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Remarks on ill-posedness for the Dirac-Klein-Gordon system

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Abstract. We show ill-posedness of the Cauchy problem for the Dirac-Klein-Gordon system in one spatial dimension with some indices of the Sobolev spaces which the initial data belong to. By combining with the existing papers [**10, 11**], we define the entire range of those indices for well-posedness or ill-posedness with the exception of one point. At this point, it is still unsolved whether well-posedness holds or not with respect to Sobolev spaces. We introduce one solvability for the problem of this point by giving the result of the unique existence of solution in the corresponding Lebesgue spaces [**16**].

1. Introduction

We consider the Cauchy problem for the Dirac-Klein-Gordon system:

$$\begin{cases} (i\gamma_0\partial_t + \gamma_1\partial_x)\psi + m\psi = \phi\psi, \\ (\partial_t^2 - \partial_x^2 + M^2)\phi = \psi^*\gamma^0\psi, \\ \psi(0,x) = \psi_0(x),\ \phi(0,x) = \phi_0(x),\ \partial_t\phi(0,x) = \phi_1(x), \end{cases} \tag{1.1}$$

where $\psi = \binom{\psi_1}{\psi_2} : \mathbb{R}^{1+1} \to \mathbb{C}^2$ and $\phi : \mathbb{R}^{1+1} \to \mathbb{R}$ are unknown functions of $(t,x) \in \mathbb{R}^{1+1}$, $\psi_0 = \binom{\psi_{0,1}}{\psi_{0,2}} : \mathbb{R} \to \mathbb{C}^2$ and $\phi_0, \phi_1 : \mathbb{R} \to \mathbb{R}$ are given functions of $x \in \mathbb{R}$, m and M are nonnegative constants, and γ_0, γ_1 are 2×2 Hermitian matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.2}$$

which satisfy the anticommutation relations which leads $(i\gamma_0\partial_t + \gamma_1\partial_x)^2 = (-\partial_t^2 + \partial_x^2)I_2$, where I_2 is the 2×2 identity matrix, ψ^* denotes the conjugate transpose of ψ.

In this paper we are interested in well-posedness of this problem, mainly, with respect to the Sobolev spaces:

$$(\psi(t,\cdot), \phi(t,\cdot), \partial_t\phi(t,\cdot)) \in H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R}), \quad 0 \le t < T \tag{1.3}$$

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with some positive T. We consider defining the region $(s, r) \in \mathbb{R}^2$ for well-posedness and for ill-posedness of this Cauchy problem. There are a lot of results on this problem (see [**4, 13, 6, 7, 2, 3, 14, 8, 15, 10**] and references cited therein). As the latest and best result for well-posedness, in [**10**] Nakanishi, Tsugawa and the first author proved that (1.1) is well-posed in $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R})$ if $|s| \le r \le s+1$ and $(s, r) \neq (-1/2, 1/2)$. They also proved that (1.1) is ill-posed if $s > \max(r, 0)$ or $r > \max(s+1, 1/2)$. Subsequently, the authors [**11**] extended the ill-posedness result with the additional range satisfying $s < 0, r < 1/2$ and $s + r < 0$. Therefore the following two lines in $(s, r) \in \mathbb{R}^2$ have been left unsolved:

- $s \le -1/2,\ r = 1/2$,
- $s = 0,\ r < 0$.

The following is the main result in this paper.

Theorem 1. *Let* $(s, r) \in \mathbb{R}^2$ *be on the either lines* $s < -1/2,\ r = 1/2$ *or* $s = 0,\ r < 0$. *Then the Cauchy problem* (1.1) *is ill-posed in* $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R})$, *more precisely, the solution map of* (1.1) *is discontinuous.*

Here we give several remarks on this theorem. In the proof, we will show ill-posedness in $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R})$ with $s = 0,\ r < 0$, and independently with $s < \min(-r, 0)$. The latter range includes the line $s < -1/2,\ r = 1/2$ which is for Theorem 1. These two ill-posedness mean that the solution map of (1.1) from $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R})$ to $C(\mathbb{R}; H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R}))$ is discontinuous. Here we recall the previous work in [**11**] which showed an ill-posedness result for the index $s < \min(-r, 0)$ and $r < 1/2$. We remove the condition $r < 1/2$ from the earlier work [**11**] to have the current result in this paper. In [**11**] the authors did apply a similar argument as in the paper by Bejenaru and Tao [**1**] in which the well-posedness in $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R})$ with $-1/2 < s = -r < 0$ played an important role for the proof. However, the same argument does not work to show ill-posedness in $H^s(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ with $s < -1/2$ because well-posedness in $H^{-1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ has not been obtained, that is still open problem. Therefore we need another technique. Iwabuchi and Ogawa [**9**] showed ill-posedness in $H^s(\mathbb{R}^2)$ with $s \le -1$ for the Schrödinger equation with quadratic nonlinearity in two spatial dimension even in the situation well-posedness in $H^{-1}(\mathbb{R}^2)$ is unknown. We apply thier technique, with some modification, to our problem. Indeed the fact that the Klein-Gordon equation is a second order differential equation with respect to t causes some difficulties to show well-posedness in the modulation space, although the easier treatment for the first order differential equation (Schrödinger equation) in the modulation spaces which was shown in [**9**]. We apply the argument in [**9**] not so directly. Moreover we use the fact that the Dirac-Klein-Gordon system can be reduced to the single Dirac equation with a potential if the initial data satisfy some special conditions (see [**4**] and [**13**]). We consider ill-posedness for this single Dirac equation. We extract the worst part of the Dirac equation with a potential, which is easier than the Dirac-Klein-Gordon system. We also use this observation to show that the Cauchy problem (1.1) is ill-posed on the line $s = 0, r < 0$.

