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On the hyperbolicity properties of inertial manifolds of reaction–diffusion equations

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Abstract. For 3D reaction–diffusion equations, we study the problem of existence or nonexistence of an inertial manifold that is normally hyperbolic or absolutely normally hyperbolic. We present a system of two coupled equations with a cubic nonlinearity which does not admit a normally hyperbolic inertial manifold. An example separating the classes of such equations admitting an inertial manifold and a normally hyperbolic inertial manifold is constructed. Similar questions concerning absolutely normally hyperbolic inertial manifolds are discussed.

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Introduction

The existence of a smooth inertial manifold $\mathcal{M}$ for the dissipative parabolic equation in the infinite-dimensional Hilbert space implies [**15**, **19**, **20**] that its final dynamics (as $t \to +\infty$) is controlled by finitely many parameters. The additional property of normal hyperbolicity of the inertial manifold $\mathcal{M}$ guarantees the structural stability of this manifold. The stronger property of absolute normal hyperbolicity means one and the same hyperbolicity parameters for the entire $\mathcal{M}$. So far, the

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To the memory of my teacher Gennadi Henkin.

existence of an inertial C^1-manifold has been established for a rather narrow class of semilinear parabolic equations, while known examples of its nonexistence [**2**, **16**, **17**] seem to be somewhat artificial and are not related to problems of mathematical physics.

The present paper deals with necessary conditions for the existence of the above-mentioned two types of inertial manifolds of scalar and vector reaction–diffusion equations. For the 3D chemical kinetics equations with a cubic nonlinearity, we strive for constructing examples separating the classes of problems admitting an *inertial manifold*, a *normally hyperbolic inertial manifold*, and an *absolutely normally hyperbolic inertial manifold*. An example separating the first two possibilities is obtained for two-component systems. Namely, in Proposition 3.5 we construct an (uncoupled) system of such equations that has an inertial manifold but does not admit a normally hyperbolic inertial manifold. In particular, this system provides an example of an inertial manifold that is not normally hyperbolic. On the other hand, we present a system of two *coupled* reaction–diffusion equations of this type that do not admit a normally hyperbolic inertial manifold in the natural state space (Proposition 3.4). An example of a scalar 3D equation with a cubic nonlinearity without an absolutely normally hyperbolic inertial manifold is constructed. Note that the order of the polynomial nonlinearity in the chemical kinetics equations corresponds to the *reaction order*, which usually does not exceed 3. We also discuss how close the well-known sufficient conditions (the *spectral jump condition* and the *spatial averaging principle*) for the existence of strongly and weakly normally hyperbolic inertial manifolds are to being necessary.

The paper is organized as follows. Section 1 contains elementary information about abstract semilinear parabolic equations. The necessary and sufficient conditions, known so far, for the existence of a smooth inertial manifold are stated in Section 2. The main results on the existence and nonexistence of various inertial manifolds for the reaction–diffusion equations are presented in Sections 3–4. Section 5 discusses problem on the relationship between spectral properties of the linear part of the equation and the existence or nonexistence of various types of inertial manifolds.

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1. Preliminaries

A semilinear parabolic equation in a real separable infinite-dimensional Hilbert space $(X, \|\cdot\|)$ has the form

$$\partial_t u = -Au + F(u). \tag{1.1}$$

Here we assume that

(i) $A\colon \mathcal{D}(A) \to X$ is a linear positive definite self-adjoint operator with compact inverse A^{-1}.

(ii) $F \in C^1(X^\theta, X)$ is a nonlinear function with domain $X^\theta = \mathcal{D}(A^\theta)$, $0 \le \theta < 1$, $\|u\|_\theta = \|A^\theta u\|$, such that

$$\|F(u_1) - F(u_2)\| \le L(r)\|u_1 - u_2\|_\theta \tag{1.2}$$

on the balls $\mathcal{B}_r = \{u \in X^\theta : \|u\|_\theta < r\}$.

(iii) There exists a dissipative phase semiflow $\{\Phi_t\}_{t\ge 0}$ on X^θ.

We refer to the number θ as the *nonlinearity exponent* of Eq. (1.1) and set $X^0 = X$. The space X will be called the *main* space. Dissipativity is understood as the existence of an absorbing ball $\mathcal{B}_r \subset X^\theta$ (see [**15**,**20**]). Under these conditions [**4**], the phase semiflow proves to be smooth, and the evolution operators $\Phi_t\colon X^\theta \to X^\theta$, $t > 0$, are compact. The *parabolic smoothing* property guarantees the inclusion $\Phi_t X^\theta \subset X^1 = \mathcal{D}(A)$ for $t > 0$.

