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Infinite energy solutions for a 1D transport equation with nonlocal velocity

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ABSTRACT. We study a one dimensional dissipative transport equation with nonlocal velocity and critical dissipation. We consider the Cauchy problem for initial values with infinite energy. The control we shall use involves some weighted Lebesgue or Sobolev spaces. More precisely, we consider the family of weights given by $w_\beta(x) = (1+|x|^2)^{-\beta/2}$ where β is a real parameter in $(0,1)$ and we treat the Cauchy problem for the cases $\theta_0 \in H^{1/2}(w_\beta)$ and $\theta_0 \in H^1(w_\beta)$ for which we prove global existence results (under smallness assumptions on the L^∞ norm of θ_0). The key step in the proof of our theorems is based on the use of two new commutator estimates involving fractional differential operators and the family of Muckenhoupt weights.

CONTENTS

Introduction

In this article, we are interested in the following 1D transport equation with nonlocal velocity which has been introduced by Córdoba, Córdoba and Fontelos in

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[14] :

$$(\mathcal{T}_\alpha) \ : \begin{cases} \partial_t \theta + \theta_x \mathcal{H}\theta + \nu \Lambda^\alpha \theta = 0 \\ \theta(0,x) = \theta_0(x). \end{cases}$$

Here, $\mathcal{H}$ denotes the Hilbert transform, defined by

$$\mathcal{H}\theta \equiv \frac{1}{\pi} PV \int \frac{\theta(y)}{y-x} \, dy, \tag{0.1}$$

and the operator Λ^α is defined (in 1D) as follows

$$\Lambda^\alpha \theta \equiv (-\Delta)^{\alpha/2}\theta = C_\alpha P.V. \int_{\mathbb{R}} \frac{\theta(x) - \theta(x-y)}{|y|^{1+\alpha}} dy$$

where $C_\alpha > 0$ is a positive constant which depends only on α and $0 < \alpha < 2$ is a real parameter. Note that with this convention in 0.1, we have $\partial_x \mathcal{H} = -\Lambda$

This equation can be viewed as a toy model for several equations coming from problems in fluid dynamics, in particular it models the 3D Euler equation written in vorticity form (see e.g. [**9**], [**1**], [**15**], [**29**] where other 1D models for 3D Euler equation are studied).

One can observe that this equation is a one dimensional model for the 2D dissipative Surface-Quasi-Gesotrophic $(SQG)_\alpha$ equation (see [**10**]) written in a non-divergence form (see also [**4**], [**5**], [**6**], [**27**] where the divergence form equation is studied). The 2D dissipative SQG equation reads as follows

$$(SQG)_\alpha : \begin{cases} \partial_t \theta(x,t) + u(\theta).\nabla\theta + \nu \Lambda^\alpha \theta = 0 \\ \theta(0,x) = \theta_0(x), \end{cases}$$

where the velocity $u(\theta) = \mathcal{R}^\perp \theta$ is given by the Riesz transforms $\mathcal{R}_1\theta$ and $\mathcal{R}_2\theta$ of θ as

$$u(\theta) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) = (-\partial_{x_2}\Lambda^{-1}\theta, \partial_{x_1}\Lambda^{-1}\theta).$$

Obviously the velocity $u(\theta)$ is divergence free. In 1D, we lose this divergence free condition, while the analogue of the Riesz transforms is the Hilbert transform; one gets the equation $(\mathcal{T}_\alpha)$.

One can also see this equation as an analogue of the fractional Burgers equation with the nonlocal velocity $u(\theta) = \mathcal{H}\theta$ instead of $u(\theta) = \theta$. However, the nonlocal character of the velocity makes the $(\mathcal{T}_\alpha)$ equation more complicated to deal with comparing to the fractional Burgers equations which is now quite well understood (see [**22**], [**7**], [**20**]). Finally, let us mention that this equation also shares some similarities with the Birkhoff-Rott equation which modelises the evolution of a vortex patch, we refer to [**14**], [**1**] for more details regarding this analogy.

It is easy to guess that this kind of fractional transport equation admits an L^∞ maximum principle (due to the diffusive character of $-\Lambda^\alpha$ and the presence of the derivative θ_x in the advection term). For $\theta \in L^\infty$, one thus may view $\theta_x \mathcal{H}\theta$ as a term of order 1, while Λ^α is of order α; thus, one has to consider 3 cases depending on the value of α, namely $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha \in (1,2)$. They are respectively called supercritical, critical and sub-critical cases.

