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Periodic solutions for a class of one-dimensional Boussinesq systems

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Abstract. In this paper we show the local and global well-posedness for the periodic Cauchy problem associated with a special class of 1D Boussinesq systems that emerges in the study of the evolution of long water waves with small amplitude in the presence of surface tension. By a variational approach, we establish the existence of periodic travelling waves. We see that those periodic solutions are characterized as critical points of some functional, for which the existence of critical points follows as a consequence of the Arzela-Ascoli Theorem and the fact that the action functional associated is coercive and is (sequentially) weakly lower semi-continuous in an appropriate set.

Contents

1. Introduction

It has been established that the evolution of 2D long water waves with small amplitude is reduced to studying solutions (η, Φ) of the 1D-Boussinesq type system $(p = 1)$

$$(1.1)\qquad \begin{cases} \left(I - a\mu\partial_x^2\right)\eta_t + \partial_x^2\Phi - b\mu\partial_x^4\Phi + \epsilon\partial_x\left(\eta\left(\partial_x\Phi\right)^p\right) &= 0,\\ \left(I - c\mu\partial_x^2\right)\Phi_t + \eta - d\mu\partial_x^2\eta + \frac{\epsilon}{p+1}\left(\partial_x\Phi\right)^{p+1} &= 0,\end{cases}$$

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where ϵ is the amplitude parameter (nonlinearity coefficient), μ is the long-wave parameter (dispersion coefficient), constants $a \geq 0$, $c \geq 0$, $b > 0$, and $d > 0$ are such that

$$a + c - (b + d) = \frac{1}{3} - \sigma,$$

where σ^{-1} is known as the Bond number (associated with the surface tension), and p is a rational number of the form $p = \frac{p_1}{p_2}$ with $(p_1, p_2) = 1$ and p_2 an odd number. The variable $\Phi = \Phi(x,t)$ represents the rescale nondimensional velocity potential on the bottom $z = 0$, and the variable $\eta = \eta(x,t)$ corresponds the rescaled free surface elevation.

The model considered in this work is the 1D version of some Boussinesq system obtained by J. Quintero and A. Montes in [**19**] in the case $a = c = \frac{1}{2}, b = \frac{2}{3}, d = \sigma$ (see also A. Montes, [**10**]) and by J. Quintero in [**14**] in the case $a = c = 0, b = \frac{1}{6}, d = \sigma - \frac{1}{2}$, which appear when looking at the evolution of long water waves with small amplitude in the presence of surface tension. Among the results for the two-dimensional version of the Boussinesq system (1.1), we want to mention [**10**], [**12**], [**14**], [**15**], [**18**], [**19**]. For instance, in the cases $a = \frac{1}{2} = c, b = \frac{2}{3}, d = \sigma$ and $a = c = 0, b = \frac{1}{6}, d = \sigma - \frac{1}{2}$, well-posedness for the Cauchy problem for $s \geq 2$ and $p \geq 1$ were obtained by J. Quintero and A. Montes in work in revision and by J. Quintero in [**15**], respectively, and the existence results of solitons (finite energy travelling wave solutions) were obtained by J. Quintero and A. Montes in [**19**] and by J. Quintero in [**14**], respectively. An interesting review in the case of existence of periodic 2D travelling waves for the full Euler equations (doubly periodic or periodic in one direction) appears in the work of M. Groves [**7**]. Results for some models as the generalized 2D-Benney-Luke equation or the KP equation in the periodic case can be see in [**11**], [**13**], [**16**], [**17**], [**21**].

We notice that taking $\psi = \partial_x \Phi$, $p = 1$ and $\mu = 1$, the Boussinesq system(1.1) is related with the system considered by J. Bona, M. Chen and J. Saut (see [**1**], [**2**], [**3**], [**9**])

$$\left\{ \begin{aligned} \left(I - \tilde{d}\partial_x^2\right)\eta_t + \partial_x\left(I + \tilde{e}\partial_x^2\right)\psi + \epsilon\partial_x\left(\eta\psi\right) &= 0, \\ \left(I - \tilde{b}\partial_x^2\right)\psi_t + \partial_x\left(I + \tilde{a}\partial_x^2\right)\eta + \tfrac{\epsilon}{2}\partial_x(\psi^2) &= 0, \end{aligned} \right. \tag{1.2}$$

where

$$\tilde{a} + \tilde{b} + \tilde{e} + \tilde{d} = \frac{1}{3} - \sigma,$$

with $\tilde{a}, \tilde{e} < 0$ and $\tilde{b}, \tilde{d} \geq 0$, in the case of non zero surface tension. In recent works, M. Chen, N. Nguyen, and S. Sun in [**5**] and [**6**] established existence and orbital stability of travelling solutions for the Boussinesq system (1.2) for $\sigma > \frac{1}{3}$, $\tilde{b} = \tilde{d} > 0$ and $\tilde{a}\tilde{e} = \tilde{d}^2$ in [**5**], and existence of travelling solutions for $\sigma > \frac{1}{3}$, $\tilde{b} = \tilde{d} \geq 0$ in [**6**].

In this paper, we will establish the local well-posedness for the Cauchy problem associated with the system (1.1) in the space $H_k^s \times \mathcal{V}_k^{s+1}$, where $H_k^s = H_k^s(\mathbb{R})$ is the usual Sobolev space of order s of k-periodic functions and $\mathcal{V}_k^{s+1}$ is defined by the norm $\|\psi\|_{\mathcal{V}_k^{s+1}} = \|\psi'\|_{H_k^s}$. We also show global well-posedness for the Cauchy problem in the energy space $H_k^1 \times \mathcal{V}_k^2$ when the initial date is small enough. We will see as usual that local well-posedness for the Cauchy problem associated with the system (1.1) follows by the Banach fixed point Theorem and appropriate linear and nonlinear estimates using different results, which are considered in two mayor case: a) for $a, c > 0$, we will use a bilinear estimative obtained by D. Rouмégoux in [**22**]; b) for $a = c = 0$, we will use the well known estimates for Kato's commutator used successfully in the KdV and KP models (see the work by T. Kato and G. Ponce [**8**]). The global existence for $a = c$ follows from the local existence, the conservation in time of the Hamiltonian, a Sobolev type inequality and the use of energy estimates.

Even though there is not a connection with the Boussinesq system considered in this work, we establish existence of periodic travelling wave solutions by following the approach by H. Brezis and J. Mawhin (see [**4**]) in a recent work related with the existence of periodic classical solutions for a differential equation

$$\phi(u') - g(x, u) = h(x),$$

where $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism, g is a Charatéodory function k-periodic with respect to x, 2π-periodic with respect to u, of mean value zero on $[0, k]$, and $h \in L_{loc}(\mathbb{R})$ is k-periodic and has mean value zero. A special case of this interesting model is the relativistic forced pendulum differential equation

$$\left(\frac{u'}{\sqrt{1-u^2}}\right) + A\sin(u) = h(x).$$

The paper is organized as follows. In section 2, using semigroup estimates and nonlinear estimates, we show a local existence and uniqueness result for the Boussinesq system (1.1), via a standard fixed point argument. In section 3, from a variational approach which involves the characterization of invariant sets under the flow for the Boussinesq system (1.1), we obtain the global existence result for initial data small enough, in the case $a = c$. In section 4, we will use the direct method of the calculus of variations to prove the existence of $k-$periodic travelling wave solutions following H. Brezis and J. Mawhin (see [**4**]). Throughout this work, if not specified, we denote by K a generic constant varying line by line.

2. Local periodic well-posedness

In this section we study the local well-posedness for the initial value problem

$$\left\{\begin{array}{rl} \left(I - a\mu\partial_x^2\right)\eta_t + \partial_x^2\left(I - b\mu\partial_x^2\right)\Phi + \epsilon\partial_x\left(\eta\left(\partial_x\Phi\right)^p\right) & = 0, \\ \left(I - c\mu\partial_x^2\right)\Phi_t + (I - d\mu\partial_x^2)\eta + \frac{\epsilon}{p+1}\left(\partial_x\Phi\right)^{p+1} & = 0, \\ (\eta(0,\cdot), \Phi(0,\cdot)) = (\eta_0, \Phi_0), & \end{array}\right. \tag{2.1}$$

for $a, c \geq 0$ and $b, d > 0$. First we define the appropriate spaces:

2.1. Notation. The L^2-based Sobolev space of k-periodic functions is defined as follow. Let C_k^∞ denote the collection of all functions $f : \mathbb{R} \to \mathbb{R}$ which are C^∞ and periodic with period $k > 0$. The collection $(C_k^\infty)'$ of all continuous linear functionals from C_k^∞ into $\mathbb{R}$ is the set of periodic distributions. If $\Upsilon \in (C_k^\infty)'$ and $\phi \in C_k^\infty$, we denote the value of Υ at ϕ by $\langle\Upsilon, \phi\rangle$. If we define the functions $\Theta_m(x) = \exp(i\pi mx/k)$ for $m \in \mathbb{Z}$ and $x \in \mathbb{R}$, then the Fourier transform of Υ is the function $\widehat{\Upsilon}:\mathbb{Z}\to\mathbb{C}$ defined by $\widehat{\Upsilon}(m)=\frac{1}{k}\langle\Upsilon,\Theta_m\rangle$, for all $m\in\mathbb{Z}$. So, if Υ is a periodic function with period k, we have

$$\widehat{\Upsilon}(m)=\frac{1}{k}\int_0^k \Upsilon(x)e^{\frac{-2\pi imx}{k}}dx.$$