The global attractor $\mathcal{A}$ is defined as the union of all complete bounded trajectories of the equation; in our case, it is a compact subset of X^θ. An *inertial manifold* of Eq. (1.1) is a *smooth* (C^1) finite-dimensional positively invariant surface $\mathcal{M} \subset X^\theta$ containing the attractor $\mathcal{A}$ and attracting all trajectories $u(t)$ with exponential tracking as $t \to +\infty$. An inertial manifold usually has a Cartesian structure and is diffeomorphic to a ball in $\mathbb{R}^n$. The restriction of Eq. (1.1) to $\mathcal{M}$ gives an inertial form (an ordinary differential equation in $\mathbb{R}^n$, $n = \dim \mathcal{M}$), which completely reproduces the final dynamics of the original equation. There is a vast literature dealing with the theory of inertial manifolds (see [**15**, **19**–**21**] and references therein); moreover, one often considers *Lipschitz* (nonsmooth) inertial manifolds.

2. Inertial manifold: existence conditions

The dissipativity of the evolution system (1.1) permits one to change the function $F(u)$ outside $\mathcal{B}_r$ with the preservation of C^1-regularity in such a way that the new function $\widetilde{F}(u)$ is identically zero outside the ball $\mathcal{B}_{r+1}$. This "truncation" procedure (e.g., see [**20**]) permits one to proceed to the equation

$$u_t = -Au + \widetilde{F}(u), \tag{2.1}$$

which inherits the final dynamics of the original problem. One has $L(r) \equiv L$ in the estimate (1.2) for $\widetilde{F}(u)$. It is well known [**1**, **18**, **19**] that the existence of a smooth n-dimensional inertial manifold $\mathcal{M} \subset X^\theta$ of Eq. (2.1) in the phase space X^θ is guaranteed by the spectral jump condition $\mu_{n+1} - \mu_n > cL(\mu_{n+1}^\theta + \mu_n^\theta)$, where $0 < \mu_1 \le \mu_2 \cdots$ are the eigenvalues of the operator A arranged in nondescending order (counting multiplicities) and $c > 0$ is an absolute constant. The manifold $\mathcal{M}$ also proves to be an inertial manifold of the original parabolic equation. Thus, the *spectrum sparseness* condition

$$\sup_{n\ge 1} \frac{\mu_{n+1} - \mu_n}{\mu_{n+1}^\theta + \mu_n^\theta} = \infty \tag{2.2}$$

is sufficient for the existence of an inertial C^1-manifold $\mathcal{M} \subset X^\theta$ of the dissipative equation (1.1) with given linear part $-A$ for an arbitrary nonlinear function $F\colon X^\theta \to X$ with properties (ii).

Now consider the scalar reaction–diffusion equation

$$\partial_t u = \nu \Delta u + f(x, u), \qquad \nu > 0, \tag{2.3}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$ with one of the standard boundary conditions (D), (N), or (P) and with a sufficiently smooth function $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfying the *sign condition* $v \cdot f(x, v) < 0$ for $x \in \Omega$ and $|v| \ge r > 0$. In this case, there exists a dissipative phase semiflow on $X = L^2(\Omega)$ [**20**, Chap. 3]. Let us extend $f(\cdot, v)$ from $[-r, r]$ with the preservation of smoothness to a Lipschitz function $\widetilde{f}(\cdot, v)$ vanishing for $|v| \ge r + 1$. By the maximum principle, the partial differential equation (2.3) with $f(x, u)$ replaced by $\widetilde{f}(x, u)$ inherits the limit modes of the original problem and admits the interpretation (1.1) with nonlinearity exponent $\theta = 0$ and with $X^1 \subset H^2(\Omega)$. To this end, one should set $Au = u - \nu\Delta u$ and $F(u) = u + \widetilde{f}(x, u)$.

