Spatial decay of rotating waves in reaction diffusion systems

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ABSTRACT. In this paper we study nonlinear problems for Ornstein-Uhlenbeck operators

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2,$$

where the matrix $A \in \mathbb{R}^{N,N}$ is diagonalizable and has eigenvalues with positive real part, the map $f : \mathbb{R}^N \to \mathbb{R}^N$ is sufficiently smooth and the matrix $S \in \mathbb{R}^{d,d}$ in the unbounded drift term is skew-symmetric. Nonlinear problems of this form appear as stationary equations for rotating waves in time-dependent reaction diffusion systems. We prove under appropriate conditions that every bounded classical solution $v_\star$ of the nonlinear problem, which falls below a certain threshold at infinity, already decays exponentially in space, in the sense that $v_\star$ belongs to an exponentially weighted Sobolev space $W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$. Several extensions of this basic result are presented: to complex-valued systems, to exponential decay in higher order Sobolev spaces and to pointwise estimates. We also prove that every bounded classical solution v of the eigenvalue problem

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2,$$

decays exponentially in space, provided $\operatorname{Re}\lambda$ lies to the right of the essential spectrum. As an application we analyze spinning soliton solutions which occur in the Ginzburg-Landau equation. Our results form the basis for investigating nonlinear stability of rotating waves in higher space dimensions and truncations to bounded domains.

CONTENTS

1. Introduction

In the present paper we study systems of reaction-diffusion equations

$$
\text{(1.1)} \qquad
\begin{aligned}
u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)), \ t > 0, \ x \in \mathbb{R}^d, \ d \geqslant 2, \\
u(x,0) &= u_0(x) \qquad\qquad\qquad\quad , \ t = 0, \ x \in \mathbb{R}^d,
\end{aligned}
$$

where $A \in \mathbb{R}^{N,N}$ is a diffusion matrix, $f : \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth nonlinearity, $u_0 : \mathbb{R}^d \to \mathbb{R}^N$ are the initial data and $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^N$ denotes a vector-valued solution.

We are mainly interested in rotating wave solutions of (1.1) which are of the form

$$
\text{(1.2)} \qquad u_\star(x,t) = v_\star(e^{-tS}x), \ t \geqslant 0, \ x \in \mathbb{R}^d, \ d \geqslant 2
$$

with space-dependent profile $v_\star : \mathbb{R}^d \to \mathbb{R}^N$ and skew-symmetric matrix $S \in \mathbb{R}^{d,d}$. The skew-symmetry of S implies that e^{-tS} describes a rotation in $\mathbb{R}^d$, and hence $u_\star$ is a solution rotating at constant velocity while maintaining its shape determined by $v_\star$. The profile $v_\star$ is called (exponentially) localized, if it tends (exponentially) to some constant vector $v_\infty \in \mathbb{R}^N$ as $|x| \to \infty$.

Transforming (1.1) via $u(x,t) = v(e^{-tS}x,t)$ into a co-rotating frame yields the evolution equation

$$
\text{(1.3)} \qquad
\begin{aligned}
v_t(x,t) &= A\triangle v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)), \ t > 0, \ x \in \mathbb{R}^d, \ d \geqslant 2, \\
v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\qquad\quad , \ t = 0, \ x \in \mathbb{R}^d.
\end{aligned}
$$

The diffusion and drift term are given by

$$
\text{(1.4)} \qquad A\triangle v(x) := A\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}v(x) \quad \text{and} \quad \langle Sx, \nabla v(x)\rangle := \sum_{i=1}^d \sum_{j=1}^d S_{ij}x_j D_i v(x).
$$

The pattern $v_\star$ itself appears as a stationary solution of (1.3), i.e. $v_\star$ solves the steady state problem

$$
\text{(1.5)} \qquad A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d, \ d \geqslant 2.
$$

We may write (1.5) as $[\mathcal{L}_0 v_\star](x) + f(v_\star(x)) = 0$ by introducing the Ornstein-Uhlenbeck operator

$$
\text{(1.6)} \qquad [\mathcal{L}_0 v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \ x \in \mathbb{R}^d.
$$

By the skew-symmetry of S we can write the drift term in terms of angular derivatives as follows

$$
\text{(1.7)} \qquad \langle Sx, \nabla v(x)\rangle = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij}\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)v(x).
$$

The aim of this paper is to derive suitable conditions guaranteeing that every localized rotating wave of (1.1) is already exponentially localized. More precisely, the main theorem states the following: if the difference $v_\star - v_\infty$ of a rotating wave

to its far field value falls below a certain threshold at infinity, then it decays exponentially in space. The decay is specified by showing that $v_\star - v_\infty$ belongs to some exponentially weighted Sobolev space $W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$, $1 < p < \infty$. Our key assumption requires all eigenvalues of the Jacobian $Df(v_\infty)$ to have negative real part.

We extend this result to complex-valued systems and then apply it to prove exponential decay of localized spinning solitons arising in the cubic-quintic complex Ginzburg-Landau equation (QCGL), [**14**]. Figure 1(a) shows the real part of a spinning soliton $v_\star$ in two space dimensions, while Figure 1(b) shows the isosurfaces of the real part of a spinning soliton in three space dimensions. Both of these rotating waves are exponentially localized, as our results will show. Two nonlocalized rotating waves are illustrated in Figure 1(c)-(d). Figure 1(c) shows the real part of a spiral wave in two space dimensions and Figure 1(d) the isosurfaces of the real part of an untwisted scroll wave. In Section 6 below we will discuss this example in more detail.
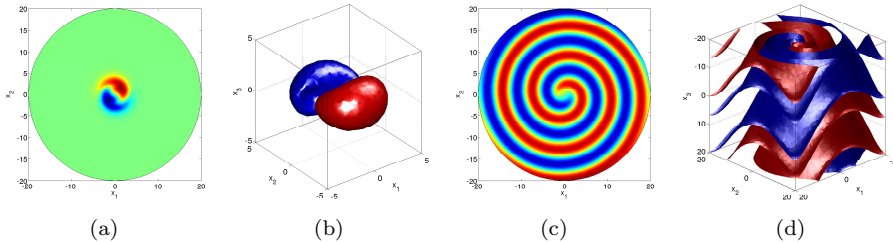

(a)   (b)   (c)   (d)

FIGURE 1. Rotating waves of QCGL (6.1). (a) Spinning solitons for $d = 2$ with colorbar reaching from -1.6 (blue) to 1.6 (red), (b) spinning soliton for $d = 3$ with isosurfaces at values -0.5 (blue) and 0.5 (red), (c) spiral wave for $d = 2$ with colorbar reaching from -1.7 (blue) to 1.7 (red), and (d) scroll wave for $d = 3$ with isosurfaces at values -0.5 (blue) and 0.5 (red)

An important issue is to investigate *nonlinear stability of rotating waves* (more precisely, *stability with asymptotic phase*) in reaction diffusion systems, see [**5**]. A well known task is to derive nonlinear stability from linear stability of the linearized operator

(1.8) $$[\mathcal{L}v](x) := [\mathcal{L}_0 v](x) + Df(v_\star(x))v(x), \ x \in \mathbb{R}^d.$$

By *linear stability* (also called *strong spectral stability*) we mean that the essential spectrum and the isolated eigenvalues of $\mathcal{L}$ lie strictly to the left of the imaginary axis, except for those on the imaginary axis caused by Euclidean equivariance, [**26**, Ch.9]. This requires to study isolated eigenvalues $\lambda \in \mathbb{C}$ of the problem

(1.9) $$[(\lambda I - \mathcal{L})v](x) = 0, \ x \in \mathbb{R}^d.$$

A further aim of this paper is to prove that every bounded eigenfunction v of the linearized operator $\mathcal{L}$ decays exponentially in space, provided the real parts of the associated (isolated) eigenvalues λ lie to the right of the essential spectrum. To be more precise, we show that for such values of λ, every bounded classical solution v of the eigenvalue problem (1.9) belongs to some exponentially weighted

Sobolev space $W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$ for some $1 < p < \infty$. In particular, we prove that the eigenfunction $v(x) = \langle Sx, \nabla v_\star(x) \rangle$ associated to the eigenvalue $\lambda = 0$ decays exponentially in space.

A nonlinear stability result for two dimensional localized rotating patterns was proved by Beyn and Lorenz in [5]. Their proof requires three essential assumptions: The matrix $Df(v_\infty)$ is stable, meaning that all its eigenvalues have a negative real part. Moreover, strong spectral stability in the sense above is assumed. And finally, the profile $v_\star$ of the rotating wave and its derivatives up to order 2 decay to zero at infinity. Their analysis shows that the decay of the rotating wave itself and the spectrum of the linearization are both crucial for investigating nonlinear stability. A corresponding result on nonlinear stability of nonlocalized rotating waves, such as spiral waves and scroll waves, is still an open problem. The difficulty is related to the fact that the essential spectrum touches the imaginary axis at infinitely many points. The spectrum of the linearization at (nonlocalized) spiral waves is well-known and has been extensively studied by Sandstede, Scheel and Fiedler in [16, 31, 32].

For numerical computations it is essential to truncate the equations (1.1), (1.3) and (1.9) to a bounded domain, so that standard approximations, e.g. with finite elements, apply. The truncation error arising in this process, depends on the boundary conditions. Assuming that a rotating wave is (exponentially) localized, we can expect the truncation error to be (exponentially) small as well. For this reason, the *exponential decay of rotating waves* plays a fundamental role when estimating errors caused by *approximations of rotating waves on bounded domains*.

We consider our results on the decay of rotating waves for (1.1) on the whole $\mathbb{R}^d$ as a first step in studying such truncation errors. Despite numerous numerical simulations of spiral behavior on bounded domains, a rigorous analysis of the errors caused by spatial truncation seems not to be available.

We emphasize that the results from Section 3-6 are extensions of the results from the PhD thesis [26]. One major improvement refers to the fact that our main result Theorem 2.8 avoids the additional assumption $v_\star \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ from [26, Thm.1.8] by using ideas from the work [6].

2. Assumptions and main result

2.1. Assumptions and main result. Consider the steady state problem

$$(2.1) \qquad A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2,$$

with *diffusion matrix* $A \in \mathbb{K}^{N,N}$ and a function $f : \mathbb{K}^N \to \mathbb{K}^N$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Recall the Ornstein-Uhlenbeck operator from (1.6) with drift and diffusion term specified in (1.4).

We define a rotating wave $u_\star$ as follows:

DEFINITION 2.1. A function $u_\star : \mathbb{R}^d \times [0, \infty) \to \mathbb{K}^N$ is called a *rotating wave* (or *rotating pattern*) if it has the form

$$(2.2) \qquad u_\star(x,t) = v_\star(e^{-tS}(x - x_\star)),\ x \in \mathbb{R}^d,\ t \in [0, \infty),$$

with *profile* (or *pattern*) $v_\star : \mathbb{R}^d \to \mathbb{K}^N$, a skew-symmetric matrix $0 \neq S \in \mathbb{R}^{d,d}$ and $x_\star \in \mathbb{R}^d$. A rotating wave $u_\star$ is called *localized* (*exponentially localized with decay*

rate η) if it satisfies

(2.3)
$$\lim_{|x| \to \infty} e^{\eta|x|} |v_\star(x) - v_\infty| = 0 \text{ for some } v_\infty \in \mathbb{K}^N$$

and for $\eta = 0$ ($\eta > 0$). It is called *nonlocalized*, if it is not localized in the sense above.

The vector $x_\star \in \mathbb{R}^d$ can be considered as the center of rotation for $d = 2$ and as the support vector of the axis of rotation for $d = 3$. In case $d \in \{2, 3\}$, S can be considered as the angular velocity tensor associated to the angular velocity vector $\omega \in \mathbb{R}^{\frac{d(d-1)}{2}}$ containing S_{ij}, $i = 1, \ldots, d - 1$, $j = i + 1, \ldots, d$. Some examples of rotating patterns are illustrated in Figure 1 and will be treated in Section 6 below.

In the following we will impose various restrictions on the matrix A:

ASSUMPTION 2.2. For $A \in \mathbb{K}^{N,N}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $1 < p < \infty$ consider the conditions

(A1) A is diagonalizable (over $\mathbb{C}$),

(A2) $\text{Re}\,\sigma(A) > 0$,

(A3) $\text{Re}\,\langle w, Aw \rangle \geqslant \beta_A \; \forall\, w \in \mathbb{K}^N$, $|w| = 1$ for some $\beta_A > 0$,

(A4) There exists $\gamma_A > 0$ such that for all $z, w \in \mathbb{K}^N$

$$|z|^2 \text{Re}\,\langle w, Aw \rangle + (p-2)\text{Re}\,\langle w, z \rangle\,\text{Re}\,\langle z, Aw \rangle \geqslant \gamma_A |z|^2 |w|^2,$$

(A5) A is invertible and $\mu_1(A) > \dfrac{|p-2|}{p}$,

(to be read as $A = a > 0$ in case $N = 1$, $\mathbb{K} = \mathbb{R}$).

Assumption (A1) is a *system condition* and ensures that all results for scalar equations can be extended to system cases. This condition is independent of (A2)-(A5) and is used in [26, 27] to derive an explicit formula for the heat kernel of $\mathcal{L}_0$. A typical case where (A1) holds, is a scalar complex-valued equation when transformed into a real-valued system of dimension 2 . The *positivity condition* (A2) guarantees that the diffusion part $A\triangle$ is an elliptic operator. All eigenvalues $\lambda \in \sigma(A)$ of A lie in the open right half-plane $\{\lambda \in \mathbb{C} \mid \text{Re}\,\lambda > 0\}$. Condition (A2) guarantees that A^{-1} exists and that $-A$ is a stable matrix. The *strict accretivity condition* (A3) is more restrictive than (A2). In (A3) we use $\langle u, v \rangle := \overline{u}^T v$ to denote the standard inner product on $\mathbb{K}^N$. Recall that condition (A2) is satisfied iff there exists an inner product $[\cdot, \cdot]$ and some $\beta_A > 0$ such that $\text{Re}\,[w, Aw] \geqslant \beta_A$ forall $w \in \mathbb{K}^N$ with $[w, w] = 1$. Condition (A3) ensures that the differential operator $\mathcal{L}_0$ is closed on its (local) domain $\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$, see Theorem 2.12 below. The L^p-*dissipativity condition* (A4) is more restrictive than (A3) and imposes additional requirements on the spectrum of A. This condition, which comes originally from [11, 12], is used in [26, 29] to prove L^p-resolvent estimates for $\mathcal{L}_0$. A geometrical meaning of (A4) can be given in terms of the antieigenvalues of the diffusion matrix A. In [26, 28], it is proved that condition (A4) is equivalent to the L^p-*antieigenvalue condition* (A5). Condition (A5) requires that the *first antieigenvalue of A* (see [17, 18]), defined by

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{K}^N \\ w \neq 0 \\ Aw \neq 0}} \frac{\text{Re}\,\langle w, Aw \rangle}{|w||Aw|} = \inf_{\substack{w \in \mathbb{K}^N \\ |w|=1 \\ Aw \neq 0}} \frac{\text{Re}\,\langle w, Aw \rangle}{|Aw|},$$

is bounded from below by a non-negative p-dependent constant. Condition (A5) is also equivalent to the following p-dependent upper bound for the (*real*) *angle of A* (cf. [**17**]),

$$\Phi_{\mathbb{R}}(A) := \cos^{-1}(\mu_1(A)) < \cos^{-1}\left(\frac{|p-2|}{p}\right) \in \left(0, \frac{\pi}{2}\right], \quad 1 < p < \infty.$$

Therefore, the first antieigenvalue $\mu_1(A)$ can be considered as the cosine of the maximal (real) turning angle of vectors mapped by the matrix A. Some special cases in which the first antieigenvalue can be given explicitly are treated in [**28**]. We summarize the relationship of (A2)–(A5):

$$(2.4) \qquad A \text{ invertible} \Longleftarrow (A2) \Longleftarrow (A3) \Longleftarrow (A4) \Longleftrightarrow (A5).$$

We continue with the *rotational condition* (A6) and a *smoothness condition* (A7),

ASSUMPTION 2.3. The matrix $S \in \mathbb{R}^{d,d}$ satisfies

(A6) S is skew-symmetric, i.e. $S = -S^T$.

ASSUMPTION 2.4. The function $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies

(A7) $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$.

Later on we apply our results to complex-valued nonlinearities of the form

$$(2.5) \qquad f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u,$$

where $g : \mathbb{R} \to \mathbb{C}^{N,N}$ is a sufficiently smooth function. Such nonlinearities arise for example in Ginzburg-Landau equations, Schrödinger equations, $\lambda - \omega$ systems and many other equations from physical sciences, see Section 6. Note, that in this case, the function f is not holomorphic in $\mathbb{C}$, but its real-valued version in $\mathbb{R}^2$ satisfies (A7) if $g \in C^2$. For differentiable functions $f : \mathbb{R}^N \to \mathbb{R}^N$ we denote by Df the Jacobian matrix in the real sense.

ASSUMPTION 2.5. For $v_\infty \in \mathbb{R}^N$ consider the following conditions:

(A8) $f(v_\infty) = 0$,

(A9) $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ are simultaneously diagonalizable (over $\mathbb{C}$),

(A10) $\operatorname{Re}\sigma(Df(v_\infty)) < 0$,

(A11) There exists $\beta_\infty > 0$ such that for all $w \in \mathbb{K}^N$ with $|w| = 1$
$$\operatorname{Re}\langle w, -Df(v_\infty)w \rangle \geqslant \beta_\infty.$$

The *constant asymptotic state condition* (A8) requires v_∞ to be a steady state of the nonlinear equation. The *system condition* (A9) is an extension of Assumption (A1), and the *coercivity condition* (A11) is again more restrictive than the *spectral condition* (A10).

DEFINITION 2.6. A function $v_\star : \mathbb{R}^d \to \mathbb{K}^N$ is called a *classical solution of* (2.1) if

$$(2.6) \qquad v_\star \in C^2(\mathbb{R}^d, \mathbb{K}^N)$$

and $v_\star$ solves (2.1) pointwise.

Later on, we will consider classical solutions $v_\star$ which are even bounded, i.e. $v_\star \in C_b(\mathbb{R}^d, \mathbb{K}^N)$. For matrices $C \in \mathbb{K}^{N,N}$ with spectrum $\sigma(C)$ we denote by $\rho(C) := \max_{\lambda \in \sigma(C)} |\lambda|$ its *spectral radius* and by $s(C) := \max_{\lambda \in \sigma(C)} \operatorname{Re} \lambda$ its *spectral abscissa* (or *spectral bound*). With this notation, we define the following constants which appear in the linear theory from [**26**, **27**, **29**]:

$$a_{\min} := \left(\rho\left(A^{-1}\right)\right)^{-1}, \quad a_{\max} := \rho(A), \qquad a_0 := -s(-A),$$

(2.7)
$$a_1 := \left(\frac{a_{\max}^2}{a_{\min} a_0}\right)^{\frac{d}{2}}, \quad b_0 := -s(Df(v_\infty)).$$

Recall the relations $0 < a_0 \leq \beta_A$ and $0 < b_0 \leq \beta_\infty$ to the coercivity constants from (A3),(A11). Our main tool for investigating exponential decay in space are exponentially weighted function spaces. For the choice of weight function we follow [**40**, Def.3.1]:

DEFINITION 2.7. (**1**) A function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a *weight function of exponential growth rate* $\eta \geqslant 0$ provided that

(W1) $\theta(x) > 0 \ \forall \, x \in \mathbb{R}^d$,

(W2) $\exists \, C_\theta > 0 : \ \theta(x + y) \leqslant C_\theta \theta(x) e^{\eta |y|} \ \forall \, x, y \in \mathbb{R}^d$.

(**2**) A weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called *radial* if

(W3) $\exists \, \phi : [0, \infty) \to \mathbb{R} : \ \theta(x) = \phi(|x|) \ \forall \, x \in \mathbb{R}^d$.

(**3**) A radial weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called *nondecreasing* (or *monotonically increasing*) provided that

(W4) $\theta(x) \leqslant \theta(y) \ \forall \, x, y \in \mathbb{R}^d$ with $|x| \leqslant |y|$.

Standard examples of radial weight functions are

$$\theta_1(x) = \exp\left(\mu |x|\right) \quad \text{and} \quad \theta_2(x) = \cosh\left(\mu |x|\right),$$

as well as their smooth analogs

$$\theta_3(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right) \quad \text{and} \quad \theta_4(x) = \cosh\left(\mu \sqrt{|x|^2 + 1}\right),$$

for $x \in \mathbb{R}^d$ and $\mu \in \mathbb{R}$. Obviously, all these functions are radial weight functions of exponential growth rate $\eta = |\mu|$ with $C_\theta = 1$. Moreover, θ_2, θ_4 are nondecreasing for any $\mu \in \mathbb{R}$ and θ_1, θ_3 if $\mu \geqslant 0$.

With every weight function of exponential growth rate we associate *exponentially weighted Lebesgue* and *Sobolev spaces*

$$L_\theta^p(\mathbb{R}^d, \mathbb{K}^N) := \{u \in L_{\text{loc}}^1(\mathbb{R}^d, \mathbb{K}^N) \mid \|\theta u\|_{L^p} < \infty\},$$

$$W_\theta^{k,p}(\mathbb{R}^d, \mathbb{K}^N) := \{u \in L_\theta^p(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in L_\theta^p(\mathbb{R}^d, \mathbb{K}^N) \ \forall \, |\beta| \leqslant k\},$$

for every $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}_0$.

With these preparations we can formulate the main result of our paper.

THEOREM 2.8 (Exponential decay of $v_\star$). *Let the assumptions* (A4), (A6)–(A9) *and* (A11) *be satisfied for* $\mathbb{K} = \mathbb{R}$ *and for some* $1 < p < \infty$. *Moreover, let* $a_{\max} = \rho(A)$ *denote the spectral radius of* A, $-a_0 = s(-A)$ *the spectral bound of* $-A$ *and* $-b_0 = s(Df(v_\infty))$ *the spectral bound of* $Df(v_\infty)$. *Further, let*

$\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ *denote a weight function for* $\mu \in \mathbb{R}$. *Then, for every* $0 < \varepsilon < 1$ *there is a constant* $K_1 = K_1(A, f, v_\infty, d, p, \varepsilon) > 0$ *with the following property: Every classical solution* $v_\star$ *of*

$$(2.8) \qquad A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0, \; x \in \mathbb{R}^d,$$

such that

$$(2.9) \qquad \sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0,$$

satisfies

$$v_\star - v_\infty \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$$

for every exponential decay rate

$$(2.10) \qquad 0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.$$

Roughly speaking, Theorem 2.8 states that every bounded classical solution $v_\star$ which is sufficiently close to the steady state v_∞ at infinity, see (2.9), must decay exponentially in space. The exponential decay is expressed by the fact, that $v_\star - v_\infty$ belongs to an exponentially weighted Sobolev space. Moreover, the theorem gives an explicit bound for the exponential growth rate, that depends only on p, the spectral radius of A, and the spectral abscissas of $-A$ and $Df(v_\infty)$. The role of ε becomes clear upon noting that $K_1 \to 0$ as $\varepsilon \to 1$ whereas $K_1 \to K_1^0 > 0$ as $\varepsilon \to 0$. The stronger the exponential rate, the closer the solution $v_\star$ has to approach v_∞ at infinity.

2.2. Outline of proof: Decomposition of linear differential operators. In the following we explain the decomposition of differential operators that leads to the proof of Theorem 2.8.

Far-Field Linearization. Consider the nonlinear problem

$$(2.11) \qquad A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0, \; x \in \mathbb{R}^d, \; d \geqslant 2.$$

Let $v_\infty \in \mathbb{R}^N$ be the constant asymptotic state satisfying (A8) and let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. By the Mean Value Theorem we can write

$$f(v_\star(x)) = \underbrace{f(v_\infty)}_{=0} + \underbrace{\int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt(v_\star(x) - v_\infty)}_{=:a(x)}, \; x \in \mathbb{R}^d.$$

From $v_\star \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^N)$ we deduce $a \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$. Moreover, since the classical solution $v_\star$ solves (2.11) pointwise and $v_\infty \in \mathbb{R}^N$ is constant, the difference $w_\star := v_\star - v_\infty$ belongs to $C^2(\mathbb{R}^d, \mathbb{R}^N) \cap C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^N)$ and satisfies the linearized equation

$$(2.12) \qquad [\mathcal{L}w_\star](x) = A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + a(x)w_\star(x) = 0, \; x \in \mathbb{R}^d.$$

In order to study the behavior of solutions to (2.11) as $|x| \to \infty$, we decompose the variable coefficient $a(x)$ in (2.12).