The inviscid case (i.e. $\nu = 0$) was first studied by Córdoba, Córdoba and Fontelos in [**14**] where the authors proved that blow-up of regular solutions may occur. They proved that there exists a family of smooth, compactly supported, even and positive initial data for which the associated solution blows up in finite time. By adapting the method used in [**14**] along with the use of new nonlocal inequalities obtained in [**13**], Li and Rodrigo [**26**] proved that blow-up of smooth solutions also holds in the viscous case, in the range $\alpha \in (0, 1/2)$. Using a different method, Kiselev [**20**] was able to prove that singularities may appear in the case $\alpha \in [0, 1/2)$ (where the case $\alpha = 0$ conventionnally designs the inviscid case $\nu = 0$). In this latter range, that is $\alpha \in [0, 1/2)$, Silvestre and Vicol [**32**] gave four differents proofs of the same results as [**14**], [**26**], [**32**], namely they proved the existence of singularities for classical ($\mathcal{C}^1$) solutions starting from a well chosen class of initial data. In [**16**], T. Do showed eventual regularization in the supercritical case and global regularity for the slightly supercritical version of equation $\mathcal{T}_\alpha$, in the spirit of what was done for the SQG equation in [**31**], [**20**]. One can also see the articles [**17**] and [**2**] where local existence results are obtained in this regime. In the range $\alpha \in [1/2, 1)$, the question about blow-up or global existence of regular solutions remains open.

The critical and the sub-critical cases are well understood. Indeed, by adapting methods introduced in [**23**], [**3**], [**11**], one recovers all the results known for the critical SQG equation, under an extra positiveness assumption on the initial data (see [**21**]). The first global existence results are those of Córdoba, Córdoba and Fontelos [**14**]. They obtained global existence results for non-negative data in H^1 and $H^{1/2}$ in the subcritical case and also in the critical case under a smallness assumption of the L^∞ norm of the initial data. In [**17**], Dong treated the critical case and obtained the global well-posedness for data in H^s where $s > 3/2 - \alpha$ and without sign conditions on the initial data. In the critical case, Kiselev proved in [**20**] that there exists a unique global smooth solution for all $\theta_0 \in H^{1/2}$.

In this article, we will focus on the critical case ($\alpha = 1$) and without loss of generality we shall fix $\nu = 1$. Futhermore, in contrast with [**14**] and [**17**], we shall not assume that θ decays at infinity fast enough to ensure that $\|\theta\|_2 < +\infty$. It is worth pointing out that, our solutions being of infinite energy, one cannot directly use methods coming from L^∞-critical case used for instance in [**3**]. However, in the case of an infinite-energy data, one can still use energy estimates (in the spirit of [**14**]) to prove global existence results provided that θ decreases only at a slow rate, namely

$$\int |\theta(x,t)|^2 \frac{dx}{(1+|x|^2)^{\beta/2}} < +\infty$$

The weight we consider is therefore given by $w_\beta(x) = (1 + |x|^2)^{-\beta/2}$. Motivated by the work done in [**14**], we will study the cases of small data in L^∞ which belong moreover to $H^{1/2}(w_\beta)$ or $H^1(w_\beta)$, although one can generalize to a higher regularity class of initial data (we think that it should be even easier to treat). When the initial data lies in $H^{1/2}(w_\beta)$ or $H^1(w_\beta)$ we prove global existence of weighted Leray-Hopf type solutions but we require the L^∞ norm of the initial data to be small enough. As one may expect, in the subcritical case one can prove the existence of global solutions without smallness assumption. This is done by the first author in [**24**] using Littlewood-Paley theory along with a suitable commutators estimates.

He also treated the supercritical case where he obtained local existence results for arbitrary big initial data [**24**].

The construction of a solution is based on an energy method and amounts to control some nontrivial commutators involving the weight w_β along with some classical harmonic analysis tools such as the use of the Hardy-Littlewood maximal function and Hedberg's inequality for instance (see [**18**], [**30**]); such tools are motivated by the fact w_β is a Muckenhoupt weight. The new commutator estimates can be used to prove existence of infinite energy solutions for other nonlinear transport equations with fractional diffusion such as the 2D dissipative quasi-geostrophic equation as well as the fractional porous media equation for instance.