For the case $(s, r) = (-1/2, 1/2)$, the worst interaction occurs in the nonlinear part of the Klein-Gordon equation. Indeed, Nakanishi, Tsugawa and the first author proved in [**10**] that an irregular flow map exists. Furthermore, the same iteration argument as in [**10**] works for the reduced Dirac equation with initial data

$(\psi_0, \phi_0, \phi_1) \in H^{-1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. Accordingly, our argument is not effective to consider ill-posedness at $(-1/2, 1/2)$.

Even with Theorem 1, the one point $(s, r) = (-1/2, 1/2) \in \mathbb{R}^2$ is still open for well-posed or not for the Cauchy problem (1.1) in the Sobolev spaces $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R})$. Shiota [**16**] reported an interesting result which corresponds to the problem of this point. We have decided to introduce this theorem in this paper because this is from Shiota's Master's Thesis written in Japanese and we thought this will never be published anywhere else. We also introduce the proof of this theorem in section 5.

THEOREM 2 (Shiota [**16**]). *For any initial data* $(\psi_0, \phi_0, \phi_1) \in L^1(\mathbb{R}) \times L^\infty(\mathbb{R}) \times L^1(\mathbb{R})$, *there exists a time local unique solution* $(\psi, \phi) \in C([0, T] : L^1(\mathbb{R}) \times L^\infty(\mathbb{R}))$ *to the Cauchy problem* (1.1). *Moreover the solution map is continuous.*

We remark that $L^1(\mathbb{R}) \times L^\infty(\mathbb{R})$ has the same scale as $H^{-1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, in the meanwhile, unfortunately it is difficult to compare those two well-posed results since there is no inclusion relation between $L^1(\mathbb{R})$ and $H^{-1/2}(\mathbb{R})$, or between $L^\infty(\mathbb{R})$ and $H^{1/2}(\mathbb{R})$.

2. Preliminary

Here we introduce the work by Chadam and Glassey [**4**], and also Ozawa and Yamauchi [**13**]. They found the following conservation law for the Dirac-Klein-Gordon system (1.1):

$$\int_{\mathbb{R}} |\psi_1(t,x) - \overline{\psi_2}(t,x)|^2 dx = \int_{\mathbb{R}} |\psi_{0,1}(x) - \overline{\psi_{0,2}}(x)|^2 dx, \quad t > 0.$$

Therefore if we set the initial data as follows

$$\psi_{0,1} = \overline{\psi_{0,2}}, \tag{2.1}$$

then we have $\psi_1(t,x) = \overline{\psi_2}(t,x)$ for almost everywhere $x \in \mathbb{R}$ and any $t > 0$. On the other hand, we calculate the nonlinear term of the Klein-Gordon equation in (1.1) with the matrix in (1.2),

$$\psi^* \gamma^0 \psi = |\psi_1|^2 - |\psi_2|^2, \tag{2.2}$$

and so we have $(\psi^*\gamma^0\psi)(t) = 0$, $t > 0$ under the initial condition (2.1). We denote by K_M the evolution operator for the free Klein-Gordon system, that is $\phi = K_M[\phi_0, \phi_1]$ satisfies

$$(\partial_t^2 - \partial_x^2 + M^2)\phi = 0, \quad \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x).$$

By using this, under the condition (2.1), (1.1) can be reduced to the following inhomogeneous but linear equation with respect to unknown functions:

$$(i\gamma_0\partial_t + \gamma_1\partial_x)\psi + m\psi = K_M[\phi_0, \phi_1]\psi, \quad \psi(0,x) = \psi_0(x). \tag{2.3}$$

We denote by S_m the evolution operator of the Dirac system. Then, (2.3) is equivalent to

$$\psi(t,x) = S_m(t)\psi_0(x) - i\gamma_0 \int_0^t S_m(t-t')(K_M[\phi_0, \phi_1]\psi)(t', x)dt'. \tag{2.4}$$

We shall write $\Psi_{\phi_0,\phi_1}[\psi]$ for the right hand side of the equation (2.4), and show below that this map is a contraction for a proof of the existence of solutions in the scaled modulation spaces. After this, we investigate the norm inflation or the discontinuity of the solution maps with respect to those solutions in the Sobolev spaces for Theorem 1. These arguments were used for ill-posedness for Schrödinger equations ([**9**]) and the massless Chern-Simons-Dirac system ([**11**]).

We introduce the definition and some properties for modulation spaces. For more information, see Feichtinger [**5**].

Definition 3. *Let A be a dyadic number. Define the space $M_A(\mathbb{R}) = (M^0_{2,1})_A(\mathbb{R})$ as the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm*

$$\|f\|_{M_A} = \sum_{k\in\mathbb{Z}} \|\widehat{f}\|_{L^2([(k-1)A,(k+1)A])}.$$

The following embeddings and bilinear estimate are well-known (see, for example, [**5**, Section 6]).

Proposition 4. (1) $H^{1/2+\varepsilon}(\mathbb{R}) \hookrightarrow M_A(\mathbb{R}) \sim_A M_1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$.

(2) *There exists a constant $C > 0$ such that for $f, g \in M_A$*

$$\|fg\|_{M_A} \le CA^{1/2}\|f\|_{M_A}\|g\|_{M_A}$$

holds.

We may say the modulation space is a Banach algebra from the property of (2) in this proposition. This property will be useful for the proof of the contraction mapping principle for our problem. Now we give the well-posedness for DKG (1.1) in the modulation spaces under the special condition for the initial data (2.1).

Lemma 5. *Let $A \in 2^{\mathbb{Z}}$. Then, the Cauchy problem* (2.1)-(2.3) *is locally in time well-posed in $M_A(\mathbb{R})$, where the existence time T depends on the initial data (ϕ_0, ϕ_1), but does not depend on the initial data ψ_0.*

We remark here, actually the Cauchy problem (2.1)-(2.3) is time globally well-posed in $M_A(\mathbb{R})$ since, as we see (2.3), the problem consists of the linear equations for ψ with the function $K_M[\phi_0, \phi_1]$ which is the definite (time global) solution for the free Klein-Gordon system. We mean by T here an existence time for the contraction mapping argument applying to this problem below.