It is well known [**8**, Sec. 20] that the nonlinear Nemytskii operator $F : X \to X$ is not differentiable. However, $|\tilde{f}| + |\tilde{f}_v| + |\tilde{f}_{vv}| \le \text{const}$ in our case and we have $F \in C^1(L^2(\Omega), L^q(\Omega))$ for $1 \le q < 2$. Moreover, $\|F'(u)\|_{\text{op}} \le \text{const}$ for $u \in X$ and $F'(u) : X \to L^q(\Omega)$. The embedding theorems in [**5**, Chap. 1] with $1 \le m \le 3$ give $X^\alpha \subset L^3(\Omega)$ for $\frac{1}{4} < \alpha < 1$. For the dual spaces, we find that $L^{3/2}(\Omega) \subset X^{-\alpha}$, and hence $F \in C^1(X, X^{-\alpha})$. If one takes $X^{-\alpha}$, $\|u\|_{-\alpha} = \|A^{-\alpha}u\|$, for the main space, then $\|F'(u)\|_{\text{op}} \le \text{const}$ for $F'(u) : X \to X^{-\alpha}$. It follows that $F \in \text{Lip}(X, X^{-\alpha})$ and the phase semiflow in X proves to be smooth.

If the spectrum is $\sigma(-\Delta) = \{0 \le \lambda_1 \le \lambda_2 \le \cdots\}$, then condition (2.2) is reduced to the relation

$$\sup_{n \ge 1}(\lambda_{n+1} - \lambda_n) = \infty, \tag{2.4}$$

which seems to be rather restrictive in view of the Weyl asymptotics $\lambda_n \sim c\, n^{2/m}$. We point out that Eq. (2.4) holds for $m = 1$ as well as for some domains $\Omega \subset \mathbb{R}^2$. These domains include rectangles with rational squared side ratio [**12**], but in general the description of planar domains for which $\sigma(-\Delta)$ is sparse remains a mystery. Already for $m = 3$, one has $\lambda_n \sim cn^{2/3}$, and condition (2.4) seems to be exotic.

In this connection, the following property of the Laplace operator in a domain $\Omega \subset \mathbb{R}^m$, $m \le 3$, was stated in [**11**, **12**], which was referred there to as the *principle of spatial averaging*. Set

$$(B_h u)(x) = h(x)u(x), \qquad \overline{h} = (\text{vol}\,\Omega)^{-1} \int_\Omega h(x)\,dx$$

for $h \in H^2(\Omega) \subset L^\infty(\Omega)$ and $u \in L^2(\Omega)$. Let P_λ be the spectral projection of the self-adjoint operator $-\Delta$ corresponding to the part of the spectrum in $[0, \lambda]$, and let $I = \text{id}$.

Definition 2.1. The Laplace operator Δ with a given standard boundary condition in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$, $m \le 3$, satisfies the *principle of spatial averaging* if there exists a $\rho > 0$ such that for any $\varepsilon > 0$ and $k > 0$ there exists an arbitrarily large $\lambda > k$ such that $\lambda \in [\lambda_n, \lambda_{n+1})$, $\lambda_{n+1} - \lambda_n \ge \rho$, and

$$\|(P_{\lambda+k} - P_{\lambda-k})(B_h - \overline{h}I)(P_{\lambda+k} - P_{\lambda-k})\|_{\text{op}} \le \varepsilon \|h\|_{H^2} \qquad \forall\, h \in H^2(\Omega), \tag{2.5}$$

where $\|\cdot\|_{\text{op}}$ is the norm on $\text{End}\, L^2(\Omega)$.

Essentially, one speaks of an arbitrarily good approximation, for any $h \in H^2(\Omega)$, to the Schrödinger operator $\Delta + h(x)I$ by a shifted Laplace operator $\Delta + \overline{h}I$ in an arbitrarily wide range of eigenmodes of the Laplace operator. Here one assumes that

$$\limsup_{n \to \infty}(\lambda_{n+1} - \lambda_n) > 0,$$

which is always the case for $m \le 2$. This principle follows from the sparseness of the spectrum (but not vice versa!) and ensures [**12**, p. 846] the existence of a smooth inertial manifold of Eq. (2.3) with $f \in C^3$. In particular, the principle of spatial averaging holds for an *arbitrary* rectangle $\Omega_2 \subset \mathbb{R}^2$ and for a cube $\Omega_3 \subset \mathbb{R}^3$ [**12**], although condition (2.4) is not guaranteed for the former and is violated for the latter. In [**9**], the existence of a (Lipschitz) inertial manifold of Eq. (2.3) is derived from less restrictive conditions: the number $\lambda > k$ may depend on bounded sets $\mathcal{B} \subset H^2(\Omega)$, and (2.5) is replaced by the estimate

$$\|(P_{\lambda+k} - P_{\lambda-k})(B_h - \overline{h}I)(P_{\lambda+k} - P_{\lambda-k})\|_{\mathrm{op}} \le \varepsilon \qquad \forall\, h \in \mathcal{B}.$$

