such that for $t \geq t^*$,*

$$|\alpha(X(a,t),t)| \leq \frac{|\alpha(X(a,t^*),t^*)|}{1+\epsilon^*(t-t^*)|\alpha(X(a,t^*),t^*)|}.$$

(ii) (A) *(Ψ vaguely increasing) If $a \in \Sigma^{+-}(0)$, $\Psi(a,t) < 1$ for $t \geq 0$ and $\limsup_{t\to\infty} \Psi(t) = 1$.*

(B) *(Ψ vaguely decreasing) If $a \in \Sigma^{-+}(0)$, $\Psi(a,t) > 1$ for $t \geq 0$ and $\liminf_{t\to\infty} \Psi(t) = 1$.*

We make four remarks on the options. First, although we work with α and (1.21) in this theorem, the blow up referred to in 1(a) and 2(a) are that of the solution (v,θ). The blow-up of α along $X(a,t)$, i.e. $\lim_{T\to\tilde{T}} \sup_{t\in[0,T]} |\alpha(X(a,t),t)| = \infty$ for some $\tilde{T} < \infty$, is not our focus. The latter implies that either

$$\lim_{T\to\tilde{T}} \sup_{t\in[0,T]} |g(X(a,t),t)| = \infty$$

(then in all of Case I to Case III, the solution blows up no later than $\tilde{T}$) or $\lim_{T\to\tilde{T}} \inf_{t\in[0,T]} f(X(a,t),t) = 0$ (whereas the solution can remain smooth). Hence the blowing up of α along $X(a,t)$ is contained in the options 1(a), 1(b)i and 2(a). Second, there is no option in scenario 1 corresponding to option 2(b)i, the finite time extinction of α. Indeed, we claim that for α to vanish, f or $\Psi - 1$ must have already vanished (i.e. 1(b)i or 1(b)ii has already happened). To see this, suppose that $a \in \Sigma^{++}(0)$. Suppose that α vanishes at $t_1 > 0$ but not before, and f does not vanish on $[0,t_1)$. Then Proposition 3.2 implies that α and α' are continuous over there. Temporarily, we write $\alpha(t)$ for $\alpha(X(a,t),t)$ and similarly for α'. As $\alpha(0) > 0$ and $\alpha(t_1) = 0$, there is a $t_2 \in (0,t_1)$ such that $\alpha'(t_2) < 0$. As $\alpha'(0) > 0$ (from (1.21)), there is a $t_3 \in (0,t_2)$ such that $\alpha'(t_3) = 0$. Then (1.21) implies that $\Psi(a,t_3) - 1 = 0$, i.e. 1(b)ii has already happened at t_3 before t_1. For $a \in \Sigma^{--}(0)$, the argument is similar. Our claim is proved. Third, there is no option in scenario 2 corresponding to option 1(b)i, the finite time extinction of f. In fact, we claim that f can vanish before some other options happen only if $a \in \Sigma^{--}(0)$. To see this, suppose that $f(t_1) := f(X(a,t_1),t_1) = 0$ for some $t_1 > 0$, $f > 0$ on $[0,t_1)$ and no other options happen before t_1. Then Proposition 3.2 implies that α is continuous on $[0,t_1)$. First suppose that $a \in \Sigma^{++}(0)$. From the previous remark (the second remark after Theorem 1.2), α does not vanish on $[0,t_1)$. Hence $\alpha > 0$ over there and (1.18) implies that $f(t_1) > 0$, contradictory to our assumption. Next suppose that $a \in \Sigma^{+-}(0)$ in scenario 2. As we are assuming that 2(b)i does not happen on $[0,t_1)$, $\alpha > 0$ over there. Again (1.18) gives the contradiction $f(t_1) > 0$. Finally suppose that $a \in \Sigma^{-+}(0)$. As 2(b)i does not happen in $[0,t_1)$, $\alpha = g/f < 0$ over there. Then $g < 0$ on $[0,t_1)$ and Proposition 3.2 implies that $\Psi - 1$ is continuous over there. Hence $\Psi - 1 > 0$ over $[0,t_1)$ as it is positive at $t = 0$. Then (1.21) implies that $\alpha' > 0$ and hence $\alpha > \alpha_0(a)$ on $[0,t_1)$. Consequently (1.18) gives $f(t_1) = \exp\{\int_0^{t_1} \alpha(X(a,s),s)ds\} f_0(a) > 0$, yielding a contradiction. This proves our claim. Fourth, as explained in the third remark after Theorem 1.1, the vanishing of f may (but not necessarily) cause α to change sign and hence a change of scenario. Hence 1(b)i is a scenario change option. Of course, the vanishing of α and $\Psi - 1$ are scenario change options.