The rest of the paper is organized into five sections. In the first section, we state our main theorems. In the second section we recall some results concerning the Muckenhoupt weights. In the third and fourth section, we respectively establish *a priori* estimates and prove our main results. In the last section we revisit the construction of regular enough solutions.

1. Main theorems

In the case of a weighted $H^{1/2}$ data we have the following theorem,

THEOREM 1.1. *Let* $0 < \beta < 1$ *and* $w_\beta(x) = (1+x^2)^{-\beta/2}$. *There exists a constant* $C_\beta > 0$ *such that, whenever* θ_0 *satisfies the conditions*

- θ_0 *is bounded and small enough* : $|\theta_0| \le C_\beta$
- $\int |\theta_0|^2 w_\beta(x)\, dx < \infty$ *and* $\int |\Lambda^{1/2}\theta_0|^2 w_\beta(x)\, dx < \infty$,

there exists a solution θ *to equation* $\mathcal{T}_1$ *such that, for every* $T > 0$, *we have*

- $\sup_{0<t<T} \int |\theta(t,x)|^2 w_\beta(x)\, dx < \infty$
- $\sup_{0<t<T} \int |\Lambda^{1/2}\theta(t,x)|^2 w_\beta(x)\, dx < \infty$
- $\int_0^T \int |\Lambda\theta(t,x)|^2 w_\beta(x)\, dx\, dt < \infty$

A similar result holds for higher regularity (weighted H^1 data).

THEOREM 1.2. *Let* $0 < \beta < 1$ *and* $w_\beta(x) = (1+x^2)^{-\beta/2}$. *There exists* $C_\beta > 0$ *such that, whenever* θ_0 *satisfies the conditions*

- θ_0 *is bounded and small enough* : $|\theta_0| \le C_\beta$
- $\int |\theta_0|^2 w_\beta(x)\, dx < \infty$ *and* $\int |\Lambda\theta_0|^2 w_\beta(x)\, dx < \infty$,

there exists a solution θ *to equation* $\mathcal{T}_1$ *such that, for every* $T > 0$, *we have*

- $\sup_{0<t<T} \int |\theta(t,x)|^2 w_\beta(x)\, dx < \infty$
- $\sup_{0<t<T} \int |\Lambda\theta(t,x)|^2 w_\beta(x)\, dx < \infty$

- $\int_0^T \int |\Lambda^{3/2}\theta(t,x)|^2 w_\beta(x)\ dx\ dt < \infty$

2. Preliminaries on the Muckenhoupt weights.

In this section, we briefly recall the tools and the notations we shall use throughout the article. We first recall some basic facts and notations on weighted Lebesgue or Sobolev spaces. A weight w is a positive and locally integrable function. A measurable function θ on $\mathbb{R}$ belongs to the weighted Lebesgue spaces $L^p(wdx)$ with $1 \le p < \infty$ if and only if

$$\|\theta\|_{L^p(wdx)} = \left(\int |\theta(x)|^p\ w(x)\ dx\right)^{1/p} < \infty.$$

An important class of weights is the so-called Muckenhoupt class $\mathcal{A}_p$ for $1 < p < \infty$. A weight is said to be in the $\mathcal{A}_p$ class of Muckenhoupt (with $p \in (1,\infty)$) if and only if there exists a constant $C(w,p)$ such that we have the reverse Hölder inequality

$$\sup_{r>0, x_0\in\mathbb{R}} \left(\frac{1}{2r}\int_{[x_0-r,x_0+r]} w(x)\ dx\right)\left(\frac{1}{2r}\int_{[x_0-r,x_0+r]} w(x)^{-\frac{1}{p-1}}\ dx\right)^{p-1} \le C(w,p).$$

In particular, if $0 < \beta < 1$, then the weight $w_\beta(x) = (1+|x|^2)^{-\beta/2}$ belongs to the $\mathcal{A}_p$ class for all $1 < p < \infty$.

