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Fractional abstract Cauchy problem with order $\alpha \in (1, 2)$

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Abstract. In this paper, we deal with a class of fractional abstract Cauchy problems of order $\alpha \in (1, 2)$ by introducing an operator S_α which is defined in terms of the Mittag-Leffler function and the curve integral. Some nice properties of the operator S_α are presented. Based on these properties, the existence and uniqueness of mild solution and classical solution to the inhomogeneous linear and semilinear fractional abstract Cauchy problems is established accordingly. The regularity of mild solution of the semilinear fractional Cauchy problem is also discussed.

Contents

1. Introduction

In history, the fractional calculus has drawn the attention from many pioneering mathematicians such as Euler, Laplace, Fourier, Liouville, Riemann, Laurant, Hardy, and Riesz etc [**14, 23, 25, 29**]. The applications of the fractional calculus in physics were initially undertaken by Abel and Heaviside [**14, 29**]. Nowadays, it has been seen that fractional differential equations have been widely applied in almost every scientific field, and in many realistic applications it appears to have

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better effects than the classical ones. Qualitative theory and its applications in physics, engineering, economics, biology and ecology are extensively discussed and demonstrated in [**2, 5, 9, 11, 14, 20, 21, 22, 23, 29, 30**] and the references therein.

In the past decades, considerable attention has been attracted to the time fractional diffusion-wave equations which arise in electromagnetic, acoustic and mechanical phenomena etc [**20**], and are derived from the classical diffusion or wave equations by replacing the first- or second-order time derivative by a fractional derivative of order α with $\alpha \in (0, 2]$. For $\alpha \in (0, 1]$, it is a fractional diffusion equation, which was explicitly applied to physics by Nigmatullin [**24**] to describe diffusion in media with fractal geometry (special types of porous media). For $\alpha \in (1, 2]$, it is a fractional wave equation, which governs the propagation of mechanical diffusive waves in viscoelatics media that reveals a power-law creep and thus provides us a physical interpretation in the framework of dynamical viscoelaticity [**20, 21, 28**].

Let us briefly recall some known methods and results on fractional abstract Cauchy problems. Bajlekova [**3**] applied the solution operator (fractional resolvent) to investigate abstract Volterra integral equations [**27**] and to discuss the associated fractional abstract Cauchy problem. Under certain conditions that the coefficient operator is the generator of a solution operator, the existence and uniqueness of mild solution of an inhomogeneous abstract Cauchy problem with order α was established in [**10, 15, 16, 17, 18**]. However, as Bajlekova [**3**] pointed out, the necessary and sufficient condition under which an operator generates a solution operator appears too strong. In many partial differential models of real-life problems, partial differential operators with respect to space variables usually generate the associated C_0-semigroups in the given functional spaces. So if the coefficient operator generates a C_0-semigroup, one needs to construct a corresponding operator to deal with the fractional abstract Cauchy problem, which is usually described by the C_0-semigroup and the probability density function $\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k+1-\alpha)}$, where $\alpha \in (0, 1)$. By using properties of the probability density function $\phi_\alpha(z)$ and the C_0-semigroup, the mild solution of fractional Cauchy problems of order $\alpha \in (0, 1)$ was extensively investigated [**6, 7, 8, 31, 32, 33, 34**].

Note that the order of the fractional derivative is frequently considered between 0 and 1 in the literature, because $\phi_\alpha(z)$ is a probability density function defined for $\alpha \in (0, 1)$. As far as our knowledge goes, not much has been undertaken with mild solutions and classical solutions of fractional evolution equations of order $\alpha \in (1, 2)$ [**19**]. However, some physical phenomena in nature can be modelled and described by equations and systems with fractional derivatives of $\alpha \in (1, 2)$. For example, particles in turbulent flows have long jump step and rapid diffusion speed. In the diffusion equation, one uses $\alpha \in (1, 2)$ to indicate superdiffusion. So studying such fractional partial differential models will enable us to better understand how the diffusion flux goes from regions of higher concentration to regions of lower concentration.

In the present paper, following the previous work [**19**], our purpose is to introduce an operator in terms of the generalized Mittag-Leffler function and the curve integral, and present its properties which will be used to discuss the fractional Cauchy problem. Under certain assumptions on the linear term, we explore the existence and uniqueness of classical solution to the linear inhomogeneous fractional

Cauchy problem

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t), \ t \in (0,T], \\ u(0) = x_0, \quad u'(0) = x_1. \end{cases} \tag{1.1}$$

By means of the Banach fixed point theorem and the Schauder fixed point theorem, we consider the existence and uniqueness of mild solution and classical solution to the semilinear fractional Cauchy problem

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t, u(t)), \ t \in (0,T], \\ u(0) = x_0, \quad u'(0) = x_1, \end{cases} \tag{1.2}$$

where $\alpha \in (1,2)$ in (1.1) and (1.2), D_t^α represents the Caputo fractional derivative of order α, A is a sectorial operator of angle $\theta \in \left[0, \left(1 - \frac{\alpha}{2}\right)\pi\right)$, and x_0, $x_1 \in X$, where X is a Banach space.

The rest of this paper is organized as follows. After presenting some preliminaries on fractional calculus and the Mittag-Leffler function in Section 2, we construct an operator $S_\alpha(t)$ and discuss its properties in Section 3. Section 4 presents the existence of classical solution of the problem (1.1). Section 5 is dedicated to the existence and regularity of mild solution of the problem (1.2). In Section 6, an example is illustrated.

2. Preliminaries

For convenience of statement, we let X be a Banach space and $B(X)$ denote the space of all bounded linear operators from X to X. If A is a closed linear operator, $\rho(A)$ and $\sigma(A)$ denote the resolvent set and the spectral set of A, respectively, and $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. $L^1(\mathbb{R}^+, X)$ represents the Banach space of X-valued Bochner integrable functions, namely, u: $\mathbb{R}^+ \to X$ with the norm

$$\|u\|_{L^1(\mathbb{R}^+,X)} = \int_0^\infty \|u(t)\| dt.$$

Let us recall some notations and definitions. We use '$*$' to define the convolution of functions by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \quad t \geq 0.$$

The Laplace transform of the convolution $f*g$ is equal to the product of the Laplace transforms of these two functions provided both of their Laplace transforms exist. Let $g_\alpha (\alpha > 0)$ denote the function

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $g_0(t) = \delta_0(t)$, is the Dirac delta function.

The Riemann-Liouville fractional integral of order $\alpha \geq 0$ of the function f is defined by

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)ds.$$

The Caputo fractional derivative of order $\alpha > 0$ of f is defined by

$$D_t^\alpha f(t) = \left(g_{m-\alpha} * \frac{d^m}{dt^m} f\right)(t), \quad t \in [0,T]$$

while $f \in C^m([0,T],X)$, where m is the smallest integer greater than or equal to α. The condition of $f \in C^m([0,T],X)$ can be weakened to $f \in C^{m-1}([0,T],X)$ and $g_{m-\alpha} * \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t)\right) \in C^m([0,T],X)$. For more details about the fractional calculus, we refer to [**13, 23, 26**] etc.

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - z} d\mu, \quad \alpha, \beta > 0,\ z \in \mathbb{C},$$

where the path $\mathcal{C}$ is a loop which starts and ends at $-\infty$, and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ in the positive direction. Usually, we denote $E_{\alpha}(z) = E_{\alpha,1}(z)$. As we know, a Mittag-Leffler function has properties as follows [**26**]:

$$\frac{d^m}{dt^m} E_{\alpha}(\mu t^{\alpha}) = \mu t^{\alpha-m} E_{\alpha,\alpha-m+1}(\mu t^{\alpha}), \quad m \in \mathbb{Z}^+, \tag{2.1}$$

$$g_{\alpha-1} * E_{\alpha}(\mu t^{\alpha}) = t^{\alpha-1} E_{\alpha,\alpha}(\mu t^{\alpha}), \tag{2.2}$$

$$\int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\mu t^{\alpha}) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \mu}, \quad Re\lambda > |\mu|^{\frac{1}{\alpha}}. \tag{2.3}$$

The following lemma is regarding the asymptotic formula and estimates of the Mittag-Leffler functions.