Proof. We use the following notation in this paper: For a function $u(t) : \mathbb{R} \to X$ with some function space X for the variable x, and $1 \le p \le \infty$,

$$\|u\|_{L^p_T X} = \left(\int_0^T \|u(t,\cdot)\|^p_X dt \right)^{\frac{1}{p}}.$$

We consider the restriction $0 < T < 1$. From Definition 3, it is easy to have

$$\|K_M[\phi_0,\phi_1]\|_{L^\infty_T M_A} \le C(\|\phi_0\|_{M_A} + \|\phi_1\|_{M_A}), \quad \|S_m(t)F\|_{L^\infty_T M_A} \le C\|F\|_{M_A}.$$

From those and Proposition 4, we have

$$\begin{aligned}\|\psi\|_{L^\infty_T M_A} &\le \|\psi_0\|_{M_A} + CT\|K_M[\phi_0,\phi_1]\psi\|_{L^\infty_T M_A} \\ &\le \|\psi_0\| + CTA^{1/2}\|K_M[\phi_0,\phi_1]\|_{L^\infty_T M_A}\|\psi\|_{L^\infty_T M_A} \\ &\le \|\psi_0\| + CTA^{1/2}(\|\phi_0\|_{M_A} + \|\phi_1\|_{M_A})\|\psi\|_{L^\infty_T M_A}.\end{aligned}$$

We can similarly estimate the difference as follows:

$$\begin{aligned}\|\psi - \psi'\|_{L^\infty_T M_A} &\le CT\|K_M[\phi_0,\phi_1](\psi - \psi')\|_{L^\infty_T M_A} \\ &\le CTA^{1/2}(\|\phi_0\|_{M_A} + \|\phi_1\|_{M_A})\|\psi - \psi'\|_{L^\infty_T M_A}.\end{aligned}$$

Let T satisfy the bound

$$CTA^{1/2}(\|\phi_0\|_{M_A} + \|\phi_1\|_{M_A}) < 1/4. \tag{2.5}$$

We define the iteration space X_T:

$$X_T := \{\psi \in C([0,T]; M_A(\mathbb{R})) : \|\psi\|_{L_T^\infty M_A} \le 2\|\psi_0\|_{M_A}\}.$$

We have shown that the flow map Ψ_{ϕ_0,ϕ_1} is a contraction mapping on X_T for $(\phi_0, \phi_1) \in M_A(\mathbb{R})^2$. We obtain a fixed point of Ψ_{ϕ_0,ϕ_1} which is a solution to (2.3).

Next, we show that Ψ_{ϕ_0,ϕ_1} is continuous with respect to $(\phi_0, \phi_1, \psi_0) \in M_A(\mathbb{R})^3$. Assume that

$$\phi_{0,n} \to \phi_0, \quad \phi_{1,n} \to \phi_1, \quad \psi_{0,n} \to \psi_0 \quad \text{in } M_A(\mathbb{R}),$$

and $\psi_{0,n}$ satisfies (2.1). Let $\psi_n = \Psi_{\phi_{0,n},\phi_{1,n}}[\psi_{0,n}]$, $\psi = \Psi_{\phi_0,\phi_1}[\psi_0]$. From (2.5) and the estimates as above, we have

$$\|\psi_n\|_{L_T^\infty M_A} \le 2\|\psi_{0,n}\| \le 4\|\psi_0\|, \quad \|\psi\|_{L_T^\infty M_A} \le 2\|\psi_0\|$$

for sufficiently large n. Since the difference $\psi_n - \psi$ satisfies

$$\begin{aligned}&(i\gamma_0\partial_t + \gamma_1\partial_x + m)(\psi_n - \psi)\\ &= K_M[\phi_{0,n}, \phi_{1,n}](\psi_n - \psi) + K_M[\phi_{0,n} - \phi_0, \phi_{1,n} - \phi_1]\psi, \ (\psi_n - \psi)(0,x)\\ &= \psi_{0,n} - \psi_0,\end{aligned}$$

it is similarly handled:

$$\begin{aligned}&\|\psi_n - \psi\|_{L_T^\infty M_A}\\ &\le \|\psi_{0,n} - \psi_0\|_{M_A} + CTA^{1/2}\left(\|K_M[\phi_{0,n}, \phi_{1,n}]\|_{L_T^\infty M_A}\|\psi_n - \psi\|_{L_T^\infty M_A}\right.\\ &\qquad \left.+\|K_M[\phi_{0,n} - \phi_0, \phi_{1,n} - \phi_1]\|_{L_T^\infty M_A}\|\psi\|_{L_T^\infty M_A}\right)\\ &\le \|\psi_{0,n} - \psi_0\|_{M_A} + CTA^{1/2}\left((\|\phi_0\|_{M_A} + \|\phi_1\|_{M_A})\|\psi_n - \psi\|_{L_T^\infty M_A}\right.\\ &\qquad \left.+2(\|\phi_{0,n} - \phi_0\|_{M_A} + \|\phi_{1,n} - \phi_1\|_{M_A})\|\psi_0\|_{M_A}\right)\\ &\le \|\psi_{0,n} - \psi_0\|_{M_A} + \frac{1}{4}\|\psi_n - \psi\|_{L_T^\infty M_A}\\ &\qquad + 2CTA^{1/2}(\|\phi_{0,n} - \phi_0\|_{M_A} + \|\phi_{1,n} - \phi_1\|_{M_A})\|\psi_0\|_{M_A},\end{aligned}$$

which leads to the following estimate.

$$\begin{aligned}&\|\psi_n - \psi\|_{L_T^\infty M_A}\\ &\le \frac{4}{3}\left(\|\psi_{0,n} - \psi_0\|_{M_A} + 2CTA^{1/2}(\|\phi_{0,n} - \phi_0\|_{M_A} + \|\phi_{1,n} - \phi_1\|_{M_A})\|\psi_0\|_{M_A}\right).\end{aligned}$$