In the framework of this approach, the existence of an inertial manifold was proved for Eq. (2.2) in some 2D and 3D polyhedra [**9**,**10**]. The principle of spatial averaging has only been proved to hold in some model cases, and unfortunately, this principle practically does not apply to systems of reaction–diffusion equations, because in this case the operator corresponding to the componentwise multiplier is the operator of multiplication by a matrix of numbers that is diagonal *but not scalar*.

Recently, Zelik [**21**] suggested an abstract form of the principle of spatial averaging, which generalizes the constructions in [**9**–**12**] and ensures the existence of a smooth inertial manifold of Eq. (1.1). This approach was further developed in [**6**, **7**]. The corresponding technique permitted establishing the existence of an inertial manifold $\mathcal{M} \in C^{1+\varepsilon}$ for the Cahn–Hilliard equation [**6**] and of an inertial manifold $\mathcal{M} \in \mathrm{Lip}$ for the modified Leray α-model of the Navier–Stokes equations on the three-dimensional torus [**7**].

So far, little is known about the cases of nonexistence of an inertial manifold for parabolic problems. A system of two coupled one-dimensional parabolic pseudodifferential equations that does not admit a smooth inertial manifold was constructed in [**16**]. A general construction of abstract equations (1.1) with nonlinearity exponent $\theta = 0$ and without a smooth inertial manifold is described in [**2**]. A more natural story is considered in [**17**], where an integro-differential parabolic equation with nonlocal diffusion on the circle is presented which does not have an inertial manifold in the chosen state space.

All these examples are based on the following argument. Since the phase semiflow is dissipative and compact, it follows that the stationary point set $E = \{u \in X^1 \colon F(u) - Au = 0\}$ of Eq. (1.1) is nonempty. Since $E \subset \mathcal{A}$, we see that E is contained in the inertial manifold, provided that the latter exists. Since the operator A^{-1} is compact and, by [**4**, Chap. 1], the linear operator $-S_u = A - F'(u)$ on X is sectorial, it follows that the spectrum $\sigma(S_u)$, $u \in E$, consists of eigenvalues λ of finite multiplicity, and the number $l(u)$ (counting multiplicities) of positive λ in $\sigma(S_u)$ is finite. Let $E_- = \{u \in E \colon \sigma(S_u) \cap (-\infty, 0] = \phi\}$.

Now we can state a necessary condition for the existence of an inertial manifold as follows.

LEMMA 2.2 ([**16**]). *If Eq.* (1.1) *admits a smooth inertial manifold* $\mathcal{M} \subset X^\theta$, *then the number* $l(u_0) - l(u_1)$ *is even for any* $u_0, u_1 \in E_-$.

PROOF. Let $Y = \mathcal{T}_u\mathcal{M}$ be the tangent space to $\mathcal{M}$ at a point $u \in E_-$; then $S_uY \subset Y$. Since $\mathcal{A} \subset \mathcal{M}$ and the attractor $\mathcal{A}$ contains the unstable manifold of the point u, the subspace Y contains the invariant subspace of S corresponding to the part $\lambda > 0$ of the spectrum. Thus, $l(u) \le n$, and $\sigma(S|Y)$ contains exactly $l(u)$ real values. Since the space X is real, it follows that the number $n - l(u)$ is even. □

To apply the lemma, one usually constructs a nonlinearity F such that Eq. (1.1) has stationary solutions $u_0, u_1 \in E_-$ with $l(u_0) = 0$ and $l(u_1) = 1$.

3. Normally hyperbolic inertial manifolds

Unfortunately, so far there are no examples physically more meaningful than those given above of parabolic equations without inertial manifolds. At the same time, such examples were obtained in [**13**, **16**] for the case in which one speaks of inertial manifolds with additional hyperbolicity properties.