Decomposition of a. Let $a(x) = Df(v_\infty) + Q(x)$ with Q defined by

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)\, dt, \; x \in \mathbb{R}^d.$$

This yields $Q \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ and (2.12) reads as

$$(2.13) \qquad A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + (Df(v_\infty) + Q(x))\, w_\star(x) = 0,\ x \in \mathbb{R}^d.$$

Decomposition of Q. Let $Q(x) = Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)$, where $Q_{\mathrm{s}} \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ is small w.r.t. $\|\cdot\|_\infty$ and $Q_{\mathrm{c}} \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N})$ is compactly supported on $\mathbb{R}^d$, see Figure 2. Then, we arrive at

$$(2.14) \qquad A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + (Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x))\, w_\star(x) = 0$$

for $x \in \mathbb{R}^d$. If we omit the term $Q_{\mathrm{s}} + Q_{\mathrm{c}}$ in (2.14), the equation (2.14) is called the *far-field linearization*.
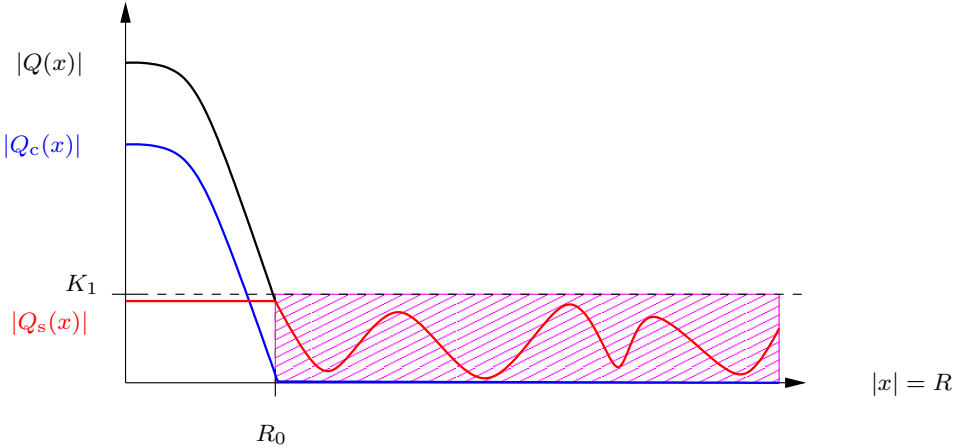

FIGURE 2. Decomposition of Q with data R_0 and K_1 from Theorem 2.8

Perturbations of Ornstein-Uhlenbeck operator. In order to show exponential decay for the solution $v_\star$ of the nonlinear steady state problem (2.11), it is sufficient to analyze solutions of the linear system (2.14). Abbreviating $B_\infty := -Df(v_\infty)$, we will study the following linear differential operators:

$$(2.15) \qquad \begin{aligned}
\mathcal{L}_{\mathrm{c}} v &= A\triangle v + \langle Sx, \nabla v\rangle - B_\infty v + Q_{\mathrm{s}}(x)v + Q_{\mathrm{c}}(x)v, \\
\mathcal{L}_{\mathrm{s}} v &= A\triangle v + \langle Sx, \nabla v\rangle - B_\infty v + Q_{\mathrm{s}}(x)v, \\
\mathcal{L}_\infty v &= A\triangle v + \langle Sx, \nabla v\rangle - B_\infty v, \\
\mathcal{L}_0 v &= A\triangle v + \langle Sx, \nabla v\rangle.
\end{aligned}$$

Recall that the drift term $\langle Sx, \nabla v(x)\rangle,\, x \in \mathbb{R}^d$, in the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ has unbounded coefficients and cannot be considered as a lower order term. Later on, it will be convenient to allow complex coefficients for the operators $\mathcal{L}_0$, $\mathcal{L}_\infty$, $\mathcal{L}_{\mathrm{s}}$ and $\mathcal{L}_{\mathrm{c}}$. Therefore, we rewrite the assumptions (A9)–(A11) as follows:

ASSUMPTION 2.9. For $B_\infty \in \mathbb{K}^{N,N}$ consider the conditions

$(\mathrm{A9}_{B_\infty})$ $A, B_\infty \in \mathbb{K}^{N,N}$ are simultaneously diagonalizable (over $\mathbb{C}$), i.e.

$$\exists Y \in \mathbb{C}^{N,N} \text{ invertible}: \ Y^{-1}AY = \Lambda_A \text{ and } Y^{-1}B_\infty Y = \Lambda_{B_\infty},$$

with $\Lambda_A = \mathrm{diag}(\lambda_1^A, \ldots, \lambda_N^A), \Lambda_{B_\infty} = \mathrm{diag}(\lambda_1^{B_\infty}, \ldots, \lambda_N^{B_\infty}) \in \mathbb{C}^{N,N}$,

$(\mathrm{A10}_{B_\infty})$ $\mathrm{Re}\,\sigma(B_\infty) > 0$,

(A11$_{B_\infty}$) There exists $\beta_\infty > 0$ such that for all $w \in \mathbb{K}^N$ with $|w| = 1$

$$\operatorname{Re} \langle w, B_\infty w \rangle \geqslant \beta_\infty.$$

Similar comments as those following (A9)–(A11) apply. In addition to (2.7), we need the constants

$$(2.16) \qquad b_0 := -s(-B_\infty), \quad \kappa := \operatorname{cond}(Y) \quad \text{(with } Y \text{ from (A9}_{B_\infty})),$$

where $\operatorname{cond}(Y) := \|Y^{-1}\|\|Y\|$ denotes the condition number of Y.

2.3. Constant coefficient perturbations of Ornstein-Uhlenbeck operators.

In the first step we review and collect results from [**26, 27, 29, 28**] for the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ and its constant coefficient perturbation $\mathcal{L}_\infty$.

Assuming (A2), (A6), (A9$_{B_\infty}$) for $\mathbb{K} = \mathbb{C}$ it is shown in [**26**, Thm.4.2-4.4], [**27**, Thm.3.1] that the function $H_\infty : \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty) \to \mathbb{C}^{N,N}$ defined by

$$(2.17) \qquad H_\infty(x, \xi, t) = (4\pi t A)^{-\frac{d}{2}} \exp\left(-B_\infty t - (4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right),$$

is a heat kernel of the perturbed Ornstein-Uhlenbeck operator $\mathcal{L}_\infty$ from (2.15). Under the same assumptions it is proved in [**27**, Thm.5.3] that the family of mappings

$$(2.18) \qquad [T_\infty(t)v](x) := \begin{cases} \int_{\mathbb{R}^d} H_\infty(x, \xi, t)v(\xi)d\xi & , t > 0 \\ v(x) & , t = 0 \end{cases} \quad , x \in \mathbb{R}^d,$$

generates a strongly continuous semigroup $T_\infty(t) : L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$, $t \geqslant 0$, for each $1 \leqslant p < \infty$, which satisfies the following estimate (see (2.7), (2.16) for the constants)

$$(2.19) \qquad \|T_\infty(t)v\|_{L^p} \leqslant \kappa a_1 e^{-b_0 t} \|v\|_{L^p} \ \forall t \geqslant 0.$$

The semigroup $(T_\infty(t))_{t \geqslant 0}$ is called the Ornstein-Uhlenbeck semigroup if $B_\infty = 0$. Otherwise, $(T_\infty(t))_{t \geqslant 0}$ is a perturbed Ornstein-Uhlenbeck semigroup. The strong continuity of the semigroup justifies to introduce its infinitesimal generator $\mathcal{A}_p : L^p(\mathbb{R}^d, \mathbb{C}^N) \supseteq \mathcal{D}(\mathcal{A}_p) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ via

$$\mathcal{D}(\mathcal{A}_p) := \left\{ v \in L^p(\mathbb{R}^d, \mathbb{C}^N) \mid \mathcal{A}_p v := \lim_{t \downarrow 0} \frac{T_\infty(t)v - v}{t} \text{ exists in } L^p(\mathbb{R}^d, \mathbb{C}^N) \right\}.$$

An application of abstract semigroup theory yields the unique solvability of the resolvent equation

$$(2.20) \quad (\lambda I - \mathcal{A}_p)v = g, \text{ for all } g \in L^p(\mathbb{R}^d, \mathbb{C}^N), \lambda \in \mathbb{C}, \operatorname{Re} \lambda > -b_0 = s(-B_\infty)$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$, [**26**, Cor.6.7], [**27**, Cor.5.5]. Combining (2.18) with the representation $(\lambda I - \mathcal{A}_p)^{-1} g = \int_0^\infty e^{-\lambda s} T_\infty(s)g ds$, then the solution $v \in \mathcal{D}(\mathcal{A}_p)$ of (2.20) satisfies

$$(2.21) \qquad v = (\lambda I - \mathcal{A}_p)^{-1} g = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda s} H_\infty(\cdot, \xi, s)g(\xi)d\xi ds.$$

The following a-priori estimate in exponentially weighted L^p-spaces is based on the integral expression (2.21) and is taken from [**27**, Thm.5.7].

THEOREM 2.10 (Existence and uniqueness in weighted $W^{1,p}$-spaces). *Let the assumptions* (A2), (A6) *and* ($A9_{B_\infty}$) *be satisfied for* $\mathbb{K} = \mathbb{C}$, $1 \leqslant p < \infty$, *and let* $0 < \varepsilon < 1$ *and* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > -b_0$ *be given. Moreover, let* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *be a radially nondecreasing weight function of exponential growth rate* $\eta \geqslant 0$ *with*

$$0 \leqslant \eta^2 \leqslant \varepsilon \frac{a_0(\operatorname{Re}\lambda + b_0)}{a_{\max}^2 p^2}.$$

Then, for every $g \in L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$ *there exists a unique solution* $v \in \mathcal{D}(\mathcal{A}_p)$ *of the resolvent equation*

$$(\lambda I - \mathcal{A}_p)v = g, \quad in \ L^p(\mathbb{R}^d, \mathbb{C}^N).$$

The solution satisfies $v \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ *and the following estimates*

$$\|v\|_{L_\theta^p} \leqslant \frac{C_{0,\varepsilon}}{\operatorname{Re}\lambda + b_0}\|g\|_{L_\theta^p}, \tag{2.22}$$

$$\|D_i v\|_{L_\theta^p} \leqslant \frac{C_{1,\varepsilon}}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}}\|g\|_{L_\theta^p}, \ i = 1, \ldots, d, \tag{2.23}$$

where the λ*-independent constants* $C_{0,\varepsilon}$, $C_{1,\varepsilon}$ *are given by*

$$C_{0,\varepsilon} = C_\theta \kappa a_1 \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\pi\varepsilon)^{\frac{1}{2}}(1-\varepsilon)^{-\frac{d+1}{2}} + {}_2F_1\left(\frac{d}{2}, 1; \frac{1}{2}; \varepsilon\right) \right)^{\frac{1}{p}},$$

$$C_{1,\varepsilon} = C_\theta \kappa \frac{a_1^{\frac{d+1}{d}}\pi^{\frac{1}{2}}}{a_{\min}^{\frac{1}{2}}} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(1-\varepsilon)^{-\frac{d+1}{2}} + \frac{d\varepsilon^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}{}_2F_1\left(\frac{d+1}{2}, 1; \frac{3}{2}; \varepsilon\right) \right)^{\frac{1}{p}},$$

with constants $a_0, a_1, a_{\min}, a_{\max}$ *from* (2.7)*,* b_0, κ *from* (2.16) *and* C_θ *from* (W2).

REMARK 2.11. Above we used the hypergeometric function ${}_2F_1$, see [**25**, 15.4]. Moreover, we modified the original constants from [**27**, Thm.5.7] by using ${}_2F_1(a, b; b; z) = (1-z)^{-a}$ from [**25**, (15.4.6)] and the Pfaff transformation ${}_2F_1(a, b; c, z) = (1-z)^{-b}{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right)$ for $z \in \mathbb{C} \setminus [1, \infty)$. Note that both quantities ${}_2F_1\left(\frac{d}{2}, 1; \frac{1}{2}; \varepsilon\right)$ and ${}_2F_1\left(\frac{d+1}{2}, 1; \frac{3}{2}; \varepsilon\right)$ behave like $(1-\varepsilon)^{-\frac{d+1}{2}}$ as $\varepsilon \to 1$ ([**25**, (15.4.23)]), which then also determines the behavior of the constants $C_{0,\varepsilon}$ and $C_{1,\varepsilon}$.

So far, we neither have an explicit representation for the maximal domain $\mathcal{D}(\mathcal{A}_p)$ in terms of Sobolev spaces, nor do we have the relation between the generator $\mathcal{A}_p$ and the differential operator $\mathcal{L}_\infty$. For this purpose, one has to solve the identification problem, which has been done in [**29**]. Assuming (A2), (A6) and ($A9_{B_\infty}$) for $\mathbb{K} = \mathbb{C}$, it is proved in [**29**, Thm.3.2] that the Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ is a core of the infinitesimal generator $(\mathcal{A}_p, \mathcal{D}(\mathcal{A}_p))$ for any $1 \leqslant p < \infty$. Next, one considers $\mathcal{L}_\infty : L^p(\mathbb{R}^d, \mathbb{C}^N) \supseteq \mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ on its domain

$$\mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0) := \left\{ v \in W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p(\mathbb{R}^d, \mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N) \right\}. \tag{2.24}$$

Under the assumption (A3) for $\mathbb{K} = \mathbb{C}$, it is shown in [**29**, Lem.4.1] that $(\mathcal{L}_\infty, \mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0))$ is a closed operator in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for any $1 < p < \infty$. Then the L^p-dissipativity condition (A4) is the key assumption which leads to an energy estimate for the resolvent with respect to the L^p-norm, see [**29**, Thm.4.4]. The same argument reappears in Theorem 3.4 below which is an extension of [**29**, Thm.4.4]. As a direct consequence, the operator $\mathcal{L}_\infty$ is dissipative in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, provided β_∞ from

Assumption (A11$_{B_\infty}$) satisfies $\beta_\infty \leqslant 0$, [**29**, Cor.4.6]. Combining these results one can solve the identification problem for $\mathcal{L}_\infty$ as follows (see [**29**, Thm.5.1]).

THEOREM 2.12 (Maximal domain, local version). *Let the assumptions* (A4), (A6) *and* (A9$_{B_\infty}$) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$, *then*

$$\mathcal{D}(\mathcal{A}_p) = \mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$$

is the maximal domain of $\mathcal{A}_p$, *where* $\mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ *is defined by* (2.24). *In particular,* $\mathcal{A}_p$ *is the maximal realization of* $\mathcal{L}_\infty$ *in* $L^p(\mathbb{R}^d, \mathbb{C}^N)$, *i.e.*

$$\mathcal{A}_p v = \mathcal{L}_\infty v \quad \forall\, v \in \mathcal{D}(\mathcal{A}_p).$$

Theorem 2.12 shows that, if we restrict $1 < p < \infty$ and replace (A2) by the stronger assumption (A4) in Theorem 2.10, we can write $\mathcal{L}_\infty$ and $\mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ instead of $\mathcal{A}_p$ and $\mathcal{D}(\mathcal{A}_p)$. This will be crucial in the proof of Theorem 3.2 below. Moreover, we stress again that it is this theorem into which the L^p-dissipativity condition (A4) enters, see the comments following Assumption 2.2.

2.4. Bootstrapping and regularity. In Section 3 we study the variable co-efficient operator

$$(2.25) \qquad [\mathcal{L}_Q v](x) = A \triangle v(x) + \langle Sx, \nabla v(x) \rangle - B_\infty v(x) + Q(x)v(x),\ x \in \mathbb{R}^d,$$

and its resolvent equation

$$(2.26) \qquad\qquad (\lambda I - \mathcal{L}_Q)\, v = g, \quad \text{in } L^p(\mathbb{R}^d, \mathbb{C}^N)$$

for $1 < p < \infty$ and for different choices of $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$. In Section 3.1, we first derive an existence and uniqueness result for the resolvent equation (2.26) in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for general Q (Theorem 3.1). The proof uses the standard bounded perturbation theorem from abstract semigroup theory as well as Theorem 2.12. In Section 3.2, we then analyze the resolvent equation (2.26) for perturbations $Q = Q_{\mathrm{s}}$ which are small w.r.t. $\|\cdot\|_{L^\infty}$. We prove that the unique solution of (2.26) in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ decays exponentially if the inhomogeneity g does (Theorem 3.2). The proof is based on a fixed point argument and uses the results from Theorem 3.1 and Theorem 2.10. In Section 3.3, we study differential operators of the form

$$[\mathcal{L}_B v](x) = A \triangle v(x) + \langle Sx, \nabla v(x) \rangle - B(x)v(x),\ x \in \mathbb{R}^d,$$

where the matrix-valued function $B \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ satisfies

$$\mathrm{Re}\, \langle w, B(x)w \rangle \geqslant c_B |w|^2 \ \forall\, x \in \mathbb{R}^d \ \forall\, w \in \mathbb{C}^N$$

for some constant $c_B \in \mathbb{R}$. We consider two different weight functions θ_1, θ_2 satisfying $\theta_1 \leq C\theta_2$, so that $L_{\theta_2}^p(\mathbb{R}^d, \mathbb{C}^N) \subseteq L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$, e.g. θ_2 may grow while θ_1 decays. Then we prove uniqueness of solutions v of $(\lambda I - \mathcal{L}_B)v = g$ in the large space $W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$ if g is in the small space $L_{\theta_2}^p(\mathbb{R}^d, \mathbb{C}^N)$, and we derive resolvent estimates (Theorem 3.4). The proof generalizes the approach from [**26**, Thm.5.13] to variable coefficient perturbations and weighted spaces. In Section 3.4 we study the resolvent equation (2.26) for asymptotically small variable coefficient matrices Q. We prove that if $|Q(x)|$ falls below a certain threshold at infinity, then every solution $v \in W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$ of (2.26) in $L_{\mathrm{loc}}^p(\mathbb{R}^d, \mathbb{C}^N)$ already belongs to the small space $W_{\theta_2}^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ if $g \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{C}^N)$, and $\mathrm{Re}\,\lambda > -\beta_\infty$ (Theorem 3.5). The idea of the proof is to decompose Q into $Q = Q_{\mathrm{s}} + Q_{\mathrm{c}}$, where $Q_{\mathrm{s}} \in L^\infty(\mathbb{R}^d, \mathbb{C}^N)$ is small w.r.t. $\|\cdot\|_{L^\infty}$ and Q_{c} is compactly supported on

$\mathbb{R}^d$. Then, Theorem 3.2 implies the existence of a solution in the smaller space $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$ and Theorem 3.4 yields the uniqueness in the larger space $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$. Note that Theorem 3.5 is the core theorem which allows us to analyze exponential decay for both, solutions of the nonlinear problem and solutions of the eigenvalue problem for $\mathcal{L}$.

In Section 4 we prove spatial exponential decay for bounded solutions of the nonlinear problem (2.11) by employing a bootstrapping argument to the linear equation (2.12). Shifting the term with the compactly supported coefficient to the right-hand side, we obtain an inhomogeneity which lies in any weighted L^p-space. Applying the previous linear theory then provides exponential decay in space provided the difference $|v_\star(x) - v_\infty|$ falls below a certain threshold at infinity. In a second step, assuming additional regularity of the nonlinearity f and the solution $v_\star$, we show that the higher order derivatives also decay exponentially in space (Corollary 4.1, Remark 4.2)
(2.27)
$$v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N), \quad \text{if } f \in C^{\max\{2,k-1\}}(\mathbb{R}^N, \mathbb{R}^N), \ v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N).$$

This holds for $k \in \mathbb{N}$ and $p \geqslant \frac{d}{2}$ in case $k \geqslant 3$, where p is from (A4). In Section 4.3 we combine this result with Sobolev embeddings to deduce that $v_\star - v_\infty$ satisfies exponentially weighted pointwise estimates (Corollary 4.3)

$$(2.28) \quad |D^\alpha(v_\star(x) - v_\infty)| \leqslant C \exp(-\mu\sqrt{|x|^2 + 1}) \quad \forall\, x \in \mathbb{R}^d, \ 0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}$$

and for every multi-index $\alpha \in \mathbb{N}_0^d$ with $d < (k - |\alpha|)p$. In Section 4.4 we extend our main result from Theorem 2.8, Corollary 4.1 and Corollary 4.3 to complex-valued systems with f as in (2.5) (Corollary 4.5).

In Section 5 we study spatial exponential decay for solutions of the eigenvalue problem

$$(2.29) \quad A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\star(x))v(x) = \lambda v(x), \ x \in \mathbb{R}^d, \ d \geqslant 2.$$

In Section 5.1 we show that every bounded classical solution v of (2.29) decays exponentially in space, in the sense that v belongs to $W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$, provided that its associated eigenvalue $\lambda \in \mathbb{C}$ satisfies $\mathrm{Re}\,\lambda > -\beta_\infty$. In Section 5.2 we apply our result from Section 5.1 to those eigenfunctions which belong to eigenvalues on the imaginary axis. These eigenfunctions are due to equivariance with respect to the action of the Euclidean group and can be calculated explicitly in terms of the profile $v_\star$, see Theorem 5.4. In particular, this yields exponential decay of the eigenfunction $v(x) = \langle Sx, \nabla v_\star(x) \rangle, x \in \mathbb{R}^d$ associated with the eigenvalue $\lambda = 0$. As in the nonlinear case we proceed with proving exponential decay of derivatives of eigenfunctions, first in Sobolev spaces and then in a pointwise sense as in (2.28), see Theorem 5.1.

In Section 6 we apply the theory to so called spinning solitons of the cubic-quintic complex Ginzburg-Laundau equation (QCGL)

$$u_t = \alpha\triangle u + u\left(\delta + \beta|u|^2 + \gamma|u|^4\right)u,$$

where $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{C}$, $d \in \{2, 3\}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$. We derive suitable conditions on the parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that Theorem 2.10 and Corollary 4.5 apply. In Section 6.1 we compute the profile and (angular) speed of the spinning solitons. In Section 6.2 we compute spectra and eigenfunctions of

the associated eigenvalue problem. In Section 6.3 we compare in a final step the theoretical decay rates with numerical rates obtained from numerical data on a large ball. It turns out that the theoretical bounds are surprisingly close to the values found from numerical computations.

3. Variable coefficient complex Ornstein-Uhlenbeck operators

In this section we analyze the resolvent equation of the differential operator

$$(3.1) \qquad [\mathcal{L}_Q v](x) = A \triangle v(x) + \langle Sx, \nabla v(x) \rangle - B_\infty v(x) + Q(x)v(x), \ x \in \mathbb{R}^d,$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 < p < \infty$ and for different choices of $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$.