Theorem 1.2 gives more information on the evolutions of the main quantities of the system. From (1.21), if the system is in scenario 1 at time t, whether a is in $\Sigma^{++}(t)$ or $\Sigma^{--}(t)$, $|\alpha(X(a,t),t)|$ is increasing. If 1(b)i happens, α may change sign. If 1(b)ii happens, $\Psi(a,t)-1$ may change sign. In either case, the system leaves scenario 1 and enters scenario 2. If the system is in scenario 2 at time t, whether a is in $\Sigma^{+-}(t)$ or $\Sigma^{-+}(t)$, $|\alpha(X(a,t),t)|$ is decreasing. If any one of the options in 2(b) happens, the system may enter scenario 1 from 2. Hence along a particle trajectory, the system can go back and forth among the two scenarios. If it stays in scenario 1 after a certain time, then $|\alpha(X(a,t),t)|$ keeps on increasing. The solution can blow up (possible only if $\nu=\kappa=0$ also) or f and g are such that Ψ is vaguely monotonic (decreasing when $a\in\Sigma^{++}(t)$, increasing if $a\in\Sigma^{--}(t)$). If it stays in scenario 2, then $|\alpha(X(a,t),t)|$ keeps on decreasing. Either the solution blows up (possible only when $\nu=\kappa=0$), or α decays at the rate $1/t$ or Ψ is vaguely monotonic (increasing if $a\in\Sigma^{+-}(t)$, decreasing if $a\in\Sigma^{-+}(t)$). Notice that the relative behaviors of f and g are given by the behavior of Ψ.

We prove the Theorem 1.1 and 1.2 in Section 2. In Section 3, we prove that the regularity requirements on the solution in the theorems ensure the smooth evolutions of various quantities along the trajectories, providing the foundations for the arguments in Section 2 and the remarks following the theorems in this section.

2. Proof of the theorems on the dynamics of the 2D Boussinesq system

Recall that $\alpha_0(a)$ and $f_0(a)$ stand for
sumalpha$(a,0)$ and $f(a,0)$ respectively. We first prove the sufficient condition for vague monotonicity mentioned after Definition 1.

PROPOSITION 2.1. Let $\Phi:[0,\infty)\to\mathbb{R}$ be continuous and strictly positive (negative), with $\liminf_{t\to\infty}\Phi(t)=0$ ($\limsup_{t\to\infty}\Phi(t)=0$). Then Φ is vaguely decreasing (increasing).

PROOF. Suppose that Φ satisfies the hypothesis of the theorem (the strictly positive part). Let $t_0=0$. Suppose t_{n-1} has been chosen. We claim that there is a $t_n\geq t_{n-1}+1$ such that $\Phi(t_n)\leq\Phi(t_{n-1})/2$ and $\Phi(t_n)=\min_{s\in[0,t_n]}\Phi(s)$. Indeed, as $\min_{s\in[0,t_{n-1}+1]}\Phi(s)>0$ and $\liminf_{t\to\infty}\Phi(t)=0$, there is a $t'>t_{n-1}+1$ such that

$$\frac{1}{2}\min_{s\in[0,t_{n-1}+1]}\Phi(s)>\Phi(t')\geq\min_{s\in[t_{n-1}+1,t']}\Phi(s)>0.$$

As Φ is continuous, there is a $t_n\in[t_{n-1}+1,t']$ such that $\Phi(t_n)=\min_{s\in[t_{n-1}+1,t']}\Phi(s)$. Then $t_n\geq t_{n-1}+1$ and

$$\Phi(t_n)=\min_{s\in[0,t_n]}\Phi(s)=\min_{s\in[0,t']}\Phi(s)<\frac{1}{2}\min_{s\in[0,t_{n-1}+1]}\Phi(s)\leq\frac{1}{2}\Phi(t_{n-1}).$$

Our claim is proved. Consequently, $0=t_0<t_1<\cdots<t_n<t_{n+1}\to\infty$ as $n\to\infty$ and $\Phi(t_j)>\Phi(t_{j+1})$ for $j=0,1,2,\ldots$. Hence Φ is vaguely decreasing. The corresponding sufficient condition for Φ being vaguely increasing can be proved similarly. □

PROOF OF THEOREM 1.1. Suppose $a\in\Sigma^{+}(0)$. Suppose 1(a) and 1(b) do not hold. From the second remark after Theorem 1.1, $f(X(a,t),t)>0$ for all $t\geq0$. Then from Proposition 3.2, f', α and Φ are continuous along $(X(a,t),t)$ for all $t\geq0$. That 1(b) does not hold implies that for all $t\geq0$, $\alpha(X(a,t),t)>0$ and hence from (1.18) $f(X(a,t),t)>0$. Therefore $\Phi=\alpha/f>0$ and $\liminf_{t\to\infty}\Phi(a,t)\geq0$. We claim that $\liminf_{t\to\infty}\Phi(a,t)=0$. To see this, suppose on the contrary that

$\liminf_{t\to\infty} \Phi(a,t) = I > 0$. Then there is some $T_1 > 0$ such that for $t \geq T_1$, $\Phi(a,t) > I/2$ and (1.18) gives $\frac{Df}{Dt}\frac{1}{f^2} = \frac{\alpha}{f} = \Phi > I/2$. Integrating along $X(a,\cdot)$ from T_1 to t, we get

$$f(X(a,t),t) > \frac{2f(X(a,T_1),T_1)}{2 - I(t-T_1)f(X(a,T_1),T_1)}$$

and f blows up along $X(a,t)$ no later than $T_1 + 2/If(X(a,T_1),T_1)$. Then the solution also blows up no later than that time, contradicting the assumption that 1(a) does not happen. Our claim is proved. That is, 1(c) holds.