Let us recall that the Hardy-Littlewood maximal function of a locally integrable function f on $\mathbb{R}$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{2r}\int_{[x-r,x+r]} |f(y)|\ dy.$$

We have the following other characterization of the $\mathcal{A}_p$ class [**8**], [**28**] : a weight w belongs to $\mathcal{A}_p$ if and only if there exists a constant $C_{p,w}$ such that for every $f \in L^p(w\ dx)$, we have

$$\|\mathcal{M}f(x)\|_{L^p(w\ dx)} \le C_{p,w}\|f\|_{L^p(wdx)}.$$

Another important property of Muckenhoupt weights is that Calderón-Zygmund type operators are bounded on $L^p(w\ dx)$ when $w \in \mathcal{A}_p$ and $1 < p < \infty$. We shall use this property in the case of the Hilbert transform $\mathcal{H}$ and in the case of the truncated Hilbert transform, defined by

$$\mathcal{H}_\# f(x) = \frac{1}{\pi} P.V. \int \frac{\alpha(x-y)}{x-y} f(y)\ dy \tag{2.1}$$

where α is an even, smooth and compactly supported function such that $\alpha(x) = 1$ if $|x| < 1$ and $\alpha(x) = 0$ if $|x| > 2$. We refer for instance to [**30**] or [**19**] for more details.

We now recall the definition of the weighted Sobolev spaces $H^1(wdx)$ and $H^{1/2}(wdx)$. The space $H^1(w\ dx)$ is defined by

$$f \in H^1(wdx) \Leftrightarrow f \in L^2(wdx) \text{ and } \partial_x f \in L^2(wdx).$$

Note that, due to 0.1, we have

$$\mathcal{H}\partial_x = \Lambda \text{ and } \mathcal{H}\Lambda = \partial_x,$$

we see that, when $w \in \mathcal{A}_2$, the semi-norm $\|\partial_x f\|_{L^2(w\ dx)}$ is equivalent to the semi-norm $\|\Lambda f\|_{L^2(wdx)}$. Therefore, when $w \in \mathcal{A}_2$, we have the following equivalence

$$f \in H^1(wdx) \Leftrightarrow (1-\partial_x^2)^{1/2} f \in L^2(wdx) \Leftrightarrow f \in L^2(wdx) \text{ and } \Lambda f \in L^2(wdx).$$

Analogously, we define the spaces $H^{1/2}(wdx)$ as

$$f \in H^{1/2}(w\ dx) \Leftrightarrow (1-\partial_x^2)^{1/4} f \in L^2(wdx) \Leftrightarrow f \in L^2(wdx) \text{ and } \Lambda^{1/2} f \in L^2(wdx).$$

The following useful property will be used several times (see [**30**], p.57). Fix an integrable nonnegative and radially decreasing function ϕ such that its integral over $\mathbb{R}$ is equal to 1. We set, $\phi_k(x) = k^{-1}\phi(xk^{-1})$ for all $k > 0$, then

$$\sup_{k>0} |f * \phi_k(x)| \leq \mathcal{M}f(x). \tag{2.2}$$

In the sequel, we shall use Gagliardo-Nirenberg type inequalities in the weighted setting. Let us first note that, provided f vanishes at infinity (in the sense that $\lim_{t\to+\infty} e^{t\Delta} f = 0$ in $\mathcal{S}'$), one may write

$$-f = \int_0^\infty e^{t\Delta}\Delta f\ dt.$$

where $f \mapsto e^{t\Delta} f$ is the heat kernel operator defined by $e^{t\Delta} f = G(x,t) * f$ where $*$ is the convolution with respect to the x variable and $G(x,t) = (4\pi t)^{-1/2} e^{-\frac{x^2}{4t}}$ which verifies the heat equation $\partial_t G(x,t) = \Delta G(x,t)$.