3.1. Solvability and uniqueness of the resolvent equation. Let us assume (A2), (A6) and (A9$_{B_\infty}$) for $\mathbb{K} = \mathbb{C}$ and let $(\mathcal{A}_p, \mathcal{D}(\mathcal{A}_p))$ denote the generator of the strongly continuous semigroup $(T_\infty(t))_{t \geqslant 0}$ from Section 2.3 on $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for some $1 \leqslant p < \infty$. Let us introduce the bounded operator

$$\mathcal{Q}_p : L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N) \quad \text{with} \quad [\mathcal{Q}_p v](x) := Q(x)v(x), \ x \in \mathbb{R}^d.$$

Then the bounded perturbation theorem [15, III.1.3] implies that

$$\mathcal{B}_p := \mathcal{A}_p + \mathcal{Q}_p \quad \text{with} \quad \mathcal{D}(\mathcal{B}_p) := \mathcal{D}(\mathcal{A}_p)$$

generates a strongly continuous semigroup $(T_Q(t))_{t \geqslant 0}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ satisfying

$$\|T_Q(t)v\|_{L^p} \leqslant \kappa a_1 e^{(-b_0 + \kappa a_1 \|\mathcal{Q}_p\|)t} \|v\|_{L^p} \leqslant \kappa a_1 e^{(-b_0 + \kappa a_1 \|Q\|_{L^\infty})t} \|v\|_{L^p} \ \forall t \geqslant 0,$$

where we used (2.19) and the estimate $\|\mathcal{Q}_p\| \leqslant \|Q\|_{L^\infty}$ of the L^p-operator norm. Then an application of [15, II.1.10] yields that the resolvent equation

$$(3.2) \qquad\qquad (\lambda I - \mathcal{B}_p)v = g, \text{ in } L^p(\mathbb{R}^d, \mathbb{C}^N)$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -b_0 + \kappa a_1 \|Q\|_{L^\infty}$ and $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ admits a unique solution $v \in \mathcal{D}(\mathcal{A}_p)$ which satisfies the resolvent estimate

$$\|v\|_{L^p} \leqslant \frac{\kappa a_1}{\operatorname{Re}\lambda - (-b_0 + \kappa a_1 \|Q\|_{L^\infty})} \|g\|_{L^p}.$$

If we restrict $1 < p < \infty$ and assume the stronger assumption (A4) (or equivalently (A5)) instead of (A2), an application of Theorem 2.12 yields that $\mathcal{D}(\mathcal{B}_p) := \mathcal{D}(\mathcal{A}_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and $\mathcal{B}_p v := \mathcal{A}_p v + \mathcal{Q}_p v = \mathcal{L}_\infty v + Qv = \mathcal{L}_Q$ for all $v \in \mathcal{D}(\mathcal{B}_p)$. Therefore we can write in the following $\mathcal{L}_Q$ and $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ instead of $\mathcal{B}_p$ and $\mathcal{D}(\mathcal{B}_p)$. Summarizing, we obtain the following result.

THEOREM 3.1 (Existence and uniqueness in weighted L^p-spaces). *Let the assumptions* (A4), (A6), (A9$_{B_\infty}$) *and* $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$. *Moreover, with constants* a_1 *from* (2.7), b_0, κ *from* (2.16), *let* $\omega := -b_0 + \kappa a_1 \|Q\|_{L^\infty}$ *and* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *be given. Then, for every* $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *the resolvent equation*

$$(\lambda I - \mathcal{L}_Q)v = g$$

admits a unique solution $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. *Moreover, the following resolvent estimate holds:*

$$\|v\|_{L^p} \leqslant \frac{\kappa a_1}{\operatorname{Re}\lambda - \omega} \|g\|_{L^p}.$$

3.2. Exponential decay for small perturbations. In the following we use the constants $a_0, a_1, a_{\max}, a_{\min}$ from (2.7), b_0, κ from (2.16) and $C_{0,\varepsilon}, C_{1,\varepsilon}$ from Theorem 2.10 without further reference.

THEOREM 3.2 (Existence and uniqueness in weighted $W^{1,p}$-spaces). *Let the assumptions* (A4)*,* (A6)*,* (A9$_{B_\infty}$) *and* (A10$_{B_\infty}$) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$. *Moreover, let* $0 < \varepsilon < 1$, $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *be a radially nondecreasing weight function of exponential growth rate*

$$(3.3) \qquad 0 \leqslant \eta \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p},$$

and let $Q_{\mathrm{s}} \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ *satisfy*

$$(3.4) \qquad \|Q_{\mathrm{s}}\|_{L^\infty} \leqslant \frac{\varepsilon b_0}{2} \min\left\{\frac{1}{\kappa a_1}, \frac{1}{C_{0,\varepsilon}}\right\}.$$

Further, let $\lambda \in \mathbb{C}$ *with* $\operatorname{Re} \lambda \geqslant -(1 - \varepsilon) b_0$ *and* $g \in L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$.
Then there exists a unique solution $v \in \mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ *of the resolvent equation*

$$(\lambda I - \mathcal{L}_{\mathrm{s}})v = g, \quad in \ L^p(\mathbb{R}^d, \mathbb{C}^N),$$

which satisfies $v \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$. *Moreover, the following estimates hold:*

$$(3.5) \qquad \|v\|_{L_\theta^p} \leqslant \frac{2C_{0,\varepsilon}}{\operatorname{Re}\lambda + b_0} \|g\|_{L_\theta^p},$$

$$(3.6) \qquad \|D_i v\|_{L_\theta^p} \leqslant \frac{2C_{1,\varepsilon}}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}} \|g\|_{L_\theta^p}, \ i = 1, \dots, d.$$

PROOF. Our proof proceeds in three steps.

1. Existence and uniqueness in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ (by Theorem 3.1): Since θ is nondecreasing we have $g \in L_\theta^p(\mathbb{R}^d, \mathbb{C}^N) \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N)$, and due to $\operatorname{Re}\lambda \geqslant -(1 - \varepsilon) b_0$ and (3.4) we have

$$\operatorname{Re}\lambda \geqslant -(1 - \varepsilon) b_0 \geqslant -b_0 + \frac{\varepsilon}{2} b_0 + \kappa a_1 \|Q_{\mathrm{s}}\|_{L^\infty} > -b_0 + \kappa a_1 \|Q_{\mathrm{s}}\|_{L^\infty}.$$

Thus, an application of Theorem 3.1 implies that there exists a unique solution $v_1 \in \mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ of $(\lambda I - \mathcal{L}_{\mathrm{s}})v = g$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. In order to verify that v_1 belongs to $W_\theta^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ and satisfies the inequalities (3.5) and (3.6) we must analyze $(\lambda I - \mathcal{L}_{\mathrm{s}})v = g$ in $L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$.

2. Existence in $L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$ (by a fixed point argument): Our aim is to show that the equation

$$(3.7) \qquad v = (\lambda I - \mathcal{L}_\infty)^{-1} g + (\lambda I - \mathcal{L}_\infty)^{-1} Q_{\mathrm{s}} v =: Fv$$

in $L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$ has a unique fixed point $v_2 \in L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$ which even belongs to $\mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ and agrees with v_1. For this purpose, consider in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ the equation

$$(3.8) \qquad (\lambda I - \mathcal{L}_\infty) u = g + Q_{\mathrm{s}} v, \quad given \quad v \in L_\theta^p(\mathbb{R}^d, \mathbb{C}^N).$$

First note, that the assumptions of Theorem 2.12 are satisfied. This allows us to write $\mathcal{L}_\infty$ and $\mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ instead of $\mathcal{A}_p$ and $\mathcal{D}(\mathcal{A}_p)$ in Theorem 2.10. Further, $\operatorname{Re}\lambda \geqslant -(1 - \varepsilon) b_0$ and equation (3.3) imply

$$0 \leqslant \eta^2 \leqslant \varepsilon^2 \frac{a_0 b_0}{a_{\max}^2 p^2} \leqslant \varepsilon \frac{a_0 (\operatorname{Re}\lambda + b_0)}{a_{\max}^2 p^2}.$$

Then Theorem 2.10 yields a unique solution $u \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ of (3.8) which satisfies $u \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. This shows that F maps $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ into itself and satisfies $Fv \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ for every $v \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. Applying $(\lambda I - \mathcal{L}_\infty)^{-1}$ to both sides in (3.8) shows $u = Fv$ with F defined in (3.7). Moreover, $Fv \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \cap L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. The linear part of F is a contraction due to (2.22) and (3.4)

$$\left\| (\lambda I - \mathcal{L}_\infty)^{-1} Q_{\mathrm{s}} v \right\|_{L^p_\theta} \leqslant q \, \|v\|_{L^p_\theta} \quad \forall \, v \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$$

with Lipschitz constant

$$(3.9) \qquad 0 \leqslant q := \frac{C_{0,\varepsilon}}{\mathrm{Re}\,\lambda + b_0} \, \|Q_{\mathrm{s}}\|_{L^\infty} \leqslant \frac{C_{0,\varepsilon}}{\varepsilon b_0} \, \|Q_{\mathrm{s}}\|_{L^\infty} \leqslant \frac{1}{2} < 1.$$

Consequently, F is a contraction in $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$. Thus, F has a unique fixed point $v_2 \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$ satisfying $v_2 = Fv_2 \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. Since $L^p_\theta(\mathbb{R}^d, \mathbb{C}^N) \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N)$, the equality $Fv_2 = v_2$ holds in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ as well, and applying $(\lambda I - \mathcal{L}_\infty)$ to both sides yields $(\lambda I - \mathcal{L}_{\mathrm{s}})v_2 = g$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. By the unique solvability of this equation we conclude $v := v_1 = v_2 \in L^p_\theta(\mathbb{R}^d, \mathbb{C}^N)$.

3. L^p_θ- and $W^{1,p}_\theta$-estimates (by contraction mapping principle and bootstrapping): The L^p_θ-estimate follows from the contraction mapping principle and the estimates (2.22), (3.9)

$$\|v\|_{L^p_\theta} \leqslant \frac{1}{1-q} \, \|F0\| \leqslant \frac{2C_{0,\varepsilon}}{\mathrm{Re}\,\lambda + b_0} \, \|g\|_{L^p_\theta} \,.$$

Finally, the $W^{1,p}_\theta$-estimate is proved by bootstrapping using the L^p_θ-estimate (3.5), the smallness condition (3.4) and (2.23) for every $i = 1, \ldots, d$

$$\begin{aligned}
\|D_i v\|_{L^p_\theta} &\leqslant \frac{C_{1,\varepsilon}}{(\mathrm{Re}\,\lambda + b_0)^{\frac{1}{2}}} \left(\|g\|_{L^p_\theta} + \|Q_{\mathrm{s}}\|_{L^\infty} \|v\|_{L^p_\theta} \right) \\
&\leqslant \frac{C_{1,\varepsilon}}{(\mathrm{Re}\,\lambda + b_0)^{\frac{1}{2}}} (1 + 2q) \, \|g\|_{L^p_\theta} \leqslant \frac{2C_{1,\varepsilon}}{(\mathrm{Re}\,\lambda + b_0)^{\frac{1}{2}}} \, \|g\|_{L^p_\theta} \,.
\end{aligned}$$

$\square$

3.3. Exponentially weighted resolvent estimates for variable coefficient operators.
Consider the differential operator

$$[\mathcal{L}_B v](x) := A \triangle v(x) + \langle Sx, \nabla v(x) \rangle - B(x)v(x), \, x \in \mathbb{R}^d.$$

The following Lemma 3.3 is crucial to derive energy estimates for $\mathcal{L}_B$ in exponentially weighted L^p-spaces, see Theorem 3.4 below. The result is proved in [29, Lem.4.2], [26, Lem.5.12], it is a vector-valued and complex-valued version of [20, Lem.2.1].

LEMMA 3.3. *Let the assumption* (A3) *be satisfied for* $\mathbb{K} = \mathbb{C}$. *Moreover, let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with a* C^2-*boundary or* $\Omega = \mathbb{R}^d$, $1 < p < \infty$, $v \in$

$W^{2,p}(\Omega, \mathbb{C}^N) \cap W_0^{1,p}(\Omega, \mathbb{C}^N)$ *and* $\eta \in C_b^1(\Omega, \mathbb{R})$ *be nonnegative, then*

$$-\mathrm{Re} \int_\Omega \eta \overline{v}^T |v|^{p-2} A \triangle v \geqslant \mathrm{Re} \int_\Omega \eta |v|^{p-2} \sum_{j=1}^d \overline{D_j v}^T A D_j v \mathbf{1}_{\{v \neq 0\}}$$

$$+ \mathrm{Re} \int_\Omega \overline{v}^T |v|^{p-2} \sum_{j=1}^d D_j \eta A D_j v$$

$$+ (p-2) \mathrm{Re} \int_\Omega \eta |v|^{p-4} \sum_{j=1}^d \mathrm{Re} \left(\overline{D_j v}^T v \right) \overline{v}^T A D_j v \mathbf{1}_{\{v \neq 0\}}.$$

Some care has to be taken when using this estimate. By a slight abuse of notation, the term $|v|^q \mathbf{1}_{\{v \neq 0\}}$ in the integrands should be read for powers $q < 0$ as follows

$$\left[|v|^q \mathbf{1}_{\{v \neq 0\}} \right](x) = \begin{cases} |v(x)|^q, & |v(x)| > 0, \\ 0, & v(x) = 0. \end{cases}$$

The proof of Lemma 3.3 shows by using Lebesgue's dominated convergence and Fatou's lemma that the integrals involving $\mathbf{1}_{\{v \neq 0\}}$ exist for $1 < p < \infty$, which is nontrivial in case $1 < p < 2$.

In the following theorem we prove resolvent estimates for $\mathcal{L}_B$ in exponentially weighted L^p-spaces. The theorem extends [**26**, Thm.5.13] to variable coefficient perturbations of $\mathcal{L}_0$ and to weighted L^p-spaces. Later on, in Theorem 3.5 we apply Theorem 3.4 to $B(x) = B_\infty - Q_s(x)$, so that $\mathcal{L}_B$ agrees with $\mathcal{L}_s$ from (2.15).

THEOREM 3.4 (Resolvent estimates in weighted L^p-spaces). *Let the assumptions* (A4) *and* (A6) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$. *Moreover, let* $B \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ *satisfy the strict accretivity condition*

$$(3.10) \qquad \mathrm{Re} \langle w, B(x)w \rangle \geqslant c_B |w|^2 \ \forall \, x \in \mathbb{R}^d \ \forall \, w \in \mathbb{C}^N, \text{ for some } c_B \in \mathbb{R},$$

let $\lambda \in \mathbb{C}$ *with* $\mathrm{Re} \, \lambda + c_B > 0$ *be given, and let* $\theta_1, \theta_2 \in C(\mathbb{R}^d, \mathbb{R})$ *be positive weight functions satisfying*

$$(3.11) \qquad \theta_1(x) = \exp\left(-\mu_1 \sqrt{|x|^2 + 1} \right) \quad \text{with} \quad 0 \leqslant |\mu_1| \leqslant \sqrt{\frac{(\mathrm{Re} \, \lambda + c_B)\gamma_A}{d|A|^2}},$$

and

$$(3.12) \qquad \theta_1(x) \leqslant C\theta_2(x) \ \forall \, x \in \mathbb{R}^d \text{ for some } C > 0.$$

Finally, let $g \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{C}^N)$ *and let* $v \in W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$ *be a solution of*

$$(3.13) \qquad (\lambda I - \mathcal{L}_B) v = g \quad \text{in } L_{\mathrm{loc}}^p(\mathbb{R}^d, \mathbb{C}^N).$$

Then, v is the unique solution of (3.13) *in* $W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$ *and satisfies the estimate*

$$(3.14) \qquad \|v\|_{L_{\theta_1}^p} \leqslant \frac{2C^{\frac{1}{p}}}{\mathrm{Re} \, \lambda + c_B} \|g\|_{L_{\theta_2}^p}.$$

In addition, for $1 < p \leqslant 2$ *the following gradient estimate holds*

$$(3.15) \qquad \|D_i v\|_{L_{\theta_1}^p} \leqslant \frac{2C^{\frac{1}{p}} \gamma_A^{-\frac{1}{2}}}{(\mathrm{Re} \, \lambda + c_B)^{\frac{1}{2}}} \|g\|_{L_{\theta_2}^p}, \quad i = 1, \dots, d,$$

with C from (3.12), γ_A from (A4) and c_B from (3.10).

PROOF. Consider $v \in W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$ which satisfies (3.13) for some $g \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{C}^N)$. For $n \in \mathbb{R}$ with $n > 0$ let us define the cut-off functions

(3.16)

$$\chi_n(x) = \chi_1\left(\frac{x}{n}\right), \ \chi_1 \in C_c^\infty(\mathbb{R}^d, \mathbb{R}), \ \chi_1(x) = \begin{cases} 1 & , |x| \leqslant 1 \\ \in [0,1], \text{ smooth} & , 1 < |x| < 2 \\ 0 & , |x| \geqslant 2 \end{cases}.$$

1. We multiply (3.13) from left by $\chi_n^2 \theta_1 \overline{v}^T |v|^{p-2}$, $n \in \mathbb{N}$, integrate over $\mathbb{R}^d$ and take real parts,

$$\mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} \overline{v}^T g$$

(3.17)
$$= (\mathrm{Re}\,\lambda) \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p - \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^T |v|^{p-2} A \triangle v$$

$$- \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^T |v|^{p-2} \sum_{j=1}^d (Sx)_j D_j v + \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^T |v|^{p-2} Bv.$$

2. Let us rewrite the third term on the right-hand side by using the formula

$$D_j\left(|v|^p\right) = p|v|^{p-2} \mathrm{Re}\left(\overline{D_j v}^T v\right)$$

and the following identity obtained from (A6) and integration by parts,

$$0 = \frac{1}{p} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \left(\sum_{j=1}^d S_{jj}\right) |v|^p = \frac{1}{p} \sum_{j=1}^d \int_{\mathbb{R}^d} \chi_n^2 D_j\left((Sx)_j\right) \theta_1 |v|^p$$

$$= -\frac{2}{p} \sum_{j=1}^d \int_{\mathbb{R}^d} \chi_n(D_j\chi_n)(Sx)_j \theta_1 |v|^p - \sum_{j=1}^d \int_{\mathbb{R}^d} \chi_n^2 \theta_1 (Sx)_j \mathrm{Re}\left(\overline{D_j v}^T v\right) |v|^{p-2}$$

$$- \frac{1}{p} \sum_{j=1}^d \int_{\mathbb{R}^d} \chi_n^2 (Sx)_j (D_j \theta_1) |v|^p$$

$$= -\frac{2}{p} \int_{\mathbb{R}^d} \chi_n \theta_1 |v|^p \sum_{j=1}^d (D_j\chi_n)(Sx)_j - \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^T |v|^{p-2} \sum_{j=1}^d (Sx)_j D_j v$$

$$- \frac{1}{p} \int_{\mathbb{R}^d} \chi_n^2 |v|^p \sum_{j=1}^d (Sx)_j (D_j \theta_1).$$

We insert this into (3.17) and apply Lemma 3.3 to the second term with $\Omega = B_{2n}(0)$, $\eta = \chi_n^2 \theta_1$

$$\mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} \overline{v}^T g = (\mathrm{Re}\,\lambda) \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p - \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^T |v|^{p-2} A \triangle v$$

$$+ \frac{2}{p} \int_{\mathbb{R}^d} \chi_n \theta_1 |v|^p \sum_{j=1}^d (D_j\chi_n)(Sx)_j + \frac{1}{p} \int_{\mathbb{R}^d} \chi_n^2 |v|^p \sum_{j=1}^d (D_j\theta_1)(Sx)_j$$

$$+ \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^T |v|^{p-2} Bv$$

$$\geqslant (\operatorname{Re}\lambda) \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^p + \frac{2}{p} \int_{\mathbb{R}^d} \chi_n \theta_1 \, |v|^p \sum_{j=1}^d (D_j \chi_n)(Sx)_j$$

$$+ \frac{1}{p} \int_{\mathbb{R}^d} \chi_n^2 \, |v|^p \sum_{j=1}^d (D_j \theta_1)(Sx)_j + \operatorname{Re} \int_{\mathbb{R}^d} 2\chi_n \theta_1 \bar{v}^T \, |v|^{p-2} \sum_{j=1}^d D_j \chi_n A D_j v$$

$$+ \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \bar{v}^T \, |v|^{p-2} \sum_{j=1}^d (D_j \theta_1) A D_j v$$

$$+ (p-2)\operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-4} \sum_{j=1}^d \operatorname{Re}\left(\overline{D_j v}^T v\right) \bar{v}^T A D_j v \mathbf{1}_{\{v\neq 0\}}$$

$$+ \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-2} \sum_{j=1}^d \overline{D_j v}^T A D_j v \mathbf{1}_{\{v\neq 0\}} + \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \bar{v}^T \, |v|^{p-2} \, Bv.$$

3. Subtracting the 2nd, 3rd, 4th and 5th term of the right hand side, yields the upper bound

$$(\operatorname{Re}\lambda) \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^p + \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-2} \sum_{j=1}^d \overline{D_j v}^T A D_j v \mathbf{1}_{\{v\neq 0\}}$$

$$+ (p-2)\operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-4} \sum_{j=1}^d \operatorname{Re}\left(\overline{D_j v}^T v\right) \bar{v}^T A D_j v \mathbf{1}_{\{v\neq 0\}}$$

$$+ \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \bar{v}^T \, |v|^{p-2} \, Bv$$

$$\leqslant \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-2} \, \bar{v}^T g - \operatorname{Re} \int_{\mathbb{R}^d} 2\chi_n \theta_1 \bar{v}^T \, |v|^{p-2} \sum_{j=1}^d D_j \chi_n A D_j v$$

$$- \frac{2}{p} \int_{\mathbb{R}^d} \chi_n \theta_1 \, |v|^p \sum_{j=1}^d (D_j \chi_n)(Sx)_j - \frac{1}{p} \int_{\mathbb{R}^d} \chi_n^2 \, |v|^p \sum_{j=1}^d (D_j \theta_1)(Sx)_j$$

$$- \operatorname{Re} \int_{\mathbb{R}^d} \chi_n^2 \bar{v}^T \, |v|^{p-2} \sum_{j=1}^d (D_j \theta_1) A D_j v =: T_1 + T_2 + T_3 + T_4 + T_5.$$

We estimate the terms successively. Using $\operatorname{Re} z \leqslant |z|$ and (3.12), Hölder's inequality yields

$$T_1 = \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-2} \operatorname{Re}\left(\bar{v}^T g\right) \leqslant \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^{p-1} \, |g|$$

$$\leqslant \left(\int_{\mathbb{R}^d} \left(\chi_n^{\frac{2(p-1)}{p}} \theta_1^{\frac{p-1}{p}} \, |v|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \left(\chi_n^{\frac{2}{p}} \theta_1^{\frac{1}{p}} \, |g|\right)^p\right)^{\frac{1}{p}}$$

$$\leqslant C^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 \, |v|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_2 \, |g|^p\right)^{\frac{1}{p}}.$$

For the 2nd term we use Hölder's inequality with $p = q = 2$ and Young's inequality with $\delta > 0$

$$
\begin{aligned}
T_2 \leqslant & 2|A| \int_{\mathbb{R}^d} \chi_n \theta_1 |v|^{p-1} \sum_{j=1}^{d} |D_j \chi_n| |D_j v| \\
\leqslant & \frac{2|A| \|\chi_1\|_{1,\infty}}{n} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n \theta_1 |D_j v| |v|^{p-1} \\
\leqslant & \frac{2|A| \|\chi_1\|_{1,\infty}}{n} \sum_{j=1}^{d} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 |D_j v|^2 |v|^{p-2} \mathbf{1}_{\{v \neq 0\}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \theta_1 |v|^p \right)^{\frac{1}{2}} \\
\leqslant & \frac{2|A| \|\chi_1\|_{1,\infty}}{n} \left[\delta \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |D_j v|^2 |v|^{p-2} \mathbf{1}_{\{v \neq 0\}} + \frac{d}{4\delta} \int_{\mathbb{R}^d} \theta_1 |v|^p \right].
\end{aligned}
$$

Here we used that for every $x \in \mathbb{R}^d$ and $j = 1, \ldots, d$

$$
|D_j \chi_n(x)| = \left| D_j \left(\chi_1 \left(\frac{x}{n} \right) \right) \right| \leqslant \frac{1}{n} \max_{j=1,\ldots,d} \max_{y \in \mathbb{R}^d} |D_j \chi_1(y)| = \frac{\|\chi_1\|_{1,\infty}}{n}.
$$

For the 3rd term we use $\chi_n(x) = 0$ for $|x| \geqslant 2n$ and $D_j \chi_n(x) = 0$ for $|x| \leqslant n$ to obtain

$$
\begin{aligned}
T_3 \leqslant & \frac{2}{p} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n \theta_1 |v|^p |(Sx)_j| |D_j \chi_n| \\
= & \frac{2}{p} \sum_{j=1}^{d} \int_{n \leqslant |x| \leqslant 2n} \chi_n \theta_1 |v|^p |(Sx)_j| |D_j \chi_n| \leqslant \frac{4d |S| \|\chi_1\|_{1,\infty}}{p} \int_{n \leqslant |x| \leqslant 2n} \theta_1 |v|^p.
\end{aligned}
$$