Lemma 2.1. [**26**, Theorem 1.4, Theorem 1.6] *If $0 < \alpha < 2$ and $\beta \in \mathbb{R}$, then for an arbitrary integer $N \geq 1$ there holds*

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad \frac{\pi\alpha}{2} < |\arg z| \leq \pi \tag{2.4}$$

as $|z| \to \infty$, and

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \frac{\pi\alpha}{2} < |\arg z| \leq \pi, \tag{2.5}$$

where C is a real constant.

Remark 2.2. Since $\frac{1}{\Gamma(-n)} = 0$ $(n = 0, 1, 2, \ldots)$, from (2.4) we know if $\beta - \alpha = -n$ $(n = 0, 1, 2, \ldots)$, then there holds

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|^2}, \quad \frac{\pi\alpha}{2} < |\arg z| \leq \pi, \tag{2.6}$$

where C is a real constant.

Definition 2.3. [**4**, Definition 1.2.1] Let A be a densely defined and closed linear operator in the Banach space X, then A is called a sectorial operator of angle $\omega \in [0, \pi)$, denoted by $A \in Sect(w)$, provided that

(1) $\sigma(A) \subseteq \overline{\Sigma_{\omega}}$, where

$$\Sigma_{\omega} := \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |argz| < \omega\},\ \omega > 0, \\ (0, \infty), \quad \omega = 0. \end{cases}$$

(2) For every $\omega' \in (\omega, \pi)$, $\sup\left\{\|zR(z,A)\| : z \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\right\} < \infty$.

For a closed linear operator A in the Banach space X, it has

LEMMA 2.4. [**1**, Proposition 1.1.7] *Let A be a closed linear operator on X and I be an interval in $\mathbb{R}$. Let $f\colon I \to X$ be Bochner integrable. Suppose that $f(t) \in D(A)$ for all $t \in I$ and $Af\colon I \to X$ is Bochner integrable. Then we have*

$$\int_I f(t)dt \in D(A) \quad \textit{and} \quad A\int_I f(t)dt = \int_I Af(t)dt.$$

In the sequel, the letter C may be used to represent various positive constants and C_α is used to represent various positive constants depending on α.

3. Properties of Operator $S_\alpha(t)$

Define the operator family $\{S_\alpha(t)\}_{t\geq 0}$ by

$$S_\alpha(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} E_\alpha(\mu t^\alpha)(\mu I + A)^{-1}d\mu, \ t > 0, \\ I, \qquad t = 0, \end{cases} \tag{3.1}$$

where the integral path $\Gamma_{\pi-\theta} := \left\{\mathbb{R}^+ e^{i(\pi-\theta)}\right\} \cup \left\{\mathbb{R}^+ e^{-i(\pi-\theta)}\right\}$ is oriented in the counterclockwise direction and A is a sectorial operator of angle $\theta \in [0, (1-\frac{\alpha}{2})\pi)$ with $0 \in \rho(A)$. Here we only restrict our attention to the case of $\alpha \in (1, 2)$.

Let us present some fundamental properties of $\{S_\alpha(t)\}_{t\geq 0}$ which will be used in the next two sections.

THEOREM 3.1. *For every $t > 0$, $S_\alpha(t)$ is well defined and $\{S_\alpha(t)\}_{t\geq 0}$ is a family of uniformly bounded linear operators on X. It indicates that there exists a constant $M \geq 1$ such that $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$.*

PROOF. For $\mu \in \Gamma_{\pi-\theta}, \ t > 0$ and $\theta \in [0, (1-\frac{\alpha}{2})\pi)$, we know that

$$|\arg(\mu t^\alpha)| = \pi - \theta > \frac{\pi\alpha}{2}.$$

By (2.5) there exists a constant $C > 0$ such that

$$|E_\alpha(\mu t^\alpha)| \leq \frac{C}{1 + |\mu t^\alpha|}. \tag{3.2}$$

In view of $A \in Sect(\theta)$, we see that

$$\Sigma_{\pi-\theta} \subset \rho(-A) \text{ and } \|(\mu I + A)^{-1}\| \leq \frac{C}{|\mu|}, \quad \mu \in \Gamma_{\pi-\theta} \setminus \{0\}. \tag{3.3}$$

It follows from (3.2) and (3.3) that $S_\alpha(t)$ is well defined. For $\mu \in \Gamma_{\pi-\theta}$, since $(\mu I + A)^{-1}$ is a linear operator, it is easy to see that $S_\alpha(t)$ is also a linear operator.

Let $t > 0$ be fixed. From the Cauchy's integral theorem, since $0 \in \rho(A)$, we may rewrite the contour $\Gamma_{\pi-\theta}$ in (3.1) as $\Gamma = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$, where

$$\begin{aligned} \Gamma^1 &= \{re^{i(\pi-\theta)}, r \geq t^{-\alpha}\}, \\ \Gamma^2 &= \{t^{-\alpha}e^{i\varphi}, -(\pi-\theta) < \varphi < \pi - \theta\}, \\ \Gamma^3 &= \{re^{-i(\pi-\theta)}, r \geq t^{-\alpha}\}. \end{aligned}$$

By (3.2) and (3.3), we can estimate the integral on Γ^1 as follows:

$$\begin{aligned}
&\left\|\int_{\Gamma^1} E_\alpha(\mu t^\alpha)(\mu I + A)^{-1} d\mu\right\| \\
\leq &C \int_{\Gamma^1} |E_\alpha(\mu t^\alpha)| \frac{1}{|\mu|} |d\mu| \\
\leq &C \int_{t^{-\alpha}}^{\infty} \left|E_\alpha(re^{i(\pi-\theta)} t^\alpha)\right| \frac{1}{r} dr \\
\leq &C \int_{t^{-\alpha}}^{\infty} \frac{1}{1+rt^\alpha} \frac{1}{r} dr \\
\leq &C \int_{t^{-\alpha}}^{\infty} r^{-2} t^{-\alpha} dr \\
= &C.
\end{aligned}$$

A similar estimate holds for the integral on Γ^3. For the integral on Γ^2, we have

$$\begin{aligned}
&\left\|\int_{\Gamma^2} E_\alpha(\mu t^\alpha)(\mu I + A)^{-1} d\mu\right\| \\
\leq &C \int_{\Gamma^2} |E_\alpha(\mu t^\alpha)| \frac{1}{|\mu|} |d\mu| \\
\leq &C \int_{-(\pi-\theta)}^{\pi-\theta} |E_\alpha(e^{i\varphi})| d\varphi \\
\leq &C E_\alpha(1).
\end{aligned}$$

Since $\|S_\alpha(0)\| = 1$, there exists $M \geq 1$ such that $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$. $\square$

Let $\theta_0 \in \left(\frac{\pi}{2}, \frac{\pi-\theta}{\alpha}\right)$, $\rho > 0$, and

$$l_{\theta_0} := \{re^{-i\theta_0},\ \rho \leq r < \infty\} \cup \{\rho e^{i\varphi},\ |\varphi| < \theta_0\} \cup \{re^{i\theta_0},\ \rho \leq r < \infty\}$$

be oriented in the counterclockwise direction.

THEOREM 3.2. *If $t > 0$, then the integral $\int_{l_{\theta_0}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I + A)^{-1} d\lambda$ converges in the uniform operator topology, and*

$$S_\alpha(t) = \frac{1}{2\pi i} \int_{l_{\theta_0}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I + A)^{-1} d\lambda, \quad t > 0. \tag{3.4}$$

PROOF. We know that $A \in Sect(\theta)$ implies that $\|(\lambda^\alpha I + A)^{-1}\| \leq \frac{C}{|\lambda^\alpha|}$ for $\lambda \in \Gamma_{\theta_0}$, and

$$\begin{aligned}
&\left\|\int_{l_{\theta_0}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I + A)^{-1} d\lambda\right\| \\
\leq &\ C \int_{l_{\theta_0}} e^{Re(\lambda t)} \frac{1}{|\lambda|} |d\lambda| \\
\leq &\ C \int_{\rho}^{\infty} e^{tr\cos\theta_0} \frac{1}{r} dr + C \int_{-\theta_0}^{\theta_0} e^{t\rho\cos\varphi} \frac{1}{\rho} d\varphi.
\end{aligned}$$

As $\theta_0 \in \left(\frac{\pi}{2},\ \frac{\pi-\theta}{\alpha}\right)$, the integrals $\int_\rho^\infty e^{tr\cos\theta_0} \frac{1}{r} dr$ and $\int_{-\theta_0}^{\theta_0} e^{t\rho\cos\varphi} \frac{1}{\rho} d\varphi$ converge for $t > 0$. So the integral $\int_{l_{\theta_0}} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I + A)^{-1} d\lambda$ converges for $t > 0$ in the uniform operator topology.