For $s\in\mathbb{R}$, the Sobolev space of k-periodic functions of order s, denoted by $H_k^s=H_k^s(\mathbb{R})$ is the set of all $f\in(C_k^\infty)'$ such that $(1+m^2)^{\frac{s}{2}}\widehat{f}(m)\in l^2(\mathbb{Z})$, with norm

$$\|f\|_{H_k^s}^2=k\sum_{m=-\infty}^{\infty}\left(1+m^2\right)^s|\widehat{f}(m)|^2.$$

We note that H_k^s is a Hilbert space with respect to the inner product

$$\langle f,g\rangle_s=k\sum_{m=-\infty}^{\infty}\left(1+m^2\right)^s\widehat{f}(m)\overline{\widehat{g}(m)}.$$

In the case $s=0$, H_k^0 is a Hilbert space that is isometrically isomorphic to $L^2[0,k]$ and

$$\langle f,g\rangle_0=\int_0^k f(x)g(x)dx.$$

The space H_k^0 will be denoted by L_k^2 and its norm will be $\|\cdot\|_{L_k^2}$. Note that $H_k^s\subset L_k^2$ for any $s\geq 0$. Finally, we define the space $\mathcal{V}_k^s$ as the closure of C_k^∞ with respect to the norm given by

$$\|f\|_{\mathcal{V}_k^s}=\|f'\|_{H_k^{s-1}}.$$

Note that $\mathcal{V}_k^s$ is a Hilbert space with inner product

$$(f,g)_{\mathcal{V}_k^s}=(f',g')_{H_k^{s-1}}.$$

Moreover,

$$\|f\|_{\mathcal{V}_k^s}^2=k\sum_{m=-\infty}^{\infty}\left(1+m^2\right)^{s-1}m^2|\widehat{f}(m)|^2.$$

Now, note that we can define the operator $L=I-l\mu\partial_x^2$ via the Fourier series as

$$\widehat{Lf}(m)=\left(1+l\mu m^2\right)\widehat{f}(m).$$

In particular, for any $l>0$, the operator L is invertible with

$$\widehat{L^{-1}f}(m)=\frac{\widehat{f}(m)}{1+l\mu m^2}.$$

2.2. Local well-posedness. It is easy to see, using the notation in previous section, that the system (2.1) can be rewritten as

$$\begin{pmatrix}\eta\\ \Phi\end{pmatrix}_t+M\begin{pmatrix}\eta\\ \Phi\end{pmatrix}+F\begin{pmatrix}\eta\\ \Phi\end{pmatrix}=0, \tag{2.2}$$

where M is a linear operator and F corresponds to the nonlinear part,

$$M=\begin{pmatrix}0 & \partial_x^2A^{-1}B\\ C^{-1}D & 0\end{pmatrix},\qquad F\begin{pmatrix}\eta\\ \Phi\end{pmatrix}=\epsilon\begin{pmatrix}\partial_xA^{-1}\left(\eta(\partial_x\Phi)^p\right)\\ \frac{1}{p+1}C^{-1}\left(\partial_x\Phi\right)^{p+1}\end{pmatrix},$$

with $A=I-a\mu\partial_x^2$, $B=I-b\mu\partial_x^2$, $C=I-c\mu\partial_x^2$ and $D=I-d\mu\partial_x^2$.

If we consider the Sobolev type space $\mathcal{X}_k^s=H_k^s\times\mathcal{V}_k^{s+1}$ with norm given by

$$\|(\eta,\Phi)\|_{\mathcal{X}_k^s}^2=\|\eta\|_{H_k^s}^2+\|\Phi\|_{\mathcal{V}_k^{s+1}}^2.$$

Then we can show that,

LEMMA 2.1. *for $s \in \mathbb{R}$, we have that $M : \mathcal{X}_k^s \longrightarrow \mathcal{X}_k^{s-1}$ is a bounded linear operator.*

PROOF. For $(\eta, \Phi) \in \mathcal{X}_k^s$ we have that

$$\begin{aligned}
\|(\partial_x^2 A^{-1} B)\Phi\|_{H_k^{s-1}}^2 &= \sum_{m=-\infty}^{\infty} m^4 \left(1+m^2\right)^{s-1} \frac{\left(1+b\mu m^2\right)^2}{\left(1+a\mu m^2\right)^2} |\widehat{\Phi}(m)|^2 \\
&\leq K_1(a,b) \sum_{m=-\infty}^{\infty} \left(1+m^2\right)^s m^2 |\widehat{\Phi}(m)|^2 \\
&\leq K_1(a,b) \|\Phi\|_{\mathcal{V}_k^{s+1}}^2 .
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\|C^{-1} D\eta\|_{\mathcal{V}_k^s}^2 &= \sum_{m=-\infty}^{\infty} m^2 \left(1+m^2\right)^{s-1} \frac{\left(1+d\mu m^2\right)^2}{\left(1+c\mu m^2\right)^2} |\widehat{\eta}(m)|^2 \\
&\leq K_2(c,d) \sum_{m=-\infty}^{\infty} \left(1+m^2\right)^s |\widehat{\eta}(m)|^2 \\
&\leq K_2(c,d) \|\eta\|_{H_k^s}^2 .
\end{aligned}$$

From these two facts we conclude that

$$\begin{aligned}
\|M(\eta,u)\|_{\mathcal{X}_k^{s-1}}^2 &= \|(\partial_x^2 A^{-1} B)\Phi\|_{H_k^{s-1}}^2 + \|C^{-1} D\eta\|_{\mathcal{V}_k^s}^2 \\
&\leq K_3 \left(\|\eta\|_{H_k^s}^2 + \|\Phi\|_{\mathcal{V}_k^{s+1}}^2 \right) \\
&\leq K_3 \|(\eta,\Phi)\|_{\mathcal{X}_k^s}^2 ,
\end{aligned}$$

as claimed. $\square$

In order to consider the initial value problem, we need to describe the semigroup $S(t)$ associated with the linear problem

$$\begin{pmatrix} \eta \\ \Phi \end{pmatrix}_t + M \begin{pmatrix} \eta \\ \Phi \end{pmatrix} = 0. \tag{2.3}$$

A simple calculation shows that the unique solution of the linear problem (2.3) with the initial condition

$$(\eta(0,\cdot), \Phi(0,\cdot)) = (\eta_0, \Phi_0) = \Psi_0 \in \mathcal{X}_k^s, \tag{2.4}$$

is given by

$$\Psi(t) = (\eta(t), \Phi(t)) = S(t)\Psi_0,$$

where $S(t)$ is defined as

$$\widehat{S(t)(\Psi)})(m) = \begin{pmatrix} \cos\left(m\rho(m)t\right) & m\varphi(m)\sin(m\rho(m)t) \\ -\frac{\sin(m\rho(m)t)}{m\varphi(m)} & \cos\left(m\rho(m)t\right) \end{pmatrix} \widehat{\Psi}(m), \tag{2.5}$$

and the functions φ and ρ are given by

$$\varphi^2(m) = \frac{\left(1+bm^2\right)\left(1+cm^2\right)}{(1+am^2)(1+dm^2)}, \quad \rho^2(m) = \frac{\left(1+bm^2\right)\left(1+dm^2\right)}{(1+am^2)(1+cm^2)}.$$

It is convenient to set

$$Q(t)(\widehat{\eta}, \widehat{\Phi}) = \big(Q_1(t), Q_2(t)\big)(\widehat{\eta}, \widehat{\Phi}),$$

where

$$\begin{aligned}
\left[Q_1(t)(\widehat{\eta}, \widehat{\Phi})\right](m) &= \cos\left(m\rho(m)t\right)\widehat{\eta}(m) + m\varphi(m)\sin(|\xi|\rho(m)t)\widehat{\Phi}(m),\\
\left[Q_2(t)(\widehat{\eta}, \widehat{\Phi})\right](m) &= -\frac{\sin\left(m\rho(m)t\right)\widehat{\eta}(m)}{m\varphi(m)} + \cos\left(m\rho(m)t\right)\widehat{\Phi}(m).
\end{aligned}$$

Then we have that

$$S(t)(\Psi) = \left(\mathcal{F}^{-1}\big[Q_1(t)(\widehat{\Psi})\big], \mathcal{F}^{-1}\big[Q_2(t)(\widehat{\Psi})\big]\right).$$

On the other hand, it is known that the Duhamel's principle implies that if Ψ is a solution of (2.2) with the initial condition (2.4), then this solution satisfies the integral equation

$$\Psi(t) = S(t)\Psi_0 - \int_0^t S(t-\tau)\, F(\Psi)(\tau)\, d\tau. \tag{2.6}$$

Hereafter, we refer to a $\Psi \in C([0,T], \mathcal{X}_k^s)$ satisfying the integral equation (2.6) as a mild solution for the initial value problem (2.1). Now, we will establish the existence of mild solutions. For this, we use some linear and nonlinear estimates. Let us start with the following result.