We therefore obtain the continuity of Ψ_{ϕ_0,ϕ_1} with respect to ψ_0, ϕ_0, and ϕ_1. □

Remark 6. From Lemma 5, we have the following expansion:

$$\psi = \sum_{k=1}^{\infty} \psi^{(k)} \text{ in } L_T^\infty M_A, \tag{2.6}$$

where $\psi^{(1)} := S_m(t)\psi_0$ and

$$\psi^{(k)} := -i\gamma_0 \int_0^t S_m(t-t')(K_M[\phi_0, \phi_1]\psi^{(k-1)})(t')dt', \quad k = 2, 3, \ldots.$$

Here T satisfies the inequality (2.5) which is defined by $\|\phi_0\|_{M_A}, \|\phi_1\|_{M_A}$. The fact that T does not depend on the initial data ψ_0 will be important in the next section.

We will treat the sequence of initial data $\psi_{0,n}$ whose norm of the Modulation space goes to infinity $\|\psi_{0,n}\|_{M_A} \to \infty$.

3. Proof of Theorem 1 with $s < -\frac{1}{2},\ r = \frac{1}{2}$

In this section we give a proof of ill-posedness for $s < \min(-r, 0)$ which includes the line $s < -1/2, r = 1/2$ as we desired. We firstly consider the massless case $m = M = 0$. By putting

$$u_\pm := \psi_1 \mp \psi_2,$$

we rewrite (2.3) as follows:

$$(\partial_t \pm \partial_x) u_\pm = -iK_0[\phi_0, \phi_1] u_\mp, \quad u_\pm(0, x) = u_{\pm,0}(x).$$

From $\psi_1 = \frac{u_+ + u_-}{2}$, $\psi_2 = \frac{-u_+ + u_-}{2}$, we have

$$\psi^* \gamma_0 \psi = |\psi_1|^2 - |\psi_2|^2 = \frac{1}{4}(|u_+ + u_-|^2 - |u_+ - u_-|^2) = \Re(u_+ \overline{u_-}).$$

Here, the condition (2.1) is equivalent to

$$\psi_1 = \overline{\psi_2} \Longleftrightarrow u_+ + u_- = -\overline{u_+} + \overline{u_-} \Longleftrightarrow \Re u_+ = -i\Im u_- \Longleftrightarrow \Re u_+ = \Im u_- = 0.$$

We rewrite $\Im u_+$ and $\Re u_-$ by u_+ and u_- respectively, and (2.3) is equivalent to

$$(\partial_t \pm \partial_x) u_\pm = \mp K_0[\phi_0, \phi_1] u_\mp. \tag{3.1}$$

We define by $u_\pm^{(k)}$ the k-th iteration part:

$$u_\pm^{(1)}(t, x) := u_{\pm,0}(x \mp t), \quad u_\pm^{(k)}(t, x) := \mp \int_0^t (K_0[\phi_0, \phi_1] u_\mp^{(k-1)})(t', x \mp (t - t'))dt'.$$

We take the initial data as follows

$$\begin{aligned} u_{+,0} &= (\log N)^{-1} N^{-s} \mathcal{F}^{-1}[\chi_{I_N}], \quad u_{-,0} = 0, \\ \phi_0 &= (\log N)^{-1} N^{-r} \mathcal{F}^{-1}[\chi_{I_N}], \quad \phi_1 = 0, \end{aligned} \tag{3.2}$$

where $I_N := [-N-1, -N+1] \cup [N-1, N+1]$. A direct calculation shows

$$\|u_{+,0}\|_{H^s} \sim \|\phi_0\|_{H^r} \sim (\log N)^{-1},$$

and they go to 0 as $N \to \infty$. We also have

$$\|u_{+,0}\|_{M_1} \sim (\log N)^{-1} N^{-s}, \quad \|\phi_0\|_{M_1} \sim (\log N)^{-1} N^{-r}.$$

Here $\|u_{+,0}\|_{M_1}$ goes to infinity as $N \to \infty$ although it is not so troublesome as we will see below. We see also $\|\phi_0\|_{M_1}$ goes to 0.

We use the expression of $K_0[\phi_0, 0]$ to have

$$u_-^{(2)}(t, x) = \frac{1}{2}\left(\int_0^t (\phi_0 u_{+,0})(x + t - 2t')dt' + \phi_0(x + t)\int_0^t u_{+,0}(x + t - 2t')dt'\right).$$

Then we have its Fourier transform

$$\begin{aligned} &\mathcal{F}[u_-^{(2)}](t, \xi) \\ &= \frac{1}{2}\left(\int_0^t e^{i(t-2t')\xi} dt' \mathcal{F}[\phi_0 u_{+,0}](\xi) + \int e^{it(\xi-\eta)} \int_0^t e^{i(t-2t')\eta} dt' \widehat{\phi_0}(\xi - \eta)\widehat{u_{+,0}}(\eta)d\eta\right) \\ &= \frac{1}{2} e^{it\xi}\left(\frac{e^{-2it\xi} - 1}{-2i\xi} \mathcal{F}[\phi_0 u_{+,0}](\xi) + \int \frac{e^{-2it\eta} - 1}{-2i\eta} \widehat{\phi_0}(\xi - \eta)\widehat{u_{+,0}}(\eta)d\eta\right). \end{aligned}$$

We estimate the each terms for $t \leq 1$,

$$\left\| \langle \xi \rangle^s \frac{e^{-2it\xi}-1}{-2i\xi} \mathcal{F}[\phi_0 u_{+,0}](\xi) \right\|_{L^2_\xi([-1,1])} \gtrsim t N^{-(s+r)} (\log N)^{-2},$$

$$\left\| \langle \xi \rangle^s \int \frac{e^{-2it\eta}-1}{-2i\eta} \widehat{\phi_0}(\xi-\eta) \widehat{u_{+,0}}(\eta) d\eta \right\|_{L^2_\xi([-1,1])} \lesssim N^{-1} N^{-(s+r)} (\log N)^{-2}.$$

Therefore we get

$$\|u_-^{(2)}(t,\cdot)\|_{H^s} \geq \|\langle \cdot \rangle^s \mathcal{F}[u_-^{(2)}](t,\cdot)\|_{L^2([-1,1])} \gtrsim (t - N^{-1}) N^{-(s+r)} (\log N)^{-2} \tag{3.3}$$

for $tN \geq 2$.