On the other hand, if $A \in Sect(\theta)$, then $\Sigma_{\pi-\theta} \subset \rho(-A)$. If $\lambda \in \Sigma_{\pi-\theta}$, then $(\lambda I + A)^{-1}$ exists and is a bounded linear operator in X. Since $\theta_0 < \frac{\pi-\theta}{\alpha}$, we see that $\lambda^\alpha \in \Sigma_{\pi-\theta}$ and $(\lambda^\alpha I + A)^{-1}$ exists for $\lambda \in l_{\theta_0}$. It follows from the Cauchy's integral formula that

$$(\lambda^\alpha I + A)^{-1} = \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} (\mu I + A)^{-1}(\lambda^\alpha - \mu)^{-1} d\mu,\ \lambda \in l_{\theta_0}. \tag{3.5}$$

For $\mu \in \Gamma_{\pi-\theta}$ and $\lambda \in l_{\theta_0}$, by using (2.3) we find

$$E_\alpha(\mu t^\alpha) = \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - \mu)^{-1} d\lambda. \tag{3.6}$$

As $t > 0$, combining (3.6) with (3.5) leads to

$$\begin{aligned} S_\alpha(t) &= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} E_\alpha(\mu t^\alpha)(\mu I + A)^{-1} d\mu \\ &= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - \mu)^{-1} d\lambda (\mu I + A)^{-1} d\mu \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}\frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} (\lambda^\alpha - \mu)^{-1}(\mu I + A)^{-1} d\mu d\lambda \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1} d\lambda. \end{aligned}$$

□

REMARK 3.3. By the Cauchy's integral theorem and (3.4), one can see that the inverse Laplace transform of $\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}$ ($\lambda^\alpha \in \rho(-A)$) is $S_\alpha(t)$ and there holds

$$\int_0^\infty e^{-\lambda t} S_\alpha(t) dt = \lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}, \quad \lambda^\alpha \in \rho(-A).$$

THEOREM 3.4. *$\{S_\alpha(t)\}_{t\geq 0}$ is a family of strongly continuous operators. That is, for each $x \in X$, the mapping $t \to S_\alpha(t)x$ is continuous from $[0,\infty)$ into X.*

PROOF. In view of Theorem 3.2, fixing $t_0 > 0$, for $t > 0$ and $x \in X$ we have

$$S_\alpha(t)x - S_\alpha(t_0)x = \frac{1}{2\pi i}\int_{l_{\theta_0}} (e^{\lambda t} - e^{\lambda t_0})\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1} x d\lambda.$$

By the dominated convergence theorem, it gives $\lim_{t\to t_0} S_\alpha(t)x = S_\alpha(t_0)x$.

To prove $\|S_\alpha(t)x - x\| \to 0$ as $t \to 0^+$, we choose $\theta_0 \in \left(\frac{\pi}{2}, \frac{\pi-\theta}{\alpha}\right)$. For $x \in D(A)$, by Theorem 3.2 we have

$$\begin{aligned} S_\alpha(t)x - x &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1} x d\lambda - x \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\left[\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1} - \lambda^{-1}\right] x d\lambda \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} \lambda^{-1} e^{\lambda t}(\lambda^\alpha I + A)^{-1} d\lambda (-A)x. \end{aligned}$$

Let $\lambda t = \mu$, then l_{θ_0} turns to be l'_{θ_0}, and

$$\begin{aligned}\|S_\alpha(t)x - x\| &\le \frac{1}{2\pi}\int_{l'_{\theta_0}} \frac{|e^\mu|}{|\mu|}\left[\left(\frac{\mu}{t}\right)^\alpha I + A\right]^{-1}|d\mu|\|Ax\| \\ &\le C\int_{l'_{\theta_0}} \frac{|e^\mu|}{|\mu|^{\alpha+1}}|d\mu|\|Ax\|t^\alpha \\ &\le C_\alpha\|Ax\|t^\alpha,\end{aligned}$$

where the integral $\int_{l'_{\theta_0}} \frac{|e^\mu|}{|\mu|^{\alpha+1}}|d\mu|$ exists because of $\theta_0 \in \left(\frac{\pi}{2}, \frac{\pi-\theta}{\alpha}\right)$. Thus, we get

$$\lim_{t\to 0^+} S_\alpha(t)x = x \text{ for } x \in D(A).$$

Using $\|S_\alpha(t)\| \le M$ and $\overline{D(A)} = X$, we obatin

$$\lim_{t\to 0^+} S_\alpha(t)x = x \text{ for all } x \in X.$$

□

THEOREM 3.5. *For $x \in X$, the following three statements are true.*

(i) *If $t > 0$, then $S_\alpha(t)x \in D(A)$ and $\|AS_\alpha(t)\| \le \frac{C_\alpha}{t^\alpha}$.*
(ii) *If $t \ge 0$, then $(g_\alpha * S_\alpha)(t)x \in D(A)$ and $A(g_\alpha * S_\alpha)(t)x = x - S_\alpha(t)x$.*
(iii) *If $t > 0$, then $S_\alpha(t)x \in C^\infty((0,\infty), X)$ and $D_t^\alpha S_\alpha(t)x = -AS_\alpha(t)x$.*

PROOF. (*i*) Since A is a sectorial operator of angle $\theta < \left(1 - \frac{\alpha}{2}\right)\pi$, we have

$$\frac{\pi - \theta}{\alpha} > \frac{\pi}{2}$$

and

$$\|(zI + A)^{-1}\| \le \frac{C}{|z|}, \quad z \in \Sigma_{\pi-\theta} \setminus \{0\}. \tag{3.7}$$

Choose $\theta_0 \in (\frac{\pi}{2}, \frac{\pi-\theta}{\alpha})$ for $\lambda \in \Gamma_{\theta_0}$ and let $\lambda t = \mu$. Then l_{θ_0} changes to l'_{θ_0}. Using the identity $(\lambda^\alpha I + A)(\lambda^\alpha I + A)^{-1} = I$, we deduce that

$$\begin{aligned}AS_\alpha(t) &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}A(\lambda^\alpha I + A)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}d\lambda - \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}\lambda^\alpha(\lambda^\alpha I + A)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{l'_{\theta_0}} e^\mu\left(\frac{\mu}{t}\right)^{\alpha-1}\frac{1}{t}d\mu - \frac{1}{2\pi i}\int_{l'_{\theta_0}} e^\mu\left(\frac{\mu}{t}\right)^{2\alpha-1}\left[\left(\frac{\mu}{t}\right)^\alpha I + A\right]^{-1}\frac{1}{t}d\mu \\ &= \frac{1}{t^\alpha}\frac{1}{2\pi i}\int_{l'_{\theta_0}} e^\mu\mu^{\alpha-1}d\mu - \frac{1}{2\pi i}\int_{l'_{\theta_0}} e^\mu\mu^{2\alpha-1}t^{-2\alpha}\left[\left(\frac{\mu}{t}\right)^\alpha I + A\right]^{-1}d\mu.\end{aligned}$$

It follows from (3.7) that

$$\|AS_\alpha(t)\| \le \frac{C}{t^\alpha}\int_{l'_{\theta_0}} |e^\mu||\mu^{\alpha-1}||d\mu| \le \frac{C_\alpha}{t^\alpha}.$$

(*ii*) From Part (*i*) and using the closeness of A, we see $A(g_\alpha * S_\alpha)(t)x = (g_\alpha * AS_\alpha)(t)x$. In view of the identity $A(\lambda^\alpha I + A)^{-1} = I - \lambda^\alpha(\lambda^\alpha I + A)^{-1}$, the

Laplace transform property of the convolution, as well as Theorem 3.2, we have

$$\begin{aligned}
A(g_\alpha * S_\alpha)(t)x &= (g_\alpha * AS_\alpha)(t)x \\
&= g_\alpha * \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}A(\lambda^\alpha I + A)^{-1}x d\lambda \\
&= g_\alpha * \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}x d\lambda - g_\alpha * \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}\lambda^\alpha(\lambda^\alpha I + A)^{-1}x d\lambda \\
&= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{-\alpha}\lambda^{\alpha-1}x d\lambda - \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{-\alpha}\lambda^{\alpha-1}\lambda^\alpha(\lambda^\alpha I + A)^{-1}x d\lambda \\
&= x - S_\alpha(t)x.
\end{aligned}$$

(*iii*) By (2.1) and the dominated convergence theorem, we get

$$\frac{d^m}{dt^m}S_\alpha(t)x = \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} \mu t^{\alpha-m}E_{\alpha,\alpha-m+1}(\mu t^\alpha)(\mu I + A)^{-1}x d\mu, \quad (m \in \mathbb{Z}^+),$$

which is convergent. So it means $S_\alpha(t)x \in C^\infty((0,\infty),X)$. Taking the α-th differentiation with respect to t on both sides of

$$(g_\alpha * AS_\alpha)(t)x = A(g_\alpha * S_\alpha)(t)x = x - S_\alpha(t)x,$$

one can see that $D_t^\alpha S_\alpha(t)x = -AS_\alpha(t)x$.