LEMMA 2.2. *Suppose $s \in \mathbb{R}$. Then for all $t \in \mathbb{R}$, $S(t)$ is a bounded linear operator from $\mathcal{X}_k^s$ into $\mathcal{X}_k^s$. Moreover, there exists $K_1 > 0$ such that for all $t \in \mathbb{R}$,*

$$\|S(t)(\Psi)\|_{\mathcal{X}_k^s} \le K_1 \|\Psi\|_{\mathcal{X}_k^s}.$$

PROOF. First note that there are constants (independent of m) $c_1, c_2 > 0$ such that $c_1 \le \varphi^2 \le c_2$. Then we have that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}[Q_1(t)(\widehat{\eta}, \widehat{\Phi})]\right\|^2_{H_k^s}\\
&\le K \sum_{m=-\infty}^{\infty} \left(1+m^2\right)^s \left|\cos(m\rho(m)t)\right|^2 |\widehat{\eta}(m)|^2\\
&+ K \sum_{m=-\infty}^{\infty} \left(1+m^2\right)^s m^2|\varphi(m)|^2|\sin\left(m\rho(m)t\right)|^2|\widehat{\Phi}(m)|^2\\
&\le K\left(\sum_{m=-\infty}^{\infty} \left(1+m^2\right)^s |\widehat{\eta}(m)|^2 + c_2 \sum_{m=-\infty}^{\infty} \left(1+m^2\right)^s m^2|\widehat{\Phi}(\xi)|^2\right)\\
&\le K\left(\|\eta\|^2_{H_k^s} + \|\Phi\|^2_{\mathcal{V}_k^{s+1}}\right).
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}[Q_2(t)(\widehat{\eta},\widehat{\Phi})]\right\|^2_{\mathcal{V}_k^{s+1}}\\
&\le K\sum_{m=-\infty}^{\infty}(1+m^2)^s\frac{|\sin(|m|\rho(m)t)|^2}{|\varphi(m)|^2}|\widehat{\eta}(m)|^2\\
&+K\sum_{m=-\infty}^{\infty}(1+m^2)^s m^2|\cos(m\rho(m)t)|^2|\widehat{\Phi}(m)|^2\\
&\le K\left(\sum_{m=-\infty}^{\infty}(1+m^2)^s|\widehat{\eta}(m)|^2+\sum_{m=-\infty}^{\infty}\left(1+m^2\right)^s m^2|\widehat{\Phi}(m)|^2\right)\\
&\le K\left(\|\eta\|^2_{H_k^s}+\|\Phi\|^2_{\mathcal{V}_k^{s+1}}\right).
\end{aligned}$$

So, if $\Psi=(\eta,\Phi)$ we obtain that

$$\begin{aligned}
\|S(t)\Psi\|^2_{\mathcal{X}_k^s}&=\|S(t)(\eta,\Phi)\|^2_{\mathcal{X}_k^s}\\
&=\left\|\mathcal{F}^{-1}[Q_1(t)(\widehat{\eta},\widehat{\Phi})]\right\|^2_{H_k^s}+\left\|\mathcal{F}^{-1}[Q_2(t)(\widehat{\eta},\widehat{\Phi})]\right\|^2_{\mathcal{V}_k^{s+1}}\\
&\le K\left(\|\eta\|^2_{H_k^s}+\|\Phi\|^2_{\mathcal{V}_k^{s+1}}\right)\\
&\le K\|(\eta,\Phi)\|^2_{\mathcal{X}_k^s}\\
&=K\|\Psi\|^2_{\mathcal{X}_k^s},
\end{aligned}$$

and $S(t)$ have the required property. □

Next, we want to perform the estimates for nonlinear terms of system (2.2) (Lemma 2.5), which will follow by an estimate obtained by D. Roumégoux (Lemma 3.1 in [**22**], with $r=r'=s\ge 0$) in the case $a,c>0$ and the well known estimates for the periodic commutator of Kato in the case $a=c=0$.

Lemma 2.3. *(D. Roumégoux, [**22**]) Let $l>0$, $s\ge 0$. Then there exists a constant $K(l)>0$ such that*

$$\|L^{-1}\partial_x(uv)\|_{H_k^s}\le K(l)\|u\|_{H_k^s}\|v\|_{H_k^s}.$$

Now, let $J=\left(I-\partial_x^2\right)^{1/2}$ be the operator defined by

$$\widehat{Jf}=(1+m^2)^{1/2}\widehat{f},$$

and let [,] be the commutator defined by

$$[J^s,u]\,v=J^s(uv)-uJ^sv.$$

Lemma 2.4. *(T. Kato, [**8**]) Suppose $s>\frac{3}{2}$ and $t>\frac{1}{2}$. Then there exists a constant $K>0$ such that*

(1) $\|[J^s,u]w\|_{L_k^2}\le K\|u\|_{H_k^s}\|w\|_{H_k^{s-1}}$.

(2) $\|u\partial_x w\|_{L_k^2}\le K\|\partial_x u\|_{H_k^t}\|w\|_{L_k^2}$.

Next, we will establish the nonlinear estimates.

Lemma 2.5. *Suppose a,c, p and s are such that* (*a*) $a, c > 0$, $p = 1, s \geq 0$, *or*

(*b*) $a, c > 0$, $p > 1$, $s > \frac{1}{2}$, *or*

(*c*) $a = c = 0$, $p \geq 1$, $s > \frac{3}{2}$.

Then there are constants $K_2, K_3 > 0$ *such that*

(1) $\|F(\Psi)\|_{\mathcal{X}_k^s} \leq K_2 \|\Psi\|_{\mathcal{X}_k^s}^{p+1}$.
(2) $\|F(\Psi) - F(\Psi_1)\|_{\mathcal{X}_k^s} \leq K_3 \|\Psi - \Psi_1\|_{\mathcal{X}_k^s} \left(\|\Psi\|_{\mathcal{X}_k^s} + \|\Psi_1\|_{\mathcal{X}_k^s}\right)^p$.

Proof. We write $F = \epsilon \left(F_1, \frac{1}{p+1} F_2\right)$ where

$$F_1(\Psi) = F_1(\eta, \Phi) = A^{-1}\partial_x \left(\eta(\partial_x \Phi)^p\right), \quad F_2(\Psi) = F_2(\eta, \Phi) = C^{-1} \left(\partial_x \Phi\right)^{p+1}.$$

First we assume that $a, c > 0$, $p = 1$ and $s \geq 0$. Using the Lemma 2.3 we have that

$$\begin{aligned}
\|F_1(\eta, \Phi)\|_{H_k^s} &= \|A^{-1}\partial_x \left(\eta \partial_x \Phi\right) \|_{H_k^s} \\
&\leq K(a) \|\eta\|_{H_k^s} \|\partial_x \Phi\|_{H_k^s} \\
&\leq K(a) \|\eta\|_{H_k^s} \|\Phi\|_{\mathcal{V}_k^{s+1}} \\
&\leq K(a) \left(\|\eta\|_{H_k^s}^2 + \|\Phi\|_{\mathcal{V}_k^{s+1}}^2\right) \\
&= K(a) \|(\eta, u)\|_{\mathcal{X}_k^s}^2.
\end{aligned}$$

Similarly we have that

$$\begin{aligned}
\|F_2(\eta, \Phi)\|_{\mathcal{V}_k^{s+1}} &= \|C^{-1} \left(\partial_x \Phi\right)^2 \|_{\mathcal{V}_k^{s+1}} \\
&= \|C^{-1} \partial_x \left(\partial_x \Phi\right)^2 \|_{H_k^s} \\
&\leq K(c) \|\partial_x \Phi\|_{H_k^s}^2 \\
&\leq K(c) \|\Phi\|_{\mathcal{V}_k^{s+1}}^2 \\
&\leq K(c) \|(\eta, \Phi)\|_{\mathcal{X}_k^s}^2.
\end{aligned}$$

In other words, we have established estimate (1). Now we prove estimate (2). In fact,

$$\begin{aligned}
&\|F_1(\eta, \Phi) - F_1(\eta_1, \Phi_1)\|_{H_k^s} \\
&= \|A^{-1}\partial_x \left(\eta \partial_x \Phi - \eta_1 \partial_x \Phi_1\right) \|_{H_k^s} \\
&\leq \|A^{-1}\partial_x (\eta(\partial_x \Phi - \partial_x \Phi_1))\|_{H_k^s} + \|A^{-1}\partial_x (\eta - \eta_1) \partial_x \Phi_1\|_{H_k^s} \\
&\leq K(a) \left(\|\eta\|_{H_k^s} \|\partial_x \Phi - \partial_x \Phi_1\|_{H_k^s} + \|\eta - \eta_1\|_{H_k^s} \|\partial_x \Phi_1\|_{H_k^s}\right) \\
&\leq K(a) \left(\|\eta\|_{H_k^s} + \|\partial_x \Phi_1\|_{H_k^s}\right) \left(\|\eta - \eta_1\|_{H_k^s} + \|\partial_x \Phi - \partial_x \Phi_1\|_{H_k^s}\right) \\
&\leq K(a) \left(\|\eta\|_{H_k^s} + \|\Phi_1\|_{\mathcal{V}_k^{s+1}}\right) \left(\|\eta - \eta_1\|_{H_k^s} + \|\Phi - \Phi_1\|_{\mathcal{V}_k^{s+1}}\right) \\
&\leq K(a) \left(\|(\eta, \Phi)\|_{\mathcal{X}_k^s} + \|(\eta_1, \Phi_1)\|_{\mathcal{X}_k^s}\right) \|(\eta, \Phi) - (\eta_1, \Phi_1)\|_{\mathcal{X}_k^s}.
\end{aligned}$$