We show the following by induction argument

$$\|u_\pm^{(k)}(t,\cdot)\|_{M_1} \leq (Ct\|\phi_0\|_{M_1})^{k-1} \|u_{+,0}\|_{M_1}, \quad k = 1,2,3,\dots \tag{3.4}$$

This estimate with $k=1$ is obvious. Assuming that (3.4) holds up to $k-1$, we have

$$\begin{aligned}
\|u_\pm^{(k)}(t,\cdot)\|_{M_1} &\leq \int_0^t \|K_0[\phi_0,0] u_\mp^{(k-1)}(t',\cdot)\|_{M_1} dt' \\
&\leq C \int_0^t \|K_0[\phi_0,0](t',\cdot)\|_{M_1} \|u_\mp^{(k-1)}(t',\cdot)\|_{M_1} dt' \\
&\leq (Ct\|\phi_0\|_{M_1})^{k-1} \|u_{+,0}\|_{M_1},
\end{aligned}$$

which concludes that (3.4) holds for any positive integer k.

Owing to (3.2) and (3.4),

$$\|u_\pm^{(k)}(t,\cdot)\|_{M_1} \leq (CtN^{-r}(\log N)^{-1})^{k-1} N^{-s} (\log N)^{-1}.$$

We get for $s \leq 0$

$$\begin{aligned}
\|u_\pm^{(k)}(t,\cdot)\|_{H^s} &\leq \|\langle \cdot \rangle^s \mathcal{F}[u_\pm^{(k)}(t,\cdot)]\|_{L^2} \leq \sup_{\xi \in \mathbb{R}} \langle \xi \rangle^s \times \|\mathcal{F}[u_\pm^{(k)}(t,\cdot)]\|_{L^2} \\
&\leq \|u_\pm^{(k)}(t,\cdot)\|_{M_1} \leq (CtN^{-r}(\log N)^{-1})^{k-1} N^{-s} (\log N)^{-1}.
\end{aligned}$$

Let $T=1$. For $r \geq 0$, from

$$TN^{-r}(\log N)^{-1} \ll 1 \tag{3.5}$$

and Lemma 5, the well-posedness in the modulation space $M_1(\mathbb{R})$ holds. Furthermore, by

$$\sum_{k=3}^\infty \|u_\pm^{(k)}\|_{L_T^\infty H^s} \leq \sum_{k=3}^\infty (CN^{-r}(\log N)^{-1})^{k-1} N^{-s} (\log N)^{-1} \sim N^{-s-2r} (\log N)^{-3},$$

(2.6) and (3.3) yield

$$\begin{aligned}
\|u_-\|_{L_T^\infty H^s} + \|u_+\|_{L_T^\infty H^s} &\geq \|u_-^{(2)}\|_{L_T^\infty H^s} - \|u_+^{(1)}\|_{L_T^\infty H^s} - \sum_{k=3}^\infty \|u_\pm^{(k)}\|_{L_T^\infty H^s} \\
&\gtrsim N^{-(s+r)} (\log N)^{-2} - (\log N)^{-1} - N^{-s-2r} (\log N)^{-3} \\
&\sim N^{-(s+r)} (\log N)^{-2},
\end{aligned}$$

which leads the norm inflation if $s < 0$, $r \geq 0$, $s + r < 0$.

As we mentioned in Introduction, we may also apply the same argument above to the problem in the range $s<0$, $r<0$, where we take $r'\in\mathbb{R}$ satisfying $s<-r'<0$ and initial data

$$u_{+,0}=(\log N)^{-1}N^{-s}\mathcal{F}^{-1}[\chi_{I_N}],\quad u_{-,0}=0,\\ \phi_0=(\log N)^{-1}N^{-r'}\mathcal{F}^{-1}[\chi_{I_N}],\quad \phi_1=0. \tag{3.6}$$

Then we have

$$\|\phi_0\|_{H^r}\sim N^{r-r'}(\log N)^{-1},$$

and the norm of the solution diverges.

We next consider the massive case $m>0$ or $M>0$. By Lemma 5 and $M_1(\mathbb{R})\hookrightarrow L^2(\mathbb{R})\hookrightarrow H^s(\mathbb{R})$, the difference of the solutions between massive and massless cases is small (see, for example, [**12**]). It shows that the norm inflation for the massive case, which concludes the proof.

4. Proof of Theorem 1 with $s=0,\ r<0$

We only consider the massless case because the difference of the solutions between massive and massless cases is small by a similar argument as in §3. We take the following initial data:

$$u_{+,0}=\delta\mathcal{F}^{-1}[\chi_{[-1,1]}],\quad u_{-,0}=0,\\ \phi_{0,N}=\delta\mathcal{F}^{-1}[\chi_{[-N-1,-N+1]\cup[N-1,N+1]}],\quad \phi_1=0. \tag{4.1}$$