$\square$

THEOREM 3.6. *When $t > 0$, we have*

(i) $\|(g_{\alpha-1} * S_\alpha)(t)\| \le C_\alpha t^{\alpha-1}$;
(ii) *for every $x \in X$, $(g_{\alpha-1} * S_\alpha)(t)x \in D(A)$ and $\|A(g_{\alpha-1} * S_\alpha)(t)\| \le \frac{C}{t}$; and*
(iii) $S'_\alpha(t) = -A(g_{\alpha-1} * S_\alpha)(t)$.

PROOF. (*i*) In view of (2.3), for $\mu \in \Gamma_{\pi-\theta}$ and $\lambda \in l_{\theta_0}$ $\left(\theta_0 \in \left(\frac{\pi}{2}, \frac{\pi-\theta}{\alpha}\right)\right)$, the inverse Laplace transform of $(\lambda^\alpha - \mu)^{-1}$ has the following form:

$$t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha) = \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha - \mu)^{-1}d\lambda. \tag{3.8}$$

By virtue of the Fubini theorem and the Cauchy's integral formula, using (2.2) and (3.8) leads to

$$\begin{aligned}
(g_{\alpha-1} * S_\alpha)(t) &= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} \left[g_{\alpha-1}(t) * E_\alpha(\mu t^\alpha)\right](\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha)(\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha - \mu)^{-1}d\lambda(\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} (\lambda^\alpha - \mu)^{-1}(\mu I + A)^{-1}d\mu d\lambda \\
&= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha I + A)^{-1}d\lambda.
\end{aligned} \tag{3.9}$$

So it further gives

$$\begin{aligned}\|(g_{\alpha-1}*S_\alpha)(t)\| &= \left\|\frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha I+A)^{-1}d\lambda\right\| \\ &= \frac{1}{2\pi}\left\|\int_{l_{\theta_0'}} e^{\mu}\left[\left(\frac{\mu}{t}\right)^\alpha I+A\right]^{-1}\frac{1}{t}d\mu\right\| \\ &\le \frac{C}{2\pi}\int_{l_{\theta_0'}} |e^\mu|\frac{t^{\alpha-1}}{|\mu|^\alpha}|d\mu| \\ &= C_\alpha t^{\alpha-1}.\end{aligned}$$

(ii) For $x\in X$ and $t>0$, by (3.9) and Lemma 2.4 we have

$$(g_{\alpha-1}*S_\alpha)(t)x\in D(A)$$

and

$$\begin{aligned}A(g_{\alpha-1}*S_\alpha)(t) &= (g_{\alpha-1}*AS_\alpha)(t) \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}A(\lambda^\alpha I+A)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}d\lambda-\frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^\alpha(\lambda^\alpha I+A)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{l'_{\theta_0}} e^{\mu}\frac{1}{t}d\mu-\frac{1}{2\pi i}\int_{l'_{\theta_0}} e^{\mu}\left(\frac{\mu}{t}\right)^\alpha\left[\left(\frac{\mu}{t}\right)^\alpha I+A\right]^{-1}\frac{1}{t}d\mu.\end{aligned}$$

Combining this equality with (3.7) gives

$$\|A(g_{\alpha-1}*S_\alpha)(t)\|\le\frac{C}{t}\int_{l'_{\theta_0}}|e^\mu||d\mu|\le\frac{C}{t}.$$

(iii) Using the Cauchy's integral theorem, for any $R>0$ we have

$$\begin{aligned}&\int_{\Gamma_{\pi-\theta}} E_{\alpha,\alpha}\left(\mu t^\alpha\right)d\mu \\ &= \int_R^\infty E_{\alpha,\alpha}\left(re^{i(\pi-\theta)}t^\alpha\right)e^{i(\pi-\theta)}dr+\int_{-(\pi-\theta)}^{\pi-\theta} E_{\alpha,\alpha}\left(Re^{i\varphi}t^\alpha\right)iRe^{i\varphi}d\varphi \\ &\quad+\int_R^\infty E_{\alpha,\alpha}\left(re^{-i(\pi-\theta)}t^\alpha\right)e^{-i(\pi-\theta)}dr.\end{aligned}$$

From Remark 2.2, we see that

$$\begin{aligned}&\left|\int_R^\infty E_{\alpha,\alpha}\left(re^{i(\pi-\theta)}t^\alpha\right)e^{i(\pi-\theta)}dr+\int_R^\infty E_{\alpha,\alpha}\left(re^{-i(\pi-\theta)}t^\alpha\right)e^{-i(\pi-\theta)}dr\right| \\ &\le\int_R^\infty\frac{C}{1+r^2t^{2\alpha}}dr \\ &\to 0,\ \text{as } R\to\infty,\end{aligned}$$

and

$$\left|\int_{-(\pi-\theta)}^{\pi-\theta} E_{\alpha,\alpha}\left(Re^{i\varphi}t^\alpha\right)iRe^{i\varphi}d\varphi\right|\le\frac{CR}{1+R^2t^{2\alpha}}\to 0,\ \text{as } R\to\infty.$$

Thus, we obtain

$$\int_{\Gamma_{\pi-\theta}} E_{\alpha,\alpha}\left(\mu t^{\alpha}\right) d\mu = 0. \tag{3.10}$$

By the dominated convergence theorem, using (2.1) and (3.10) we have

$$\begin{aligned}
S'_{\alpha}(t) &= \frac{d}{dt}\left(\frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} E_{\alpha}(\mu t^{\alpha})(\mu I + A)^{-1} d\mu\right) \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} \mu t^{\alpha-1} E_{\alpha,\alpha}(\mu t^{\alpha})(\mu I + A)^{-1} d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-1} E_{\alpha,\alpha}(\mu t^{\alpha}) d\mu - \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} A t^{\alpha-1} E_{\alpha,\alpha}(\mu t^{\alpha})(\mu I + A)^{-1} d\mu \\
&= -\frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} A\left(g_{\alpha-1} * E_{\alpha}(\mu t^{\alpha})\right)(\mu I + A)^{-1} d\mu \\
&= -A(g_{\alpha-1} * S_{\alpha})(t).
\end{aligned}$$

□

The following result is regarding the compact criterion.

Theorem 3.7. *If $(\lambda I + A)^{-1}$ is a compact operator for $\lambda \in \rho(-A)$, then $S_{\alpha}(t)$ is also a compact operator for $t > 0$.*

Proof. Let $\Omega \subset X$ be a bounded set. For $t > 0$, we need to prove that $S_{\alpha}(t)\Omega$ is compact. Since $(\lambda I + A)^{-1}$ is compact for $\lambda \in \rho(-A)$, it is easy to see that $\lambda(\lambda I + A)^{-1} S_{\alpha}(t)\Omega$ is also compact. On the other hand, by Theorem 3.5, we know $S_{\alpha}(t)x \in D(A)$ for $x \in \Omega$. Then for $\lambda \in \rho(-A)$, we have

$$\begin{aligned}
&\lambda(\lambda I + A)^{-1} S_{\alpha}(t)x - S_{\alpha}(t)x \\
=& - A(\lambda I + A)^{-1} S_{\alpha}(t)x \\
=& - (\lambda I + A)^{-1} A S_{\alpha}(t)x.
\end{aligned}$$

Thus, we further have

$$\begin{aligned}
&\|\lambda(\lambda I + A)^{-1} S_{\alpha}(t)x - S_{\alpha}(t)x\| \\
=&\|(\lambda I + A)^{-1} A S_{\alpha}(t)x\| \\
\le&\frac{C}{|\lambda|}\|A S_{\alpha}(t)x\| \to 0, \text{ as } |\lambda| \to \infty.
\end{aligned}$$

Consequently, given the compactness of $\lambda(\lambda I + A)^{-1} S_{\alpha}(t)\Omega$, we know that $S_{\alpha}(t)\Omega$ is also compact. □

Remark 3.8. From Theorem 3.7, we see that, if A generates a compact C_0-semigroup, then $S_{\alpha}(t)$ is a compact operator for $t > 0$.