DEFINITION 3.1. A smooth inertial manifold $\mathcal{M} \subset X^\theta$ of Eq. (1.1) is said to be *normally hyperbolic* if, for some vector bundle $\mathcal{T}_\mathcal{M} X^\theta = \mathcal{T}\mathcal{M} \oplus \mathcal{N}$ invariant with respect to the linearization $\{\Phi_t'\}$ of the semiflow $\{\Phi_t\}_{t\ge 0}$, where $\mathcal{T}\mathcal{M}$ is the tangent bundle of $\mathcal{M}$, one has the estimates

$$\begin{aligned} &\|\Phi_t'(u)h\|_\theta \ge M^{-1}e^{-\gamma_1 t}\|h\|_\theta \quad (h \in \mathcal{T}_u\mathcal{M}),\\ &\|\Phi_t'(u)h\|_\theta \le Me^{-\gamma_2 t}\|h\|_\theta \qquad (h \in \mathcal{N}_u) \end{aligned} \tag{3.1}$$

with constants $M > 0$ and $0 < \gamma_1 < \gamma_2$ depending on $\mathcal{M}$ and $u \in \mathcal{M}$. If these constants are independent of $u \in \mathcal{M}$, then the manifold is said to be *absolutely normally hyperbolic*.

We point out that the normally hyperbolic invariant manifolds of finite- and infinite-dimensional dynamical systems are structurally stable [**5**, **14**].

The existence of a normally hyperbolic inertial manifold for the reaction–diffusion equations (2.3) under the assumption that the spatial averaging principle holds was announced as early as in [**11**] and [**12**, p. 830]. Apparently, this can be proved by the methods in [**6**].

The known necessary conditions for the existence of an inertial manifold $\mathcal{M} \subset X^\theta$ with hyperbolicity properties amount to analyzing the spectrum of the linearization of the vector field $F(u) - Au$ of Eq. (1.1) on the stationary point set $E \subset X^1$. For $\gamma \in \mathbb{R}$ and $u \in E$, let $Y(u,\gamma)$ be the finite-dimensional invariant subspace of the operator $S_u = F'(u) - A$ corresponding to the part of the spectrum $\sigma(S_u)$ with $\operatorname{Re}\lambda \ge \gamma$.

LEMMA 3.2 ([**13**,**16**]). *If the inertial manifold $\mathcal{M} \subset X^\theta$ of Eq. (1.1) is normally hyperbolic, then*

$$\forall\, u \in E \quad \exists\, \gamma = \gamma(u;\mathcal{M}) < 0\colon\ \dim Y(u,\gamma) = \dim \mathcal{M}.$$

In the case of absolutely normal hyperbolicity of $\mathcal{M} \subset X^\theta$, one has $\gamma = \gamma(\mathcal{M})$.

Here $\gamma = -(\gamma_1 + \gamma_2)/2$, where $0 < \gamma_1 < \gamma_2$ are the numbers in Definition 3.1. For $u \in E$, the invariant subspaces $\mathcal{T}_u\mathcal{M}$ and $\mathcal{N}_u$ of the operator S_u correspond to the parts of the spectrum $\sigma(S_u)$ with $\operatorname{Re}\lambda \ge -\gamma_1$ and $\operatorname{Re}\lambda \le -\gamma_2$, respectively; moreover, $\Phi_t'(u) = \exp(-tS_u)$, $t > 0$.

The lemma was used to obtain the well-known example [**13**, Th. 2.5] of Eq. (2.3) in the cube $\Omega = (0,\pi)^4$ with the Neumann condition on $\partial\Omega$ and with a real-analytic function $f(x,u)$ (polynomial in u) for which there does not exist a normally hyperbolic inertial manifold $\mathcal{M} \subset L^2(\Omega)$. However, the function f was not constructed in closed form in this example. Furthermore, it would be of interest to obtain similar examples for 3D reaction–diffusion equations with a homogeneous polynomial nonlinearity $f(u)$. Moreover, from the viewpoint of applications to chemical kinetics, the degrees of the polynomials should not exceed 3.

Consider the two-component system

$$\partial_t u_1 = \Delta u_1 + f_1(u_1,u_2), \qquad \partial_t u_2 = \Delta u_2 + f_2(u_1,u_2) \tag{3.2}$$

in the cube $\Omega = (0,\pi)^3$ with the Neumann condition (N) on $\partial\Omega$ and with a C^3-function $f = (f_1, f_2)$, $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$. Then, just as above, system (3.2) can be reduced to the abstract dissipative problem (1.1) with $X = L^2(\Omega;\mathbb{R}^2)$ and with the nonlinearity exponent $\theta = 0$ under the assumption that there exists an *invariant region* [**20**, Chap. 3] for the ordinary differential equation $v_t = f(v)$, $v \in \mathbb{R}^2$.