Let $(u_{\pm,N},\phi_N)$ be the solution to (3.1) with this initial data. A direct calculation shows

$$\|u_{+,0}\|_{L^2}\sim\delta,\quad \|\phi_{0,N}\|_{L^2}\sim\delta,\quad \|\phi_{0,N}\|_{H^r}\sim\delta N^r,$$

which yields $\phi_{0,N}\to 0$ in $H^r(\mathbb{R})$ as $N\to\infty$ if $r<0$. Let $(u_\pm,\phi)$ be the solution to (3.1) with $(u_{+,0},0,0,0)$. We may write those by

$$u_+(t,x)=u_{+,0}(x-t),\quad u_-(t,x)=\phi(t,x)=0.$$

We will study the difference of those two solutions $(u_{\pm,N},\phi_N)$ and $(u_\pm,\phi)$. We have the following condition for the initial data

$$\|u_{\pm,N}(0,\cdot)-u_\pm(0,\cdot)\|_{L^2}=0,\qquad \|\phi_N(0,\cdot)-\phi(0,\cdot)\|_{H^r}\sim\delta N^r. \tag{4.2}$$

By setting

$$u_+^{(1)}(t,x)=u_+(t,x)=u_0(x-t),\ u_-^{(1)}(t,x)=u_{+,N}^{(2)}(t,x)=0,\\ u_{-,N}^{(2)}(t,x)=\int_0^t(\phi_N u_+^{(1)})(t',x+(t-t'))dt',$$

and $v_{\pm,N}=u_{\pm,N}-u_\pm^{(1)}-u_{\pm,N}^{(2)}$, we have

$$(\partial_t\pm\partial_x)v_{\pm,N}=\mp\phi_N(u_{\mp,N}-u_\mp^{(1)})=\mp\phi_N v_{\mp,N}\mp\phi_N u_{\mp,N}^{(2)},\quad v_{\pm,N}(0,\cdot)=0.$$

We follow the argument in [**10**, §2.5]. There they wrote the variables $\alpha=t+x$, $\beta=t-x$ and used the smooth cut-off function $\chi_T(\alpha,\beta)$ which is supported on

$|\alpha| + |\beta| \le T$. In a similar way to [**10**, (2.82)], we estimate

$$\begin{aligned}
&\|\chi_T(\alpha,\beta)v_{+,N}\|_{L^2_\beta L^\infty_\alpha} \\
&\lesssim \|\chi_{2T}(\alpha,\beta)(\phi_N\chi_T(\alpha,\beta)v_{-,N} + \phi_N\chi_T(\alpha,\beta)u^{(2)}_{-,N})\|_{L^2_\beta L^1_\alpha} \\
&\lesssim \|\chi_{2T}(\alpha,\beta)\phi_N\|_{L^2_\alpha L^2_\beta}(\|\chi_T(\alpha,\beta)v_{-,N}\|_{L^\infty_\beta L^2_\alpha} + \|\chi_T(\alpha,\beta)u^{(2)}_{-,N}\|_{L^\infty_\beta L^2_\alpha}) \\
&\lesssim \delta T^{1/2}(\|\chi_T(\alpha,\beta)v_{-,N}\|_{L^2_\alpha L^\infty_\beta} + \|\chi_{2T}(\alpha,\beta)\phi_N\|_{L^2_\alpha L^2_\beta}\|\chi_{2T}(\alpha,\beta)u^{(1)}_{+}\|_{L^2_\beta L^\infty_\alpha}) \\
&\lesssim \delta T^{1/2}(\|\chi_T(\alpha,\beta)v_{-,N}\|_{L^2_\alpha L^\infty_\beta} + T^{1/2}\delta^2).
\end{aligned}$$

The estimate for the opposite sign follows in the same manner, and then the sum of them satisfies the following, for sufficiently small $\delta T^{1/2}$,

$$\|\chi_T(\alpha,\beta)v_{+,N}\|_{L^2_\beta L^\infty_\alpha} + \|\chi_T(\alpha,\beta)v_{-,N}\|_{L^2_\alpha L^\infty_\beta} \lesssim T\delta^3.$$

We use the boundedness for (t,x) norm by (α,β) norm (see (2.84) in [**10**]) as

$$\|v_{+,N}\|_{L^\infty_T L^2_x} + \|v_{-,N}\|_{L^\infty_T L^2_x} \lesssim \|v_{+,N}\|_{L^2_\beta L^\infty_\alpha} + \|v_{-,N}\|_{L^2_\alpha L^\infty_\beta} \lesssim T\delta^3. \tag{4.3}$$

On the other hand, by the expression

$$u^{(2)}_{-,N}(t,x) = \frac{1}{2}\left(\int_0^t (\phi_{0,N}u_{+,0})(x+t-2t')dt' + \phi_{0,N}(x+t)\int_0^t u_{+,0}(x+t-2t')dt'\right),$$

we get

$$\begin{aligned}
&\mathcal{F}[u^{(2)}_{-,N}](t,\xi) \\
&= \frac{1}{2}\Bigg(\int_0^t e^{i(t-2t')\xi}dt'\mathcal{F}[\phi_{0,N}u_{+,0}](\xi) \\
&+ \int e^{it(\xi-\eta)}\int_0^t e^{i(t-2t')\eta}dt'\widehat{\phi_{0,N}}(\xi-\eta)\widehat{u_{+,0}}(\eta)d\eta\Bigg) \\
&= \frac{1}{2}e^{it\xi}\left(\frac{e^{-2it\xi}-1}{-2i\xi}\mathcal{F}[\phi_{0,N}u_{+,0}](\xi) + \int \frac{e^{-2it\eta}-1}{-2i\eta}\widehat{\phi_{0,N}}(\xi-\eta)\widehat{u_{+,0}}(\eta)d\eta\right).
\end{aligned}$$