In a similar fashion we have that

$$\begin{aligned}
&\|F_2(\eta,\Phi)-F_2(\eta_1,\Phi_1)\|_{\mathcal{V}_k^{s+1}}\\
&=\|C^{-1}\left((\partial_x\Phi)^2-(\partial_x\Phi_1)^2\right)\|_{\mathcal{V}_k^{s+1}}\\
&=\|C^{-1}\partial_x\left((\partial_x\Phi)^2-(\partial_x\Phi_1)^2\right)\|_{H_k^s}\\
&=\|C^{-1}\partial_x\left((\partial_x\Phi+\partial_x\Phi_1)(\partial_x\Phi-\partial_x\Phi_1)\right)\|_{H_k^s}\\
&\le K(c)\|\partial_x\Phi+\partial_x\Phi_1\|_{H_k^s}\|\partial_x\Phi-\partial_x\Phi_1\|_{H_k^s}\\
&\le K(c)\|\Phi+\Phi_1\|_{\mathcal{V}_k^{s+1}}\|\Phi-\Phi_1\|_{\mathcal{V}_k^{s+1}}\\
&\le K(c)\left(\|(\eta,\Phi)\|_{\mathcal{X}_k^s}+\|(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}\right)\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
&\|F(\eta,\Phi)-F(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}\\
&\qquad\le K\left(\|F_1(\eta,\Phi)-F_1(\eta_1,\Phi_1)\|_{H_k^s}+\|F_2(\eta,\Phi)-F_2(\eta_1,\Phi_1)\|_{\mathcal{V}_k^{s+1}}\right)\\
&\qquad\le K\left(\|(\eta,\Phi)\|_{\mathcal{X}_k^s}+\|(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}\right)\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}.
\end{aligned}$$

Now we suppose that $s>\frac{1}{2}$ and $p>1$. Using the Lemma 2.3 and that $H_k^s(\mathbb{R})$ is an algebra we obtain that

$$\begin{aligned}
\|F_1(\eta,\Phi)\|_{H_k^s}&=\|A^{-1}\partial_x\left(\eta(\partial_x\Phi)^p\right)\|_{H_k^s}\\
&\le K(a)\|\eta(\partial_x\Phi)^p\|_{H_k^s}\\
&\le K(a)\|\eta\|_{H_k^s}\|\partial_x\Phi\|_{H_k^s}^p\\
&\le K(a)\|\eta\|_{H_k^s}\|\Phi\|_{\mathcal{V}_k^{s+1}}^p\\
&\le K(a)\|(\eta,\Phi)\|_{\mathcal{X}_k^s}^{p+1}.
\end{aligned}$$

And also that

$$\begin{aligned}
\|F_2(\eta,u)\|_{\mathcal{V}_k^{s+1}}&=\|C^{-1}\left(\partial_x\Phi\right)^{p+1}\|_{\mathcal{V}_k^{s+1}}\\
&=\|C^{-1}\partial_x\left(\partial_x\Phi\right)^{p+1}\|_{H_k^s}\\
&\le K(c)\|\partial_x\Phi\|_{H_k^s}^{p+1}\\
&\le K(c)\|\Phi\|_{\mathcal{V}_k^{s+1}}^{p+1}\\
&\le K(c)\|(\eta,\Phi)\|_{\mathcal{X}_k^s}^{p+1}.
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
&\|F_1(\eta,\Phi)-F_1(\eta_1,\Phi_1)\|_{H_k^s}\\
&\qquad\le\|A^{-1}\partial_x\left(\eta\left((\partial_x\Phi)^p-(\partial_x\Phi_1)^p\right)\right)\|_{H_k^s}+\|A^{-1}\partial_x\left((\eta-\eta_1)(\partial_x\Phi_1)^p\right)\|_{H_k^s}\\
&\qquad\le K(a)\|\eta\|_{H_k^s}\|(\partial_x\Phi)^p-(\partial_x\Phi_1)^p\|_{H_k^s}+\|\eta-\eta_1\|_{H_k^s}\|\partial_x\Phi_1\|_{H_k^s}^p.
\end{aligned}$$

But a simple calculation shows that

$$\begin{aligned}
\|(\partial_x\Phi)^p-(\partial_x\Phi_1)^p\|_{H_k^s}&\le K(p)\|\partial_x\Phi-\partial_x\Phi_1\|_{H_k^s}\left(\|\partial_x\Phi\|_{H_k^s}+\|\partial_x\Phi_1\|_{H_k^s}\right)^{p-1}\\
&=K(p)\|\Phi-\Phi_1\|_{\mathcal{V}_k^{s+1}}\left(\|\Phi\|_{\mathcal{V}_k^{s+1}}+\|\Phi_1\|_{\mathcal{V}_k^{s+1}}\right)^{p-1}.
\end{aligned}$$

Then we have that

$$\begin{aligned}&\|F_1(\eta,\Phi)-F_1(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}\\&\qquad\leq K(a,c,p)\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}\left(\|(\eta,\Phi)\|_{\mathcal{X}_k^s}+\|(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}\right)^p.\end{aligned}$$

In a similar fashion we obtain the same estimate for $\|F_2(\eta,\Phi)-F_2(\eta_1,\Phi_1)\|_{\mathcal{X}_k^s}$ and then (1) and (2) hold.

We assume now that $a=c=0$, $p\geq 1$ and $s>\frac{3}{2}$. In this case we notice that $A=C=I$, i.e., the identity operator. First we will show that there exists $K>0$ such that if $v,\partial_x w\in H_k^s$, then

$$\|v\partial_x w\|_{H_k^s}\leq K\|v\|_{H_k^s}\|w\|_{H_k^s}. \tag{2.7}$$

In fact, from Lemma 2.4 we see that

$$\begin{aligned}\|v\partial_x w\|_{H_k^s}&=\|J^s(v\partial_x w)\|_{L_k^2}\\&\leq\|\,[J^s,v]\partial_x w\,\|_{L_k^2}+\|v\partial_x J^s w\|_{L_k^2}\\&\leq K\left(\|v\|_{H_k^s}\|\partial_x w\|_{H_k^{s-1}}+\|\partial_x v\|_{H_k^{s-1}}\|J^s w\|_{L_k^2}\right)\\&\leq K\|v\|_{H_k^s}\|w\|_{H_k^s}.\end{aligned}$$

Then, using (2.7) and that $H_k^s(\mathbb{R})$ is an algebra, we have that

$$\begin{aligned}\|F_1(\eta,u)\|_{H_k^s}&\leq K(p)\left(\|\partial_x\eta(\partial_x\Phi)^p\|_{H_k^s}+\|\eta(\partial_x\Phi)^{p-1}\partial_x^2\Phi\|_{H_k^s}\right)\\&\leq K(p)\left(\|\eta\|_{H_k^s}\|(\partial_x\Phi)^p\|_{H_k^s}+\|\partial_x\Phi\|_{H_k^s}\|\eta(\partial_x\Phi)^{p-1}\|_{H_k^s}\right)\\&\leq K(p)\|\eta\|_{H_k^s}\|\partial_x\Phi\|_{H_k^s}^p\\&\leq K(p)\|(\eta,\Phi)\|_{\mathcal{X}_k^s}^{p+1}.\end{aligned}$$

And also that

$$\begin{aligned}\|F_2(\eta,\Phi)\|_{\mathcal{V}_k^{s+1}}&=K(p)\|(\partial_x\Phi)^p\partial_x^2\Phi\|_{H_k^s}\\&\leq K(p)\|(\partial_x\Phi)^p\|_{H_k^s}\|\partial_x\Phi\|_{H_k^s}\\&\leq K(p)\|\partial_x\Phi\|_{H_k^s}^{p+1}\\&\leq K(p)\|(\eta,\Phi)\|_{\mathcal{X}_k^s}^{p+1}.\end{aligned}$$

Thus, we conclude that there exists $K>0$ such that

$$\|F(\eta,u)\|_{\mathcal{X}_k^s}\leq K\|(\eta,\Phi)\|_{\mathcal{X}_k^s}^{p+1}.$$

In a similar way we obtain the part (2) and then the theorem follows. □

Next, we establish the local well-posedness for the system (2.1) in the space $\mathcal{X}_k^s$. For this we will show the existence of a solution for the integral equation (2.6), using the Banach fixed point Theorem.

THEOREM 2.1. *Let a,c,p and s be as in Lemma 2.5. Then for all $(\eta_0,\Phi_0)\in\mathcal{X}_k^s$ there exists a time $T>0$ which depends only on $\|(\eta_0,\Phi_0)\|_{\mathcal{X}_k^s}$ such that the initial value problem (2.1) has a unique solution (η,Φ) satisfying*

$$(\eta,\Phi)\in C\left([0,T],\mathcal{X}_k^s\right)\cap C^1\left([0,T],\mathcal{X}_k^{s-1}\right).$$

Moreover, for all $0<T'<T$ there exists a neighborhood $\mathbb{V}$ of (η_0,Φ_0) in $\mathcal{X}_k^s$ such that the correspondence $(\tilde\eta_0,\tilde\Phi_0)\longrightarrow(\tilde\eta(\cdot),\tilde\Phi(\cdot))$, that associates to $(\tilde\eta_0,\tilde\Phi_0)$

the solution $(\tilde{\eta}(\cdot), \tilde{\Phi}(\cdot))$ *of the problem (2.1) with initial condition* $(\tilde{\eta}_0, \tilde{\Phi}_0)$ *is a Lipschitz mapping from* $\mathbb{V}$ *in* $C([0,T'], \mathcal{X}^s_k)$.