For a fixed point $p \in \mathbb{R}^2$ of the vector field f, we set $\delta(p) = |\operatorname{Re}(\xi_1 - \xi_2)|$, where ξ_1 and ξ_2 are the eigenvalues of the Jacobian matrix $f'(p)$. Note that $\delta(p) = 0$ in the case of multiple or complex eigenvalues of the matrix $f'(p)$.

LEMMA 3.3 ([**16**]). *The dissipative system* (3.2) *does not have a normally hyperbolic inertial manifold in the state space X if the vector field f has four fixed points $p_i \in \mathbb{R}^2$ such that $\delta(p_i) = i$ for $i = 0, 1, 2, 3$.*

The proof uses the necessary condition given by Lemma 3.2. The existence of a smooth vector field f with the desired properties on $\mathbb{R}^2$ is obvious. Our aim is to construct a third-order polynomial field of this kind. Set

$$f_1(v_1, v_2) = kv_1(1 - av_1^2 + v_2^2), \qquad f_2(v_1, v_2) = kv_2(1 - bv_2^2 - v_1^2) \tag{3.3}$$

with some constants $k, a, b > 0$.

We have $v \cdot f(v) \le 0$ for $|v|^2 \ge r_0^2 = 2/\min(a, b)$, and the dissipativity of system (3.2) with the vector field (3.3) is ensured by the positive invariance of the disks $|v| \le r$ with $r \ge r_0$ for the ordinary differential equation $v_t = f(v)$, $v \in \mathbb{R}^2$. Furthermore, the condition $b^{-1} \le c^2 \le a - 1$ implies the positive invariance of the region $D_c = \{v \in \mathbb{R}^2 : 0 \le v_1 \le 1,\ 0 \le v_2 \le c\}$ for the equation $v_t = f(v)$, which in its turn implies [**20**] the preservation of this region for the components u_1, u_2 in system (3.2).

PROPOSITION 3.4. *There exist positive k, a and b with $a \ge 1 + b^{-1} \ge b$ such that the dissipative coupled system* (3.2) *with the vector field* (3.3) *and with $\Omega = (0,\pi)^3$ does not have a normally hyperbolic inertial manifold $\mathcal{M} \subset X$.*

PROOF. Assuming that $a > 1$, let us single out four fixed points

$$p_0 = (0,0), \quad p_1 = \left(\frac{1}{\sqrt{a}}, 0\right), \quad p_2 = \left(\sqrt{\frac{b+1}{ab+1}}, \sqrt{\frac{a-1}{ab+1}}\right), \qquad p_3 = \left(0, \frac{1}{\sqrt{b}}\right)$$

of the vector field f on $\mathbb{R}^2$. Here

$$f'(v) = k\begin{pmatrix} 1 - 3av_1^2 + v_2^2 & 2v_1v_2 \\ -2v_1v_2 & 1 - v_1^2 - 3bv_2^2 \end{pmatrix}$$

for $v \in \mathbb{R}^2$, and

$$f'(p_0) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \qquad f'(p_1) = \begin{pmatrix} -2k & 0 \\ 0 & k - k/a \end{pmatrix},$$

$$f'(p_2) = k\begin{pmatrix} \frac{-2ab-2a}{ab+1} & \frac{2((a-1)(b+1))^{1/2}}{ab+1} \\ \frac{-2((a-1)(b+1))^{1/2}}{ab+1} & \frac{-2ab+2b}{ab+1} \end{pmatrix}, \qquad f'(p_3) = \begin{pmatrix} k + k/b & 0 \\ 0 & -2k \end{pmatrix}.$$

Set $\delta_i = \delta(p_i)$, $0 \le i \le 3$. We have $\delta_0 = 0$, $\delta_1 = k(3 - a^{-1})$, $\delta_3 = k(3 + b^{-1})$, and

$$\delta_2^2 = \frac{4k^2(a+b)^2}{(ab+1)^2} - 16k^2\frac{(a-1)(b+1)}{(ab+1)^2}, \qquad \delta_2 = \frac{2k}{ab+1} \cdot |a - b - 2|.$$