From (4.1), we have for $0 < t \le T \le 1$

$$\begin{aligned}
&|\mathcal{F}[u^{(2)}_{-,N}](t,\xi)| \\
&\ge \frac{1}{2}\left(\left|\int \frac{e^{-2it\eta}-1}{-2i\eta}\widehat{\phi_{0,N}}(\xi-\eta)\widehat{u_{+,0}}(\eta)d\eta\right| - \left|\frac{e^{-2it\xi}-1}{-2i\xi}\mathcal{F}[\phi_{0,N}u_{+,0}](\xi)\right|\right) \\
&\gtrsim t\delta^2\left(\chi_{[N-1,N+1]}(\xi) - \frac{1}{N}\chi_{[-N-2,-N+2]\cup[N-2,N+2]}(\xi)\right),
\end{aligned}$$

which yields

$$\|u^{(2)}_{-,N}(t,\cdot)\|_{L^2} \ge \|\mathcal{F}[u^{(2)}_{-,N}](t,\cdot)\|_{L^2([N-1,N+1])} \gtrsim t\delta^2(1-N^{-1}) \gtrsim t\delta^2. \tag{4.4}$$

Now we estimate for

$$u_{\pm,N} - u_\pm = v_{\pm,N} + u^{(2)}_{\pm,N}$$

by using (4.3) and (4.4) as follows

$$\|u_{-,N}(t,\cdot) - u_-(t,\cdot)\|_{L^2} \ge \|u^{(2)}_{-,N}(t,\cdot)\|_{L^2} - \|v_{-,N}\|_{L^\infty_t L^2} \ge C_1 t\delta^2 - C_2 t\delta^3.$$

Here with small $\delta > 0$, we may say that the right hand side is positive, that is $C_1 t\delta^2 - C_2 t\delta^3 > 0$, for any positive $t > 0$. This means the discontinuity of the solution map: $L^2(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R}) \to C([0,T] : L^2(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R}))$ for any $T > 0$.

5. Well-posedness result in $(\psi, \phi) \in L^1(\mathbb{R}) \times L^\infty(\mathbb{R})$

In this section, we prove Theorem 2 which implies well-posedness of (1.1) in $L^1(\mathbb{R}) \times L^\infty(\mathbb{R})$. This result was derived by Shiota [**16**]. We set $u_\pm := \psi_1 \mp \psi_2$ as we did in the former sections. We consider the Dirac-Klein-Gordon system in the following form:

$$(5.1)\qquad \begin{cases} (\partial_t \pm \partial_x)u_\pm = i(m-\phi)u_\mp, \\ (\partial_t^2 - \partial_x^2 + M^2)\phi = 2\Re(u_+\bar{u}_-), \\ u_\pm(0,x) = u_{\pm,0}(x),\ \phi(0,x) = \phi_0(x),\ \partial_t\phi(0,x) = \phi_1(x). \end{cases}$$

We follow the argument in [**10**, §2.5] here again, using the variables $\alpha = t + x$, $\beta = t - x$ and the smooth cut-off function $\chi_T(\alpha, \beta)$. We assume the initial data condition $u_{\pm,0} \in L^1(\mathbb{R}), \phi_0 \in L^\infty(\mathbb{R}), \phi_1 \in L^1(\mathbb{R})$. Then we show the existence of the following unique solution for (5.1):

$$u_\pm \in C([0,T] : L^1(\mathbb{R})), \qquad \phi \in C([0,T] : L^\infty(\mathbb{R})),$$
$$\chi_{[0,T]}(t)u_+ \in L^1_\beta L^\infty_\alpha, \qquad \chi_{[0,T]}(t)u_- \in L^1_\alpha L^\infty_\beta, \qquad \chi_{[0,T]}(t)\phi \in L^\infty_\beta L^\infty_\alpha.$$

We remark that the smooth cut-off $\chi_{[0,T]}(t)$ means the cut-off with respect to the both finite time $0 < t < T$ and finite space $|x - a| \lesssim T$ for each $a \in \mathbb{R}$ from the finite speed of propagation. Therefore we may treat the solutions as functions restricted on the local domain $|\alpha| + |\beta| \lesssim T$. We estimate on the integral equation corresponding to (5.1) in those spaces. We define the following complete metric space:

$$(5.2)\qquad X = X(T, M_1, M_2) = \{\|u_+\|_{L^1_\beta L^\infty_\alpha} + \|u_-\|_{L^1_\alpha L^\infty_\beta} \le M_1,\ \|\phi\|_{L^\infty_\alpha L^\infty_\beta} \le M_2\}$$

where we set $M_1 = 2(\|u_{+,0}\|_{L^1} + \|u_{-,0}\|_{L^1})$, $M_2 = 2(\|\phi_0\|_{L^\infty} + \|\phi_1\|_{L^1})$. Here we may think M_1 is sufficiently small from the restriction on the local domain $|x - a| \lesssim T$ for $a \in \mathbb{R}$ if we take $T > 0$ small. We show the following map $\Psi(u_\pm, \phi) = (u^\sharp_\pm, \phi^\sharp)$ is a contraction on X:

$$u^\sharp_\pm(t,x) = u_{\pm,0}(x \mp t) + i\int_0^t (mu_\mp - \phi u_\mp)(t', x \mp (t - t'))dt',$$
$$\phi^\sharp(t,x) = K_M[\phi_0, \phi_1] + 2\int_0^t W_M(t - t')\Re(u_+\bar{u}_-)(t', x)dt'$$

where W_M denotes the evolution operator of the Klein-Gordon equation with mass $M \ge 0$. We estimate

$$\begin{aligned} \|\chi_T(\alpha,\beta)u^\sharp_+\|_{L^1_\beta L^\infty_\alpha} &\lesssim \|u_{+,0}\|_{L^1} + \|\chi_{2T}(\alpha,\beta)(mu_- - \phi u_-)\|_{L^1_\beta L^1_\alpha} \\ &\lesssim \|u_{+,0}\|_{L^1} + \|u_-\|_{L^1_\alpha L^1_\beta} + \|\phi\|_{L^\infty_\alpha L^\infty_\beta}\|u_-\|_{L^1_\alpha L^1_\beta} \\ &\lesssim \|u_{+,0}\|_{L^1} + T(\|u_-\|_{L^1_\alpha L^\infty_\beta} + \|\phi\|_{L^\infty_\alpha L^\infty_\beta}\|u_-\|_{L^1_\alpha L^\infty_\beta}). \end{aligned}$$