For the last estimate note that $\chi_n(x) \leqslant 1$ and

$$
\begin{aligned}
|(Sx)_j| |D_j \chi_n(x)| = & \frac{1}{n} |(Sx)_j| \left| (D_j \chi_1) \left(\frac{x}{n} \right) \right| \leqslant \frac{1}{n} |S| |x| \left| (D_j \chi_1) \left(\frac{x}{n} \right) \right| \\
\leqslant & \frac{|S|}{n} \left(\sup_{n \leqslant |\xi| \leqslant 2n} |\xi| \right) \max_{j=1,\ldots,d} \max_{y \in \mathbb{R}^d} |D_j \chi_1(y)| = 2|S| \|\chi_1\|_{1,\infty}.
\end{aligned}
$$

The 4th term vanishes, as follows from (3.11) and (A6),

$$
T_4 = -\frac{1}{p} \int_{\mathbb{R}^d} \chi_n^2 \frac{-\mu_1}{\sqrt{|x|^2 + 1}} \theta_1 |v|^p \sum_{j=1}^{d} x_j (Sx)_j = 0.
$$

For the 5th term we use again $\operatorname{Re} z \leqslant |z|$, Hölder's inequality with $p = q = 2$ and Young's inequality with some $\rho > 0$, (3.11) and $|\mu_1| \leqslant \mu_0$ for some $\mu_0 \geqslant 0$ that will be specified below

$$
\begin{aligned}
T_5 \leqslant & \int_{\mathbb{R}^d} \chi_n^2 |v|^{p-1} \sum_{j=1}^{d} \left| \frac{-\mu_1 x_j}{\sqrt{|x|^2 + 1}} \right| \theta_1 |A| |D_j v| \leqslant |\mu_1| |A| \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-1} |D_j v| \\
\leqslant & |\mu_1| |A| \sum_{j=1}^{d} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} |D_j v|^2 \mathbf{1}_{\{v \neq 0\}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p \right)^{\frac{1}{2}} \\
\leqslant & \frac{\mu_0 |A|}{4\rho} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} |D_j v|^2 \mathbf{1}_{\{v \neq 0\}} + \mu_0 |A| \rho d \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p.
\end{aligned}
$$

Summarizing, we arrive at the following estimate

$$
(\operatorname{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p + \int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\operatorname{Re}\langle v,Bv\rangle + \int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-4}\,\mathbf{1}_{\{v\neq0\}}
$$

$$
\times\sum_{j=1}^d\left[|v|^2\operatorname{Re}\langle D_jv,AD_jv\rangle + (p-2)\operatorname{Re}\langle D_jv,v\rangle\operatorname{Re}\langle v,AD_jv\rangle\right]
$$

$$
\leqslant C^{\frac1p}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\,|g|^p\right)^{\frac1p} + \frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\,|v|^p
$$

$$
+2\,\|\chi_1\|_{1,\infty}\left[\frac{2d\,|S|}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\,|v|^p + \frac{|A|\delta}{n}\sum_{j=1}^d\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|D_jv|^2\,|v|^{p-2}\,\mathbf{1}_{\{v\neq0\}}\right]
$$

$$
+\frac{\mu_0|A|}{4\rho}\sum_{j=1}^d\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,|D_jv|^2\,\mathbf{1}_{\{v\neq0\}} + \mu_0|A|\rho d\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p.
$$

4. The L^p-dissipativity assumption (A4) guarantees positivity of the term appearing in brackets $[\cdots]$ and (3.10) provides a lower bound for $\operatorname{Re}\langle v,Bv\rangle$. Therefore, putting the last 3 terms from the right-hand to the left-hand side in the last inequality from step 3, we obtain

$$
\left(\gamma_A - \frac{\mu_0|A|}{4\rho} - \frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\right)\sum_{j=1}^d\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|D_jv|^2\,|v|^{p-2}\,\mathbf{1}_{\{v\neq0\}}
$$

$$
+ (\operatorname{Re}\lambda + c_B - \mu_0|A|\rho d)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p
$$

$$
\leqslant C^{\frac1p}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\,|g|^p\right)^{\frac1p} + \frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\,|v|^p
$$

$$
+ \frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\,|v|^p.
$$

Now, we choose $\rho=\sqrt{\frac{\operatorname{Re}\lambda+c_B}{4d\gamma_A}}$, $\mu_0=\sqrt{\frac{(\operatorname{Re}\lambda+c_B)\gamma_A}{d|A|^2}}$ so that

$$
\operatorname{Re}\lambda + c_B - \mu_0|A|\rho d = \frac{\operatorname{Re}\lambda+c_B}{2}\quad\text{and}\quad \gamma_A - \frac{\mu_0|A|}{4\rho}=\frac{\gamma_A}{2}.
$$

Then our estimate reads

$$
\left(\frac{\gamma_A}{2} - \frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\right)\sum_{j=1}^d\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|D_jv|^2\,|v|^{p-2}\,\mathbf{1}_{\{v\neq0\}}
$$

$$
+ \frac{\operatorname{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p
$$

(3.18)

$$
\leqslant C^{\frac1p}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\,|g|^p\right)^{\frac1p} + \frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\,|v|^p
$$

$$
+ \frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\,|v|^p.
$$

5. Let us choose $\delta>0$ such that $\frac{\gamma_A}{2} - 2|A|\,\|\chi_1\|_{1,\infty}\,\delta>0$. Then we apply Fatou's Lemma to (3.18) and take the limit inferior $n\to\infty$. First observe that the terms

$\chi_n^2 \theta_1 |v|^p$ and $\chi_n^2 \theta_1 \left(\frac{\gamma_A}{2} - \frac{2|A| \|\eta\|_{1,\infty} \delta}{n} \right) |D_j v|^2 |v|^{p-2} \mathbf{1}_{\{v \neq 0\}}$ on the left-hand side are positive functions in $L^1(\mathbb{R}^d, \mathbb{R})$ and converge pointwise. The convergence of the integrals on the right-hand side of (3.18) is justified by Lebesgue's dominated convergence theorem. We have the pointwise convergence $\chi_n^2 \theta_1 |v|^p \to \theta_1 |v|^p$, $\chi_n^2 \theta_2 |g|^p \to \theta_2 |g|^p$, $\frac{1}{n} \theta_1 |v|^p \to 0$ and $\theta_1 |v|^p \mathbf{1}_{\{n \leqslant |x| \leqslant 2n\}} \to 0$ for almost every $x \in \mathbb{R}^d$ as $n \to \infty$. They are dominated by $|\chi_n^2 \theta_1 |v|^p| \leqslant \theta_1 |v|^p$, $|\chi_n^2 \theta_2 |g|^p| \leqslant \theta_2 |g|^p$, $\frac{1}{n} \theta_1 |v|^p \leqslant \theta_1 |v|^p$, $\theta_1 |v|^p \mathbf{1}_{\{n \leqslant |x| \leqslant 2n\}} \leqslant \theta_1 |v|^p$, and the bounds belong to $L^1(\mathbb{R}^d, \mathbb{R})$ since

$v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ and $g \in L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$. Thus we arrive at

$$\frac{\operatorname{Re} \lambda + c_B}{2} \|v\|_{L^p_{\theta_1}}^p \leqslant \frac{\operatorname{Re} \lambda + c_B}{2} \int_{\mathbb{R}^d} \theta_1 |v|^p + \frac{\gamma_A}{2} \sum_{j=1}^d \int_{\mathbb{R}^d} \theta_1 |D_j v|^2 |v|^{p-2} \mathbf{1}_{\{v \neq 0\}}$$

$$\leqslant C^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \theta_1 |v|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \theta_2 |g|^p \right)^{\frac{1}{p}} = C^{\frac{1}{p}} \|v\|_{L^p_{\theta_1}}^{p-1} \|g\|_{L^p_{\theta_2}}.$$

The $L^p_{\theta_1}$–resolvent estimate (3.14) follows by dividing both sides by $\frac{\operatorname{Re} \lambda + c_B}{2}$ and $\|v\|_{L^p_{\theta_1}}^{p-1}$.

6. The unique solvability of the resolvent equation $(\lambda I - \mathcal{L}_B)v = g$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ clearly follows from the resolvent estimate (3.14). From step 5 we obtain for every $j = 1, \dots, N$

$$\int_{\mathbb{R}^d} \theta_1 |D_j v|^2 |v|^{p-2} \mathbf{1}_{\{v \neq 0\}} \leqslant \frac{2C^{\frac{1}{p}}}{\gamma_A} \|v\|_{L^p_{\theta_1}}^{p-1} \|g\|_{L^p_{\theta_2}}.$$

We take into account that $|D_j v| = |D_j v| \mathbf{1}_{\{v \neq 0\}}$ a.e. (see e.g. [41, Cor.2.1.8]) and use the L^p–resolvent estimate (3.14) to deduce from Hölder's inequality for $1 < p \leqslant 2$

$$\|D_j v\|_{L^p_{\theta_1}}^p = \int_{\mathbb{R}^d} \theta_1^{\frac{p}{2}} |D_j v|^p |v|^{-\frac{p(2-p)}{2}} \mathbf{1}_{\{v \neq 0\}} \theta_1^{\frac{2-p}{2}} |v|^{\frac{p(2-p)}{2}}$$

$$\leqslant \left(\int_{\mathbb{R}^d} \theta_1 |D_j v|^2 |v|^{p-2} \mathbf{1}_{\{v \neq 0\}} \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^d} \theta_1 |v|^p \right)^{\frac{2-p}{2}}$$

$$\leqslant \left(\frac{4C^{\frac{2}{p}}}{(\operatorname{Re} \lambda + c_B)\gamma_A} \right)^{\frac{p}{2}} \|g\|_{L^p_{\theta_2}}^p.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

3.4. Exponential decay for asymptotically small perturbations. In this section we combine the results of Theorems 3.2 and 3.4 to obtain exponential a-priori estimates of solutions to variable coefficient equations when the coefficients become small at infinity.

THEOREM 3.5 (A-priori estimates in weighted L^p-spaces). *Let the assumptions* (A4), (A6), (A9$_{B_\infty}$) *and* (A11$_{B_\infty}$) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and some* $1 < p < \infty$. *Consider the radial weight functions*

$$(3.19) \qquad\qquad \theta_j(x) = \exp\left(\mu_j \sqrt{|x|^2 + 1} \right), \ x \in \mathbb{R}^d, \ j = 1, 2,$$

with $\mu_1, \mu_2 \in \mathbb{R}$,

$$(3.20) \qquad -\sqrt{\varepsilon \frac{\gamma_A \beta_\infty}{2d|A|^2}} \leqslant \mu_1 \leqslant 0 \leqslant \mu_2 \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}$$

for some $0 < \varepsilon < 1$. Moreover, let $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ with

$$(3.21) \qquad \operatorname*{ess\,sup}_{|x| \geqslant R_0} |Q(x)| \leqslant \frac{\varepsilon}{2} \min\left\{ \frac{b_0}{\kappa a_1}, \frac{b_0}{C_{0,\varepsilon}}, \beta_\infty \right\} \text{ for some } R_0 > 0,$$

let $g \in L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$, and let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant -(1-\varepsilon)\beta_\infty$ be given. Then every solution $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ of the resolvent equation

$$(\lambda I - \mathcal{L}_Q) \, v = g, \quad in \; L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N)$$

satisfies $v \in W^{1,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$. Moreover, the following estimates hold:

$$(3.22) \qquad \|v\|_{L^p_{\theta_2}} \leqslant \frac{2C_{0,\varepsilon}}{\operatorname{Re}\lambda + b_0} \left(C \|v\|_{L^p_{\theta_1}} + \|g\|_{L^p_{\theta_2}} \right),$$

$$(3.23) \qquad \|D_i v\|_{L^p_{\theta_2}} \leqslant \frac{2C_{1,\varepsilon}}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}} \left(C \|v\|_{L^p_{\theta_1}} + \|g\|_{L^p_{\theta_2}} \right), \quad i = 1, \dots, d,$$

with constants $a_0, a_1, a_{\max}$ from (2.7), γ_A from (A4), b_0, κ from (2.16), β_∞ from $(A11_{B_\infty})$, $C := \exp((\mu_2 - \mu_1)(4R_0^2 + 1)^{\frac{1}{2}}) \|Q\|_{L^\infty}$, and $C_{0,\varepsilon}, C_{1,\varepsilon}$ from Theorem 2.10 (with $C_\theta = 1$).

REMARK 3.6. Note that the exponential decay rate μ_2 in (3.20) depends on the spectral data $a_0, b_0, a_{\max}$, while the growth rate θ_1 allowing uniqueness, depends on the norm and accretivity data $\gamma_A, \beta_\infty, |A|$.

PROOF. The proof is structured as follows: First we decompose of Q into the sum of Q_s and Q_c, where Q_s is small according to (3.21) and Q_c is compactly supported on $\mathbb{R}^d$ (step 1). We then consider $(\lambda I - \mathcal{L}_s) u = Q_c v + g$ and prove existence of a solution in the smaller space $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$ by an application of Theorem 3.2 (step 2) and uniqueness in the larger space $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ by an application of Theorem 3.4 (step 3).

1. Decomposition of Q: With the cut-off function χ_{R_0} from (3.16) and R_0 from (3.21) let us write

$$Q(x) = Q_s(x) + Q_c(x), \quad Q_s(x) := (1 - \chi_{R_0}(x))Q(x), \quad Q_c(x) := \chi_{R_0}(x)Q(x).$$

Then Q_c is compactly supported and Q_s satisfies due to (3.21)

$$\|Q_s\|_{L^\infty} \leqslant \|1 - \chi_{R_0}\|_\infty \|Q\|_{L^\infty(\mathbb{R}^d \setminus B_{R_0}, \mathbb{C}^{N,N})} \leqslant \frac{\varepsilon}{2} \min\left\{ \frac{b_0}{\kappa a_1}, \frac{b_0}{C_{0,\varepsilon}}, \beta_\infty \right\}.$$

Let $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ be a solution of $(\lambda I - \mathcal{L}_Q) \, v = g$ with $\operatorname{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty$ and $g \in L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$. Then v satisfies the equation $(\lambda I - \mathcal{L}_s) \, v = Q_c v + g$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N)$. Therefore, we consider the problem

$$(3.24) \qquad (\lambda I - \mathcal{L}_s) \, u = Q_c v + g, \text{ in } L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N) \text{ and in } L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N).$$

2. Existence in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p_{\theta_2}(\mathbb{R}^d,\mathbb{C}^N)$: Let us apply Theorem 3.2 to (3.24) with $\theta=\theta_2$, $\eta=|\mu_2|$ and $Q_{\mathrm{c}}v+g$ instead of g. Note that $Q_{\mathrm{c}}v+g$ belongs to $L^p_{\theta_2}(\mathbb{R}^d,\mathbb{C}^N)$ since

$$
(3.25)\qquad
\begin{aligned}
\|Q_{\mathrm{c}}v+g\|_{L^p_{\theta_2}} &\leqslant \|\theta_2 Q_{\mathrm{c}}v\|_{L^p}+\|g\|_{L^p_{\theta_2}}\\
&\leqslant \left\|\theta_2\theta_1^{-1}\right\|_{L^\infty(B_{2R_0},\mathbb{R})}\|Q\|_{L^\infty}\|v\|_{L^p_{\theta_1}}+\|g\|_{L^p_{\theta_2}}=C\|v\|_{L^p_{\theta_1}}+\|g\|_{L^p_{\theta_2}},
\end{aligned}
$$

with $C:=\exp((\mu_2-\mu_1)(4R_0^2+1)^{\frac12})\|Q\|_{L^\infty}$. Further, $(\mathrm{A}10_{B_\infty})$ follows from $(\mathrm{A}11_{B_\infty})$, $\beta_\infty\leqslant b_0$ and $\varepsilon<1$ imply $\mathrm{Re}\,\lambda\geqslant-(1-\varepsilon)b_0$ and θ_2 is radially nondecreasing since $\mu_2\geqslant0$. Theorem 3.2 yields that there exists a (unique) $u\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\subset W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N)$ which solves (3.24) in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, Theorem 3.2 assures $u\in W^{1,p}_{\theta_2}(\mathbb{R}^d,\mathbb{C}^N)$ as well as the estimates (3.5) and (3.6).

3. Uniqueness in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^N)$: Consider (3.24) in $L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^N)$. We apply Theorem 3.4 with $B(x)=B_\infty-Q_{\mathrm{s}}(x)$ and $Q_{\mathrm{c}}v+g$ instead of g. First, $B\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ follows from $B_\infty\in\mathbb{C}^{N,N}$ and $Q_{\mathrm{s}}\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. Then, strict accretivity (3.10) with $c_B=\left(1-\frac{\varepsilon}{2}\right)\beta_\infty$ is a consequence of $(\mathrm{A}11_{B_\infty})$ and $\|Q_{\mathrm{s}}\|_{L^\infty}\leqslant\varepsilon\frac{\beta_\infty}{2}$,

$$
\mathrm{Re}\,\langle w,B(x)w\rangle\geqslant(\beta_\infty-\|Q_{\mathrm{s}}\|_{L^\infty})\,|w|^2\geqslant\left(1-\frac{\varepsilon}{2}\right)\beta_\infty|w|^2\quad\forall\,w\in\mathbb{C}^N\ \forall\,x\in\mathbb{R}^d.
$$

Moreover, we have $\mathrm{Re}\,\lambda\geqslant-(1-\varepsilon)\beta_\infty\geqslant-\left(1-\frac{\varepsilon}{2}\right)\beta_\infty=-c_B$. The growth bound in (3.11) is implied by (3.20) and $\mathrm{Re}\,\lambda+c_B\geqslant\frac{\varepsilon}{2}\beta_\infty$,

$$
0\leqslant|\mu_1|\leqslant\sqrt{\varepsilon\frac{\beta_\infty\gamma_A}{2d|A|^2}}\leqslant\sqrt{\frac{(\mathrm{Re}\,\lambda+c_B)\gamma_A}{d|A|^2}}.
$$

Finally, inequality (3.12) is obvious with $C=1$, since $\mu_1\leqslant0\leqslant\mu_2$. Let $u\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p_{\theta_2}(\mathbb{R}^d,\mathbb{C}^N)\subseteq W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^N)$ denote the solution from step 2 of the equation $(\lambda I-\mathcal{L}_{\mathrm{s}})u=Q_{\mathrm{c}}v+g$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)$. Since the given $v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^N)$ solves the same equation, the difference $w=u-v$ solves the homogeneous equation $(\lambda I-\mathcal{L}_{\mathrm{c}})w=0$ in $L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^N)$, and Theorem 3.4 implies $\|w\|_{L^p_{\theta_1}}=0$. Therefore, we obtain $v=u\in W^{1,p}_{\theta_2}(\mathbb{R}^d,\mathbb{C}^N)$.

4. $L^p_{\theta_2}$- and $W^{1,p}_{\theta_2}$-estimates: The $L^p_{\theta_2}$-estimate follows from (3.5) and (3.25)

$$
\|v\|_{L^p_{\theta_2}}=\|u\|_{L^p_{\theta_2}}\leqslant\frac{2C_{0,\varepsilon}}{\mathrm{Re}\,\lambda+b_0}\|Q_{\mathrm{c}}v+g\|_{L^p_{\theta_2}}\leqslant\frac{2C_{0,\varepsilon}}{\mathrm{Re}\,\lambda+b_0}\left(C\|v\|_{L^p_{\theta_1}}+\|g\|_{L^p_{\theta_2}}\right).
$$

Analogously, the $W^{1,p}_{\theta_2}$-estimate follows from (3.6) and (3.25). $\qquad\square$

4. Exponential decay of rotating nonlinear waves

4.1. Proof of main result.

PROOF (OF THEOREM 2.8). The proof is structured as follows: First we decompose the nonlinearity $f(v_\star(x))$ and derive an equation $\mathcal{L}_Q w_\star=0$ solved by $w_\star=v_\star-v_\infty$ (step 1). Then we apply Theorem 3.5 (step 2) and check its assumptions (steps 3,4).

1. Let $v_\star$ be a classical solution of (2.8) satisfying (2.9). Note that this implies $v_\star \in C_b(\mathbb{R}^d, \mathbb{R}^N)$. Using (A7) and (A8) we obtain from the mean value theorem

$$\begin{aligned} f(v_\star(x)) =& f(v_\infty) + Df(v_\infty)\,(v_\star(x) - v_\infty) \\ & + \int_0^1 (Df(v_\infty + t(v_\star(x) - v_\infty)) - Df(v_\infty))\,dt\,(v_\star(x) - v_\infty) \\ =& -B_\infty\,(v_\star(x) - v_\infty) + Q(x)\,(v_\star(x) - v_\infty)\,,\ x \in \mathbb{R}^d \end{aligned}$$

with

(4.1) $$B_\infty := -Df(v_\infty),\ Q(x) := \int_0^1 (Df(v_\infty + t(v_\star(x) - v_\infty)) - Df(v_\infty))\,dt.$$

For $w_\star := v_\star - v_\infty$, we have $w_\star \in C^2(\mathbb{R}^d, \mathbb{R}^N) \cap C_b(\mathbb{R}^d, \mathbb{R}^N)$ and

$$\begin{aligned} 0 =& A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) \\ =& A\triangle\,(v_\star(x) - v_\infty) + \langle Sx, \nabla\,(v_\star(x) - v_\infty) \rangle \\ & - B_\infty\,(v_\star(x) - v_\infty) + Q(x)\,(v_\star(x) - v_\infty) \\ =& A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle - B_\infty w_\star(x) + Q(x)w_\star(x) \\ =& [\mathcal{L}_Q w_\star]\,(x),\ x \in \mathbb{R}^d. \end{aligned}$$

2. Let us apply Theorem 3.5 with B_∞, Q from (4.1), $\theta_2 = \theta$, $\mu_2 = \mu$, $\mu_1 < 0$, $\lambda = 0$ and $g = 0$. For this purpose, we have to check the assumptions: Assumptions (A4) and (A6) are directly satisfied, $(A9_{B_\infty})$ follows from (A9), and $(A11_{B_\infty})$ from (A11), using the relation $B_\infty = -Df(v_\infty)$. In the following let $0 < \varepsilon < 1$ be fixed and let $\theta_1, \theta_2 \in C(\mathbb{R}^d, \mathbb{R})$ be given by (3.19) satisfying $\mu_1 < 0$ and (3.20). First, note that $w_\star \in W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ follows from $w_\star \in C^2(\mathbb{R}^d, \mathbb{R}^N) \cap C_b(\mathbb{R}^d, \mathbb{R}^N)$ and $C_b(\mathbb{R}^d, \mathbb{R}^N) \subset L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ due to $\mu_1 < 0$. It remains to verify that $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ (step 3) and that (3.21) is satisfied.

3. Since $w_\star \in C_b(\mathbb{R}^d, \mathbb{R}^N)$ we obtain

$$|v_\infty + tw_\star(x)| \leqslant |v_\infty| + t\,|w_\star(x)| \leqslant |v_\infty| + \|w_\star\|_\infty =: R_1$$

for every $x \in \mathbb{R}^d$ and $0 \leqslant t \leqslant 1$. Due to (A7), we have $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ which implies

$$\begin{aligned} |Q(x)| \leqslant& \int_0^1 |Df(v_\infty + tw_\star(x))| + |Df(v_\infty)|\,dt \\ \leqslant& \sup_{z \in B_{R_1}(0)} |Df(z)| + |Df(v_\infty)| < \infty \end{aligned}$$

for all $x \in \mathbb{R}^d$. We deduce $Q \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \subset L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ by taking the suprema over $x \in \mathbb{R}^d$.