Next suppose that $a \in \Sigma^-(0)$. Suppose 2(a) and both options in 2(b) do not happen. Then f is strictly positive along $X(a,t)$ for all $t \geq 0$ (2(b)ii does not happen) and Proposition 3.2 guarantees that f', α and Φ are continuous along $(X(a,t),t)$. Hence for all $t \geq 0$, $\alpha(X(a,t),t) < 0$ (as 2(b)i does not hold) and $\Phi = \alpha/f < 0$. Consequently, $\limsup_{t\to\infty} \Phi(a,t) = S \leq 0$. In case $S < 0$, there is a $T_2 > 0$ such that for $t \geq T_2$, $S/2 > \Phi(a,t) = \frac{Df}{Dt}\frac{1}{f^2}(X(a,t),t)$(by (1.18)). Integrating along $X(a,\cdot)$ from T_2 to t, we get

$$f(X(a,t),t) < \frac{2f(X(a,T_2),T_2)}{2 - S(t-T_2)f(X(a,T_2),T_2)}.$$

As $S < 0$, 2(c)i holds. In case $S = 0$, 2(c)ii holds. The theorem is proved. □

We prove Theorem 1.2 below. Recall that the blowing up of α along $X(a,t)$ is not equivalent to the blowing up of the solution (v,θ), but is contained in the options of blowing up of the solution (1(a) or 2(a)) and the vanishing of f somewhere along $X(a,t)$ (1(b)i) in Theorem 1.2. See the first remark after the theorem.

Proof of Theorem 1.2. Suppose $a \in \Sigma^{++}(0)$. Suppose that 1(a) and 1(b) do not happen and we will show that 1(c) holds. Recall that $a \in \Sigma^{++}(0)$ implies that f does not vanish along $(X(a,t),t)$ (see the third remark after Theorem 1.2, or anyway excluded by our assumption that 1(b) do not happen). Hence α is defined there. Now α cannot vanish along $(X(a,t),t)$ as we are assuming that 1(b)i and 1(b)ii do not happen (see the second remark after Theorem 1.2). Hence $g \neq 0$ along $(X(a,t),t)$. Then Proposition 3.2 implies that α, α' and Ψ are continuous along $(X(a,t),t)$. It follows that for all $t \geq 0$, $\alpha(X(a,t),t)$ and $\Psi(a,t)-1$ remain positive. Hence $\liminf_{t\to\infty}(\Psi(a,t)-1) = I \geq 0$. We claim that $I = 0$. Suppose that $I > 0$. Then there is a $T_1 > 0$ such that for all $t \geq T_1$, $I/2 \leq \Psi - 1 = \frac{1}{\alpha^2}\frac{D\alpha}{Dt}$ (by (1.21)). Integrating from T_1 to t gives

$$\alpha(X(a,t),t) \geq \frac{2\alpha(X(a,T_1),T_1)}{2 - I\alpha(X(a,T_1),T_1)(t-T_1)}.$$

Hence $\alpha(X(a,t),t) \to \infty$ no later than $T_1 + 2/(I\alpha(X(a,T_1),T_1))$. As f cannot vanish for $a \in \Sigma^{++}(0)$ (third remark after Theorem 1.2), $g(X(a,t),t) \to \infty$ no later than that time and hence the solution blows up in finite time, contradicting the assumption that 1(a) does not happen. Our claim is proved and 1(c)i holds.

Suppose $a \in \Sigma^{--}(0)$ and 1(a) and 1(b) do not happen. Then f remains positive along $(X(a,t),t)$ (1(b)i does not happen). Hence α is continuous along the trajectory (Prop. 3.2) and never vanishes (second remark after Theorem 1.2). Then g never vanishes too and $\Psi - 1$ is continuous along $(X(a,t),t)$ (Prop. 3.2). Hence α and $\Psi - 1$ remains negative along $(X(a,t),t)$ (α never vanishes and 1(b)ii does not happen). Hence $\limsup_{t\to\infty}(\Psi(a,t)-1) = S \leq 0$. Similar to the above paragraph, if $S < 0$, $\alpha(X(a,t),t) \to -\infty$ in finite time, implying that either $g \to -\infty$ (solution blows up and 1(a) holds) or $f \to 0$ (1(b)i holds), contradicting our assumption. Hence $S = 0$ and 1(c)ii holds.