Similarly

$$\|\chi_T(\alpha,\beta)u^\sharp_-\|_{L^1_\alpha L^\infty_\beta} \lesssim \|u_{-,0}\|_{L^1} + T(\|u_+\|_{L^1_\beta L^\infty_\alpha} + \|\phi\|_{L^\infty_\alpha L^\infty_\beta}\|u_+\|_{L^1_\beta L^\infty_\alpha}).$$

Therefore

$$\|\chi_T(\alpha,\beta)u_+^\sharp\|_{L^1_\beta L^\infty_\alpha} + \|\chi_T(\alpha,\beta)u_-^\sharp\|_{L^1_\alpha L^\infty_\beta} \leq \frac{M_1}{2} + CT(M_1 + M_1M_2) \leq M_1$$

for sufficiently small $T > 0$. We estimate

$$\begin{aligned}\|\chi_T(\alpha,\beta)\phi^\sharp\|_{L^\infty_\beta L^\infty_\alpha} &\lesssim \|\phi_0\|_{L^\infty} + \|\phi_1\|_{L^1} + \|\chi_{2T}(\alpha,\beta)u_+u_-\|_{L^1_\beta L^1_\alpha}\\ &\lesssim \|\phi_0\|_{L^\infty} + \|\phi_1\|_{L^1} + \|u_+\|_{L^1_\beta L^\infty_\alpha}\|u_-\|_{L^1_\alpha L^\infty_\beta}\\ &\leq \frac{M_2}{2} + CM_1^2 \leq M_2\end{aligned}$$

for sufficiently small M_1. This conclude the contraction mapping principle and we obtain the unique solution.

Next, we show that the solutions $(u_+, u_-, \phi) \in C([0,T] : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^\infty(\mathbb{R}))$ above depend continuously on the initial data $(u_{+,0}, u_{0,-}, \phi_0, \phi_1) \in L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^\infty(\mathbb{R}) \times L^1(\mathbb{R})$. Assume that

$$u_{+,0,n} \to u_{+,0},\ u_{-,0,n} \to u_{-,0} \text{ in } L^1, \qquad \phi_{0,n} \to \phi_0 \text{ in } L^\infty, \qquad \phi_{1,n} \to \phi_1 \text{ in } L^1.$$

We write the corresponding solutions $(u_{+,n}, u_{-,n}, \phi_n)$ and (u_+, u_-, ϕ). We have the uniform boundedness with respect to $n = 1, 2, \dots$

$$\|u_{+,n}\|_{L^1_\beta L^\infty_\alpha} + \|u_{-,n}\|_{L^1_\alpha L^\infty_\beta} \leq M_1, \qquad \|\phi_n\|_{L^\infty_\alpha L^\infty_\beta} \leq M_2.$$

Then we estimate the difference

$$\begin{aligned}&\|\chi_T(\alpha,\beta)(u_{+,n} - u_+)\|_{L^1_\beta L^\infty_\alpha}\\ &\lesssim \|u_{+,0,n} - u_{+,0}\|_{L^1} + T(\|u_{-,n} - u_-\|_{L^1_\alpha L^\infty_\beta} + \|\phi_n\|_{L^\infty_\alpha L^\infty_\beta}\|u_{-,n} - u_-\|_{L^1_\alpha L^\infty_\beta}\\ &\qquad + \|\phi_n - \phi\|_{L^\infty_\alpha L^\infty_\beta}\|u_-\|_{L^1_\alpha L^\infty_\beta})\\ &\lesssim \|u_{+,0,n} - u_{+,0}\|_{L^1} + T(\|u_{-,n} - u_-\|_{L^1_\alpha L^\infty_\beta} + \|\phi_n - \phi\|_{L^\infty_\alpha L^\infty_\beta}).\end{aligned}$$

Similarly

$$\begin{aligned}\|\chi_T(\alpha,\beta)(u_{-,n} - u_-)\|_{L^1_\alpha L^\infty_\beta} &\lesssim \|u_{-,0,n} - u_{-,0}\|_{L^1}\\ &\quad + T(\|u_{+,n} - u_+\|_{L^1_\beta L^\infty_\alpha} + \|\phi_n - \phi\|_{L^\infty_\alpha L^\infty_\beta}),\\ \|\chi_T(\alpha,\beta)(\phi_n - \phi)\|_{L^\infty_\alpha L^\infty_\beta} &\lesssim \|\phi_{0,n} - \phi_0\|_{L^\infty} + \|\phi_{1,n} - \phi_1\|_{L^1}\\ &\quad + T(\|u_{-,n} - u_-\|_{L^1_\alpha L^\infty_\beta} + \|u_{+,n} - u_+\|_{L^1_\beta L^\infty_\alpha}).\end{aligned}$$

Therefore we have for sufficiently small $T > 0$,

$$\begin{aligned}&\|\chi_T(\alpha,\beta)(u_{+,n} - u_+)\|_{L^1_\beta L^\infty_\alpha}\\ &+ \|\chi_T(\alpha,\beta)(u_{-,n} - u_-)\|_{L^1_\alpha L^\infty_\beta} + \|\chi_T(\alpha,\beta)(\phi_n - \phi)\|_{L^\infty_\alpha L^\infty_\beta}\\ &\qquad \lesssim \|u_{+,0,n} - u_{+,0}\|_{L^1} + \|u_{-,0,n} - u_{-,0}\|_{L^1} + \|\phi_{0,n} - \phi_0\|_{L^\infty} + \|\phi_{1,n} - \phi_1\|_{L^1}.\end{aligned}$$

This conclude that the solutions in $L^1_\beta L^\infty_\alpha \times L^1_\alpha L^\infty_\beta \times L^\infty_\beta L^\infty_\alpha$ depend continuously on the initial data. We use (4.3) to show that the map is also continuous for the solution in $C([0,T] : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^\infty(\mathbb{R}))$.

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