4. We finally verify (3.21): Let us choose $K_1 = K_1(A, f, v_\infty, d, p, \varepsilon) > 0$ such that

(4.2) $$K_1 \left(\sup_{z \in B_{K_1}(v_\infty)} |D^2 f(z)| \right) \leqslant \varepsilon \min \left\{ \frac{b_0}{\kappa a_1}, \frac{b_0}{C_{0,\varepsilon}}, \beta_\infty \right\} =: K(\varepsilon)$$

is satisfied, with the constants $C_{0,\varepsilon} = C_{0,\varepsilon}(A, d, p, \varepsilon, \kappa)$ from Theorem 2.10, $b_0 := -s(Df(v_\infty))$ and a_1 from (2.7), β_∞ from (A11), and

$$|D^2 f(z)| := \|D^2 f(z)\|_{\mathcal{L}(\mathbb{R}^N, \mathbb{R}^{N,N})} := \sup_{\substack{v \in \mathbb{R}^N \\ |v|=1}} |D^2 f(z)v|.$$

Since $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ by (A7), inequalities (2.9) and (4.2) lead to

$$
\begin{aligned}
|Q(x)| &= \left| \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty) dt \right| \\
&= \left| \int_0^1 \int_0^1 D^2 f(v_\infty + stw_\star(x))[tw_\star(x)] ds dt \right| \\
&\leqslant \int_0^1 \int_0^1 \sup_{|x| \geqslant R_0} \left| D^2 f(v_\infty + st(v_\star(x) - v_\infty)) \right| ds \cdot t \sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \, dt \\
&\leqslant \frac{K_1}{2} \left(\sup_{z \in B_{K_1}(v_\infty)} \left| D^2 f(z) \right| \right) \leqslant \frac{\varepsilon}{2} \min \left\{ \frac{b_0}{\kappa a_1}, \frac{b_0}{C_{0,\varepsilon}}, \beta_\infty \right\}
\end{aligned}
$$

for every $|x| \geqslant R_0$. Taking the supremum over $|x| \geqslant R_0$ yields condition (3.21). This justifies to apply Theorem 3.5 which shows $w_\star = v_\star - v_\infty \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$. $\square$

4.2. Exponential decay of higher order derivatives. For estimating higher order derivatives, recall the *Sobolev embedding* for $0 \leqslant l \leqslant k$,

(4.3) $W^{k,p}(\mathbb{R}^d, \mathbb{R}^N) \subseteq W^{l,q}(\mathbb{R}^d, \mathbb{R}^N), \quad \text{if } 1 < p < q \leqslant \infty, \quad \frac{d}{p} - k \leqslant \frac{d}{q} - l,$

where at least one of the inequalities '$\leqslant$' is strict. Moreover, the embedding is continuous, i.e.

$$\exists C_{p,q,k,l} > 0 : \|u\|_{W^{l,q}(\mathbb{R}^d)} \leqslant C_{p,q,k,l} \|u\|_{W^{k,p}(\mathbb{R}^d)} \; \forall u \in W^{k,p}(\mathbb{R}^d).$$

For the Sobolev embedding we refer to [**2**, Thm.5.4], [**24**, Ch.6], [**4**, Ch.8] as general reference, and to [**35**, Thm.3, Exer.24] for the compact version used in (4.3). For the corresponding weighted spaces it is important to note that

(4.4) $u \in W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^N) \Longrightarrow \theta D^\alpha u \in W^{k-|\alpha|,p}(\mathbb{R}^d, \mathbb{R}^N) \quad \text{for} \quad 0 \leqslant |\alpha| \leqslant k.$

Here θ is chosen as in Theorem 2.8 for some $0 < \varepsilon < 1$. By definition, $u \in W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$ implies $D^\alpha u \in W_\theta^{k-|\alpha|,p}(\mathbb{R}^d, \mathbb{R}^N)$ for every $0 \leqslant |\alpha| \leqslant k$. Since θ belongs to $C^\infty(\mathbb{R}^d, \mathbb{R})$ and satisfies $\left\| \frac{D^\gamma \theta}{\theta} \right\|_{L^\infty} \leqslant C(\gamma)$ for every $\gamma \in \mathbb{N}_0^d$, we obtain $\theta D^\alpha u \in W^{k-|\alpha|,p}(\mathbb{R}^d, \mathbb{R}^N)$ from

$$
\begin{aligned}
\|\theta D^\alpha u\|_{W^{k-|\alpha|,p}}^p &= \sum_{|\beta| \leqslant k-|\alpha|} \left\| D^\beta(\theta D^\alpha u) \right\|_{L^p}^p \\
&= \sum_{|\beta| \leqslant k-|\alpha|} \left\| \sum_{|\gamma| \leqslant |\beta|} \binom{\beta}{\gamma} (D^\gamma \theta)(D^{\alpha+\beta-\gamma} u) \right\|_{L^p}^p \\
&\leqslant \sum_{|\beta| \leqslant k-|\alpha|} \left(\sum_{|\gamma| \leqslant |\beta|} \binom{\beta}{\gamma} \left\| \frac{D^\gamma \theta}{\theta} \right\|_{L^\infty} \left\| \theta D^{\alpha+\beta-\gamma} u \right\|_{L^p} \right)^p \leqslant C \|u\|_{W_\theta^{k,p}}^p.
\end{aligned}
$$

Finally, recall the *generalized Hölder's inequality*

(4.5) $\left\| \prod_{j=1}^\ell u_j \right\|_{L^p(\mathbb{R}^d, \mathbb{R})} \leq \prod_{j=1}^\ell \|u_j\|_{L^{p_j}(\mathbb{R}^d, \mathbb{R})},$

for $u_j \in L^{p_j}(\mathbb{R}^d, \mathbb{R})$, $1 \leqslant p, p_j \leqslant \infty$ and $\sum_{j=1}^\ell \frac{1}{p_j} = \frac{1}{p}$.

The following corollary shows, that if we assume more regularity for the solution $v_\star$ of (2.8) in Theorem 2.8, i.e. $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$, then $v_\star$ even belongs to

$W_\theta^{2,p}(\mathbb{R}^d, \mathbb{R}^N)$. The proof is based on the results of Theorem 2.8 and on a further application of Theorem 3.5. For the proof it is crucial that we allow inhomogeneities g in Theorem 3.5. The argument can be continued to higher order weighted Sobolev spaces.

COROLLARY 4.1 (Exponential decay of $v_\star$ with higher regularity). *Let the assumptions* (A4), (A6)–(A9) *and* (A11) *be satisfied for* $\mathbb{K} = \mathbb{R}$ *and for some* $1 < p < \infty$. *Moreover, let* $a_{\max} = \rho(A)$ *denote the spectral radius of* A, $-a_0 = s(-A)$ *the spectral bound of* $-A$ *and* $-b_0 = s(Df(v_\infty))$ *the spectral bound of* $Df(v_\infty)$. *Further, let* $\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ *denote a weight function for some* $\mu \in \mathbb{R}$. *Then, for every* $0 < \varepsilon < 1$ *there is a constant* $K_1 = K_1(A, f, v_\infty, d, p, \varepsilon) > 0$ *with the following property: Every classical solution* $v_\star$ *of*

$$(4.6) \qquad A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0, \ x \in \mathbb{R}^d,$$

with $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ *and*

$$(4.7) \qquad \sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \ \textit{for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W_\theta^{2,p}(\mathbb{R}^d, \mathbb{R}^N)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.$$

If additionally, $p \geqslant \frac{d}{2}$, $f \in C^{k-1}(\mathbb{R}^N, \mathbb{R}^N)$, *and* $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ *for some* $k \in \mathbb{N}$ *with* $k \geqslant 3$, *then* $v_\star$ *even satisfies* $v_\star - v_\infty \in W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$.

PROOF. Let $v_\star$ be a classical solution of (4.6) satisfying (4.7). Again this implies $v_\star \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^N)$.

1. The additional regularity $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ allows us to apply $D_i = \frac{\partial}{\partial x_i}$ to equation (4.6)

$$0 = A\triangle D_i v_\star(x) + \langle Sx, \nabla D_i v_\star(x) \rangle + Df(v_\star(x))D_i v_\star(x) + \sum_{j=1}^{d} S_{ji} D_j v_\star, \ x \in \mathbb{R}^d.$$

For $w_\star := D_i v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^N)$ we obtain using $S_{ii} = 0$,

$$0 = A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + Df(v_\star(x))w_\star(x) + \sum_{\substack{j=1 \\ j \neq i}}^{d} S_{ji} D_j v_\star$$

$$= [\mathcal{L}_Q w_\star](x) + g(x), \ x \in \mathbb{R}^d$$

with the settings

$$(4.8) \quad B_\infty := -Df(v_\infty), \ Q(x) := Df(v_\star(x)) - Df(v_\infty), \ g(x) := \sum_{\substack{j=1 \\ j \neq i}}^{d} S_{ji} D_j v_\star.$$

2. We now apply Theorem 3.5 with B_∞, Q, g from (4.8), $\theta_2 = \theta$, $\mu_2 = \mu$, $\mu_1 < 0$ and $\lambda = 0$. The assumptions (A4) and (A6) are directly satisfied while (A9$_{B_\infty}$), (A11$_{B_\infty}$) follow from (A9), (A11). In the following let $0 < \varepsilon < 1$ be fixed and let $\theta_1, \theta_2 \in C(\mathbb{R}^d, \mathbb{R})$ be given by (3.19) satisfying $\mu_1 < 0 \leqslant \mu_2$ and (3.20). The relation $w_\star \in W_{\text{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^N) \cap L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^N)$ is a consequence of Theorem 2.8 which implies $D_i v_\star \in L_\theta^p(\mathbb{R}^d, \mathbb{R}^N) \subseteq L_{\theta_1}^p(\mathbb{R}^d, \mathbb{R}^N)$ since θ_1 is decreasing. Moreover, $-\mathcal{L}_Q w_\star = g$ holds in $L_{\text{loc}}^p(\mathbb{R}^d, \mathbb{C}^N)$ by construction. Then, $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ follows from (4.8) since $v_\star$ is bounded, and we also have $g \in L_\theta^p(\mathbb{R}^d, \mathbb{C}^N)$ since Theorem 2.8 shows $v_\star \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$ which leads to the estimate

$$\|g\|_{L_\theta^p} \leqslant \sum_{\substack{j=1 \\ j \neq i}}^d |S_{ji}| \, \|D_j v_\star\|_{L_\theta^p} \leqslant C.$$

3. We now verify condition (3.21): Let us choose $K_1 = K_1(A, f, v_\infty, d, p, \varepsilon) > 0$ such that (4.2) holds with $\frac{K(\varepsilon)}{2}$ instead of $K(\varepsilon)$ on the right-hand side. Since $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ by (A7), equations (2.9) and (4.2) imply for all $|x| \geqslant R_0$

$$|Q(x)| = |Df(v_\star(x)) - Df(v_\infty)|$$
$$\leqslant \int_0^1 \left| D^2 f(v_\infty + s\,(v_\star(x) - v_\infty)) \right| ds \, |v_\star(x) - v_\infty|$$
$$\leqslant K_1 \left(\sup_{z \in B_{K_1}(v_\infty)} \left| D^2 f(z) \right| \right) \leqslant \frac{\varepsilon}{2} \min \left\{ \frac{b_0}{\kappa a_1}, \frac{b_0}{C_{0,\varepsilon}}, \beta_\infty \right\}.$$

Taking the supremum over $|x| \geqslant R_0$ we have shown (3.21) with R_0 from (2.9). By applying Theorem 3.5 we obtain $w_\star = D_i v_\star \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$ for every $i = 1, \ldots, d$, thus $v_\star - v_\infty \in W_\theta^{2,p}(\mathbb{R}^d, \mathbb{R}^N)$.

4. For the final assertion we consider $f \in C^{k-1}(\mathbb{R}^N, \mathbb{R}^N)$ and $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ for some $k \in \mathbb{N}$ with $k \geqslant 3$ and show $v_\star - v_\infty \in W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$ by induction with respect to k. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index of length $|\alpha| = k - 1$ for some $k \geqslant 3$. Applying D^α to (4.6) yields that $w_\star := D^\alpha v_\star$ satisfies

$$(4.9) \qquad\qquad 0 = [\mathcal{L}_Q w_\star](x) + g(x),\, x \in \mathbb{R}^d$$

with B_∞ and $Q(x)$ as in (4.8), and $g(x) := g_1(x) + g_2(x)$ defined by

$$g_1(x) := \sum_{i=1}^d \sum_{\substack{j=1 \\ e_j \leqslant \alpha}}^d S_{ij} \binom{\alpha}{e_j} D^{\alpha - e_j + ei} v_\star(x),$$

$$(4.10)$$

$$g_2(x) := \sum_{\ell=2}^{k-1} \sum_{\pi \in \mathcal{P}_{\ell, k-1}} (D^\ell f)(v_\star(x)) \left[D^{|\pi_1|} v_\star(x) h_{\pi_1}, \ldots, D^{|\pi_\ell|} v_\star(x) h_{\pi_\ell} \right].$$

The first term g_1 arises from the Leibniz rule applied to $\langle Sx, \nabla v_\star(x)\rangle$,

$$D^\alpha(\langle Sx, \nabla v_\star(x)\rangle) = \sum_{i=1}^d \sum_{j=1}^d S_{ij} D^\alpha(x_j D_i v_\star(x))$$

(4.11)

$$= \langle Sx, \nabla D^\alpha v_\star(x)\rangle + \sum_{i=1}^d \sum_{\substack{j=1 \\ e_j \leqslant \alpha}}^d S_{ij} \binom{\alpha}{e_j} D^{\alpha - e_j + e_i} v_\star(x)$$

$$= \langle Sx, \nabla D^\alpha v_\star(x)\rangle + g_1(x).$$

The second term is obtained by applying Faá di Bruno's formula for multivariate calculus,

$$D^\alpha(f(v_\star(x))) = \sum_{\ell=1}^{k-1} \sum_{\pi \in \mathcal{P}_{\ell, k-1}} (D^\ell f)(v_\star(x)) \left[D^{|\pi|} v_\star(x) h_\pi\right]$$

(4.12)

$$= Df(v_\star(x)) D^\alpha v_\star(x) + \sum_{\ell=2}^{k-1} \sum_{\pi \in \mathcal{P}_{\ell, k-1}} (D^\ell f)(v_\star(x)) \left[D^{|\pi|} v_\star(x) h_\pi\right]$$

$$= Df(v_\star(x)) D^\alpha v_\star(x) + g_2(x),\ x \in \mathbb{R}^d.$$

Here $\mathcal{P}_{\ell, k-1}$ denotes the set of all ℓ-partitions of the set $\langle k-1 \rangle = \{1, \ldots, k-1\}$, given by

$$\mathcal{P}_{\ell, k-1} = \left\{\pi = \{\pi_1, \ldots, \pi_\ell\} \subset 2^{\langle k-1 \rangle} : \bigcup_{j=1}^\ell \pi_j = \langle k-1 \rangle,\ \pi_i \cap \pi_j = \emptyset\ \forall i \neq j\right\}.$$

Moreover, we used short-hands for multilinear terms

(4.13) $$(h_1, \ldots, h_{k-1}) = (\underbrace{e_1, \ldots, e_1}_{\alpha_1}, \ldots, \underbrace{e_d, \ldots, e_d}_{\alpha_d}),\ e_j\ j\text{-th unit vector in } \mathbb{R}^d,$$

$$h_\rho = (h_{\rho_1}, \ldots h_{\rho_\nu})\text{ for index sets } \rho = \{\rho_1, \ldots \rho_\nu\} \subseteq \langle k-1 \rangle,$$

$$[D^{|\pi|} v_\star(x) h_\pi] = [D^{|\pi_1|} v_\star(x) h_{\pi_1}, \ldots, D^{|\pi_\ell|} v_\star(x) h_{\pi_\ell}],\ \pi = \{\pi_1, \ldots, \pi_\ell\} \subset 2^{\langle k-1 \rangle}.$$

For $\ell = 1$ the only partition is $\pi_1 = \{\langle k-1 \rangle\}$ and $Df(v_\star)[D^{k-1} v_\star h_{\pi_1}]$ agrees with $Df(v_\star) D^\alpha v_\star$. Below we show $g \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$. Then Theorem 3.5 applies to (4.9) and yields $w_\star = D^\alpha v_\star \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$ and thus our assertion $v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$.

5. To verify $g \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$, consider first g_1. By the first part of the Corollary (base case $k = 3$) and by the induction hyperthesis (induction step $k > 3$) we have $v_\star - v_\infty \in W^{k-1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$. The indices $\gamma := \alpha - e_j + e_i$ with $|\alpha| = k - 1$ and $e_j \leqslant \alpha$ satisfy $|\gamma| = k - 1$, hence $D^{\alpha - e_j + e_i} v_\star \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$, and we deduce $g_1 \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)$.

Next we consider g_2. We show $\theta g_2 \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ by using the generalized Hölder's inequality (4.5) with $p_j := \frac{k-1}{|\pi_j|} p$ and $u_j := \theta \left|D^{|\pi_j|} v_\star h_{\pi_j}\right|$ for $j = 1, \ldots, \ell$. Note that $\sum_{j=1}^\ell \frac{1}{p_j} = \frac{1}{p}$ since $\sum_{j=1}^\ell |\pi_j| = k - 1$. We obtain

$$\left\|\theta(D^\ell f)(v_\star) \left[D^{|\pi_1|} v_\star h_{\pi_1}, \ldots, D^{|\pi_\ell|} v_\star h_{\pi_\ell}\right]\right\|_{L^p}$$

$$\leqslant \left\|(D^{\ell}f)(v_{\star})\right\|_{L^{\infty}} \left\|\prod_{j=1}^{\ell} \theta^{\frac{1}{j}} \left|D^{|\pi_j|}v_{\star} h_{\pi_j}\right| \right\|_{L^p}$$

$$\leqslant \left\|(D^{\ell}f)(v_{\star})\right\|_{L^{\infty}} \prod_{j=1}^{\ell} \left\|\theta^{\frac{1}{j}} D^{|\pi_j|}v_{\star} h_{\pi_j}\right\|_{L^{p_j}}$$

$$\leqslant \left\|(D^{\ell}f)(v_{\star})\right\|_{L^{\infty}} \prod_{j=1}^{\ell} \left\|\theta D^{|\pi_j|}v_{\star} h_{\pi_j}\right\|_{L^{p_j}},$$

where we used $\theta(x) \geqslant 1$ and $\theta^{\frac{1}{j}}(x) \leqslant \theta(x)$. Note that $v_{\star} \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^N)$, $f \in C^{k-1}(\mathbb{R}^N, \mathbb{R}^N)$ and $\ell \leqslant k-1$ imply the total derivative $(D^{\ell}f)(v_{\star})$ to be bounded on $\mathbb{R}^d$. It remains to verify that $\theta D^{|\pi_j|}v_{\star} h_{\pi_j} \in L^{p_j}(\mathbb{R}^d, \mathbb{R}^N)$: From $v_{\star} - v_{\infty} \in W_{\theta}^{k-1,p}(\mathbb{R}^d, \mathbb{R}^N)$, (4.4) we infer $\theta D^{\gamma}(v_{\star} - v_{\infty}) \in W^{k-1-|\gamma|,p}(\mathbb{R}^d, \mathbb{R}^N)$ for all $0 \leqslant |\gamma| \leqslant k-1$. Using $1 \leqslant |\pi_j| \leqslant k-2$ this proves the first assertion in

$$\theta D^{|\pi_j|}v_{\star} h_{\pi_j} \in W^{k-1-|\pi_j|,p}(\mathbb{R}^d, \mathbb{R}^N) \subset L^{p_j}(\mathbb{R}^d, \mathbb{R}^N).$$

The second assertion '$\subset$' follows from the Sobolev embedding (4.3), provided that $1 < p < p_j < \infty$ and $\frac{d}{p} - (k-1-|\pi_j|) \leqslant \frac{d}{p_j}$. The first inequality is implied by $1 \leqslant |\pi_j| \leqslant k-2$,

$$1 < p < \frac{k-1}{k-2}p \leqslant \frac{k-1}{|\pi_j|}p = p_j < \infty,$$

while the second is implied by our assumptions $p \geqslant \frac{d}{2}$ and $k \geqslant 3$,

$$\frac{d}{p} - \frac{d}{p_j} = \frac{d}{p}\left(1 - \frac{|\pi_j|}{k-1}\right) \leqslant 2\left(1 - \frac{|\pi_j|}{k-1}\right) \leqslant (k-1)\left(1 - \frac{|\pi_j|}{k-1}\right) = k-1-|\pi_j|.$$

$$\square$$

REMARK 4.2 (Higher regularity of the profile $v_{\star}$). Collecting the results of Theorem 2.8 and Corollary 4.1, we obtain

$$f \in C^{\max\{2,k-1\}}(\mathbb{R}^N, \mathbb{R}^N), \ v_{\star} \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N) \implies v_{\star} - v_{\infty} \in W_{\theta}^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$$

for any $k \in \mathbb{N}$, provided that $1 < p < \infty$ from (A4) satisfies $p \geqslant \frac{d}{2}$ if $k \geqslant 3$.

4.3. Pointwise exponential decay. For the pointwise estimates we use the embedding in L^{∞}. The particular choice $q = \infty$ and $l = 0$ in (4.4) leads to

(4.14) $$W^{k,p}(\mathbb{R}^d) \subseteq L^{\infty}(\mathbb{R}^d), \quad k \geqslant 0, \ 1 < p < \infty, \ d < kp$$

and to the inequality

(4.15) $$\|u\|_{L^{\infty}(\mathbb{R}^d)} \leqslant C_{p,\infty,k,0} \|u\|_{W^{k,p}(\mathbb{R}^d)} \ \forall u \in W^{k,p}(\mathbb{R}^d).$$

COROLLARY 4.3 (Pointwise exponentially decaying estimates). *Let the assumptions of Corollary 4.1 be satisfied. Moreover, let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_{\star} \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ for some $k \in \mathbb{N}$ and let $p \geqslant \frac{d}{2}$ if $k \geqslant 3$. Then the function $v_{\star} - v_{\infty} \in W_{\theta}^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$ satisfies the following estimate*

(4.16) $$|D^{\alpha}(v_{\star}(x) - v_{\infty})| \leqslant C \exp\left(-\mu\sqrt{|x|^2 + 1}\right) \quad \forall x \in \mathbb{R}^d$$

for every exponential decay rate $0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max}p}$ and for every multi-index $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k - |\alpha|)p$.

PROOF. The proof follows from (4.4) and the Sobolev embedding (4.14).   □

REMARK 4.4. In case of $d \in \{2, 3\}$ and $p = 2$ it is sufficient to choose $k = 4$ to obtain pointwise estimates for $D^\alpha v_\star$ of order $0 \leqslant |\alpha| \leqslant 2$. This requires to assume $f \in C^3(\mathbb{R}^N, \mathbb{R}^N)$ and $v_\star \in C^5(\mathbb{R}^d, \mathbb{R}^N)$. Note that the authors of [5] consider the case $d = p = 2$ and assume $f \in C^4(\mathbb{R}^N, \mathbb{R}^N)$ for their stability analysis of rotating patterns. Our results show that $f \in C^3(\mathbb{R}^N, \mathbb{R}^N)$ is sufficient to guarantee [5, Ass.1] which, therefore, can be omitted.

4.4. Application to complex-valued systems. The next corollary extends the results from Theorem 2.8, Corollary 4.1 and Corollary 4.3 to complex-valued systems of type (2.5) which appear in several applications.

COROLLARY 4.5 (Exponential decay of $v_\star$ for $\mathbb{K} = \mathbb{C}$). *Let the assumptions* (A4) *and* (A6) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$. *Moreover, let* $g : \mathbb{R} \to \mathbb{C}^{N,N}$ *satisfy the following properties*

(A7$_g$) $g \in C^2(\mathbb{R}, \mathbb{C}^{N,N})$,

(A9$_g$) $A, g(0) \in \mathbb{C}^{N,N}$ *are simultaneously diagonalizable (over* $\mathbb{C}$),

(A11$_g$)   *There exists* $\beta_\infty > 0$ *such that for all* $w \in \mathbb{C}^N$ *with* $|w| = 1$

$$\mathrm{Re} \langle w, -g(0)w \rangle \geqslant \beta_\infty,$$

and define

(4.17) $f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u.$

Further, let $a_{\max} = \rho(A)$ *denote the spectral radius of* A, $-a_0 = s(-A)$ *the spectral bound of* $-A$, $-b_0 = s(g(0))$ *the spectral bound of* $g(0)$ *and let* $\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ *a weight function with* $\mu \in \mathbb{R}$. *Then, for every* $0 < \varepsilon < 1$ *there is a constant* $K_1 = K_1(A, g, d, p, \varepsilon) > 0$ *with the following property: Every classical solution* $v_\star \in C^k(\mathbb{R}^d, \mathbb{C}^N)$ *of*

(4.18) $A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0, \ x \in \mathbb{R}^d,$

such that $g \in C^{\max\{2, k-1\}}(\mathbb{R}, \mathbb{C}^{N,N})$ *for some* $k \in \mathbb{N}$, $p \geqslant \frac{d}{2}$ *if* $k \geqslant 3$, *and*

(4.19) $\sup_{|x| \geqslant R_0} |v_\star(x)| \leqslant K_1$ *for some* $R_0 > 0$,

satisfies

$$v_\star \in W^{k,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.$$

Moreover, $v_\star$ *satisfies the following pointwise estimate*

$$|D^\alpha v_\star(x)| \leqslant C \exp\left(-\mu\sqrt{|x|^2 + 1}\right) \quad \forall \, x \in \mathbb{R}^d$$

for every exponential decay rate $0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}$ *and for every multi-index* $\alpha \in \mathbb{N}_0^d$ *satisfying* $d < (k - |\alpha|)p$.