4. Linear Problem

In this section, we apply the properties presented in the preceding section to a linear fractional Cauchy problem

$$\begin{cases} D_t^{\alpha} u(t) + A u(t) = f(t), \ t \in (0, T], \\ u(0) = x_0, \ u'(0) = x_1, \end{cases} \tag{4.1}$$

where $x_0, x_1 \in X$, $A \in Sect(\theta)$ with $\theta \in \left[0, \left(1 - \frac{\alpha}{2}\right)\pi\right)$ and $f \in L^1((0,T);X)$.

In order to present the definition of mild solution of the problem(4.1), we need the following two technical lemmas.

LEMMA 4.1. *Suppose that $u \in C([0,T];X)$ satisfies $(g_{2-\alpha} * u) \in C^2((0,T];X)$. If for $t \in [0,T]$, $u(t) \in D(A)$ satisfies the problem (4.1) and $Au \in L^1((0,T);X)$, then we have*

$$u(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + (g_{\alpha-1} * S_\alpha * f)(t). \tag{4.2}$$

PROOF. If $u(t)$ satisfies all assumptions, it can be rewritten as

$$u(t) = x_0 + tx_1 - A(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \in [0,T]. \tag{4.3}$$

For $\lambda > 0$, applying the Laplace transform to (4.3) yields

$$\hat{u}(\lambda) = \lambda^{-1}x_0 + \lambda^{-2}x_1 - \lambda^{-\alpha}A\hat{u}(\lambda) + \lambda^{-\alpha}\hat{f}(\lambda).$$

That is,

$$\hat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}x_0 + \lambda^{\alpha-2}(\lambda^\alpha I + A)^{-1}x_1 + (\lambda^\alpha I + A)^{-1}\hat{f}(\lambda), \ \lambda > 0. \tag{4.4}$$

According to Theorem 3.2, for $\lambda > 0$ it gives

$$\begin{aligned}\lambda^{\alpha-2}(\lambda^\alpha I + A)^{-1}x_1 &= \lambda^{-1}\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}x_1 \\ &= \int_0^\infty e^{-\lambda t}(1 * S_\alpha)(t)dt\end{aligned}$$

and

$$\begin{aligned}(\lambda^\alpha I + A)^{-1}\hat{f}(\lambda) &= \lambda^{1-\alpha}\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}\hat{f}(\lambda) \\ &= \int_0^\infty e^{-\lambda t}(g_{\alpha-1} * S_\alpha * f)(t)dt.\end{aligned}$$

So taking the inverse Laplace transform to (4.4), we arrive at formula (4.2). □

LEMMA 4.2. *If $f \in L^1((0,T);X)$, then the convolution $(g_{\alpha-1} * S_\alpha * f)(t)$ exists and defines a continuous function.*

PROOF. If $f \in L^1((0,T);X)$ and $g_{\alpha-1} \in L^1((0,T);\mathbb{R})$, then $(g_{\alpha-1} * f) \in L^1((0,T);X)$. By Theorem 1.3.4 in [**27**], we know that $(S_\alpha * g_{\alpha-1} * f)(t)$ exists and defines a continuous function. So $(g_{\alpha-1} * S_\alpha * f)(t) = (S_\alpha * g_{\alpha-1} * f)(t)$ exists and defines a continuous function. □

DEFINITION 4.3. A function $u \in C([0,T],X)$ given by

$$u(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + (g_{\alpha-1} * S_\alpha * f)(t)$$

is called a mild solution of the Cauchy problem (4.1).

DEFINITION 4.4. A function $u \in C([0,T],X)$ is called a classical solution of the problem (4.1) if $D_t^\alpha u \in C((0,T],X)$, and for all $t \in (0,T]$ $u(t) \in D(A)$ satisfies the problem (4.1).

According to Definition 4.3 and Lemma 4.1, for any $f \in L^1((0,T);X)$, the Cauchy problem (4.1) has a unique mild solution. Now a natural question to the problem (4.1) is that under what conditions on f, a mild solution is also a classical solution when x_0, $x_1 \in X$.

THEOREM 4.5. *Suppose that $f \in L^1((0,T);X)$ and f is Hölder continuous with an exponent $\gamma \in (0,1]$. That is, there is a constant $k > 0$ such that*

$$\|f(t) - f(s)\| \leq k|t-s|^\gamma, \quad 0 < t \text{ and } s \leq T.$$

Then $w(t) := (g_{\alpha-1} * S_\alpha * f)(t)$ is the unique classical solution of the following problem

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t), & t \in (0,T], \\ u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{4.5}$$

PROOF. From Lemma 4.2, we know that $w \in C([0,T];X)$. Firstly, we show that $w(t) \in D(A)$ for $t \in (0,T]$.

Write $w(t) = I_1(t) + I_2(t)$, where

$$I_1(t) = \int_0^t (g_{\alpha-1} * S_\alpha)(t-s)(f(s) - f(t))ds, \quad 0 < t \leq T$$

and

$$I_2(t) = \int_0^t (g_{\alpha-1} * S_\alpha)(t-s)f(t)ds, \quad 0 < t \leq T.$$

Then, it gives

$$I_2(t) = (1 * g_{\alpha-1} * S_\alpha)(t)f(t) = (g_\alpha * S_\alpha)(t)f(t).$$

By Theorem 3.5 (ii), we see that $I_2(t) \in D(A)$ and

$$AI_2(t) = A(g_\alpha * S_\alpha)(t)f(t) = f(t) - S_\alpha(t)f(t), \quad 0 < t \leq T. \tag{4.6}$$

As $t \in (0,T]$, by Theorem 3.6 (ii) and Hölder continuity of f, we find

$$\|A(g_{\alpha-1} * S_\alpha)(t-s)(f(s) - f(t))\| \leq \frac{C}{t-s}(t-s)^\gamma \in L^1(0,t).$$

Using Lemma 2.4, we have $I_1(t) \in D(A)$.

In order to verify $D_t^\alpha w \in C((0,T];X)$, we need to show that $Aw \in C((0,T];X)$ and $D_t^\alpha w = -Aw(t) + f(t)$.

Let $v(t) := -Aw(t) + f(t)$. By Theorem 3.6 (iii), it gives that

$$\begin{aligned} v(t) &= -A(g_{\alpha-1} * S_\alpha * f)(t) + f(t) \\ &= (S'_\alpha * f)(t) + f(t). \end{aligned}$$

If $Aw \in C((0,T];X)$, we know $v \in C(0,T]$. It follows from the Fubini theorem that

$$\begin{aligned} \int_0^t v(s)dt &= \int_0^t \int_0^s S'_\alpha(s-\tau)f(\tau)d\tau ds + \int_0^t f(s)ds \\ &= \int_0^t \int_\tau^t S'_\alpha(s-\tau)f(\tau)dsd\tau + \int_0^t f(s)ds \\ &= \int_0^t \int_0^{t-\tau} S'_\alpha(s)f(\tau)dsd\tau + \int_0^t f(s)ds \\ &= \int_0^t (S_\alpha(t-\tau) - I)f(\tau)ds + \int_0^t f(s)ds \\ &= (S_\alpha * f)(t), \end{aligned}$$

which implies

$$\frac{d}{dt}(S_\alpha * f)(t) = v(t). \tag{4.7}$$

Note that

$$D_t^\alpha w(t) = D_t^\alpha (g_{\alpha-1} * S_\alpha * f)(t) = \frac{d}{dt}(S_\alpha * f)(t). \tag{4.8}$$

From (4.7) and (4.8), we obtain $D_t^\alpha w = -Aw(t) + f(t)$.

Now, we show that $Aw \in C((0,T];X)$. By (4.6) and Theorem 3.1, we see that $AI_2(t)$ is continuous on $(0,T]$.