Suppose $a \in \Sigma^{+-}(0)$. Suppose 2(a) and 2(b) do not happen. Then α and hence g does not vanish along the trajectory (2(b)i does not happen). Also f does not vanish in this scenario (the third remark after the theorem). Hence Proposition 3.2 implies that α, α' and Ψ are continuous along $(X(a,t),t)$. Consequently $\alpha(X(a,t),t)$ and $\Psi(a,t) - 1$ remain positive and negative respectively for all $t \geq 0$. Hence $\limsup_{t\to\infty}(\Psi(a,t) - 1) = S \leq 0$. If $S < 0$, there is a T_1 such that for $t \geq T_1$, $S/2 \geq \Psi - 1 = \frac{1}{\alpha^2}\frac{D\alpha}{Dt}$. Integrating from T_1 to t gives

$$\alpha(X(a,t),t) \leq \frac{2\alpha(X(a,T_1),T_1)}{2 - S\alpha(X(a,T_1),T_1)(t-T_1)}.$$

Hence α decreases at least at the rate $1/t$ and 2(c)i holds. In case $S = 0$, 2(c)iiA holds.

Suppose that $a \in \Sigma^{-+}(0)$. If 2(a) and 2(b) do not hold, then similar to the last paragraph, α, α' and Ψ are continuous along $(X(a,t),t)$ for all $t \geq 0$. Hence α and $\Psi - 1$ stay negative and positive respectively. Consequently $\liminf_{t\to\infty}(\Psi(a,t) - 1) = I \geq 0$. If $I > 0$, by (1.21), there is a T_1 such that for $t \geq T_1$, $I/2 \leq \Psi - 1 = \frac{1}{\alpha^2}\frac{D\alpha}{Dt}$. Integrating from T_1 to t gives

$$\alpha(X(a,t),t) \geq \frac{2\alpha(X(a,T_1),T_1)}{2 - I\alpha(X(a,T_1),T_1)(t-T_1)}.$$

Hence $|\alpha|$ decreases at least at the rate $1/t$ and 2(c)i holds. In case $I = 0$, 2(c)iiB holds. □

3. Smooth evolutions of various quantities along the trajectories

We first show that $C([0,T]; H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$ solutions of (1.1) are smooth in time.

PROPOSITION 3.1. Let $\nu, \kappa \geq 0$. Let $(v,\theta) \in C([0,T]; H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$ with $m > 2$ and $T > 0$ be a solution to (1.1). Then $(v,\theta) \in C^1([0,T]; H^{m-2}(\mathbb{R}^2) \times H^{m-2}(\mathbb{R}^2))$.

PROOF. Let v, θ be as in the hypothesis and $m > 2$. For $0 \leq s, t \leq T$, $(1.1)_3$ gives

$$\theta_t(t) - \theta_t(s) = [v(t) - v(s)] \cdot \nabla\theta(t) + v(s) \cdot \nabla[\theta(t) - \theta(s)] + \kappa[\Delta\theta(t) - \Delta\theta(s)]. \tag{3.1}$$

We claim that $[v(t) - v(s)] \cdot \nabla\theta(t) \in H^{m-2}(\mathbb{R}^2)$. To see this, suppose first that $m \in \mathbb{Z}$. Then the Sobolev Embedding Theorem gives $v(t) - v(s) \in C^{m-2}(\mathbb{R}^2)$, and $\nabla\theta(t) \in H^{m-1}(\mathbb{R}^2)$. By the Leibniz formula, our claim holds. If $m \notin \mathbb{Z}$, the Sobolev Embedding Theorem gives $v(t) - v(s) \in C^{[m-1]}(\mathbb{R}^2)$, and $\nabla\theta(t) \in H^{[m-1]}(\mathbb{R}^2)$. Hence from the Leibniz formula, $[v(t) - v(s)] \cdot \nabla\theta(t) \in H^{[m-1]}(\mathbb{R}^2) \hookrightarrow H^{m-2}(\mathbb{R}^2)$. This proves our claim. Similarly, $v(s) \cdot \nabla[\theta(t) - \theta(s)] \in H^{m-2}(\mathbb{R}^2)$. Obviously, $\Delta\theta(t) - \Delta\theta(s) \in H^{m-2}(\mathbb{R}^2)$. It follows from (3.1) that $\theta_t(t) - \theta_t(s) \in H^{m-2}(\mathbb{R}^2)$, and

$$\begin{aligned}\|\theta_t(t) - \theta_t(s)\|_{H^{m-2}(\mathbb{R}^2)} &\leq \|[(v(t) - v(s)) \cdot \nabla]\theta(t)\|_{H^{m-2}(\mathbb{R}^2)} \\ &+ \|(v(s) \cdot \nabla)[\theta(t) - \theta(s)]\|_{H^{m-2}(\mathbb{R}^2)} + \kappa\|\Delta\theta(t) - \Delta\theta(s)\|_{H^{m-2}(\mathbb{R}^2)}.\end{aligned}$$

When $m \notin \mathbb{Z}$, further relax the first two $\|\cdot\|_{H^{m-2}}$ norms on the right hand side to $\|\cdot\|_{H^{[m-1]}}$ norms. As $(v,\theta) \in C([0,T];H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2))$, the right hand side tends to 0 as $s \to t$ by the Leibniz formula. Hence $\theta_t \in C([0,T];H^{m-2}(\mathbb{R}^2))$ and $\theta \in C^1([0,T];H^{m-2}(\mathbb{R}^2))$.