PROOF. We transform the N-dimensional complex-valued system (4.18) into the $2N$-dimensional real-valued system

$$(4.20) \qquad \mathbf{A}\triangle\mathbf{v}(x) + \langle Sx, \nabla\mathbf{v}(x)\rangle + \mathbf{f}(\mathbf{v}(x)) = 0, \ x \in \mathbb{R}^d,$$

For this purpose, we decompose $A = A_1 + iA_2$ with $A_1, A_2 \in \mathbb{R}^{N,N}$, $v = v_1 + iv_2$ with $v_1, v_2 : \mathbb{R}^d \to \mathbb{R}^N$, $f_1, f_2 : \mathbb{R}^{2N} \to \mathbb{R}^N$ with $f_1(u_1, u_2) = \mathrm{Re}\, f(u_1 + iu_2)$, $f_2(u_1, u_2) = \mathrm{Im}f(u_1 + iu_2)$, $g = g_1 + ig_2$ with $g_1, g_2 : \mathbb{R} \to \mathbb{R}^{N,N}$. Moreover, we define $\mathbf{A} \in \mathbb{R}^{2N,2N}$, $\mathbf{v} \in \mathbb{R}^{2N}$ and $\mathbf{f} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ by

$$\mathbf{A} := \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, \ \mathbf{v} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ \mathbf{f}(\mathbf{v}) := \begin{pmatrix} f_1(\mathbf{v}) \\ f_2(\mathbf{v}) \end{pmatrix} = \begin{pmatrix} g_1(|\mathbf{v}|^2) & -g_2(|\mathbf{v}|^2) \\ g_2(|\mathbf{v}|^2) & g_1(|\mathbf{v}|^2) \end{pmatrix} \mathbf{v}.$$

Let us apply Theorem 2.8 to the $2N$-dimensional problem (4.20) and check its assumptions. First, we collect the following relations of A and $\mathbf{A}$:

$$(4.21) \quad \lambda \in \sigma(A) \quad \Longleftrightarrow \quad \lambda, \overline{\lambda} \in \sigma(\mathbf{A}),$$

$$(4.22) \quad Y^{-1}AY = \Lambda_A \quad \Longleftrightarrow \quad \begin{pmatrix} iY & \overline{Y} \\ Y & -\overline{iY} \end{pmatrix}^{-1} \mathbf{A} \begin{pmatrix} iY & \overline{Y} \\ Y & -\overline{iY} \end{pmatrix} = \begin{pmatrix} \Lambda_A & 0 \\ 0 & \overline{\Lambda_A} \end{pmatrix},$$

$$(4.23) \quad \mathrm{Re}\, \langle v, Av\rangle = \langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle, \ |v| = |\mathbf{v}|, \ |Av| = |\mathbf{A}\mathbf{v}|.$$

Since A satisfies (A4) for some $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$, we deduce from (4.23), that $\mathbf{A}$ satisfies (A4) for the same $1 < p < \infty$ and $\mathbb{K} = \mathbb{R}$. In particular, we have $\gamma_A = \gamma_{\mathbf{A}}$ in (A4). Note that if A satisfies (A1), (A2), (A3) for $\mathbb{K} = \mathbb{C}$ then $\mathbf{A}$ satisfies (A1), (A2), (A3) for $\mathbb{K} = \mathbb{R}$, as follows from (4.22), (4.21), (4.23). Assumption (A6) is not affected by the transformation. From (A7$_g$) we deduce that $\mathbf{f} \in C^2(\mathbb{R}^{2N}, \mathbb{R}^{2N})$, so that assumption (A7) is satisfied for $\mathbb{K} = \mathbb{R}$. Obviously, $\mathbf{f}(v_\infty) = 0$ holds for $v_\infty = 0 \in \mathbb{R}^{2N}$, so that condition (A8) is satisfied. Since A and $g(0)$ are simultaneously diagonalizable (over $\mathbb{C}$), cf. (A9$_g$), we deduce from (4.22) that $\mathbf{A}$ and

$$D\mathbf{f}(0) = \begin{pmatrix} g_1(0) & -g_2(0) \\ g_2(0) & g_1(0) \end{pmatrix}$$

are simultaneously diagonalizable (over $\mathbb{C}$). This proves assumption (A9) for $\mathbb{K} = \mathbb{R}$. Finally, (A11$_g$) implies (A11) with β_∞ given by (A11$_g$). Every classical solution $v_\star$ of (4.18) satisfying $v_\star \in C_\mathrm{b}(\mathbb{R}^d, \mathbb{C}^N)$ and (4.19) leads to a classical solution

$$\mathbf{v}_\star := \begin{pmatrix} \mathrm{Re}\, v_\star \\ \mathrm{Im}v_\star \end{pmatrix}$$

of (4.20) satisfying $\mathbf{v}_\star \in C_\mathrm{b}(\mathbb{R}^d, \mathbb{R}^{2N})$ and (4.19). Summarizing, Theorem 2.8 yields $\mathbf{v}_\star \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^{2N})$, and thus $v_\star \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$. □

In Section 6 we will apply this result to the cubic-quintic complex Ginzburg-Landau equation which is of the form (4.17). Other examples fitting into this class include the Schrödinger and the Gross-Pitaevskii equation.

5. Exponential decay of eigenfunctions

Consider the eigenvalue problem

$$(5.1) \qquad A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x), \ x \in \mathbb{R}^d, \ d \geqslant 2.$$

We are interested in *classical solutions* (λ, v) *of* (5.1), i.e. $\lambda \in \mathbb{C}$ and $v \in C^2(\mathbb{R}^d, \mathbb{C}^N)$ solves (5.1) pointwise (cf. Definition 2.6).

5.1. Sobolev and pointwise estimates of eigenfunctions. The following theorem states that every classical solution v of the eigenvalue problem (5.1) decays exponentially in space, provided its associated (isolated) eigenvalue λ satisfies $\operatorname{Re}\lambda > -\beta_\infty$. The proof is similar to those of Theorem 2.8 and Corollary 4.1, but now it is crucial that Theorem 3.5 can be employed for cases where $\lambda \neq 0$, $g \neq 0$ and $\mathbb{K} = \mathbb{C}$.

THEOREM 5.1 (Exponential decay of eigenfunctions). (**1**) *Let the assumptions* (A4), (A6)–(A9) *and* (A11) *be satisfied for* $\mathbb{K} = \mathbb{R}$ *and for some* $1 < p < \infty$. *Moreover, let* $a_{\max} = \rho(A)$ *denote the spectral radius of* A, $-a_0 = s(-A)$ *the spectral bound of* $-A$, $-b_0 = s(Df(v_\infty))$ *the spectral bound of* $Df(v_\infty)$ *and let* β_∞ *be from* (A11). *Further, let*

$$\theta_j(x) = \exp\left(\mu_j\sqrt{|x|^2+1}\right), \quad x \in \mathbb{R}^d, \quad j = 1,2$$

denote a weight function for $\mu_1, \mu_2 \in \mathbb{R}$. *Then, for every* $0 < \varepsilon < 1$ *there is a constant* $K_1 = K_1(A, f, v_\infty, d, p, \varepsilon) > 0$ *such that for every classical solution* $v_\star$ *of* (2.8) *satisfying* (2.9) *the following property holds: Every classical solution* v *of the eigenvalue problem*

$$(5.2) \qquad A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d,$$

with $\lambda \in \mathbb{C}$ *and* $\operatorname{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty$, *such that* $v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ *for some exponential growth rate*

$$(5.3) \qquad -\sqrt{\varepsilon\frac{\gamma_A\beta_\infty}{2d|A|^2}} \leqslant \mu_1 \leqslant 0,$$

satisfies

$$v \in W^{1,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N)$$

for every exponential decay rate

$$(5.4) \qquad 0 \leqslant \mu_2 \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max}p}.$$

(**2**) *If additionally,* $p \geqslant \frac{d}{2}$, $f \in C^k(\mathbb{R}^N, \mathbb{R}^N)$, $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ *and* $v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^N)$ *for some* $k \in \mathbb{N}$ *with* $k \geqslant 2$, *then* $v \in W^{k,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$ *holds. Moreover,* v *satisfies the pointwise estimate*

$$(5.5) \qquad |D^\alpha v(x)| \leqslant C\exp\left(-\mu_2\sqrt{|x|^2+1}\right), \quad x \in \mathbb{R}^d$$

for every exponential decay rate $0 \leqslant \mu_2 \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max}p}$ *and for every multi-index* $\alpha \in \mathbb{N}_0^d$ *satisfying* $d < (k-|\alpha|)p$.

REMARK 5.2. In case of $d \in \{2,3\}$ and $p = 2$ it is sufficient to choose $k = 5$ to obtain pointwise estimates for v of order $0 \leqslant |\alpha| \leqslant 2$. This requires to assume $f \in C^5(\mathbb{R}^N, \mathbb{R}^N)$, $v_\star \in C^6(\mathbb{R}^d, \mathbb{R}^N)$ and $v \in C^6(\mathbb{R}^d, \mathbb{C}^N)$. Moreover, note that the space $L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ does not only allow bounded but even exponentially growing eigenfunctions.

PROOF. (**1**) Let $v_\star$ be a classical solution of (2.8) satisfying (2.9) and let v be a classical solution of (5.2) satisfying $v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N)$ with θ_1 from (3.19) and μ_1 such that (5.3). Then v satisfies

$$0 = \lambda v(x) - (A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - B_\infty v(x) + Q(x)v(x))$$

$$= [(\lambda I - \mathcal{L}_Q)\, v](x),\ x \in \mathbb{R}^d$$

with $B_\infty := -Df(v_\infty)$ and $Q(x) := Df(v_\star(x)) - Df(v_\infty)$ as in (4.8). Now, $v \in W_{\theta_2}^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ follows from Theorem 3.5 with $\mu_1 < 0$, $g = 0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty$. Note, that the assumptions of Theorem 3.5 are satisfied as shown in the proof of Corollary 4.1.

(**2**) Assuming more smoothness for $f, v, v_\star$, we prove $v \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^N)$ by induction on k. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index of length $|\alpha| = k - 1$. Similar to the proof of Corollary 4.1, an application of D^α to (5.2) yields an inhomogenous equation for $w := D^\alpha v$,

$$(5.6) \qquad\qquad [(\lambda I - \mathcal{L}_Q)w](x) = g(x),\ x \in \mathbb{R}^d$$

with matrices B_∞ and $Q(x)$ as in (4.8), $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty$, and $g(x) := g_1(x) + g_2(x)$ where

$$g_1 := \sum_{i=1}^d \sum_{\substack{j=1 \\ e_j \leqslant \alpha}}^d S_{ij} \binom{\alpha}{e_j} D^{\alpha - e_j + e_i} v,$$

$$g_2 := \sum_{\substack{\beta \leqslant \alpha \\ |\beta| \geqslant 1}} \binom{\alpha}{\beta} \sum_{\ell=1}^{|\beta|} \sum_{\pi \in \mathcal{P}_{\ell,|\beta|}} (D^{\ell+1} f)(v_\star) \left[D^{|\pi_1|} v_\star h_{\pi_1}, \ldots, D^{|\pi_\ell|} v_\star h_{\pi_\ell}, D^{\alpha - \beta} v \right].$$

In this expression, the multilinear argument h is defined as in (4.13) with α replaced by β. Below we prove $g_1, g_2 \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{C}^N)$, so that Theorem 3.5 implies $w = D^\alpha v \in W_{\theta_2}^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ and therefore, $v \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^N)$.

First we consider g_1: By the first part of this Corollary (base case $k = 2$) and by the induction hypethesis (induction step $k > 2$) we have $v \in W_{\theta_2}^{k-1,p}(\mathbb{R}^d, \mathbb{C}^N)$. As in the proof of Corollary 4.1, we then deduce $g_1 \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{R}^N)$. Finally, we show $g_2 \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{R}^N)$ by applying Hölder's inequality (4.5) with $p_j := \frac{k}{|\pi_j|} p$ and $u_j := \theta_2^{\frac{1}{j}} \left| D^{|\pi_j|} v_\star h_{\pi_j} \right|$ for $j = 1, \ldots, \ell$, $p_{\ell+1} := \frac{k}{k-|\beta|} p$ and $u_{\ell+1} := \left| D^{\alpha - \beta} v \right|$. Note that $\sum_{j=1}^{\ell+1} \frac{1}{p_j} = \frac{1}{p}$ follows from $\sum_{j=1}^\ell |\pi_j| = |\beta|$. We obtain

$$\left\| \theta_2 (D^{\ell+1} f)(v_\star) \left[D^{|\pi_1|} v_\star h_{\pi_1}, \ldots, D^{|\pi_\ell|} v_\star h_{\pi_\ell}, D^{\alpha-\beta} v \right] \right\|_{L^p}$$

$$\leqslant \left\| (D^{\ell+1} f)(v_\star) \right\|_{L^\infty} \left\| |D^{\alpha-\beta} v| \prod_{j=1}^\ell \theta_2^{\frac{1}{j}} \left| D^{|\pi_j|} v_\star h_{\pi_j} \right| \right\|_{L^p}$$

$$\leqslant \left\| (D^{\ell+1} f)(v_\star) \right\|_{L^\infty} \left\| D^{\alpha-\beta} v \right\|_{L^{p_{\ell+1}}} \prod_{j=1}^\ell \left\| \theta_2^{\frac{1}{j}} D^{|\pi_j|} v_\star h_{\pi_j} \right\|_{L^{p_j}}$$

$$\leqslant \left\| (D^{\ell+1} f)(v_\star) \right\|_{L^\infty} \left\| D^{\alpha-\beta} v \right\|_{L^{p_{\ell+1}}} \prod_{j=1}^\ell \left\| \theta_2 D^{|\pi_j|} v_\star h_{\pi_j} \right\|_{L^{p_j}}$$

since $\theta_2(x) \geqslant 1$ and $j \geqslant 1$ imply $\theta_2^{\frac{1}{j}}(x) \leqslant \theta_2(x)$. Note that $v_\star \in C_b(\mathbb{R}^d, \mathbb{R}^N)$, $f \in C^k(\mathbb{R}^N, \mathbb{R}^N)$ and $\ell \leqslant |\beta| \leqslant |\alpha| = k - 1$ imply the boundedness of $(D^{\ell+1} f)(v_\star)$. As in the proof of Corollary 4.1, both $\theta_2 D^{|\pi_j|} v_\star h_{\pi_j} \in L^{p_j}(\mathbb{R}^d, \mathbb{R}^N)$ and $\theta_2 D^{\alpha-\beta} v \in L^{p_{\ell+1}}(\mathbb{R}^d, \mathbb{C}^N)$ follow from the Sobolev embedding (4.3): Since $f \in C^{k-1}(\mathbb{R}^N, \mathbb{R}^N)$

and $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$, Corollary 4.1 implies that $v_\star - v_\infty \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$, therefore $\theta_2 D^{|\pi_j|} v_\star h_{\pi_j} \in W^{k-|\pi_j|,p}(\mathbb{R}^d, \mathbb{R}^N)$. The Sobolev embedding (4.3) shows

$$\theta_2 D^{|\pi_j|} v_\star h_{\pi_j} \in W^{k-|\pi_j|,p}(\mathbb{R}^d, \mathbb{R}^N) \subseteq L^{p_j}(\mathbb{R}^d, \mathbb{R}^N) \ \forall\, 1 \leqslant |\pi_j| \leqslant |\beta|,$$

provided that $1 < p < p_j < \infty$ and $\frac{d}{p} - (k - |\pi_j|) \leqslant \frac{d}{p_j}$. These conditions are obviously satisfied, since $p \geqslant \frac{d}{2}$, $1 \leqslant |\pi_j| \leqslant k - 1$. Next, since $v \in W_{\theta_2}^{k-1,p}(\mathbb{R}^d, \mathbb{C}^N)$, (4.4) implies $\theta_2 D^\gamma v \in W^{k-1-|\gamma|,p}(\mathbb{R}^d, \mathbb{C}^N)$ for all $0 \leqslant |\gamma| \leqslant k - 1$. For $\gamma = \alpha - \beta$ with $|\alpha| = k - 1$ and $\beta \leqslant \alpha$ we have $k - 1 - |\gamma| = |\beta|$ and therefore, the Sobolev embedding (4.3) implies

$$\theta_2 D^{\alpha-\beta} v \in W^{|\beta|,p}(\mathbb{R}^d, \mathbb{R}^N) \subset L^{p_{\ell+1}}(\mathbb{R}^d, \mathbb{R}^N),$$

provided that $1 < p < p_{\ell+1} < \infty$ and $\frac{d}{p} - |\beta| \leqslant \frac{d}{p_{\ell+1}}$. These conditions are satisfied by the same arguments as above. This concludes the proof of $g_2 \in L_{\theta_2}^p(\mathbb{R}^d, \mathbb{R}^N)$.

Finally, the pointwise estimates follow when combining our previous Sobolev estimates with (4.4) and the embedding (4.14), in a similar manner as in Corollary 4.3 $\square$

REMARK 5.3 (Higher regularity of the eigenfunction v). Collecting the results of Theorem 5.1 and Remark 4.2, we obtain

$$f \in C^{\max\{2,k\}}, \ v_\star, v \in C^{k+1} \implies v_\star - v_\infty, v \in W_{\theta_2}^{k,p}$$

for any $k \in \mathbb{N}$, provided that $1 < p < \infty$ from (A4) satisfies $p \geqslant \frac{d}{2}$ if $k \geqslant 2$. We also recall the role of the parameter ε in the exponential estimates. Theorem 5.1 shows that every eigenfunction associated to an eigenvalue λ with $\operatorname{Re} \lambda > -\beta_\infty$ decays exponentially in space. Usually, one expects this behavior even for $\operatorname{Re} \lambda > -b_0$. The rate of decay is controlled by $\varepsilon \in (0,1)$. If $\operatorname{Re} \lambda \geqslant -(1+\varepsilon)\beta_\infty$ is close to $-\beta_\infty$ we may take ε close to 0 and obtain a small rate μ_2 of decay according to (5.4). On the other hand, if λ is close to the imaginary axis we may take ε close to 1 and obtain a higher rate of decay.

5.2. Eigenfunctions belonging to eigenvalues on the imaginary axis. Some classical solutions of the eigenvalue problem (5.1) are due to equivariance of the underlying equations and can be expressed in terms of the rotating wave itself. The following result proved in [26, Thm.9.4], specifies these eigenfunctions.

THEOREM 5.4 (Point spectrum on the imaginary axis). *Let $S \in \mathbb{R}^{d,d}$ be skew-symmetric and let $U \in \mathbb{C}^{d,d}$ denote the unitary matrix satisfying $\Lambda_S = \bar{U}^T S U$ with $\Lambda_S = \operatorname{diag}(\lambda_1^S, \ldots, \lambda_d^S)$ and eigenvalues $\lambda_1^S, \ldots, \lambda_d^S$ of S. Moreover, let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ be a classical solution of (2.8), then $v : \mathbb{R}^d \to \mathbb{C}^N$ given by*

$$v(x) = \langle C^{\mathrm{rot}} x + C^{\mathrm{tra}}, \nabla v_\star(x) \rangle$$

(5.7)
$$= \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} C_{ij}^{\mathrm{rot}}(x_j D_i - x_i D_j) v_\star(x) + \sum_{l=1}^{d} C_l^{\mathrm{tra}} D_l v_\star(x)$$

is a classical solution of the eigenvalue problem (5.1) if $C^{\mathrm{rot}} \in \mathbb{C}^{d,d}$, $C^{\mathrm{tra}} \in \mathbb{C}^d$ either satisfy

$$\lambda = -\lambda_l^S, \quad C^{\mathrm{rot}} = 0, \quad C^{\mathrm{tra}} = U e_l$$

for some $l = 1, \ldots, d$, *or*

$$\lambda = -(\lambda_n^S + \lambda_m^S), \quad C^{\mathrm{rot}} = U(I_{nm} - I_{mn})U^T, \quad C^{\mathrm{tra}} = 0$$

for some $n = 1, \ldots, d - 1$ *and* $m = n + 1, \ldots, d$. *Here* $I_{nm} \in \mathbb{R}^{d,d}$ *denotes the matrix having the entries* 1 *at the* n-*th row and* m-*th column and* 0 *otherwise. All the eigenvalues above lie on the imaginary axis.*

A direct consequence of Theorem 5.1 and Theorem 5.4 is that the eigenfunctions v from (5.7) belong to $W_{\theta_2}^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$ and decay exponentially in space, [**26**, Thm.9.8].

COROLLARY 5.5 (Exponential decay of eigenfunctions for eigenvalues on $i\mathbb{R}$). *Let all assumptions of the statements* (**1**) *and* (**2**) *of Theorem 5.1 be satisfied. Then the classical solution*

$$v(x) = \left\langle C^{\mathrm{rot}}x + C^{\mathrm{tra}}, \nabla v_\star(x) \right\rangle, \quad x \in \mathbb{R}^d$$

of the eigenvalue problem (5.2) *with* λ, C^{rot} *and* C^{tra} *from Theorem 5.4 lies in* $W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^N)$ *for every exponential decay rate* (5.4). *Moreover, the function* $v \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^N)$ *satisfies the pointwise estimate* (5.5).

PROOF. In order to apply Theorem 5.1 to $v(x) = \langle C^{\mathrm{rot}}x + C^{\mathrm{tra}}, \nabla v_\star(x) \rangle$, we observe that the map $x \mapsto \langle C^{\mathrm{rot}}x + C^{\mathrm{tra}}, \nabla v_\star(x) \rangle$ is of class C^{k+1} since $v_\star \in C^{k+2}(\mathbb{R}^d, \mathbb{R}^N)$. In particular, $\beta_\infty > 0$ allows to deal with eigenvalues $\lambda \in i\mathbb{R}$. □

REMARK 5.6. Later on, we numerically approximate the spectrum of the linearization $\mathcal{L}$ for (2.12). For this purpose we decompose the L^p-spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ into the disjoint union of point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$ and essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$

$$(5.8) \qquad \sigma(\mathcal{L}) = \sigma_{\mathrm{point}}(\mathcal{L}) \cup \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The point spectrum of $\mathcal{L}$ is affected by the symmetries of the group action and contains the eigenvalues described in Theorem 5.4. In particular, it contains the spectrum of S and the sum of its different eigenvalues, i.e.

$$(5.9) \qquad \sigma_{\mathrm{point}}^{\mathrm{part}}(\mathcal{L}) := \sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S), \ \lambda_1 \neq \lambda_2\}. \subseteq \sigma_{\mathrm{point}}(\mathcal{L})$$

The associated eigenfunctions are explictly known, see (5.7), and they are exponentially localized as shown in Corollary 5.5. In general, $\sigma_{\mathrm{point}}(\mathcal{L})$ contains further eigenvalues. Neither these additional eigenvalues nor their associated eigenfunctions can usually be determined explicitly. The essential spectrum of $\mathcal{L}$ depends on the asymptotic behavior of the wave at infinity. Under the same assumptions as in Theorem 2.8 one derives a dispersion relation for rotating waves, see [**26**, Sec.9.5],

$$(5.10) \quad \det\left(\lambda I_N + \omega^2 A - Df(v_\infty) + i\sum_{l=1}^{m} n_l \sigma_l I_N\right) = 0 \text{ for some } \omega \in \mathbb{R}, \ n_l \in \mathbb{Z}$$

which then yields information about the essential spectrum

$$(5.11) \qquad \sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid \lambda \text{ satisfies } (5.10)\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

Here, S has the $(d - 2m)$–fold eigenvalue 0 and nontrivial eigenvalues $\pm i\sigma_l, l = 1, \ldots, m$ on the imaginary axis. For more details we refer to the examples in Section 6 and to [**26**, Ch.9,10].