PROOF. Given $T > 0$ we define the space $X^s(T) = C([0,T], \mathcal{X}^s_k)$, equipped with the norm defined by

$$\|\Psi\|_{X^s(T)} = \max_{t\in[0,T]} \|\Psi(\cdot,t)\|_{\mathcal{X}^s_k}.$$

It is easy to see that $X^s(T)$ is a Banach space. Let $B_R(T)$ be the closed ball of radius R centered at the origin in $X^s(T)$, i.e.

$$B_R(T) = \left\{\Psi \in X^s(T) \ : \ \|\Psi\|_{X^s(T)} \leq R\right\}.$$

For fixed $\Psi_0 = (\eta_0, \Phi_0) \in \mathcal{X}^s_k$, we define the map

$$\Lambda(\Psi(t)) = S(t)\Psi_0 - \int_0^t S(t-\tau)F(\Psi(\tau))\, d\tau,$$

where $\Psi = (\eta, \phi) \in X(T)$. We will show that the correspondence $\Psi(t) \mapsto \Lambda(\Psi(t))$ maps $B_R(T)$ into itself and is a contraction if R and T are well chosen. In fact, if $t \in [0,T]$ and $\Psi \in B_R(T)$, then using Lemma 2.2 and statement (1) of Lemma 2.5 we have that

$$\begin{aligned}\|\Lambda(\Psi(t))\|_{\mathcal{X}^s_k} &\leq K_1\left(\|\Psi_0\|_{\mathcal{X}^s_k} + K_2\int_0^t \|\Psi(\tau)\|^{p+1}_{\mathcal{X}^s_k} d\tau\right)\\ &\leq K_1\left(\|\Psi_0\|_{\mathcal{X}^s_k} + K_2 R^{p+1}T\right).\end{aligned}$$

Choosing $R = 2K_1\|\Psi_0\|_{\mathcal{X}^s_k}$ and $T > 0$ such that

$$(2K_1)^{p+1}K_2\|\Psi_0\|^p_{\mathcal{X}^s_k} T \leq 1,$$

we obtain that

$$\|\Lambda(\Psi(t))\|_{\mathcal{X}^s_k} \leq K_1\, \|\Psi_0\|_{\mathcal{X}^s_k}\left(1 + (2K_1)^{p+1}K_2\|\Psi_0\|^p_{\mathcal{X}^s_k} T\right) \leq 2\, K_1\|\Psi_0\|_{\mathcal{X}^s_k} = R.$$

So that Λ maps $B_R(T)$ to itself. Let us prove that Ψ is a contraction. If $\Psi, \Psi_1 \in B_R(T)$, then by the definition of Ψ we have that

$$\Lambda(\Psi(t)) - \Lambda(\Psi_1(t)) = -\int_0^t S(t-\tau)\big[F(\Psi(\tau)) - F(\Psi_1(\tau))\big]\, d\tau.$$

Then using the statement (2) of Lemma 2.5 we see that for $t \in [0,T]$,

$$\begin{aligned}\|\Lambda(\Psi(t)) - \Lambda(\Psi_1(t))\|_{\mathcal{X}^s_k} &\leq K_1K_3\int_0^t \left(\|\Psi(\tau)\|_{\mathcal{X}^s_k} + \|\Psi_1(\tau)\|_{\mathcal{X}^s_k}\right)^p \|\Psi(\tau) - \Psi_1(\tau)\|_{\mathcal{X}^s_k}\, d\tau\\ &\leq K_1K_3(2R)^pT\|\Psi - \Psi_1\|_{X^s(T)}\\ &\leq 4^pK_1^{p+1}K_3\|\Psi_0\|^p_{\mathcal{X}^s_k} T\|\Psi - \Psi_1\|_{X^s(T)}.\end{aligned}$$

We choose T enough small so that (2.2) holds and

$$\alpha = 4^pK_1^{p+1}K_3\|\Psi_0\|^p_{\mathcal{X}^s_k} T \leq \frac{1}{2}.$$

So, we conclude that

$$\|\Lambda(\Psi) - \Lambda(\Psi_1)\|_{X^s(T)} \leq \alpha\|\Psi - \Psi_1\|_{X^s(T)}.$$

Therefore Λ is a contraction, and so there exists a unique fixed point of Λ in $B_R(T)$, which is a solution of the integral equation (2.6). Now, if $\Psi(t) \in C([0,T], \mathcal{X}^s_k)$ is a mild solution, obviously $\Psi(0) = (\eta(0), \Phi(0)) = (\eta_0, \Phi_0)$. Moreover, differentiating the equation (2.6) with respect t there appears the relation (2.2). In other words, $\Psi(t) = (\eta(t), \Phi(t))$ is a local solution for the initial value problem (2.1). The uniqueness and continuous dependence of the solution are obtained by standard arguments. $\square$

3. Global periodic well-posedness for $a = c$

In this section taking advantage of the conservation in time of the Hamiltonian in the case $a = c$, we establish that any local solution in time of the system (1.1) can be extended for any $t > 0$. We only sketch the proofs since the details can be found in the proof of similar results for water wave models (see for example Section 3 in [**15**], for a 2D-dimensional version of (1.1)). The result will depends strongly on the Hamiltonian structure given by

$$\begin{aligned} \mathcal{H}(\Psi) &= \mathcal{H}(\eta, \Phi) \\ &= \frac{1}{2}\int_0^k \left(\eta^2 + d\mu\left(\partial_x\eta\right)^2 + \left(\Phi_x\right)^2 + b\mu\left(\partial_x^2\Phi\right)^2 + \frac{2\epsilon}{p+1}\eta\left(\partial_x\Phi\right)^{p+1}\right)dx \\ &= \frac{1}{2}\left(\mathcal{E}(\Psi) + G(\Psi)\right), \end{aligned} \tag{3.1}$$

where functional $\mathcal{E}$ (energy) and G are given by

$$\begin{aligned} \mathcal{E}(\Psi) &= \int_0^k \left(\eta^2 + d\mu\left(\partial_x\eta\right)^2 + \left(\partial_x\Phi\right)^2 + b\mu\left(\partial_x^2\Phi\right)^2\right)dx, \\ G(\Psi) &= \frac{2\epsilon}{p+1}\int_0^k \eta\left(\partial_x\Phi\right)^{p+1}dx. \end{aligned}$$

It is not difficult to see that the system (1.1) can be expressed in the following Hamiltonian form

$$\begin{pmatrix} \eta_t \\ \Phi_t \end{pmatrix} = \mathcal{J}\mathcal{H}'\begin{pmatrix} \eta \\ \Phi \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & \left(I - c\mu\partial_x^2\right)^{-1} \\ -\left(I - a\mu\partial_x^2\right)^{-1} & 0 \end{pmatrix}.$$

Note that for $a = c$ the operator $\mathcal{J}$ becomes skew symmetric

$$\mathcal{J} = \left(I - a\mu\partial_x^2\right)^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In addition, we see directly that the functional $\mathcal{H}$ is well defined when $\eta \in H_k^1$ and $\Phi \in \mathcal{V}_k^2$. These conditions already characterize the natural space (energy space) in which we consider the global well-posedness of the Cauchy problem and the existence of periodic travelling wave solutions (see Section 4).

The global well-posedness follows by using a variational approach and the fact that the energy $\sqrt{\mathcal{E}}$ is a norm in the space $\mathcal{X}_k^1$, since for some constant $K(b, d, \mu) > 1$,

$$K(b, d, \mu)^{-1}\|\Psi\|_{\mathcal{X}_k^1}^2 \leq \mathcal{E}(\Psi) \leq K(b, d, \mu)\|\Psi\|_{\mathcal{X}_k^1}^2. \tag{3.2}$$

A clever analysis to obtain global solutions, as done in the 2D-dimensional case, depends upon the variational characterization of the number δ_0 defined by

$$\delta_0 = \inf\left\{\sup_{\lambda\geq 0}\mathcal{H}(\lambda\Psi) : \Psi \in \mathcal{X}_k^1 \setminus \{0\}\right\} = \inf\{\sup_{\lambda\geq 0}\mathcal{H}(\lambda\Psi) : \Psi \in \mathcal{X}_k^1,\ G(\Psi) < 0\}.$$

LEMMA 3.1.

$$\delta_0 = \frac{p}{2(p+2)} \left(\frac{2}{p+2}\right)^{\frac{2}{p}} K_p^{-\frac{p+2}{p}}, \tag{3.3}$$

where K_p is defined as

$$K_p = \sup \left\{ \frac{G^{\frac{2}{p+2}}(\Psi)}{\mathcal{E}(\Psi)} \ : \ \Psi \in \mathcal{X}_k^1 \setminus \{0\} \right\}. \tag{3.4}$$

Moreover, we have the following Sobolev type inequality

$$|G(\Psi)|^{\frac{1}{p+2}} \leq K_p^{\frac{1}{2}} \sqrt{\mathcal{E}(\Psi)} \leq K(b,d,\mu)^{\frac{1}{2}} K_p^{\frac{1}{2}} \|\Psi\|_{\mathcal{X}_k^1}. \tag{3.5}$$

Before we go further, we consider the auxiliary functional $\mathcal{H}_1(\Psi) = \mathcal{H}'(\Psi)(\Psi)$, which has can be expressed as

$$\mathcal{H}_1(\Psi) = \mathcal{E}(\Psi) + \left(\frac{p+2}{2}\right) G(\Psi). \tag{3.6}$$

In particular, we have that

$$\mathcal{H}(\Psi) = \left(\frac{p}{2(p+2)}\right) \mathcal{E}(\Psi) + \left(\frac{1}{p+2}\right) \mathcal{H}_1(\Psi). \tag{3.7}$$

We will see that the global existence result is a consequence that the set

$$\mathcal{A} = \left\{ \Psi \in \mathcal{X}_k^1 : \mathcal{H}(\Psi) < \delta_0, \ \ \mathcal{H}_1(\Psi) > 0 \right\}.$$

is invariant under the flow associated with the system (1.1). First we observe that the Hamiltonian $\mathcal{H}$ is conserved in time on solutions, meaning for classical solutions that

$$\mathcal{H}(\Psi(t)) = \mathcal{H}(\Psi(0)) < \delta_0, \tag{3.8}$$

Assume by continuity that there is $t_1 \in (0, T_0)$ such that $\mathcal{H}_1(\Psi(t)) > 0$ for $0 < t < t_1$ and

$$\mathcal{H}_1(\Psi(t_1)) = 0, \quad \Psi(t_1) \neq 0. \tag{3.9}$$

Then, from (3.7), we have that

$$0 < \mathcal{E}(\Psi(t_1)) = \frac{2(p+2)}{p} \mathcal{H}(\Psi(t_1)) - \frac{2}{p} \mathcal{H}_1(\Psi(t_1)) < \frac{2(p+2)}{p} \delta_0. \tag{3.10}$$