Set $k = a/(3a-1)$ and $b = a/(6a-3)$; then $\delta_1 = 1$ and $\delta_3 = 3$. The function $\varphi\colon a \to \delta_2$ is continuous on $(1,\infty)$, and since $k(\infty) = 1/3$ and $b(\infty) = 1/6$, we have

$\varphi(7) < 2$ and $\varphi(\infty) = 4$. Thus, there exists an $a = a^* > 7$ such that $\varphi(a) = 2$. One can readily verify that (1) $a \geq 1 + b^{-1} \geq b$ and $r_0^2 = 2/b < 12$; (2) $p_i \in D_c$ and $|p_i| \leq \sqrt{7}$ for $c = \sqrt{6}$ and $0 \leq i \leq 3$. Since $\delta(p_i) = i$, $0 \leq i \leq 3$, we see that the proposition follows from Lemma 3.3. □

Now consider the vector field

$$f_1(v_1, v_2) = v_1(a - v_1)(v_1 - b), \qquad f_2(v_1, v_2) = v_2(c - v_2)(v_2 - d) \tag{3.4}$$

with $a = 2$, $b = \sqrt{3}$, $c = \sqrt{6}$, and $d = \sqrt{2}$. The dissipativity and the preservation of the positivity of solutions of the corresponding problem (3.2) is guaranteed by the sign condition with respect to each component and by the positive invariance of the quadrant $v_1 \geq 0, v_2 \geq 0$ with respect to the ordinary differential equation $v_t = f(v)$ in $\mathbb{R}^2$.

PROPOSITION 3.5. *The dissipative uncoupled system* (3.2) *with the vector field* (3.4) *and with* $\Omega = (0, \pi)^3$ *admits an inertial manifold* $\mathcal{M} \subset X$ *but does not have a normally hyperbolic inertial manifold in* X.

PROOF. Each of the scalar equations in (3.2) admits an inertial manifold $\mathcal{M}_| \subset L^2(\Omega)$, $j = 1, 2$ [**12**], and hence $\mathcal{M} = \mathcal{M}_\infty \times \mathcal{M}_\in$ is an inertial manifold of the two-component system in X. At the stationary points

$$p_0 = (0, 0), \quad p_1 = (b, d), \quad p_2 = (a, c), \quad p_3 = (b, c),$$

the Jacobian matrix of the vector field f has the form

$$f'(p_0) = \begin{pmatrix} -ab & 0 \\ 0 & -cd \end{pmatrix} = \begin{pmatrix} -2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{pmatrix},$$

$$f'(p_1) = \begin{pmatrix} b(a-b) & 0 \\ 0 & d(c-d) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} - 3 & 0 \\ 0 & 2\sqrt{3} - 2 \end{pmatrix},$$

$$f'(p_2) = \begin{pmatrix} a(b-a) & 0 \\ 0 & c(d-c) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} - 4 & 0 \\ 0 & 2\sqrt{3} - 6 \end{pmatrix},$$

$$f'(p_3) = \begin{pmatrix} b(a-b) & 0 \\ 0 & c(d-c) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} - 3 & 0 \\ 0 & 2\sqrt{3} - 6 \end{pmatrix}.$$

We see that $\delta(p_i) = i$, $0 \leq i \leq 3$, and hence this system does not admit a normally hyperbolic inertial manifold in the state space X by Lemma 3.3. □

REMARK 3.6. Thus, we have separated the classes of problems admitting inertial manifolds and normally hyperbolic inertial manifolds for 3D two-component systems of chemical kinetics equations with a cubic nonlinearity. In particular, we have obtained an inertial manifold that is not normally hyperbolic.

4. Absolutely normally hyperbolic inertial manifolds

Under assumptions (i)–(iii), the same spectrum sparseness condition (2.2) is sufficient for the existence of an absolutely normally hyperbolic inertial manifold $\mathcal{M} \subset X^\theta$ for an arbitrary nonlinear part $F(u)$ of Eq. (1.1) (see [**18**, Th. 5.6] and [**19**, Th. 81.4]).[1]

Consider scalar homogeneous equations of the form

$$\partial_t u = \nu \Delta u + f(u), \qquad \nu > 0, \tag{4.1}$$

[1]Such manifolds are called *normally hyperbolic* in [**18, 19**].