6. Rotating waves in reaction diffusion systems: The cubic-quintic complex Ginzburg-Landau equation

Consider the *cubic-quintic complex Ginzburg-Landau equation (QCGL)*, [19]

$$(6.1) \qquad u_t = \alpha \triangle u + u\left(\delta + \beta |u|^2 + \gamma |u|^4\right)$$

where $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{C}$, $d \in \{2,3\}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$ and $f : \mathbb{C} \to \mathbb{C}$ given by

$$(6.2) \qquad f(u) := u\left(\delta + \beta |u|^2 + \gamma |u|^4\right).$$

The real-valued version of (6.1) reads as follows

$$(6.3) \qquad \mathbf{u}_t = \mathbf{A}\triangle\mathbf{u} + \mathbf{f}(\mathbf{u}) \quad \text{with} \quad \mathbf{A} := \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(6.4) \qquad \mathbf{f}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} (u_1\delta_1 - u_2\delta_2) + (u_1\beta_1 - u_2\beta_2)|\mathbf{u}|^2 + (u_1\gamma_1 - u_2\gamma_2)|\mathbf{u}|^4 \\ (u_1\delta_2 + u_2\delta_1) + (u_1\beta_2 + u_2\beta_1)|\mathbf{u}|^2 + (u_1\gamma_2 + u_2\gamma_1)|\mathbf{u}|^4 \end{pmatrix},$$

where $|\mathbf{u}|^2 = u_1^2 + u_2^2$, $u = u_1 + iu_2$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\delta = \delta_1 + i\delta_2$.

This equation describes different aspects of signal propagation in heart tissue, superconductivity, superfluidity, nonlinear optical systems, see [23], photonics, plasmas, physics of lasers, Bose-Einstein condensation, liquid crystals, fluid dynamics, chemical waves, quantum field theory, granular media and is used in the study of hydrodynamic instabilities, see [21]. It shows a variety of coherent structures like stable and unstable pulses, fronts, sources and sinks in 1D, see [3, 36, 38, 39], vortex solitons, see [13], spinning solitons, see [14], dissipative ring solitons, see [33], rotating spiral waves, propagating clusters, see [30], and exploding dissipative solitons, see [34] in 2D as well as scroll waves and spinning solitons in 3D, see [22].

We are interested in exponentially localized rotating wave solutions $u_\star : \mathbb{R}^d \times [0,\infty) \to \mathbb{C}$ of (6.1) and $\mathbf{u}_\star : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^2$ of (6.3). Note that, given some skew-symmetric $S \in \mathbb{R}^{d,d}$ and some vector $x_\star \in \mathbb{R}^d$, the function $u_\star(x,t) = v_\star(e^{-tS}(x - x_\star))$ with $v_\star : \mathbb{R}^d \to \mathbb{C}$ is a rotating wave of (6.1) if and only if $\mathbf{u}_\star(x,t) = \mathbf{v}_\star(e^{-tS}(x - x_\star))$ is a rotating wave of (6.3), where $\mathbf{u}_\star = \begin{pmatrix} \operatorname{Re} u_\star \\ \operatorname{Im} u_\star \end{pmatrix}$ and $\mathbf{v}_\star = \begin{pmatrix} \operatorname{Re} v_\star \\ \operatorname{Im} v_\star \end{pmatrix}$. We are going to show that $v_\star$ (and $\mathbf{v}_\star$) are exponentially localized by applying Theorem 2.8 and Corollaries 4.1, 4.3 to the real-valued system (6.3) and Corollary 4.5 to the complex equation (6.1).

First, consider the assumptions (A1)–(A11) for $\mathbb{K} = \mathbb{R}$: With $\mathbf{A}$ from (6.3) and $\mathbf{f}$ from (6.4), Assumption (A1) is satisfied for every $\alpha \in \mathbb{C}$, since

$$(6.5) \qquad Y^{-1}\mathbf{A}Y = \Lambda_\mathbf{A}, \ \Lambda_\mathbf{A} = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \ Y = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \ Y^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}.$$

Assumption (A2) follows from $\operatorname{Re} \alpha > 0$, since $\sigma(\mathbf{A}) = \{\alpha, \overline{\alpha}\}$. Assumption (A3) holds with $\beta_\mathbf{A} = \operatorname{Re} \alpha$ if $\operatorname{Re} \alpha > 0$, since $\operatorname{Re}\langle w, \mathbf{A}w\rangle = \operatorname{Re} \alpha$ for $w \in \mathbb{R}^2$ with $|w| = 1$. Condition (A5), which is equivalent to (A4), requires $\alpha \neq 0$ and $\frac{\operatorname{Re}\alpha}{|\alpha|} = \mu_1(\mathbf{A}) >$

$\frac{|p-2|}{p}$. The latter condition is equivalent to

$$|\arg \alpha| < \arctan \left(\frac{2\sqrt{p-1}}{|p-2|} \right) \text{ for some } 1 < p < \infty,$$

or alternatively to

$$(6.6) \qquad p_{\min} := \frac{2|\alpha|}{|\alpha| + \operatorname{Re} \alpha} < p < \frac{2|\alpha|}{|\alpha| - \operatorname{Re} \alpha} =: p_{\max}.$$

The condition (A6) is satisfied with $S \in \mathbb{R}^{d,d}$ given by

$$(6.7) \qquad S = \begin{pmatrix} 0 & S_{12} \\ -S_{12} & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & S_{12} & S_{13} \\ -S_{12} & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{pmatrix}$$

for $d = 2$ and $d = 3$, respectively. Below we specify the entries $S_{12}, S_{13}, S_{23} \in \mathbb{R}$ and the point $x_\star \in \mathbb{R}^d$, that will be the center of rotation if $d = 2$ and a support vector of the axis of rotation if $d = 3$, cf. (2.2). All this information come actually from a simulation. First we simulate the original system for some time. Then we switch to the freezing method, which yields the profile $v_\star$, its center of rotation, and its rotational velocities. For more details on the computation, see [**26**, Sec.10.3]. Some general theory and applications of the freezing method may be found in [**7, 8, 9, 10, 37**]. Note that in case $d = 2$ we have a clockwise rotation, if $S_{12} > 0$, and a counter clockwise rotation, if $S_{12} < 0$. Assumption (A7) is obviously satisfied, even with $\mathbf{f} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, since every component of $\mathbf{f}$ is a polynomial. With $\mathbf{v}_\infty = (0,0)^T$, the assumption (A8) is satisfied, and for this choice we have

$$D\mathbf{f}(\mathbf{v}_\infty) = \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{pmatrix}.$$

Assumption (A9) holds for the same transformation matrix Y as in (6.5). The condition $\operatorname{Re} \delta < 0$ implies both, Assumption (A10) and Assumption (A11) with $\beta_\infty = -\operatorname{Re} \delta$.

Next we consider assumptions (A7$_g$), (A9$_g$) and (A11$_g$): Writing f as $f(u) = g(|u|^2)u$ with

$$g : \mathbb{R} \to \mathbb{C}, \quad g(v) = \delta + \beta v + \gamma v^2,$$

Assumption (A7$_g$) is obivously satisfied and we even have $g \in C^\infty(\mathbb{R}, \mathbb{C})$. Assumption (A9$_g$) is satisfied with $g(0) = \delta$ for every $\alpha, \delta \in \mathbb{C}$ and assumption (A11$_g$) with $\beta_\infty = -\operatorname{Re} \delta$ if $\operatorname{Re} \delta < 0$. The assumptions (A1)–(A5) for $A = \alpha \in \mathbb{C}$ lead to the same requirements as in the real-valued case.

Our discussion shows that if we assume

$$(6.8) \qquad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \delta < 0, \quad p_{\min} = \frac{2|\alpha|}{|\alpha| + \operatorname{Re} \alpha} < p < \frac{2|\alpha|}{|\alpha| - \operatorname{Re} \alpha} = p_{\max},$$

we can apply Theorem 2.8, Corollary 4.1 and Corollary 4.3 to the real-valued system (6.3), and Corollary 4.5 to the complex-valued equation (6.1). In both cases, the bound for the rate of the exponential decay reads

$$(6.9) \qquad 0 \leqslant \mu \leqslant \varepsilon \frac{\nu}{p}, \quad \text{for } \nu = \frac{\sqrt{\operatorname{Re} \alpha \, (-\operatorname{Re} \delta)}}{|\alpha|} \text{ and some } 0 < \varepsilon < 1,$$

since $a_0 = \operatorname{Re} \alpha$, $b_0 = -\operatorname{Re} \delta$ and $a_{\max} = |\alpha|$.

6.1. Spinning solitons. For the parameter values from [14], given by

$$(6.10) \qquad \alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \delta = -\frac{1}{2},$$

equation (6.1) exhibits so called *spinning soliton* solutions for space dimensions $d = 2$ and $d = 3$, see Figure 3. The parameter values (6.10) satisfy the requirements from (6.8), and therefore our assumptions (A1)–(A11) for every p with

$$(6.11) \qquad 1.1716 \approx \frac{4}{2 + \sqrt{2}} = p_{\min} < p < p_{\max} = \frac{4}{2 - \sqrt{2}} \approx 6.8284,$$

e.g. for $p = 2, 3, 4, 5, 6$. Therefore, the solitons (and their derivatives) are exponentially localized in the sense of Theorem 2.8 and Corollary 4.1, i.e. $\mathbf{v}_\star$ belongs to $W_\theta^{2,p}(\mathbb{R}^d, \mathbb{R}^2)$ for $p \in (p_{\min}, p_{\max})$ and for the weight function $\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ with exponential decay rate

$$(6.12) \qquad 0 \leqslant \mu \leqslant \frac{\varepsilon}{\sqrt{2}p}.$$

Corollary 4.1 implies that $\mathbf{v}_\star$ even belongs to $W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^2)$ for every $k \geqslant 0$, provided $d = 2$, $p \in (p_{\min}, p_{\max})$ or $d = 3$, $p \in [\frac{3}{2}, p_{\max})$ since $\mathbf{f} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Moreover, Corollary 4.3 shows that the solitons satisy the pointwise estimates (4.16). In Section 6.3 we will compare this with the rate of decay measured from numerical experiments.

Next, we discuss the numerical results from Figure 3 and explain how to compute the profile and velocities numerically. For all numerical computations including the eigenvalue computations we used Comsol Multiphysics 5.2, [1].

Figure 3(a) shows the spinning soliton in $\mathbb{R}^2$ as the solution of (6.1) on a circular disk of radius $R = 20$ centered at the origin at time $t = 150$. For the computation we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.25$, the BDF method of order 2 with absolute tolerance atol $= 10^{-5}$, relative tolerance rtol $= 10^{-3}$ and maximal stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0^{2D}(x_1, x_2) = \frac{1}{5}(x_1 + ix_2)\exp\left(-\frac{x_1^2 + x_2^2}{49}\right), \quad x_1^2 + x_2^2 \leq R^2.$$

Figure 3(b) shows isosurfaces of the spinning soliton in $\mathbb{R}^3$ obtained from the solution of (6.1) on a cube $[-10, 10]^3$ at time $t = 100$. For the computation we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.8$, the BDF method of order 2 with absolute tolerance atol $= 10^{-4}$, relative tolerance rtol $= 10^{-2}$ and maximal stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and (discontinuous) initial data

$$u_0^{3D}(x_1, x_2, x_3) = u_0^{2D}(x_1, x_2) \text{ for } |x_3| < 9 \text{ and } 0 \text{ otherwise.}$$

Using these solutions as initial data, the freezing method from [7, 8] provides an approximate rotating wave with profile $\mathbf{w}_\star$ in the following format

$$(6.13) \qquad \mathbf{u}_\star(x, t) = \mathbf{w}_\star(\exp(-tS)(x - tE(tS)\tau)), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R},$$

where $S \in \mathbb{R}^{d,d}$ is skew symmetric, $\tau \in \text{range}(S) \subset \mathbb{R}^d$ and E is the analytic function $E(z) = \sum_{j=1}^\infty \frac{z^{j-1}}{j!}$ satisfying $E(z)z = \exp(z) - 1, z \in \mathbb{C}$. In order to put
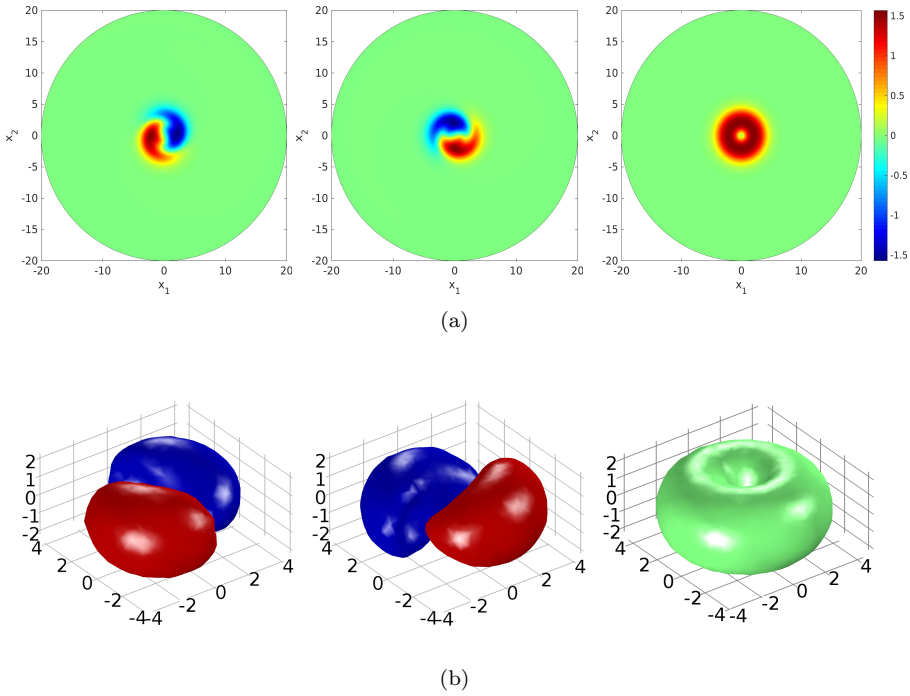
(a)

(b)

FIGURE 3. Spinning soliton of QCGL (6.1) for $d = 2$ (a) and $d = 3$ (b) with real part (left), imaginary part (middle) and absolute value (right). The colorbar in (a) reaches from -1.6 (blue) to 1.6 (red). The isosurfaces in (b) have values -0.5 (blue), 0.5 (red) and 0.5 (green).

this into the standard form (2.2) used in our theory, we determine the position vector $x_\star \in \mathbb{R}^d$ by solving

$$(6.14) \qquad\qquad\qquad Sx_\star + \tau = 0.$$

Then we may write

$$\exp(-tS)(x - tE(tS)\tau) = \exp(-tS)(x + E(tS)(tS)x_\star)$$
$$= \exp(-tS)(x - x_\star) + x_\star,$$

so that (6.13) turns into

$$(6.15) \qquad \mathbf{u}_\star(x, t) = \mathbf{w}_\star(\exp(-tS)(x - x_\star) + x_\star) = \mathbf{v}_\star(\exp(-tS)(x - x_\star)).$$

Thus the numerical profile $\mathbf{w}_\star$ is a slightly shifted version of the profile $\mathbf{v}_\star$ used for the theory. In practice, we solve (6.14) directly for $d = 2$ and by rank-deficient least squares for $d = 3$, see [**26**, Ex.10.8] for details.

Using for $d = 2$ the data at time $t = 400$, one obtains the following values for S_{12}, τ and the center of rotation $x_\star^{2D}$ of the spinning soliton

$$S_{12} = 1.0286, \quad \tau = \begin{pmatrix} -0.0054 \\ -0.0071 \end{pmatrix},$$

(6.16)

$$x_\star^{2D} := -S^{-1}\tau = \frac{1}{S_{12}} \begin{pmatrix} \tau_2 \\ -\tau_1 \end{pmatrix} = \begin{pmatrix} -0.0069 \\ 0.0052 \end{pmatrix}.$$

The rotating wave $\mathbf{u}_\star : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ satisfies

$$\mathbf{u}_\star(x, t) = \mathbf{w}_\star \left(e^{-tS}(x - x_\star^{2D}) + x_\star^{2D} \right) = \mathbf{v}_\star \left(e^{-tS}(x - x_\star^{2D}) \right).$$

In case $d = 3$ at time $t = 500$ the rotational velocities S_{12}, S_{13}, S_{23} and the translational vector τ of the spinning soliton are found to be

(6.17)

$$\begin{pmatrix} S_{12} \\ S_{13} \\ S_{23} \end{pmatrix} = \begin{pmatrix} 0.6888 \\ -0.0043 \\ -0.0043 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0.0023 \\ -0.0415 \\ 0.0005 \end{pmatrix}.$$

The axis of rotation is $\{x_\star^{3D} + r x_{\text{rot}}^{3D}, r \in \mathbb{R}\}$ with support vector $x_\star^{3D}$ and the direction x_{rot}^{3D} spanning the null space of S, is given by

$$x_{\text{rot}}^{3D} := \begin{pmatrix} S_{23} \\ -S_{13} \\ S_{12} \end{pmatrix} = \begin{pmatrix} -0.0043 \\ 0.0043 \\ 0.6888 \end{pmatrix},$$

$$x_\star^{3D} := \frac{1}{S_{12}^2 + S_{13}^2 + S_{23}^2} \begin{pmatrix} S_{12}\tau_2 + S_{13}\tau_3 \\ -S_{12}\tau_1 + S_{23}\tau_3 \\ -S_{13}\tau_1 - S_{23}\tau_2 \end{pmatrix} = \begin{pmatrix} -0.0602 \\ -0.0033 \\ -0.0004 \end{pmatrix}.$$

As above, the rotating wave $\mathbf{u}_\star : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^2$ satisfies

$$\mathbf{u}_\star(x, t) = \mathbf{w}_\star \left(e^{-tS}(x - x_\star^{3D}) + x_\star^{3D} \right) = \mathbf{v}_\star \left(e^{-tS}(x - x_\star^{3D}) \right).$$

The periods of rotation for the spinning solitons in $\mathbb{R}^2$ and $\mathbb{R}^3$ are determined by

$$T^{2D} = \frac{2\pi}{|S_{12}|} = 6.1085 \quad \text{and} \quad T^{3D} = \frac{2\pi}{\left| \sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2} \right|} = 9.1216.$$

6.2. Spectrum and eigenfunctions at spinning solitons. Consider the eigenvalue problem for the real-valued version of the QCGL-equation

(6.18) $\mathcal{L}\mathbf{v}(x) = \mathbf{A}\triangle\mathbf{v}(x) + \langle Sx, \nabla\mathbf{v}(x) \rangle + D\mathbf{f}(\mathbf{v}_\star(x))\,\mathbf{v}(x) = \lambda\mathbf{v}(x), \; x \in \mathbb{R}^d,$

where $d \in \{2, 3\}$, $\mathbf{v} : \mathbb{R}^d \to \mathbb{C}^2$, $\mathbf{A} \in \mathbb{R}^{2,2}$ is from (6.3), $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ from (6.4) and $S \in \mathbb{R}^{d,d}$ from (6.7).

Recall from Section 6.1 that the Ginzburg-Landau equation (6.3) exhibits spinning soliton solutions for space dimensions $d = 2$ and $d = 3$ and for the parameter values from (6.10).

Below we approximate solutions $(\lambda, \mathbf{v})$ of the eigenvalue problem (6.18) and apply Theorem 5.1 and Corollary 5.5 to (6.18). Instead of (6.18) we solve the eigenvalue problem

(6.19) $\mathbf{A}\triangle\mathbf{w}(x) + \langle S(x - x_\star), \nabla\mathbf{w}(x) \rangle + D\mathbf{f}(\mathbf{w}_\star(x))\,\mathbf{w}(x) = \lambda\mathbf{w}(x), \; x \in \mathbb{R}^d,$

with $d \in \{2, 3\}$, $S \in \mathbb{R}^{d,d}$ and $\tau \in \mathbb{R}^d$ from (6.16) and (6.17). Note that (6.19) is just the shifted version of (6.18) where $\mathbf{w}(x) = \mathbf{v}(x - x_\star), x \in \mathbb{R}^d$. Hence the eigenvalues are the same.

We use the following values

(6.20)
$$\mathbf{v}_\infty = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \sigma\left(D\mathbf{f}(\mathbf{v}_\infty)\right) = \{\delta, \bar{\delta}\} = \left\{-\frac{1}{2}\right\},$$
$$b_0 = -s\left(D\mathbf{f}(\mathbf{v}_\infty)\right) = -\mathrm{Re}\,\delta = \frac{1}{2}.$$

The spectrum of $S \in \mathbb{R}^{d,d}$ is given by

(6.21) $d = 2:$ $\sigma(S) = \{\pm\sigma_1 i\},$ $\sigma_1 = S_{12} = 1.0286,$

(6.22) $d = 3:$ $\sigma(S) = \{0, \pm\sigma_1 i\},$ $\sigma_1 = \sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2} = 0.6888.$

For the solution of the eigenvalue problem (6.19) we use in both cases, $d = 2$ and $d = 3$, continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.25$ (if $d = 2$) and $\triangle x = 0.8$ (if $d = 3$), homogeneous Neumann boundary conditions and the following parameters for the eigenvalue solver

(6.23) $\mathrm{neigs} = 800, \quad \sigma = -1, \quad \mathrm{etol} = 10^{-7},$

i.e. we approximate $\mathrm{neigs} = 800$ eigenvalues that are located nearest to $\sigma = -1$ and satisfy the eigenvalue tolerance $\mathrm{etol} = 10^{-7}$. As above, the profile $\mathbf{w}_\star$ and the pair $(S, x_\star)$ in (6.19) are obtained from simulating the freezing system until $t = 400$ for $d = 2$ and until $t = 500$ for $d = 3$. With the data from the last time instance we then solve the eigenvalue problem (6.19).

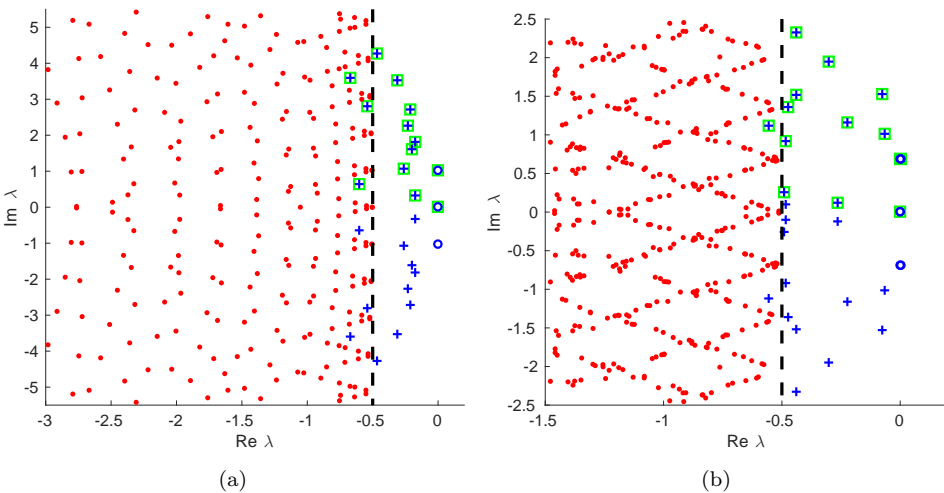(a)   (b)

FIGURE 4. Numerical spectra of QCGL (6.1) linearized about a spinning soliton for $d = 2$ (a), and $d = 3$ (b).