For v_t, project $(1.1)_2$ to the space of divergence free vector fields to get

$$v_t + \Pi(v\cdot\nabla v) = \nu\Delta v + \Pi(\theta e_2),$$

where $\Pi : H^k(\mathbb{R}^2) \to H^k(\mathbb{R}^2)$ denote the continuous projection operator, $k \geq 0$. Then using a reasoning similar to that for θ_t, we get $v \in C^1([0,T];H^{m-2}(\mathbb{R}^2))$. The proposition is proved. □

We now show that the regularity requirement $m > 4$ guarantees the smooth evolutions of various quantities along the trajectories. In Case I, the weaker requirement $m > 3$ is sufficient for the continuity of f', α and Φ.

Proposition 3.2. Let $m > 4$ and $(v,\theta) \in C([0,T];H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2)) \cap C^1([0,T];H^{m-2}(\mathbb{R}^2)\times H^{m-2}(\mathbb{R}^2))$ be a solution to (1.1). The following holds in Case I, II and III. For $a \in \mathbb{R}^2$, f is continuous along the trajectory $(X(a,t),t)$ for $t \in [0,T]$. Moreover, f', α, α' and Φ are continuous along the trajectory whenever $f(X(a,t),t) \neq 0$. Ψ is continuous along the trajectory whenever $g(X(a,t),t) \neq 0$ or equivalently $\alpha(X(a,t),t) \neq 0$. (In case I, $m > 3$ is sufficient for the continuity of f', α and Φ. f is continuous when $m > 1$ in Case I and $m > 2$ in Case II and III.)

We break up the proof of Proposition 3.2 into four lemmas.

Lemma 3.3. *(continuity of f', α and Φ in Case I) Let $\nu = 0$. Let $m > 3$ and (v,θ) be be as in Proposition 3.2. Then over $\mathbb{R}^2\times[0,T]\setminus\{(x,t) : v(x,t) = 0\}$, $D|v|/Dt$, F and $\alpha_1 = F/|v|$ are continuous. This implies the continuity in $t \in [0,T]$ along $(X(a,t),t)$ of $|v|'$, α_1 and $\Phi_1 = F/|v|^2$ if $v \neq 0$.*

Proof. If $m > 3$, $v, v_t, v_{x_i} \in C([0,T];H^{m-2}(\mathbb{R}^2)) \hookrightarrow C(\mathbb{R}^2\times[0,T];\mathbb{R}^2)$ by the Sobolev embedding theorem. Consequently over $\mathbb{R}^2\times[0,T]\setminus\{v = 0\}$, $|v|_t = (v/|v|)\cdot v_t$, $|v|_{x_i} = (v/|v|)\cdot v_{x_i}$ and hence $\frac{D|v|}{Dt} = |v|_t + (v\cdot\nabla)|v|$ are continuous. For the continuity of F and α_1, the hypothesis $m > 3$ together with the Sobolev embedding theorem gives the continuity of θe_2, v_t, v, ∇v and thus $(v\cdot\nabla)v$. Then ∇p is continuous by $(1.1)_2$. It follows that $D_\xi p$ and $\xi^2\theta$ are continuous if $v \neq 0$. Hence F and α are continuous if v is non-zero. Restricting to $(X(a,t),t)$, we get the continuity of $|v|'$, α_1 and Φ_1 along the trajectory when $v \neq 0$. □

Lemma 3.4. *(continuity of α' and Ψ in Case I) Let $\nu = 0$. Let $m > 4$ and (v,θ) be as in Proposition 3.2. Then DF/Dt is continuous on $\mathbb{R}^2\times[0,T]\setminus\{v = 0\}$. This implies the continuity in $t \in [0,T]$ along $(X(a,t),t)$ of F' and $\alpha_1' = (|v|F' - F|v|')/|v|^2$ (if $v \neq 0$) and $\Psi_1 = F'|v|/F^2$ (if $F \neq 0$).*

Proof. *Step 1.* To show that $\frac{DF}{Dt} = \frac{D}{Dt}(-D_\xi p) + \frac{D}{Dt}(\xi^2\theta)$ is space-time continuous, we will show that its two summands are. The first summand can be expanded to

$$\frac{D}{Dt}(D_\xi p) = \left(\frac{v}{|v|}\right)_t\cdot\nabla p + \frac{v}{|v|}\cdot\nabla p_t + (v\cdot\nabla)\left(\frac{v}{|v|}\cdot\nabla p\right) \tag{3.2}$$

and we will show that every term on the right hand side is continuous when $|v| \neq 0$.
Step 2. In this step, we show that the term $\frac{v}{|v|}\cdot\nabla p_t$ in (3.2) is continuous when $|v| \neq 0$. It is relatively complicate because of the presence of ∇p_t. We first show that ∇p_t is continuous. By the Sobolev embedding theorem, it suffices to show that