But from the Sobolev type inequality (3.5) we conclude that

$$|G(\Psi(t_1))| \leq K_p^{\frac{p+2}{2}} \left[\mathcal{E}(\Psi(t_1))\right]^{\frac{p}{2}} \mathcal{E}(\Psi(t_1)) < \left(\frac{2}{p+2}\right) \mathcal{E}(\Psi(t_1)),$$

which implies, by using (3.6), that we already have $\mathcal{H}_1(\ \Psi(t_1)) > 0$; but this is a contradiction. So, $\mathcal{H}_1(\Psi(t)) > 0$ for $t \in (0, T_0)$. Moreover for $t \in (0, T_0)$

$$e(t) = \sup_{r \in [0,t]} \mathcal{E}(\Psi(r)) < \left(\frac{2(p+2)}{p}\right) \delta_0.$$

THEOREM 3.1. *Assume $a = c \geq 0$ and $p \geq 1$. Let $\Psi_0 = (\eta_0, \Phi_0) \in \mathcal{X}_k^1$ be such that $\mathcal{H}(\Psi_0) < \delta_0$ and $\mathcal{H}_1(\Psi_0) > 0$. Then there exists a unique global solution for the initial value problem (2.1).*

Proof. In the case $a = c > 0$, we just use the invariance in time of the Hamiltonian and the invariance of set $\mathcal{A}$ under the flow. In fact, if $\Psi_0 \in \mathcal{X}_k^1$, by the local existence result, there is a maximal existence time $T_0 > 0$ and a unique solution $\Psi \in C([0, T_0), \mathcal{X}_k^1)$ of the initial value problem (2.1) with initial condition $\Psi(0, \cdot) = \Psi_0$. So, $\mathcal{H}(\Psi(t)) = \mathcal{H}(\Psi_0) < \delta_0$. Moreover, we also have that $\mathcal{H}_1(\Psi(t)) > 0$ and

$$\mathcal{E}(\Psi(t)) \leq \left(\frac{2(p+2)}{p}\right) \mathcal{H}(\Psi_0) < \left(\frac{2(p+2)}{p}\right) \delta_0.$$

This implie from (3.2) that for $t \in [0, T_0)$,

$$\|\Psi(t)\|^2_{\mathcal{X}_k^1} \leq K(b, d, \mu)\mathcal{E}(\Psi(t)) < \left(\frac{2(p+2)}{p}\right) K(b, d, \mu)\delta_0.$$

In other words, the solution Ψ is bounded in time on the space $\mathcal{X}_k^1$ and that for any finite $T_0 < \infty$ we are able to conclude that

$$\lim_{t \to T_0^-} \left\|\Psi(t)\right\|^2_{\mathcal{X}_k^1} < \infty.$$

In the casoe $a = c = 0$, we are going to use a density argument. Let $s_0 > \frac{3}{2}$ be fixed, then by there exists $\Psi_{0,n} \in \mathcal{X}_k^{s_0}$ such that

$$\Psi_{0,n} \to \Psi_0 \ \text{ in } \ \mathcal{X}_k^1, \ \text{ as } \ n \to \infty.$$

From the local existence result, for each $n \in \mathbb{Z}^+$ there is $T_{0,n} > 0$ and a unique solution Ψ_n of the Cauchy problem for the Boussinesq system (1.1) with initial condition $\Psi_n(0, \cdot) = \Psi_{0,n}$. On the other hand, there exists $n_0 \in \mathbb{Z}^+$ such that $\mathcal{H}(\Psi_{0,n}) < \delta_0$ and $\mathcal{H}_1(\Psi_{0,n}) > 0$ for $n \geq n_0$. Now, for $n \geq n_0$ we have that

$$\mathcal{H}(\Psi_n) = \left(\frac{2(p+2)}{p}\right) \mathcal{E}\left(\Psi_n\right) + \left(\frac{1}{p+2}\right) \mathcal{H}_1(\Psi_n) = \mathcal{H}(\Psi_{0,n}) < \delta_0.$$

From the invariant under the flow of the set $\mathcal{A}$, we have that $\mathcal{H}_1(\Psi_n) > 0$ for $n \geq n_0$. Then we also have that

$$\mathcal{E}(\Psi_n) \leq \left(\frac{2(p+2)}{p}\right) \mathcal{H}(\Psi_{0,n}) < \left(\frac{2(p+2)}{p}\right) \delta_0.$$

Then using (3.2), we get for $n \geq n_0$ and $t \in [0, T_{0,n})$ that

$$\|\Psi_n\|^2_{\mathcal{X}_k^1} \leq K\mathcal{E}(\Psi_n) < \left(\frac{2(p+2)}{p}\right) K\delta_0,$$

implying that $\{\Psi_n\}_k$ is bounded sequence in the space $\mathcal{X}_k^1$ and that for any finite $T_0 < \infty$ and $n \geq n_0$ we are able to conclude that

$$\lim_{t \to T_0^-} \left\|\Psi_n\right\|^2_{\mathcal{X}_k^1} < \infty.$$

In other words, for $n \geq n_0$ we have that Ψ_n can be extended in time. Now, since $\{\Psi_n\}$ is bounded sequence in $\mathcal{X}_k^1$, we have for some subsequence (denoted the same) that there is $\Psi \in \mathcal{X}_k^1$ such that $\Psi_n \rightharpoonup \Psi$ (weakly) in $\mathcal{X}_k^1$, as $n \to \infty$. Moreover, it is easy to see that $\Psi \in C\left([0, \infty), \mathcal{X}_k^1\right)$ is a weak solution of the Cauchy problem for the system (1.1) satisfying $\Psi(\cdot, 0) = \Psi_0$. □

As a consequence of the previous result, we are able to establish that the Cauchy problem associated with the Boussinesq system (1.1) has global solution in time for initial data $\Psi_0 \in H_k^1 \times \mathcal{V}^2$ small enough such that $\Psi_0 \neq 0$.

THEOREM 3.2. *Let $p \geq 1$. Then there exists $\delta > 0$ such that for any $\Psi_0 = (\eta_0, \Phi_0) \in \mathcal{X}_k^1$ with $\|\Psi_0\|_{\mathcal{X}_k^1} \leq \delta$, the initial value problem (2.1) has a unique global solution*

$$\Psi \in C\left([0,\infty), \mathcal{X}_k^1\right) \cap C^1\left([0,\infty), \mathcal{X}_k^0\right).$$

PROOF. If $G(\Psi_0) \geq 0$, then we have that $\mathcal{H}_1(\Psi_0) = \mathcal{E}(\Psi_0) + \left(\frac{p+2}{2}\right) G(\Psi_0) > 0$. Now, If $G(\Psi_0) < 0$, then we see from (3.2) that

$$\begin{aligned}\mathcal{H}_1(\Psi_0) &= \mathcal{E}(\Psi_0) + \left(\frac{p+2}{2}\right) G(\Psi_0)\\ &\geq K^{-1}(b,d,\mu)\left(\|\Psi_0\|^2_{\mathcal{X}_k^1} + \left(\frac{p+2}{2}\right) K(b,d,\mu) G(\Psi_0)\right).\end{aligned}$$

Thus, for $\|\Psi_0\|^2_{\mathcal{X}_k^1}$ sufficiently small we would have $\mathcal{H}_1(\Psi_0) > 0$, since

$$G(\Psi_0) = O\left(\mathcal{E}(\Psi_0)^{\frac{p+2}{2}}\right) = O\left(\|\Psi_0\|^{p+2}_{\mathcal{X}_k^1}\right).$$

From (3.2), (3.1) and (3.5) we see that there exists $K_1(b,d,\mu,\epsilon,p) > 0$ such that

$$\mathcal{H}(\Psi_0) \leq K_1(b,d,\mu,\epsilon,p)\left(1 + \|\Psi_0\|^p_{\mathcal{X}_k^1}\right)\|\Psi_0\|^2_{\mathcal{X}_k^1},$$

and from (3.2) we have that

$$\mathcal{E}(\Psi_0) \leq K(b,d,\mu)\|\Psi_0\|^2_{\mathcal{X}_k^1}.$$

Hence, we choose $\delta > 0$ in a such way that

$$K_1(b,d,\mu,\epsilon,p)\left(1+\delta^p\right)\delta^2 < \delta_0 \quad \text{and} \quad K(b,d,\mu)\delta^2 < \left(\frac{2(p+2)}{p}\right)\delta_0.$$

Let $\Psi_0 \in \mathcal{X}_k^1$ be such that $\|\Psi_0\|_{\mathcal{X}_k^1} \leq \delta$, then we see that $\mathcal{H}(\Psi_0) < \delta_0$. Moreover, from the Sobolev type inequality (3.5) we obtain that

$$\begin{aligned}|G(\Psi_0)| &\leq K_p^{\frac{p+2}{2}}\left(\mathcal{E}(\Psi_0)\right)^{\frac{p}{2}}\mathcal{E}(\Psi_0)\\ &< K_p^{\frac{p+2}{2}}\left(\frac{2(p+2)}{p}\delta_0\right)^{\frac{p}{2}}\mathcal{E}(\Psi_0)\\ &< \left(\frac{p+2}{2}\right)\mathcal{E}(\Psi_0).\end{aligned}$$