Figure 4 shows the numerical approximation $\sigma^{\mathrm{approx}}(\mathcal{L})$ of the spectrum $\sigma(\mathcal{L}) = \sigma_{\mathrm{point}}(\mathcal{L}) \cup \sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ obtained by linearizing about the spinning soliton $\mathbf{v}_\star$ for $d = 2$ (a), and $d = 3$ (b). Let us discuss the numerical spectra in more detail and compare them with our theoretical results:

a) Essential spectrum: We replace $\mathbf{v}_\star$ in $\mathcal{L}$ by its limiting value zero and find a dispersion relation (5.10), leading to

$$(6.24) \qquad \sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) = \left\{\lambda = -\omega^2\alpha_1 + \delta_1 + i\left(\mp\omega^2\alpha_2 \pm \delta_2 - n\sigma_1\right) \mid \omega \in \mathbb{R},\ n \in \mathbb{Z}\right\}$$

with σ_1 from (6.21) for $d = 2$ and from (6.22) for $d = 3$, cf. Remark 5.6 and [**26**, Thm.9.10]. Recall the inclusion $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$ and taking the parameter values (6.10) into account, (6.24) reads as

$$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) = \left\{\lambda = -\frac{1}{2}(\omega^2 + 1) + i\left(\mp\frac{1}{2}\omega^2 - n\sigma_1\right) \mid \omega \in \mathbb{R},\ n \in \mathbb{Z}\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

In both cases the part $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})$ of the essential spectrum forms a zig-zag-structure that can be considered as the union of infinitely many copies of cones. The tips of the cones $-\frac{1}{2} - in\sigma_1$, $n \in \mathbb{Z}$ lie on the line $\delta_1 + i\mathbb{R} = -\frac{1}{2} + i\mathbb{R}$. Therefore, the distance between two neighboring tips equals σ_1. The gap between the whole essential spectrum and the imaginary axis equals $b_0 = \frac{1}{2}$, since $\operatorname{Re}\sigma_{\mathrm{ess}}(\mathcal{L}) \leqslant -b_0 = \operatorname{Re}\delta = -\frac{1}{2}$. The inclusion $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$ is proved in [**27**, Thm.9.10] for the L^p-spectrum of $\mathcal{L}$. We believe that even equality holds, i.e. $\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L})$, but this has not been proved so far.

Let us now consider the numerical results: The red dots in Figure 4 represent the approximation $\sigma_{\mathrm{ess}}^{\mathrm{approx}}(\mathcal{L})$ of the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$. They approximate the collection of cones in the essential spectrum. As expected, the tips are approximatively located on $-\frac{1}{2} + i\mathbb{R}$, indicated by the black dashed line. Our results show that the distance between two neighboring tips of the cones agrees with σ_1 from (6.21) for $d = 2$ and from (6.22) for $d = 3$. In particular, the approximation suggests that we have equality in (6.24). The case $d = 2$ has been also treated in [**5**, Sec.8].

b) Point spectrum: From Theorem 5.4 and Remark 5.6 we have the relation

$$(6.25) \qquad\qquad \sigma_{\mathrm{point}}^{\mathrm{part}}(\mathcal{L}) = \{0, \pm i\sigma_1\} \subseteq \sigma_{\mathrm{point}}(\mathcal{L})$$

with σ_1 from (6.21) for $d = 2$ and (6.22) for $d = 3$ (cf. [**26**, Thm.9.4] for more details). The eigenvalues $0, \pm i\sigma_1$ are located on the imaginary axis and have (at least) algebraic multiplicities 1 for $d = 2$ and 2 for $d = 3$, respectively, see Theorem 5.4. The numerical results below suggest that we do not have equality in (6.25), i.e. in general there are further isolated eigenvalues which have to be determined numerically.

Let us now consider the associated numerical results: The blue circles and the blue plus signs in Figure 4 represent an approximation $\sigma_{\mathrm{point}}^{\mathrm{approx}}(\mathcal{L})$ of the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$. The approximate eigenvalues $0, \pm i\sigma_1$ are visualized by blue circles. The plus signs indicate further isolated eigenvalues which belong to the point spectrum, but cannot be determined explicitly. In case $d = 2$, there are 11 additional complex-conjugate pairs of isolated eigenvalues, of which 8 pairs are located to the right of the vertical black dashed line $-\frac{1}{2} + i\mathbb{R}$ and 3 pairs to the left in between the zig-zags. We emphasize that these three pairs are not numerical artifacts but are robust to spatial mesh refinement and to increase of spatial domain. Similarly, in case $d = 3$, we find 12 additonal complex-conjugate pairs of isolated eigenvalues, of which 11 pairs are located to the right of the vertical line $-\frac{1}{2} + i\mathbb{R}$ and 1 pair to the left in between the zig-zags. Numerical values of the isolated eigenvalues are given in Table 1 below.

c) Eigenfunctions: The eigenfunctions associated to eigenvalues from the essential spectrum are explicitly known and bounded but never localized, i.e. they do not
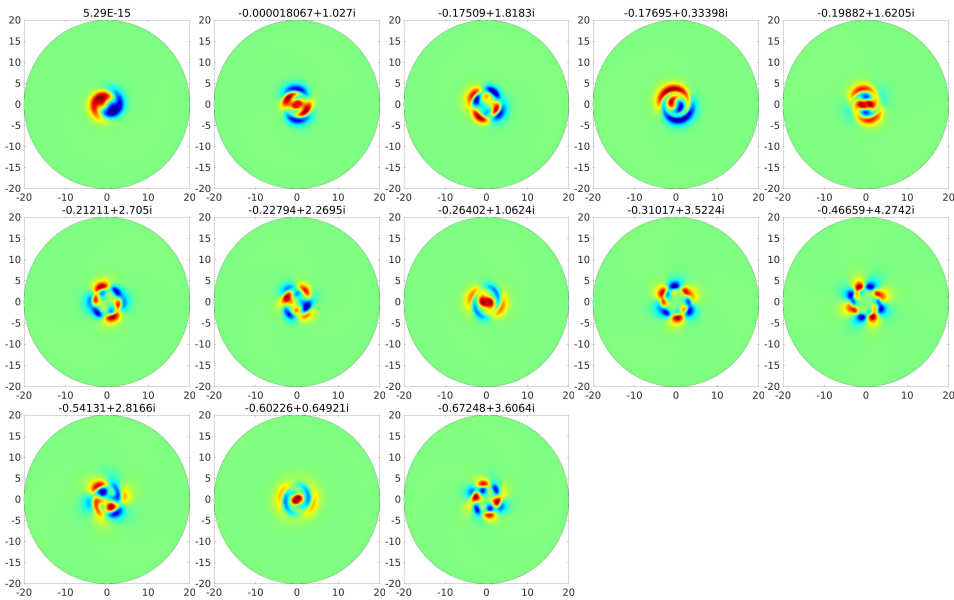
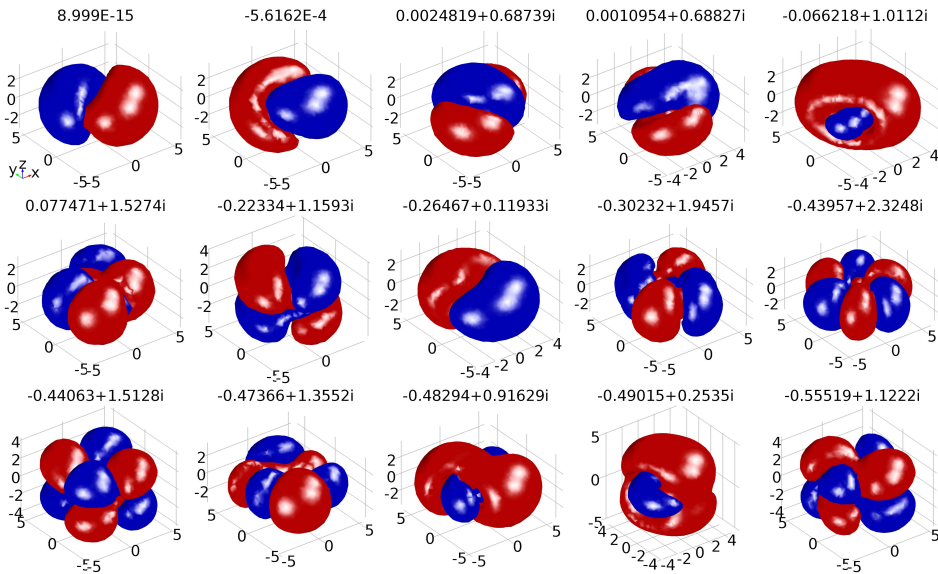FIGURE 5. Real parts of eigenfunctions of $2D$-QCGL (6.1) for a spinning soliton.

FIGURE 6. Isosurfaces of real parts of eigenfunctions of $3D$-QCGL (6.1) for a spinning soliton.

decay in space, see [26, Thm.7.9 and 9.10]. In contrast to this, all eigenfunctions associated to eigenvalues from the point spectrum, in particular those on the imaginary axis, are exponentially localized. For eigenvalues λ with $\operatorname{Re} \lambda > -\beta_\infty = -\frac{1}{2}$ this follows from our theory. To be more precise, let us introduce angular derivatives

by

$$D^{(1,2)} := x_2 D_1 - x_1 D_2, \quad D^{(1,3)} := x_3 D_1 - x_1 D_3, \quad D^{(2,3)} := x_3 D_2 - x_2 D_3.$$

Then Theorem 5.4 yields eigenfunctions associated to eigenvalues from $\sigma_{\text{point}}^{\text{part}}(\mathcal{L})$ as follows

$$(6.26) \qquad \begin{aligned} \lambda_1 &= 0, & \mathbf{v_1} &= D^{(1,2)}\mathbf{v_\star}, \\ \lambda_{2,3} &= \pm i\sigma_1, & \mathbf{v_{2,3}} &= D_1\mathbf{v_\star} \pm iD_2\mathbf{v_\star} \end{aligned}$$

for $d = 2$, see [**26**, Ex.9.6], and by

$$(6.27) \qquad \begin{aligned} \lambda_1 &= 0, & \mathbf{v_1} &= S_{12}D^{(1,2)}\mathbf{v_\star} + S_{13}D^{(1,3)}\mathbf{v_\star} + S_{23}D^{(2,3)}\mathbf{v_\star}, \\ \lambda_2 &= 0, & \mathbf{v_2} &= S_{23}D_1\mathbf{v_\star} - S_{13}D_2\mathbf{v_\star} + S_{12}D_3\mathbf{v_\star}, \\ \lambda_{3,4} &= \pm i\sigma_1, & \mathbf{v_{3,4}} &= (\sigma_1 S_{13} \pm iS_{12}S_{23})D_1\mathbf{v_\star} \\ & & &\quad + (\sigma_1 S_{23} \pm iS_{12}S_{13})D_2\mathbf{v_\star} \\ & & &\quad \pm i(S_{13}^2 + S_{23}^2)D_3\mathbf{v_\star}, \\ \lambda_{5,6} &= \pm i\sigma_1, & \mathbf{v_{5,6}} &= -(S_{13}^2 + S_{23}^2)D^{(1,2)}\mathbf{v_\star} \\ & & &\quad - (-S_{12}S_{13} \pm i\sigma_1 S_{23})D^{(1,3)}\mathbf{v_\star} \\ & & &\quad + (S_{12}S_{23} \pm i\sigma_1 S_{13})D^{(2,3)}\mathbf{v_\star} \end{aligned}$$

for $d = 3$, see [**26**, Ex.9.7]. We next study the asymptotic behavior of eigenfunctions with eigenvalues in $\sigma_{\text{point}}(\mathcal{L})$. As shown in the previous section, Theorem 5.1 and Corollary 5.5 imply that all eigenfunctions with eigenvalues $\text{Re}\,\lambda > -\frac{1}{2}$ are exponentially localized, in the L^p- and in the pointwise sense. The maximal exponential rate of decay for the eigenfunctions will depend on λ as we will see in Section 6.3 below. Note that Theorem 5.1 does not apply to eigenvalues satisfying $\text{Re}\,\lambda \leqslant -\frac{1}{2}$.

Let us now discuss the numerical results: In Figure 4, there are some isolated eigenvalues labeled by a green square. Their eigenfunctions are visualized in Figure 5 for $d = 2$ and in Figure 6 for $d = 3$. Both pictures show the real parts of the first component of the associated eigenfunction $\mathbf{w} : \mathbb{R}^d \to \mathbb{C}^2$. The first two eigenfunctions in Figure 5 are approximations of $\mathbf{v_1}, \mathbf{v_2}$ from (6.26). Their corresponding eigenvalues approximate λ_1, λ_2 from (6.26), as specified in the title of the figure. Similarly, the first four eigenfunctions in Figure 6 approximate $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_5}$ from (6.27). Their associated eigenvalues are approximations of $\lambda_1, \lambda_2, \lambda_3, \lambda_5$ from (6.27) and again specified in the title. Note that the first eigenfunction in Figure 5 and in Figure 6 agrees with a slightly shifted version of the rotational term $\mathbf{v_1}(x) = \langle Sx, \nabla\mathbf{v_\star}(x)\rangle$ which arises in the rotating wave equation (1.5). The eigenfunctions $3-10$ from Figure 5 and $5-14$ from Figure 6 belong to the eigenvalues in green boxes carrying a plus sign and satisfying $\text{Re}\,\lambda > -\frac{1}{2}$. They are ordered with decaying real parts. All eigenfunctions with eigenvalues satisfying $\text{Re}\,\lambda > -\frac{1}{2}$ seem to decay exponentially, as expected by Theorem 5.1. The last three eigenfunctions in Figure 5 and the last eigenfunction in Figure 6 show those eigenfunctions whose eigenvalues are marked by a green box but satisfy $\text{Re}\,\lambda \leqslant -\frac{1}{2}$. In this case Theorem 5.1 is not applicable. However, even these eigenfunctions seem to have exponential decay in space. Finally, we note that we found further isolated eigenvalues inside the zig-zag structure, see Figure 4(b), the eigenfunctions of which seem to decay exponentially in space as well.

6.3. Rate of exponential decay for spinning solitons and their eigenfunctions. Let us consider the rates of exponential decay for spinning solitons and their associated eigenfunctions in more detail. For this purpose we compare theoretical decay rates (short: TDR), guaranteed by Theorem 2.8 and 5.1, with numerical decay rates (short: NDR) computed from our numerical results by linear regression.
a) Decay rates of spinning solitons: The maximal rate of exponential decay for the profiles of the spinning solitons, which one obtains from Corollary 4.3, is given by, cf. (6.9), (6.6),

$$(6.28) \qquad 0 \leqslant \mu \leqslant \frac{\varepsilon\nu}{p} < \frac{\nu}{p} =: \mu^{\mathrm{pro}}(p) < \frac{\nu}{\max\left\{p_{\min}, \frac{d}{2}\right\}} =: \mu^{\mathrm{pro}}_{\max}.$$

Taking the parameter values (6.10) into account, (6.28) implies the following upper bounds for the theoretical decay rates

$$\mu^{\mathrm{pro}}(p) = \frac{1}{\sqrt{2}p} \approx \frac{0.7071}{p}, \quad \mu^{\mathrm{pro}}_{\max} = \begin{cases} \frac{\sqrt{2}+1}{4} \approx 0.6036 & , \mathrm{d}{=}2, \\ \frac{\sqrt{2}}{3} \approx 0.4714 & , \mathrm{d}{=}3. \end{cases}$$

We compare this with the numerical exponential decay rates for the profile:

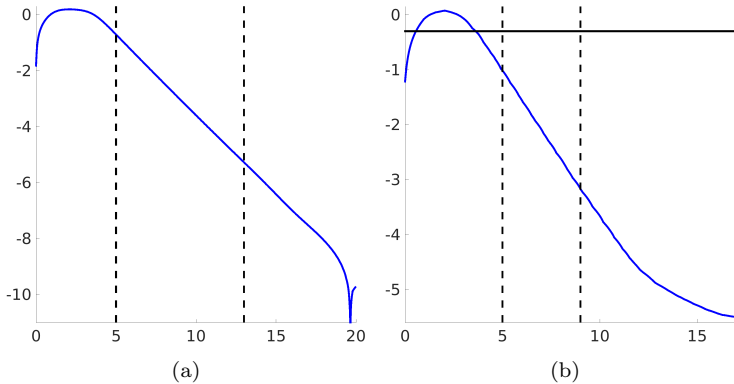(a)   (b)

FIGURE 7. Numerical exponential decay rate of the spinning soliton profiles for $d = 2$ (a) and $d = 3$ (b) arising in the QCGL (6.3). The black line indicates the level 0.5 use for the isosurfaces in Figure 3(b).

Figure 7 shows the absolute value of the spinning soliton profile along a straight line in radial direction, for $d = 2$ in (a) and $d = 3$ in (b). To be more precise, Figure 7 (a) shows the function

$$(6.29) \qquad [0, 20] \to \mathbb{R}, \quad r \mapsto \log_{10}\left|\mathbf{w}_\star\left(r\cos\frac{\pi}{2}, r\sin\frac{\pi}{2}\right)\right|$$

in case of $d = 2$. Similarly, Figure 7 (b) shows the function

$$(6.30) \qquad [0, 10\sqrt{3}] \to \mathbb{R}, \quad r \mapsto \log_{10}\left|\mathbf{w}_\star\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)\right|$$

in case of $d = 3$. The functions are almost linear at least in the regions enclosed by the black dashed lines, which are $[5, 13]$ for $d = 2$ and $[5, 9]$ for $d = 3$. In case $d = 2$ the observed NDR is slightly below the TDR. This is attributed to the fact that the NDR is affected by the size of the bounded domain and by the choice of

boundary conditions. Summarizing, this indicates that the heat kernel estimates from [**26, 27**], which form the origin of these decay rates, are quite accurate.

b) Decay rates of eigenfunctions: The maximal rate of exponential decay for the eigenfunctions, obtained from Theorem 5.1, will now depend on λ, since

$$\operatorname{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty = -(1-\varepsilon)(-\operatorname{Re}\delta) \quad \Longleftrightarrow \quad \varepsilon \leqslant \frac{\operatorname{Re}\lambda - \operatorname{Re}\delta}{-\operatorname{Re}\delta} =: \varepsilon(\lambda).$$

This gives us the bounds

$$(6.31) \qquad 0 \leqslant \frac{\varepsilon\nu}{p} \leqslant \frac{\varepsilon(\lambda)\nu}{p} =: \mu^{\mathrm{eig}}(p,\lambda) < \frac{\varepsilon(\lambda)\nu}{\max\left\{p_{\min}, \frac{d}{2}\right\}} =: \mu^{\mathrm{eig}}_{\max}(\lambda).$$

With parameter values (6.10) the bounds (6.31) lead to

$$\mu^{\mathrm{eig}}(p,\lambda) = \frac{2\left(\operatorname{Re}\lambda + \frac{1}{2}\right)}{\sqrt{2}p}, \quad \mu^{\mathrm{eig}}_{\max}(\lambda) = \frac{\sqrt{2}+1}{2}\left(\operatorname{Re}\lambda + \frac{1}{2}\right).$$

This shows, that the decay rate is maximal for eigenvalues on the imaginary axis and decreases linearly to 0 as $\operatorname{Re}\lambda$ approaches $-\frac{1}{2}$, cf. Figure 4. Recall, that Theorem 5.1 does not apply for $\operatorname{Re}\lambda \leqslant -\frac{1}{2}$. For the isolated eigenvalues labeled by a green square in Figure 4, the TDR's $\mu^{\mathrm{eig}}_{\max}(\lambda)$ of the associated eigenfunctions are given in the third columns of Table 1. We compare with the numerical exponential decay rates for the eigenfunctions:
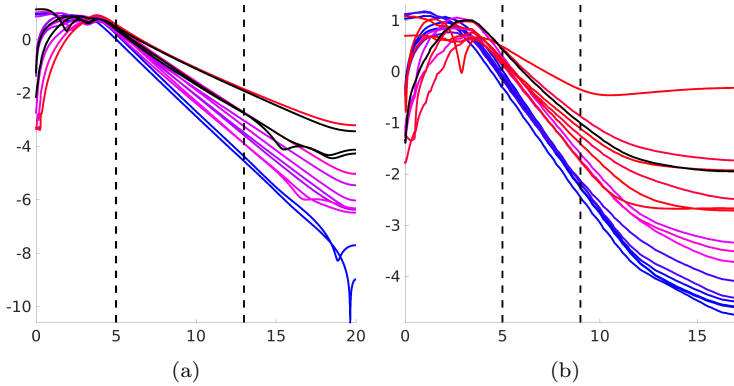

FIGURE 8. Numerical rate of exponential decay of the eigenfunctions for $d = 2$ (a) and $d = 3$ (b) of (6.19) linearized at a spinning soliton.

Figure 8 shows the absolute value of the eigenfunctions along the lines from (6.29) and (6.30) with $\mathbf{w}$ instead of $\mathbf{w}_\star$. The eigenfunctions are associated to the eigenvalues in green boxes in Figure 4. The color of the graphs vary with $\operatorname{Re}\lambda$ of the associated eigenvalue. Varying $\operatorname{Re}\lambda$ from 0 to $-\frac{1}{2}$, the graphs change color from blue to red. A red graph indicates that $\operatorname{Re}\lambda$ is near $-\frac{1}{2}$ and that the TDR is small. Finally, a black graph indicates an eigenvalue $\operatorname{Re}\lambda \leqslant -\frac{1}{2}$, in which case we do not have a TDR. All eigenfunctions are approximately linear in the regions enclosed by the black dashed lines, which are again $[5, 13]$ for $d = 2$, and $[5, 9]$ for $d = 3$. Moreover, we observe that the decay rate of the eigenfunctions decreases when the

eigenvalue moves to the left of the imaginary axis. We note that even those eigenfunctions the eigenvalues of which satisfy $\operatorname{Re} \lambda \leqslant -\frac{1}{2}$, have exponential decay. Once more, we used linear regression on 1000 radially equispaced points to estimate the NDR. The numerical values are collected in the second columns of Table 1. Again the TDR's are surprisingly close to the NDR's with difference increasing towards $\operatorname{Re} \lambda = -\frac{1}{2}$.

eigenvalue	NDR	TDR
$5.29 \cdot 10^{-15}$	0.5713	0.6036
$-0.00002 \pm 1.0270i$	0.5730	0.6035
$-0.17509 \pm 1.8183i$	0.5001	0.3922
$-0.17695 \pm 0.3340i$	0.4815	0.3900
$-0.19882 \pm 1.6205i$	0.4139	0.3636
$-0.21211 \pm 2.7050i$	0.4652	0.3475
$-0.22794 \pm 2.2695i$	0.5155	0.3284
$-0.26402 \pm 1.0624i$	0.5355	0.2849
$-0.31017 \pm 3.5224i$	0.4044	0.2291
$-0.46659 \pm 4.2742i$	0.2984	0.0403
$-0.54131 \pm 2.8166i$	0.2972	—
$-0.60226 \pm 0.6492i$	0.3982	—
$-0.67248 \pm 3.6064i$	0.3889	—

eigenvalue	NDR	TDR
$8.999 \cdot 10^{-15}$	0.5387	0.4714
$-5.6162 \cdot 10^{-4}$	0.5478	0.4714
$0.00110 \pm 0.68827i$	0.5507	0.4714
$0.00248 \pm 0.6874i$	0.5398	0.4714
$-0.06622 \pm 1.0112i$	0.4899	0.4090
$-0.07747 \pm 1.5274i$	0.5355	0.3984
$-0.22334 \pm 1.1593i$	0.4756	0.2608
$-0.26467 \pm 0.1193i$	0.4785	0.2219
$-0.30232 \pm 1.9457i$	0.4649	0.1864
$-0.43957 \pm 2.3248i$	0.3595	0.0570
$-0.44063 \pm 1.5128i$	0.3310	0.0560
$-0.47366 \pm 1.3552i$	0.4781	0.0248
$-0.48294 \pm 0.9163i$	0.4145	0.0161
$-0.48506 \pm 0.0991i$	0.2126	0.0141
$-0.49015 \pm 0.2535i$	0.3307	0.0093
$-0.55519 \pm 1.1222i$	0.3581	—

TABLE 1. Numerical (NDR) and theoretical (TDR) exponential decay rates of QCGL (6.1) for the eigenfunctions of the linearization at a spinning soliton for $d = 2$ (left) and $d = 3$ (right).

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