When $h > 0$ and $t \in (0, T-h]$, we can rewrite $AI_1(t+h) - AI_1(t)$ as

$$AI_1(t+h) - AI_1(t) = h_1 + h_2 + h_3,$$

where

$$\begin{aligned} h_1 &= \int_0^t A[(g_{\alpha-1} * S_\alpha)(t+h-s) - (g_{\alpha-1} * S_\alpha)(t-s)](f(s) - f(t))ds, \\ h_2 &= \int_0^t A(g_{\alpha-1} * S_\alpha)(t+h-s)(f(t) - f(t+h))ds, \\ h_3 &= \int_t^{t+h} A(g_{\alpha-1} * S_\alpha)(t+h-s)(f(s) - f(t+h))ds. \end{aligned}$$

For h_1, it has

$$\lim_{h\to 0} A(g_{\alpha-1} * S_\alpha)(t+h-s)(f(s) - f(t)) = A(g_{\alpha-1} * S_\alpha)(t-s)(f(s) - f(t)).$$

By Theorem 3.6 (ii), we know that

$$\begin{aligned} \|A(g_{\alpha-1} * S_\alpha)(t+h-s)(f(s) - f(t))\| &\le C(t+h-s)^{-1}(t-s)^\gamma \\ &\le C(t-s)^{\gamma-1} \in L^1(0,t). \end{aligned}$$

Using the dominated convergence theorem gives

$$h_1 \to 0 \text{ as } h \to 0.$$

Similarly, for h_2 and h_3, we have

$$\begin{aligned} \|h_2\| &= \left\| \int_0^t A(g_{\alpha-1} * S_\alpha)(t+h-s)(f(t) - f(t+h))ds \right\| \\ &\le C \int_0^t (t+h-s)^{-1} h^\gamma ds \\ &= C(\ln(t+h) - \ln h)h^\gamma \to 0 \text{ as } h \to 0, \end{aligned}$$

and

$$\begin{aligned} \|h_3\| &\le C \int_t^{t+h} (t+h-s)^{-1}(t+h-s)^\gamma ds \\ &= \frac{Ch^\gamma}{\gamma} \to 0 \text{ as } h \to 0. \end{aligned}$$

Thus, this means $Aw \in C((0,T];X)$.

Note that $w(t) = (g_{\alpha-1} * S_\alpha * f)(t)$ and $S_\alpha * f \in C^1$. Using a similar proof to (4.7), we have

$$\begin{aligned} w'(t) &= \frac{d}{dt}(g_{\alpha-1} * S_\alpha * f)(t) \\ &= [g_{\alpha-1} * (S_\alpha * f)'](t). \end{aligned}$$

It is easy to see that $w(0) = 0$ and $w'(0) = 0$. Consequently, by virtue of Lemma 4.1, w is a unique classical solution of the problem (4.5). $\square$

For x_0, $x_1 \in X$, let $\widetilde{u}(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1$ $(t \geq 0)$. It follows from Theorem 3.5 (ii) that $\widetilde{u}(t)$ is a classical solution of the following problem

$$\begin{cases} D_t^\alpha u(t) + Au(t) = 0, & t \in (0,T], \\ u(0) = x_0, \ u'(0) = x_1. \end{cases}$$

Then $w + \widetilde{u}$ is a classical solution of the problem (4.1). According to Lemma 4.1, it is unique. So we obtain the following corollary.

COROLLARY 4.6. Suppose that $f \in L^1((0,T);X)$ and f is Hölder continuous with an exponent $\gamma \in (0,1]$. Then, for x_0, $x_1 \in X$,

$$u(t) := S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + (g_{\alpha-1} * S_\alpha * f)(t)$$

is the unique classical solution of the problem (4.1).

5. Nonlinear Problem

Consider the nonlinear fractional Cauchy problem:

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t, u(t)), & t \in (0,T], \\ u(0) = x_0, \quad u'(0) = x_1. \end{cases} \tag{5.1}$$

DEFINITION 5.1. A function $u \in C([0,T];X)$ is called a mild solution of the problem (5.1) if u satisfies

$$u(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + \int_0^t (g_{\alpha-1} * S_\alpha)(t-s)f(s,u(s))ds.$$

The following theorem is regarding the existence of mild solution of the problem (5.1).

THEOREM 5.2. *Suppose that the nonlinear mapping $f(t,x)$: $[0,T] \times X \to X$ is continuous with respect to t and there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \text{ for } t \in [0,T] \text{ and } x,\, y \in X. \tag{5.2}$$

Then the Cauchy problem (5.1) *has a unique mild solution for x_0, $x_1 \in X$.*

PROOF. Consider the Banach space $C([0,T],X)$ with the norm

$$\|u\|_{C([0,T];X)} = \max_{[0,T]} \|u(t)\|.$$

Define the operator by

$$(Fu)(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + \int_0^t (g_{\alpha-1} * S_\alpha)(t-s)f(s,u(s))ds. \tag{5.3}$$

Using a similar argument to the proof of Lemma 4.2, we know that F maps $C([0,T],X)$ into itself. It suffices to prove that F has a unique fixed point, which is a unique mild solution of the problem (5.1).

Assume that u, $v \in C([0,T];X)$. Using (5.2) and Theorem 3.6 (i), for $t \in [0,T]$ we have

$$\begin{aligned}\|(Fu)(t)-(Fv)(t)\| &= \left\|\int_0^t (g_{\alpha-1} * S_\alpha)(t-s)(f(s,u(s))-f(s,v(s))ds\right\| \\ &\le \int_0^t \|(g_{\alpha-1} * S_\alpha)(t-s)\|\|(f(s,u(s))-f(s,v(s))\|ds \\ &\le \int_0^t C_\alpha (t-s)^{\alpha-1} L\|u(s)-v(s)\|ds \\ &\le \frac{C_\alpha L}{\alpha} t^\alpha \|u-v\|_{C([0,T];X)} \\ &\le \frac{C_\alpha L}{\alpha} T^{\alpha-1} t \|u-v\|_{C([0,T];X)}. \end{aligned} \tag{5.4}$$

By the mathematical induction, we derive that

$$\|(F^n u)(t)-(F^n v)(t)\| \le \frac{(C_\alpha L T^{\alpha-1} t)^n}{\alpha n!}\|u-v\|_{C([0,T];X)}, \quad t \in [0,T].$$

That is,

$$\|F^n u - F^n v\|_{C([0,T];X)} \le \frac{(C_\alpha L T^\alpha)^n}{\alpha n!}\|u-v\|_{C([0,T];X)}.$$

Apparently, $\frac{(C_\alpha L T^\alpha)^n}{\alpha n!} < 1$ as $n \to +\infty$. Hence, F^n is a contraction map and has a unique fixed point. Thus, F has a unique fixed point. □

Now, by dropping the Lipschitz continuity of f with respect to the second variable and imposing a proper condition on the operator A, we show the existence of mild solution of the problem (5.1).

THEOREM 5.3. *Suppose that A generates a compact C_0-semigroup and the nonlinear mapping f: $[0,T]\times X \to X$ is a Carathéodory function. If for any $r > 0$, there exists a function $h_r \in L^p((0,T);\mathbb{R}^+)$ with $p > 1$ such that*

$$\|f(t,x)\| \le h_r(t), \tag{5.5}$$

for a.e. $t \in [0,T]$ and $x \in X$ with $\|x\| \le r$, and

$$\liminf_{r\to\infty} \frac{\|h_r\|_{L^p(0,T)}}{r} = \delta < \infty. \tag{5.6}$$

Then for x_0, $x_1 \in X$, the problem (5.1) *has at least a mild solution provided that*

$$C_\alpha \delta \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{\frac{1}{q}} < 1, \tag{5.7}$$

where C_α is given as in Theorem 3.6 (i).

PROOF. Given x_0, $x_1 \in X$, we define a mapping F: $C([0,T];X) \to C([0,T];X)$ by

$$(Fu)(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + \int_0^t (g_{\alpha-1} * S_\alpha)(t-s) f(s,u(s))ds.$$

Let

$$B_r = \{u \in C([0,T];X) : \|u(t)\| \le r \text{ for } 0 \le t \le T\},$$

where $r > 0$ is to be determined.

Firstly, we show that there exists a constant $r > 0$ such that F maps B_r into itself.