$\nabla p_t \in C([0,T]; H^2(\mathbb{R}^2))$. For this, we will write ∇p_t, $\nabla^2 p_t$ and $\nabla^3 p_t$ as singular integrals and invoke the theory of singular integral operators to show that they are in $C([0,T]; L^2(\mathbb{R}^2))$.

We first show that $\nabla^2 p_t$ and $\nabla^3 p_t$ are in $C([0,T]; L^2(\mathbb{R}^2))$. Take the divergence of $(1.1)_2$ to get

$$-\Delta p = \sum_{i,j=1}^{2} v^i_{x_j} v^j_{x_i} - \theta_{x_2}. \tag{3.3}$$

Differentiate with respect to t to get

$$-\Delta p_t = \sum_{i,j=1}^{2} [v^i_{x_j t} v^j_{x_i} + v^i_{x_j} v^j_{x_i t}] - \theta_{x_2 t} := f. \tag{3.4}$$

We claim that $f \in C([0,T]; H^1(\mathbb{R}^2))$. To see this, suppose first that $m \in \mathbb{Z}$. Then $v^i_{x_j} \in C([0,T]; H^{m-1}(\mathbb{R}^2)) \hookrightarrow C([0,T]; C^{m-3}(\mathbb{R}^2))$, $v^j_{x_i t} \in C([0,T]; H^{m-3}(\mathbb{R}^2))$ and the Leibniz formula gives $v^i_{x_j} v^j_{x_i t} \in C([0,T]; H^{m-3}(\mathbb{R}^2)) \hookrightarrow C([0,T]; H^1(\mathbb{R}^2))$ as $m > 4$. If $m \notin \mathbb{Z}$, $v^i_{x_j} \in C([0,T]; H^{m-1}(\mathbb{R}^2)) \hookrightarrow C([0,T]; C^{[m-2]}(\mathbb{R}^2))$, $v^j_{x_i t} \in C([0,T]; H^{m-3}(\mathbb{R}^2))$ and the Leibniz formula gives $v^i_{x_j} v^j_{x_i t} \in C([0,T]; H^{[m-3]}(\mathbb{R}^2)) \hookrightarrow C([0,T]; H^1(\mathbb{R}^2))$. Obviously, $\theta_{x_2 t} \in C([0,T]; H^1(\mathbb{R}^2))$. Our claim is proved. From (3.4), $p_t(x,t) = -(2\pi)^{-1} \int_{\mathbb{R}^2} \log|x-y| f(y,t) dy$. Hence

$$\nabla p_t(x,t) = \frac{-1}{2\pi} \int_{\mathbb{R}^2} \frac{(x_1 - y_1, x_2 - y_2)^\top}{|x-y|^2} f(y,t) dy. \tag{3.5}$$

Then the components of $\nabla^2 p_t(x,t)$ are

$$\begin{cases} \partial^2_{x_i} p_t(x,t) = \frac{-1}{2\pi} \int_{\mathbb{R}^2} \left[\frac{1}{|x-y|^2} - \frac{2(x_i - y_i)^2}{|x-y|^4} \right] f(y,t) dy, \quad i = 1,2, \\ \partial_{x_1} \partial_{x_2} p_t(x,t) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{(x_1 - y_1)(x_2 - y_2)}{|x-y|^4} f(y,t) dy, \end{cases} \tag{3.6}$$

and those of $\nabla^3 p_t(x,t)$ are

$$\partial^2_{x_i} \partial_{x_k} p_t(x,t) = \frac{-1}{2\pi} \int_{\mathbb{R}^2} \left[\frac{1}{|x-y|^2} - \frac{2(x_i - y_i)^2}{|x-y|^4} \right] (\partial_{x_k} f)(y,t) dy, \quad i,k = 1,2. \tag{3.7}$$

Notice that the kernels of the singular integrals in (3.6) and (3.7) are of the form $\Omega(z)/|z|^2$, with $\Omega(z) = 1 - (2z_i^2/|z|^2)$ or $z_1 z_2/|z|^2$. As we have just shown that $f, \partial_{x_k} f \in C([0,T]; L^2(\mathbb{R}^2))$, the theory of singular integral operators gives

$$\nabla^2 p_t, \nabla^3 p_t \in C([0,T]; L^2(\mathbb{R}^2))$$

(see for instance [**14**, p. 269, Corollary 4.2.6]).