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$, $m \le 3$, with the Neumann condition (N) or the periodicity condition (P) on $\partial\Omega$ and with a function $f \in C^3(\mathbb{R}, \mathbb{R})$ satisfying the sign condition. Let $\sigma(-\Delta) = \{0 = \lambda_1 \le \lambda_2 \le \cdots\}$. Being a special case of (2.3), the dissipative equation (4.1) can be represented in the form (1.1) with $X = L^2(\Omega)$ and with the nonlinearity exponent $\theta = 0$.

Lemma 4.1 ([**16**]). *Let $\lambda_{n+1} - \lambda_n \le K$, $n \ge 1$, and let $f'(p_0) - f'(p_1) = a > 0$ for some $p_0, p_1 \in \mathbb{R}$ such that $f(p_0) = f(p_1) = 0$. Then problems* $(4.1)_\mathrm{N}$ *and* $(4.1)_\mathrm{P}$ *do not have a normally hyperbolic inertial manifold $\mathcal{M} \subset X$ for $\nu < a/K$.*

Proof. Suppose that $f'(p_j) = a_j$ and $u_j \equiv p_j$, where $u_j \in E$ for $j = 0, 1$. We have $\sigma(A - F'(u)) = \sigma(-\nu\Delta - a_j I) = \{\mu_{n,j}\}$, where $\mu_{n,j} = \nu\lambda_n - a_j$ for $n \ge 1$. If $a = a_0 - a_1 > 0$ and $\nu < a/K$, then $\nu\lambda_{n+1} - a_0 < \nu\lambda_n - a_1$ or $\mu_{n+1,0} < \mu_{n,1}$ for all $n \ge 1$. It remains to apply Lemma 3.2. □

Corollary 4.2. *If $\lambda_{n+1} - \lambda_n \le K$, $n \ge 1$, and $f(u) = u - u^3$, then Eq.* (4.1) *with the boundary condition* (N) *or* (P) *does not have an absolutely normally hyperbolic inertial manifold $\mathcal{M} \subset X$ for $\nu < 3/K$.*

Here we have $f'(v) = 1 - 3v^2$, $p_0 = 0$, $p_1 = 1$, and $a = f'(p_0) - f'(p_1) = 3$.

In the case of $\Omega = (0, \pi)^3$, the spectrum of the operator $-\Delta$ with the Neumann condition or the periodicity condition on $\partial\Omega$ consists of eigenvalues of the form $\lambda_n = l_1^2 + l_2^2 + l_3^2$, $l_j \in \mathbb{Z}$; here one always has $\lambda_{n+1} - \lambda_n \le 3$ by the Gauss theorem [**3**], and hence one can take $K = 3$ in Corollary 4.2.

Corollary 4.3. *Equation* (4.1) *with $f(u) = u - u^3$ and with one of the boundary conditions* (N) *and* (P) *in the cube $\Omega = (0, \pi)^3$ does not have an absolutely normally hyperbolic inertial manifold $\mathcal{M} \subset X$ for $\nu < 1$.*

We see that an absolutely normally hyperbolic inertial manifold may fail to exist even for very simple semilinear parabolic equations.

5. Conclusion

The problem of separating the classes of scalar reaction–diffusion equations admitting normally hyperbolic and absolutely normally hyperbolic inertial manifolds is quite topical. In view of Corollary 4.3, it suffices to derive the existence of a normally hyperbolic inertial manifold $\mathcal{M} \subset L^2(\Omega)$ for Eq. (4.1) with a cubic nonlinearity in $\Omega = (0, \pi)^3$ from the validity of the spatial averaging principle for the Laplace operator Δ_Ω.

More generally, one can also ask how the spatial averaging principle is related to the existence or nonexistence of a normally hyperbolic inertial manifold for the scalar equations of chemical kinetics. It seems to be very likely that a relationship of this kind should exist between the sparseness of the spectrum of the Laplace operator and the existence of an absolutely normally hyperbolic inertial manifold.

Example 5.1. The following properties are equivalent for problems $(4.1)_\mathrm{N}$ or $(4.1)_\mathrm{P}$ with $f(u) = u - u^3$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$, $m \le 3$.

(a) The sparseness of the spectrum of the Laplace operator Δ_Ω.

(b) The existence of an absolutely normally hyperbolic inertial manifold in an appropriate state space for an arbitrary diffusion coefficient ν.

The implication (b) ⇒ (a) follows from Corollary 4.2. The converse can be obtained by a development of the technique in [**19**, Chaps. 5, 8].

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