Suppose otherwise, then for every $r > 0$, there exists $u^r \in B_r$ and $t \in [0,T]$ such that $\|(Fu^r)(t)\| > r$. On the other hand, by Theorem 3.6 (i) and (5.5), we deduce that

$$\begin{aligned}
r &< \|(Fu^r)(t)\| \\
&\leq \|S_\alpha(t)x_0\| + \|(1 * S_\alpha)(t)x_1\| + \int_0^t \|(g_{\alpha-1} * S_\alpha)(t-s)f(s,u^r(s))\|ds \\
&\leq M\|x_0\| + MT\|x_1\| + C_\alpha \int_0^t (t-s)^{\alpha-1}h_r(s)ds \\
&\leq M\|x_0\| + MT\|x_1\| + C_\alpha \left(\int_0^t (t-s)^{(\alpha-1)q}ds\right)^{\frac{1}{q}} \|h_r\|_{L^p(0,T)} \\
&= M\|x_0\| + MT\|x_1\| + C_\alpha \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{\frac{1}{q}} \|h_r\|_{L^p(0,T)},
\end{aligned}$$

where $q = \frac{p}{p-1}$. Dividing both sides by r and taking the limit as $r \to \infty$, in view of (5.6) we have $1 \leq C_\alpha \delta \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{\frac{1}{q}}$. This yields a contradiction with (5.7). Thus, there exists $r > 0$ such that $F(B_r) \subset B_r$.

Now, we separate our discussions into three steps.

Step 1: F is continuous on B_r.

Let $\{u_n\}_{n=1}^{\infty} \subseteq B_r$ with $u_n \to u$ in $C([0,T];X)$. We have

$$f(s,u_n(s)) \to f(s,u(s)) \text{ for } \ a.e. \ s \in [0,T].$$

For $t \in (0,T]$, from Theorem 3.6 (i) and (5.5) we see that

$$\|(g_{\alpha-1} * S_\alpha)(t-s)(f(s,u_n(s)) - f(s,u(s)))\| \leq 2C_\alpha T^{\alpha-1}h_r(s) \in L^1(0,T).$$

Then, for $t \in (0,T]$, it follows from Theorem 3.6 (i) and the dominated convergence theorem that

$$\begin{aligned}
&\|(Fu_n)(t) - (Fu)(t)\| \\
\leq & \int_0^t \|(g_{\alpha-1} * S_\alpha)(t-s)(f(s,u_n(s)) - f(s,u(s)))\|ds \\
\leq & C_\alpha T^{\alpha-1} \int_0^T \|f(s,u_n(s)) - f(s,u(s))\|ds \to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \|Fu_n - Fu\|_{C([0,T];X)} = 0.$$

Step 2: $\Omega = \{(Fu), \ u \in B_r\}$ is a family of equicontinuous functions.

For $0 < t_1 < t_2 \leq T$, we know that

$$\|(Fu)(t_1) - (Fu)(t_2)\| \leq I_1 + I_2 + I_3 + I_4,$$

where

$$
\begin{aligned}
I_1 &= \|S_\alpha(t_1)x_0 - S_\alpha(t_2)x_0\|, \\
I_2 &= \|(1 * S_\alpha)(t_1)x_1 - (1 * S_\alpha)(t_2)x_1\|, \\
I_2 &= \int_{t_1}^{t_2} \|(g_{\alpha-1} * S_\alpha)(t_2 - s)\| \|f(s, u(s))\| ds, \\
I_4 &= \int_0^{t_1} \|(g_{\alpha-1} * S_\alpha)(t_1 - s) - (g_{\alpha-1} * S_\alpha)(t_2 - s)\| \|f(s, u(s))\| ds.
\end{aligned}
$$

From Theorem 3.4, we obtain that $I_1 \to 0$ as $t_1 \to t_2$.

It follows from Theorem 3.1 that

$$I_2 \le M\|x_1\|(t_2 - t_1) \to 0 \text{ as } t_1 \to t_2.$$

Using Theorem 3.6 (i) yields

$$
\begin{aligned}
I_3 &\le C_\alpha \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} h_r(s) ds \\
&\le C_\alpha \left(\frac{(t_2 - t_1)^{(\alpha-1)q+1}}{(\alpha - 1)q + 1} \right)^{\frac{1}{q}} \|h_r\|_{L^p(0,T)}.
\end{aligned} \tag{5.8}
$$

In order to estimate I_4, we start with an estimate of $\frac{d}{dt}(g_{\alpha-1} * S_\alpha)(t)$. It follows from the Laplace transform property of the convolution that

$$
\begin{aligned}
&\frac{d}{dt}(g_{\alpha-1} * S_\alpha)(t) \\
=&(g_{\alpha-1} * S'_\alpha)(t) + S_\alpha(0)g_{\alpha-1}(t) \\
=&- g_{\alpha-1} * A(g_{\alpha-1} * S_\alpha)(t) + g_{\alpha-1}(t)I \\
=&- g_{\alpha-1} * \frac{1}{2\pi i} \int_{l_{\theta_0}} e^{\lambda t} A(\lambda^\alpha I + A)^{-1} d\lambda + g_{\alpha-1}(t)I \\
=&- \frac{1}{2\pi i} \int_{l_{\theta_0}} e^{\lambda t} \lambda^{1-\alpha} A(\lambda^\alpha I + A)^{-1} d\lambda + g_{\alpha-1}(t)I \\
=&- \frac{1}{2\pi i} \int_{l_{\theta_0}} e^{\lambda t} \lambda^{1-\alpha} d\lambda + \frac{1}{2\pi i} \int_{l_{\theta_0}} e^{\lambda t} \lambda (\lambda^\alpha I + A)^{-1} d\lambda + g_{\alpha-1}(t)I \\
=&- \frac{1}{2\pi i} \int_{l'_{\theta_0}} e^{\mu} \left(\frac{\mu}{t}\right)^{1-\alpha} \frac{1}{t} d\mu + \frac{1}{2\pi i} \int_{l'_{\theta_0}} e^{\mu} \frac{\mu}{t} \left[\left(\frac{\mu}{t}\right)^{\alpha} I + A\right]^{-1} \frac{1}{t} d\mu + g_{\alpha-1}(t)I.
\end{aligned}
$$

According to (3.7), we get

$$
\begin{aligned}
\left\| \frac{d}{dt}(g_{\alpha-1} * S_\alpha)(t) \right\| &\le C_\alpha t^{\alpha-2} \int_{l'_{\theta_0}} |e^\mu| |\mu|^{1-\alpha} |d\mu| + g_{\alpha-1}(t) \\
&\le C_\alpha t^{\alpha-2}.
\end{aligned}
$$

Using the mean value theorem, we deduce that

$$
\begin{aligned}
I_4 &= \int_0^{t_2} \|(g_{\alpha-1} * S_\alpha)(t_1 - s) - (g_{\alpha-1} * S_\alpha)(t_2 - s)\| \|f(s, u(s))\| ds \\
&\le \int_0^{t_2} \left\| \frac{d}{dt}(g_{\alpha-1} * S_\alpha)(t)\Big|_{t=(t_2-s+\theta(t_1-t_2))} \right\| (t_1 - t_2)\|f(s, u(s))\| ds \\
&\le C_\alpha \int_0^{t_2} [t_2 - s + \theta(t_1 - t_2)]^{\alpha-2}(t_1 - t_2) h_r(s) ds \\
&\le C_\alpha \int_0^{t_2} [\theta(t_1 - t_2)]^{\alpha-2}(t_1 - t_2) h_r(s) ds \\
&\le C_\alpha \left(\int_0^{t_2} (t_1 - t_2)^{(\alpha-1)q} ds \right)^{\frac{1}{q}} \|h_r\|_{L^p(0,T)} \\
&\le C_\alpha \|h_r\|_{L^p(0,T)} (t_1 - t_2)^{(\alpha-1)}, \qquad (5.9)
\end{aligned}
$$

where $\theta \in [0, 1]$ is a constant. From (5.8) and (5.9), one can see that both of them are independent of $u \in B_r$ and tend to 0 as $t_2 \to t_1$.