Then from (3.6) we have that $\mathcal{H}_1(\Psi_0) > 0$ and the conclusion follows from the previous lemma. $\square$

4. Existence of periodic travelling waves via the Arzela-Ascoli Theorem

In this section we will establish the existence of periodic travelling waves for the 1D-Boussinesq system with $a = c \geq 0$, $b, d > 0$ and wave speed ω satisfying $0 < |\omega| < \omega_0$, where $\omega_0 = \min\left\{1, \frac{d}{a}, \frac{b}{a}\right\}$ for $a \neq 0$ and $\omega_0 = 1$ for $a = 0$. We will see that periodic travelling waves are characterized as critical points of some functional, for which the existence of critical points follows as a consequence of the Arzela-Ascoli Theorem, the coerciveness of action functional and the fact that the action functional is (sequentially) weakly lower semi-continuous in an appropriate subset. By a travelling wave solution we shall mean a solution (η, Φ) of (1.1) of the form

$$\eta(t,x) = \frac{1}{\epsilon^{1/p}} u\left(\frac{x-\omega t}{\sqrt{\mu}}\right), \quad \Phi(t,x) = \frac{\sqrt{\mu}}{\epsilon^{1/p}} v\left(\frac{x-\omega t}{\sqrt{\mu}}\right).$$

It is straightforward to see that the travelling wave profile (u, v) should satisfy the system

$$\left\{\begin{array}{ll} bv'''' - v'' + \omega\left(u' - au'''\right) - \left[u\left(v'\right)^p\right]' & = 0, \\ u - du'' - \omega\left(v' - av'''\right) + \frac{1}{p+1}\left(v'\right)^{p+1} & = 0. \end{array}\right. \tag{4.1}$$

We note that the existence of k-periodic travelling waves for the system (1.1) is a consequence of a variational approach in the sense that periodic solutions (u, v) of the system (4.1) are critical points of the action functional $J_{\omega,k}$ given by

$$J_{\omega,k}(u,v) = I_k(u,v) + G_{1,k}(u,v) + G_{2,k}(u,v),$$

where the functionals I_k, $G_{1,k}$, and $G_{2,k}$ are defined by

$$\begin{aligned} I_k(u,v) &= \int_0^k \left[u^2 + d(u')^2 + (v')^2 + b(v'')^2\right] d\xi, \\ G_{1,k}(u,v) &= -2\omega \int_0^k (uv' + au'v'')\, d\xi, \\ G_{2,k}(u,v) &= \frac{2}{p+1} \int_0^k u(v')^{p+1}\, d\xi, \end{aligned}$$

Hereafter, we will say that weak solutions for (4.1) are critical points of the functional $J_{\omega,k}$. A direct computation shows for $\Sigma_k = I_k + G_{1,k}$ that

$$\begin{aligned} \left\langle J'_{\omega,k}(v), v\right\rangle &= 2I_k(v) + 2G_{1,k}(v) + (p+2)G_{2,k}(v) \\ &= 2\Sigma_k(v) + (p+2)G_{2,k}(v) \\ &= 2J_{\omega,k}(v) + pG2, k(v). \end{aligned} \tag{4.2}$$

Moreover, on any critical point w, we have that

$$J_{\omega,k}(v) = \left(\frac{p}{p+2}\right)\Sigma_k(v), \tag{4.3}$$

$$J_{\omega,k}(v) = -\left(\frac{p}{2}\right)G_{2,k}(v), \tag{4.4}$$

$$\Sigma_k(v) = -\left(\frac{p+2}{2}\right)G_{2,k}(v). \tag{4.5}$$

We see that the appropriate space to look for k-periodic travelling waves is $\mathcal{X}_k := H_k^1 \times \mathcal{V}_k$, where $H_k^1 = H_k^1(\mathbb{R})$ is the space of functions k-periodic $\psi \in L_k^2(\mathbb{R})$ such that $\psi' \in L_k^2(\mathbb{R})$, and the space $\mathcal{V}_k$ as the closure of the $C_k^\infty(\mathbb{R})$ (periodic C^∞ functions of period k) with respect to the norm given by

$$\|\varphi\|_{\mathcal{V}_k}^2 := \int_0^k \left[(\varphi')^2 + (\varphi'')^2\right] d\xi.$$

Note that $\mathcal{V}_k$ and $\mathcal{X}_k$ are Hilbert spaces with inner products given respectively by

$$\begin{aligned}(u,v)_{H^1_k(\mathbb{R})} &= (u,v)_{L^2_k(\mathbb{R})} + (u',v')_{L^2_k(\mathbb{R})},\\ (u,v)_{\mathcal{V}_k} &= (u',v')_{H^1_k(\mathbb{R})},\\ (u,v)_{\mathcal{X}_k} &= (u,v)_{H^1_k(\mathbb{R})} + (u,v)_{\mathcal{V}_k}.\end{aligned}$$

In particular,

$$\|(u,v)\|^2_{\mathcal{X}_k} = \|(u,v)\|^2_{H^1_k(\mathbb{R})} + \|(u,v)\|^2_{\mathcal{V}_k}.$$

It is easy to see that the functionals I_k, $G_{1,k}$ and $G_{2,k}$ are smooth maps from $\mathcal{X}_k$ to $\mathbb{R}$. For instance, if $f \in H^1_k$ has mean zero in $[0,k]$, then for $q \geq 1$ we have that

$$|f(\xi)| \leq C(k)\|f'\|_{L^2_k}, \quad \|f\|_{L^q_k} \leq C(k)\|f\|_{H^1_k}.$$

On the other hand, from Cauchy and Young inequalities, we get that

$$\begin{aligned}|G_{1,k}(u,v)| &\leq (2+a)|\omega| \int_0^k \left(|u|^2 + |u'|^2 + |v'|^2 + |v''|^2\right) d\xi \\ &\leq |\omega| C(k,a)\|(u,v)\|^2_{\mathcal{X}_k},\end{aligned} \tag{4.6}$$

$$\begin{aligned}|G_{2,k}(u,v)| &\leq \frac{2}{p+1}\left(\int_0^k |u|^2\, d\xi\right)^{\frac{1}{2}} \left(\int_0^k |v'|^{2(p+1)}\, d\xi\right)^{\frac{1}{2}} \\ &\leq \frac{2}{p+1}\|(u,v)\|^{p+2}_{\mathcal{X}_k},\end{aligned} \tag{4.7}$$

since $v' \in H^1_k$ has trivially mean zero on $[0,k]$. Now, from (4.6), (4.7) and for $0 < |\omega| < \omega_0$, with $\omega_0 = \min\left\{1, \frac{d}{a}, \frac{b}{a}\right\}$ for $a \neq 0$ and $\omega_0 = 1$ for $a = 0$, we have that

$$\int_0^k \left\{(1-|\omega|)u^2 + (d - a|\omega|)(u')^2 + (1-|\omega|)(v')^2 + (b - a|\omega|)^2 (v'')^2\right\} d\xi \leq \Sigma_k(u,v)$$

and

$$\int_0^k \left\{(1+|\omega|)u^2 + (d + a|\omega|)(u')^2 + (1+|\omega|)(v')^2 + (b + a|\omega|)^2 (v'')^2\right\} d\xi \geq \Sigma_k(u,v),$$

which imply that $\sqrt{\Sigma_k}$ is like a norm in $\mathcal{X}_k$, since there is some positive constant $C_1(\omega,a,b,d) > 1$ such that

$$C_1^{-1}\|(u,v)\|^2_{\mathcal{X}_k} \leq \Sigma_k(u,v) \leq C_1\|(u,v)\|^2_{\mathcal{X}_k}. \tag{4.8}$$

LEMMA 4.1. *Assume that the sequence $(u_n,v_n)_n \subset \mathcal{X}_k$ converges weakly in $\mathcal{X}_k$ to $(u_0,v_0) \in \mathcal{X}_k$. If $(v'_n)_n$ converges uniformly to v'_0 on $[0,k]$, we have that*

$$\liminf_{n\to\infty} J_{c,k}(u_n,v_n) \geq J_{c,k}(u_0,v_0). \tag{4.9}$$

PROOF. Recall that $J_{c,k} = \Sigma_k + G_{2,k}$. Now, since Σ_k is like a norm in $\mathcal{X}_k$, so is convex. More exactly, for $\lambda \in (0,1)$ and $v,w \in \mathcal{X}_k$ we have that

$$\Sigma_k(u_n,v_n) \geq \Sigma_k(\lambda(u_0,v_0)) + \Sigma'_k(\lambda(u_0,v_0))(u_n - \lambda u_0, v_n - \lambda v_0). \tag{4.10}$$

On the other hand, we have that

$$\Sigma_k'(u,v)((z,w)) \\ = 2\int_0^k \left(uz + du'z' + v'w' + bv''w'' - \omega(v'z+uw') - a(u'w''+v''z')\right)\,d\xi.$$

From previous remark, we have that

$$\begin{aligned}&\Sigma_k'(\lambda(u_0,v_0))((u_n-\lambda u_0, v_n-\lambda v_0))\\ &= 2\lambda\int_0^k \Big[u_0(u_n-\lambda u_0) + du_0'(u_n'-\lambda u_0') + v'(v_n'-\lambda v_0') + bv''(v_n''-\lambda v_0'')\\ &- \omega\left(v_0'(u_n-\lambda u_0) + u_0(v_n'-\lambda v_0')\right) - a(u_0'(v_n''-\lambda v_0'') + v_0''(u_n'-\lambda u_0'))\Big]d\xi.\end{aligned}$$