We claim that ∇p_t also belongs to $C([0,T]; L^2(\mathbb{R}^2))$. To see this, write ∇p_t as a singular integral as follows (c.f. [**4**, p. 994]). Rewrite (3.3) as $-\Delta p = (v^1 v^1_{x_1} + v^2 v^1_{x_2})_{x_1} + (v^1 v^2_{x_1} + v^2 v^2_{x_2})_{x_2} - \theta_{x_2}$. Hence

$$-\Delta p_t = (v^1 v^1_{x_1} + v^2 v^1_{x_2})_{t x_1} + (v^1 v^2_{x_1} + v^2 v^2_{x_2})_{t x_2} - \theta_{t x_2}.$$

Solve the Poisson equation and integrate by parts to get

$$\begin{aligned} & p_t(x,t) \\ = & \frac{-1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|[(v^1v^1_{x_1}+v^2v^1_{x_2})_{tx_1}+(v^1v^2_{x_1}+v^2v^2_{x_2})_{tx_2}-\theta_{tx_2}](y,t)dy \\ = & -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_1-y_1}{|x-y|^2}(v^1v^1_{x_1}+v^2v^1_{x_2})_t(y,t)dy \\ & -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_2-y_2}{|x-y|^2}[(v^1v^2_{x_1}+v^2v^2_{x_2})_t-\theta_t](y,t)dy. \end{aligned} \tag{3.8}$$

Differentiating with respect to x_1 gives

$$\begin{aligned} \partial_{x_1}p_t(x,t) = & \frac{-1}{2\pi}\int_{\mathbb{R}^2}\left[\frac{1}{|x-y|^2}-\frac{2(x_1-y_1)^2}{|x-y|^4}\right](v^1v^1_{x_1}+v^2v^1_{x_2})_t(y,t)dy \\ & +\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^4}[(v^1v^2_{x_1}+v^2v^2_{x_2})_t-\theta_t](y,t)dy \end{aligned} \tag{3.9}$$

and we have a similar expression for $\partial_{x_2}p_t$. Again it is straight forward to see that $(v^1v^1_{x_1}+v^2v^1_{x_2})_t$, $(v^1v^2_{x_1}+v^2v^2_{x_2})_t$ and θ_t are in $C([0,T];L^2(\mathbb{R}^2))$. Hence the theory of singular integral operators ([**14**, p. 269, Corollary 4.2.6]) again gives $\nabla p_t \in C([0,T];L^2(\mathbb{R}^2))$. In summary, $\nabla p_t \in C([0,T];H^2(\mathbb{R}^2)) \hookrightarrow C(\mathbb{R}^2\times[0,T];\mathbb{R}^2)$. As v is continuous, $(v/|v|)\cdot\nabla p_t$ is continuous if $|v|\neq 0$.

Step 3. We now show that the first and third term on the right hand side of (3.2) are space-time continuous when $|v|\neq 0$. The continuity of the first term $(v/|v|)\cdot\nabla p$ follows from the that of ∇p (see the proof of Lemma 3.3). To see that the third term

$$(v\cdot\nabla)\left(\frac{v}{|v|}\cdot\nabla p\right)=\sum_{i,j}\left[\frac{v^jv^i}{|v|}p_{x_ix_j}+v^jp_{x_i}\partial_{x_j}\left(\frac{v^i}{|v|}\right)\right] \tag{3.10}$$

is continuous when $v\neq 0$, we look at $p_{x_ix_j}$. Write $(1.1)_2$ as

$$p_{x_i}=-v^i_t-\sum_k v^kv^i_{x_k}+\nu v_{x_ix_i}+\theta\delta_{i2},\quad i=1,2$$

and differentiate with respect to x_j to get

$$p_{x_ix_j}=-v^i_{x_jt}-\sum_k v^kv^i_{x_kx_j}-\sum_k v^k_{x_j}v^i_{x_k}+\nu v_{x_ix_ix_j}+\theta_{x_j}\delta_{i2},\quad i,j=1,2. \tag{3.11}$$

We claim that $\sum_k v^kv^i_{x_kx_j}\in C([0,T];H^{m-3}(\mathbb{R}^2))$. To see this, notice that $v^i_{x_kx_j}\in C([0,T];H^{m-2})$, while the Sobolev embedding theorem implies that

$$v^k\in C([0,T];C^{m-2})$$

when $m\in\mathbb{Z}$ and $v^k\in C([0,T;C^{[m-1]})$ when $m\notin\mathbb{Z}$. In any case our claim follows from the Leibniz rule. Next we assert that $\sum_k v^k_{x_j}v^i_{x_k}\in C([0,T];H^{m-3}(\mathbb{R}^2))$. To see this, notice that $v^i_{x_k}\in C([0,T];H^{m-1})$. By Sobolev, $v^k_{x_j}$ is in $C([0,T];C^{m-3})$ when $m\in\mathbb{Z}$ and $C([0,T;C^{[m-2]})$ when $m\notin\mathbb{Z}$. In any case $v^k_{x_j}v^i_{x_k}\in C([0,T];H^{m-3})$ by the Leibniz formula and our assertion holds. The other terms $v^i_{x_jt}$, $\nu v_{x_ix_ix_j}$ and $\theta_{x_i}\delta_{i2}$ in (3.11) are obviously in $C([0,T];H^{m-3})$. Then from (3.11), $p_{x_ix_j}$ is in $C([0,T];H^{m-3})$ and hence space-time continuous. The continuity of the third term on the right hand side of (3.2) when $v\neq 0$ follows from (3.10).