For $0 = t_2 < t_1 \le T$, we have

$$
\begin{aligned}
\|(Fu)(t_1) - (Fu)(0)\| \le &\|S_\alpha(t_1)x_0 - x_0\| + \|(1 * S_\alpha)(t_1)x_1\| \\
&+ \int_0^{t_1} \|(g_{\alpha-1} * S_\alpha)(t_1 - s)\| \cdot \|f(s, u(s))\| ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^{t_1} \|(g_{\alpha-1} * S_\alpha)(t_1 - s)\| \cdot \|f(s, u(s))\| ds \\
\le &C_\alpha \int_0^{t_1} (t_1 - s)^{\alpha-1} h_r(s) ds \\
\le &C_\alpha (\int_0^{t_1} (t_1 - s)^{(\alpha-1)q})^{\frac{1}{q}} \|h_r\|_{L^p} \\
= &C_\alpha \left(\frac{t_1^{(\alpha-1)q+1}}{(\alpha-1)q+1} \right)^{\frac{1}{q}} \|h_r\|_{L^p}.
\end{aligned}
$$

Thus, we have $(Fu)(t_1) - (Fu)(0) \to 0$ as $t_1 \to 0$.

Step 3: for each $t \in [0, T]$, $\{(Fu)(t) : u \in B_r\}$ is precompact in X.

When $t = 0$, we see that $\{(Fu)(0) : u \in B_r\} = \{x_0 : u \in B_r\}$ is precompact. Let $t > 0$ be fixed and $\varepsilon > 0$ be sufficiently small. We set

$$
(F_\varepsilon u)(t) = S_\alpha(t)x_0 + (1 * S_\alpha)(t)x_1 + \int_0^t \int_\varepsilon^{t-s} g_{\alpha-1}(t - s - \tau) S_\alpha(\tau) d\tau f(s, u(s)) ds.
$$

Since A generates a compact C_0-semigroup, by Remark 3.8 we know that $S_\alpha(t)$ is a compact operator, and then the set $\{(F_\varepsilon u)(t) : u \in B_r\}$ is precompact in X. For

$u \in B_r$, we have

$$\begin{aligned}
&\|(Fu)(t) - (F_\varepsilon u)(t)\| \\
&\leq \int_0^t \int_0^\varepsilon g_{\alpha-1}(\tau)\|S_\alpha(t-s-\tau)\|d\tau\|f(s,u(s))\|ds \\
&\leq M \int_0^t \int_0^\varepsilon g_{\alpha-1}(\tau)d\tau h_r(s)ds \\
&= M g_\alpha(\varepsilon)\int_0^t h_r(s)ds \\
&\leq M g_\alpha(\varepsilon) t^{\frac{1}{q}}\|h_r\|_{L^p}.
\end{aligned}$$

This implies that $\{(Fu)(t) : u \in B_r\}$ is bounded, i.e, it is precompact in X.

Based on Steps 1-3 and using the Ascoli theorem [**12**, Theorem 7.18], one can see that F is a compact operator. Consequently, by virtue of the Schauder's fixed point theorem, we conclude that F has a fixed point in B_r, which is a mild solution of the problem (5.1). □

The following result is regarding the regularity of the mild solution.

THEOREM 5.4. *Suppose that for any $k > 0$, there exists a constant $L(k)$ such that f: $[0,T] \times X \to X$ satisfies*

(5.10)
$$\|f(t,u) - f(s,v)\| \leq L(k)(|t-s| + \|u-v\|),\ t,s \in [0,T],\ u,v \in X \text{ with } \|u\|,\|v\| \leq k.$$

If $x_0 \in D(A)$, $x_1 \in X$, and $u \in C([0,T];X)$ is a mild solution of the problem (5.1), *then u is a classical solution of the problem* (5.1).

PROOF. Let $h > 0$ and $t \in [0, T-h]$. A direct calculation shows that

$$\begin{aligned}
u(t+h) - u(t) =& S_\alpha(t+h)x_0 - S_\alpha(t)x_0 + (1 * S_\alpha)(t+h)x_1 - (1 * S_\alpha)(t)x_1 \\
&+ \int_0^h (g_{\alpha-1} * S_\alpha)(t+h-s)f(s,u(s))ds \\
&+ \int_0^t (g_{\alpha-1} * S_\alpha)(t-s)(f(s+h,u(s+h)) - f(s,u(s)))ds.
\end{aligned}$$

Let $k = \max_{0\leq t\leq T}\|u(t)\|$. There holds

$$\|f(s+h,u(s+h)) - f(s,u(s))\| \leq L(k)(h + \|u(s+h) - u(s)\|).$$

For any $x_0 \in D(A)$, it follows from Theorem 3.5 (ii) and Theorem 3.6 (i) that

$$\begin{aligned}
\|S_\alpha(t+h)x_0 - S_\alpha(t)x_0\| &= \|(g_\alpha * S_\alpha)(t+h)Ax_0 - (g_\alpha * S_\alpha)(t)Ax_0\| \\
&\leq h\|(g_{\alpha-1} * S_\alpha)(t+\gamma h)\|\|Ax_0\| \\
&\leq C_\alpha T^{\alpha-1}\|Ax_0\|h,
\end{aligned}$$

where $\gamma \in [0,1]$ is a constant.

Using Theorem 3.1 gives

$$\|(1 * S_\alpha)(t+h)x_1 - (1 * S_\alpha)(t)x_1\| \leq M\|x_1\|h,$$

and

$$\begin{aligned}
&\|u(t+h)-u(t)\| \\
&\le C_\alpha T^{\alpha-1}\|Ax_0\|h + M\|x_1|h + C_\alpha T^{\alpha-1}h \max_{0\le s\le T}\|f(s,u(s))\| \\
&\quad + C_\alpha T^{\alpha-1}L(k)\int_0^t (h+\|u(s+h)-u(s)\|)\,ds \\
&\le C_1 h + C_2\int_0^t \|u(s+h)-u(s)\|ds.
\end{aligned}$$

According to the Gronwall's inequality, there exists $C>0$ such that

$$\|u(t+h)-u(t)\| \le Ch, \quad 0\le t\le t+h\le T.$$

That is, u: $[0,T]\to X$ is Lipschitz continuous. In view of (5.10), we know that $F(\cdot,u(\cdot))$: $[0,T]\to X$ is Lipschitz continuous too. Consequently, by Theorem 4.5, we complete the proof. □

6. An Example

Consider a fractional wave equation

$$\left\{\begin{array}{l} D_t^\alpha u - \triangle u + au = f(t,u),\ t\in(0,T],\quad x\in\mathbb{R}^n, \\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\quad x\in\mathbb{R}^n, \end{array}\right. \tag{6.1}$$

where $1<\alpha<2$ and $a>0$.

Let

$$Au = -\triangle u + au, \quad D(A) = \{u\in L^2(\mathbb{R}^n) : \triangle u \in L^2(\mathbb{R}^n)\},$$

Following Theorem 2.3.3 in [**4**], we know that $0\in\rho(A)$ and A is a sectorial operator with angle $\theta=0$.

Take $f(t,u)=g(t)\sin u$, where $g\in C([0,T];\mathbb{R}^+)$. Then we have

$$\begin{aligned}
\|f(t,u)-f(t,v)\|_{L^2(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} g^2(t)(\sin u(x)-\sin v(x))^2dx\right)^{\frac{1}{2}} \\
&\le g(t)\left(\int_{\mathbb{R}^n}(u(x)-v(x))^2dx\right)^{\frac{1}{2}} \\
&\le \|g\|_{C([0,T];\mathbb{R}^+)}\|u-v\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

which implies that the function f satisfies all conditions of Theorem 5.2, so the problem (6.1) has a unique mild solution for $u_0,\ u_1\in X$.

If we choose $f(t,u)=g(t)u$, where $g\in L^p((0,T);\mathbb{R}^+)$ with $p>1$, and take $h_r(t)=g(t)r$ with $r>0$, then we derive that

$$\liminf_{r\to\infty}\frac{\|h_r\|_{L^p(0,T)}}{r} = \|g\|_{L^p(0,T)} < \infty$$

and

$$\|f(t,u)\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} g^2(t)u^2(x)dx\right)^{\frac{1}{2}} = g(t)\|u\|_{L^2(\mathbb{R}^n)}.$$

So, for a.e. $t\in[0,T]$ and $u\in L^2(\mathbb{R}^n)$ with $\|u\|_{L^2(\mathbb{R}^n)}\le r$, it gives

$$\|f(t,u)\|_{L^2(\mathbb{R}^n)} \le h_r(t).$$

Furthermore, when the inequality

$$C_\alpha \|g\|_{L^p(0,T)} \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1} \right)^{\frac{1}{q}} < 1$$

holds, it follows from Theorem 5.3 that for u_0, $u_1 \in X$, the problem (6.1) has at least a mild solution.

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