Note using that the sequences $(u_n)_n$ and $(v_n')_n$ converge weakly in H_k^1 to u_0 and v_0' in H_k^1, we conclude have that

$$\lim_{n\to\infty}\Sigma_k'(\lambda(u_0,v_0))((u_n-\lambda u_0, v_n-\lambda v_0)) = 2\lambda(1-\lambda)\Sigma_k(u_0,v_0).$$

In other words, we have that

$$\liminf_{n\to\infty}\Sigma_k(u_n,v_n) \ge \Sigma_k(\lambda(u_0,v_0)) + 2\lambda(1-\lambda)\Sigma_k(u_0,v_0) = \lambda(2-\lambda)\Sigma_k(u_0,v_0),$$

which implies after taking $\lambda\to 1^-$ that

$$\liminf_{n\to\infty}\Sigma_k(u_n,v_n) \ge \Sigma_k(u_0,v_0). \tag{4.11}$$

Now, we need to observe that

$$G_{2,k}(u_n,v_n) = \frac{2}{p+1}\left(\int_0^k u_n\left((v_n')^{p+1} - (v_0')^{p+1}\right)d\xi + \int_0^k u_n\,(v_0')^{p+1}\,d\xi\right).$$

Since we know that $(v_n')^{p+1}, (v_0')^{p+1} \in L^2\,[0,k]$, we conclude that

$$\lim_{n\to\infty}\int_0^k u_n\,(v_0')^{p+1}\,d\xi = \int_0^k u_0\,(v_0')^{p+1}\,d\xi.$$

Moreover, using the uniform convergence of $(v_n')_n$ to v_0' we also have that

$$\begin{aligned}&\left|\int_0^k u_n\left((v_n')^{p+1} - (v_0')^{p+1}\right)d\xi\right|\\ &\le C(p)\int_0^k |u_n|\,(|v_n'|+|v_0'|)^p\,|v_n'-v_0'|\,d\xi\\ &\le C_1(p)\sup_{[0,k]}|v_n'-v_0'|\|u_n\|_{L^2}\left(\|v_n'\|_{L^{2p}}^p + \|v_0'\|_{L^{2p}}^p\right)\\ &\le C_1(p)\sup_{[0,k]}|v_n'-v_0'|\|u_n\|_{H_k^1}\left(\|v_n\|_{\mathcal{V}_k}^p + \|v_0\|_{\mathcal{V}_k}^p\right).\end{aligned}$$

which means, after recalling that the sequence $(u_n,v_n)_n$ is bounded, that

$$\lim_{n\to\infty}G_{2,k}(u_n,v_n) = \frac{2}{p+1}\int_0^k u_0\,(v_0')^{p+1}\,d\xi = G_{2,k}(u_0,v_0).$$

As a consequence of previous remarks, we conclude that

$$\liminf_{n\to\infty}J_{\omega,k}(u_n,v_n) = \liminf_{n\to\infty}\left(\Sigma_k(u_n,v_n) + G_{2,k}(u_n,v_n)\right) \ge J_{\omega,k}(u_0,v_0).$$

□

We must recall that H_k^1 is the space of absolutely continuous functions u which are k-periodic and such that $u' \in L^2_{loc}(\mathbb{R})$. For $\alpha > 0$, we consider the weakly closed subset of $\mathcal{X}_k$

$$\mathcal{X}_{k,\alpha} = \{(\psi, \varphi) \in \mathcal{X}_k : |\varphi'(\xi)| \leq \alpha, \quad \text{a. e. } \xi \in \mathbb{R}\}$$

LEMMA 4.2. *1.- There are positive constants C_1 and C_2 such that for any $\Phi \in \mathcal{X}_k$, we have that*

$$J_{\omega,k}(\Phi) \geq C_1 \|\Phi\|^2_{\mathcal{X}_k} - C_2 \|\Phi\|^{p+2}_{\mathcal{X}_k}. \tag{4.12}$$

2.- There exits $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$ the functional $J_{\omega,k}$ is coercive on $\mathcal{X}_{k,\alpha}$. More exactly, there is $C_3 > 0$ such that for $\Phi \in \mathcal{X}_{k,\alpha}$,

$$J_{\omega,k}(\Phi) \geq C_3 \|\Phi\|^2_{\mathcal{X}_k}. \tag{4.13}$$

PROOF. 1.- From inequalities (4.7) and (4.8), there are positive constants C_1 and C_2 such that

$$J_{\omega,k}(\Phi) = \Sigma_k(\Phi) + G_{2,k}(\Phi) \geq C_1^{-1} \|\Phi\|^2_{\mathcal{X}_k} - C_2 \|\Phi\|^{p+2}_{\mathcal{X}_k}.$$

2.- Let $(\psi, \varphi) \in \mathcal{X}_{k,\alpha}$. Then $|\varphi'(\xi)| \leq \alpha$ for a. e. $\xi \in \mathbb{R}$.

$$|G_{2,k}(\psi, \varphi)| \leq \frac{2}{p+1} \int_0^k |\psi|\, |\varphi'|^{p+1}\, d\xi \leq \frac{2\alpha^p}{p+1} \int_0^k |\psi|\, |\varphi'|\, d\xi \leq \alpha^p C(p) \|(\psi, \varphi)\|^2_{\mathcal{X}_k}.$$

So, using inequality (4.8) and previous one, we have that

$$J_{\omega,k}(\psi, \varphi) \geq \frac{1}{C_1} \|(\psi, \varphi)\|^2_{\mathcal{X}_k} - \alpha^p C(p) \|(\psi, \varphi)\|^2_{\mathcal{X}_k} = \left(\frac{1}{C_1} - \alpha^p C(p) \right) \|(\psi, \varphi)\|^2_{\mathcal{X}_k},$$

as desired. □

Our goal now is to show the existence of a non trivial critical point for $J_{\omega,k}$. The result will be a direct consequence of the coerciveness of $J_{\omega,k}$ and that $J_{\omega,k}$ is (sequentially) weakly lower semi-continuous on $\mathcal{X}_{k,\alpha}$ for $0 < \alpha < \alpha_0$. We will use the following result,

THEOREM 4.1. *([**23**]). Let X be a Hilbert space and let $M \subset X$ be a weakly closed subset of X. Suppose that $E : M \to \mathbb{R} \cup \{+\infty\}$ is coercive and that is (sequentially) weakly lower semi-continuous on M with respect to X, that is, suppose the following conditions are fulfilled:*

1. *$E(u) \to \infty$, as $\|u\| \to \infty$, with $u \in M$.*
2. *For any $u \in M$, any sequence $(u_n)_n$ in M such that $u_n \rightharpoonup u$ (weakly) in X there holds:*

$$E(u) \leq \liminf_{n \to \infty} E(u_n).$$

Then E is bounded below on M and attains its minimum in M.

Now, we are ready to establish the existence of a periodic travelling characterize as a critical point of the functional $J_{\omega,k}$.

THEOREM 4.2. *For $0 < \alpha < \alpha_0$, $J_{\omega,k}$ has a minimum over $\mathcal{X}_{k,\alpha}$.*

PROOF. We will verify that $J_{\omega,k}$ satisfies the hypotheses in previous Theorem. Now, it is straightforward to check that $\mathcal{X}_{k,\alpha}$ is weakly closed subset of $\mathcal{X}_k$. In fact, let $(\psi, \varphi)_n \subset \mathcal{X}_{k,\alpha}$ be a sequence that converges weakly to (ψ_0, φ_0). Then we have that the sequence $(\psi,\varphi)_n$ is bounded in $\mathcal{X}_k$. Now, since $\varphi_n' \in H_k^1$ has mean zero on $[0,k]$, we know that

$$|\varphi_n'(x)-\varphi_n'(y)| \leq \int_y^x |\varphi_n''(r)|\,dr \leq |x-y|^{\frac{1}{2}}\|(\psi_n,\varphi_n)\|_{\mathcal{X}_k} \leq M|x-y|^{\frac{1}{2}}.$$

From Arzela-Ascoli Theorem we have for some subsequence (denoted equal) that $(\varphi_n')_n$ converges uniformly to φ_0' on $[0,k]$, since we have that $|\varphi_n'(\xi)| \leq \alpha$ for a. e. $\xi \in \mathbb{R}$ and for all $n \in \mathbb{N}$. From this fact and the uniform convergence of $(\varphi_n')_n$, we conclude that $|\varphi_0'(\xi)| \leq \alpha$ for a. e. $\xi \in \mathbb{R}$. In other words, $\varphi_0 \in \mathcal{X}_{k,\alpha}$, meaning that $\mathcal{X}_{k,\alpha}$ is weakly closed subset of $\mathcal{X}_k$. Now note that the coerciveness property of $J_{\omega,k}$ and condition (1) are obtained using the inequality (4.13) in previous lemma. We need now to verify condition (2). Let $(\psi_0,\varphi_0) \in \mathcal{X}_{k,\alpha}$ and let $(\psi,\varphi)_n \subset \mathcal{X}_k$ such that $(\psi,\varphi)_n \rightharpoonup (\psi_0,\varphi_0)$ (weakly) in $\mathcal{X}_{k,\alpha}$. This sequence $(\psi,\varphi)_n$ is bounded $\mathcal{X}_k$ and the same type of arguments show that $(\varphi_n')_n$ converges uniformly to φ_0' on $[0,k]$ (up to a subsequence), so by Lemma 4.1 we conclude that

$$\liminf_{n\to\infty} J_{\omega,k}(\psi_n,\varphi_n) \geq J_{\omega,k}(\psi_0,\varphi_0).$$

On the other hand, by Lemma 4.2 part (2), we have condition (2) in Theorem 4.1. Then, from Theorem 4.1 we conclude that $J_{\omega,k}$ attains a minimum over $\mathcal{X}_{k,\alpha}$. □

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