In summary, Step 2 and 3 show that $\frac{D}{Dt}(D_\xi p)$ is space-time continuous when $v \neq 0$.

Step 4. Using the Sobolev embedding theorem and the Leibniz formula as in the reasonings above, it is straight forward to see that $\frac{D}{Dt}(\xi^2\theta) = (\partial_t + v\cdot\nabla)(v^2\theta/|v|)$ is space-time continuous when $v \neq 0$.

In conclusion, DF/Dt is continuous when $v \neq 0$. Combining this with the continuity of $D|v|/Dt$ and F from Lemma 3.3 and restricting to $(X(a,t),t)$, we get the continuity of $F(X(a,t),t)'$, $\alpha_1(X(a,t),t)'$ and $\Psi_1(a,t)$ in t whenever $v(X(a,t),t) \neq 0$. The lemma is proved. □

LEMMA 3.5. *(continuity of f', α, α', Φ and Ψ in Case II) Let $\nu = 0$. Let $m > 4$ and (v,θ) be as in Proposition 3.2. Then ω, θ_{x_1} and $D\theta_{x_1}/Dt$ are space-time continuous, and the same is true for $D|\omega|/Dt$ if $\omega \neq 0$. This implies the continuity in $t \in [0,T]$ along $(X(a,t),t)$ of $|\omega|'$, $\alpha_2 = sgn(\omega)\theta_{x_1}/|\omega|$, $\alpha_2' = [\omega(\theta_{x_1})' - \theta_{x_1}\omega']/\omega^2$ and $\Phi_2 = \alpha_2/|\omega|$ if $\omega \neq 0$, and that of $\Psi_2 = \theta_{x_1}'\omega/\theta_{x_1}^2$ if $\theta_{x_1} \neq 0$.*

PROOF. As $m > 4$, v, $\nabla\theta_{x_1}$ and $\theta_{x_1 t}$ are in $C([0,T];H^{m-3}(\mathbb{R}^2))$ and hence space-time continuous by the Sobolev embedding theorem. Therefore $D\theta_{x_1}/Dt$ is continuous on $\mathbb{R}^2 \times [0,T]$. Next $\omega, \omega_{x_i}, \omega_t \in C([0,T];H^{m-3}(\mathbb{R}^2)) \hookrightarrow C(\mathbb{R}^2 \times [0,T];\mathbb{R})$. Consequently, when $\omega \neq 0$, $|\omega|_t = \mathrm{sgn}(\omega)\omega_t$, $|\omega|_{x_i} = \mathrm{sgn}(\omega)\omega_{x_i}$ and hence $D|\omega|/Dt$ are space-time continuous. The rest of the lemma follows. □

LEMMA 3.6. *(continuity of f', α, α', Φ and Ψ in Case III) Let $\kappa = 0$. Let $m > 4$ and (v,θ) be as in Proposition 3.2. Then $D|\nabla\theta|/Dt$ and $\alpha_3 = -\eta\cdot S\eta$ are space-time continuous when $\nabla\theta \neq 0$. This implies the continuity in $t \in [0,T]$ along $(X(a,t),t)$ of $|\nabla\theta|'$, $\alpha_3 = -\eta\cdot S\eta$, $\alpha_3' = (-\eta\cdot S\eta)'$ and $\Phi_3 = \alpha_3/|\nabla\theta|$ if $\nabla\theta \neq 0$, and that of $\Psi_3 = (-\eta\cdot S\nabla\theta)'|\nabla\theta|/(\eta\cdot S\nabla\theta)^2$ if $\eta\cdot S\nabla\theta \neq 0$ (or equivalently $\alpha_3 \neq 0$).*

PROOF. The hypothesis $m > 4$ ensures that v, ∇v, $\nabla^2 v$, v_t, S, S_t, ∇S, $\nabla\theta$, $\nabla^2\theta$ and $\nabla\theta_t$ are in $C([0,T];H^{m-3}(\mathbb{R}^2))$ and hence continuous by the Sobolev embedding theorem. It follows that $\partial_t|\nabla\theta| = \eta\cdot\nabla\theta_t$, $\partial_{x_i}|\nabla\theta| = \eta\cdot\nabla\theta_{x_i}$ and hence $D|\nabla\theta|/Dt$ are continuous when $\nabla\theta \neq 0$. The same information also gives the continuity of α_3 when $\nabla\theta \neq 0$. The rest of the lemma follows from the formulas for α_3', Φ_3 and Ψ_3. □

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