Then, for all $N \in \mathbb{N}^*$ by writing $1 = \partial_t^{N-1}(\frac{t^{N-1}}{(N-1)!})$ and integrating by parts $(N-1)$ times, one obtain the following equality

$$-f = \frac{1}{(N-1)!}\int_0^\infty (-t\Delta)^N e^{t\Delta} f \frac{dt}{t}.$$

Then, for $0 < \gamma < \delta < 2N$, using the fact that the operator $\Lambda^{2N-\delta+\gamma}$ is a convolution operator with an integrable kernel which is dominated by an integrable radially decreasing function, along with the inequality

$$\sup_t |\Lambda^\gamma e^{t\Delta} f| \leq ct^{-\gamma/2}\mathcal{M}f(x)$$

allow us to get

$$|\Lambda^\gamma f(x)| \leq C \int_0^\infty \min(t^{-\gamma/2}\|f\|_\infty, t^{\frac{\delta-\gamma}{2}}\mathcal{M}(\Lambda^\delta f)(x))\frac{dt}{t}$$

Then, we recover Hedberg's inequality (see Hedberg [**18**])

$$|\Lambda^\gamma f(x)| \leq C_{\gamma,\delta}(\mathcal{M}(\Lambda^\delta f)(x)))^{\frac{\gamma}{\delta}}\|f\|_\infty^{1-\frac{\gamma}{\delta}} \tag{2.3}$$

Note that, if $\gamma \in \mathbb{N}^*$, one may replace $\Lambda^\gamma f(x)$ with $\partial_x^\gamma f(x)$. Using (2.3), one easily deduce the following Gagliardo-Nirenberg type inequalities provided that the weight $w \in \mathcal{A}_3$ (actually $w \in \mathcal{A}_2$ suffices for 2.4 and 2.5)

$$\|\Lambda^{1/2} f\|_{L^4(wdx)} \leq C\|f\|_\infty^{1/2}\|\Lambda f\|_{L^2(wdx)}^{1/2} \tag{2.4}$$

$$\|\Lambda f\|_{L^3(wdx)} \leq C\|f\|_\infty^{1/3}\|\Lambda^{3/2} f\|_{L^2(wdx)}^{2/3} \tag{2.5}$$

and

$$\|\partial_x f\|_{L^3(wdx)} \leq C\|f\|_\infty^{1/3}\|\Lambda^{3/2} f\|_{L^2(wdx)}^{2/3} \tag{2.6}$$

For instance, to prove 2.5, it suffices to set $\gamma = 1$ and $\delta = 3/2$ in 2.3 and to raise to the power 3 in both side, one obtain

$$|\Lambda f|^3 \le (\mathcal{M}(\Lambda^{3/2} f)(x))^2 \|f\|_\infty.$$

Mutiplying this latter inequality by w and integrating with respect to x give

$$\|\Lambda f\|^3_{L^3(wdx)} \le \|f\|_\infty \|\mathcal{M}(\Lambda^{3/2} f)(x)\|^2_{L^2(wdx)} \le \|f\|_\infty \|\Lambda^{3/2} f\|^2_{L^2(wdx)},$$

where, in the last inequality, we used the continuity of the maximal function $\mathcal{M}$ on $L^2(wdx)$ because $w \in \mathcal{A}_2$. Therefore, inequality 2.5 follows by taking the power 1/3 in both sides. Then, observe that 2.6 is a direct consequence of 2.5. Indeed, we have $\mathcal{H}\Lambda f = -\partial_x f$ and due to $w \in \mathcal{A}_3$ one can use the continuity on $L^3(wdx)$ of $\mathcal{H}$ to obtain the inequality

$$\|\partial_x f\|_{L^3(wdx)} = \|\mathcal{H}\Lambda f\|_{L^3(wdx)} \le \|\Lambda f\|_{L^3(wdx)},$$

therefore we recover 2.6.

The space of positive smooth functions compactly supported in an open set Ω will be denoted by $\mathcal{D}(\Omega)$. We shall use the notation $A \lesssim B$ if there exists constant $C > 0$ depending only on controlled quantities such that $A \le CB$. We shall often use the same notation to design a controlled constant although it is not the same from a line to another. Note that we shall write indifferently $\partial_x \theta$ or θ_x for the derivative as well as $\|.\|_p$ or $\|.\|_{L^p}$ for the classical Lebesgue spaces.

3. Useful lemmas

In our future estimations, we will need to control the L^p norm of some nontrivial commutators involving our weight w_β and the nonlocal operators Λ and $\Lambda^{1/2}$. Also, a control of Λw by cw will be needed. The aim of this section is to establish all those nonlocal estimates involving w and Λ. Before starting the proofs of those commutator estimates, we shall give some important remarks that will be helpful to estimate singular integrals involving the weight w. When estimating commutators involving the weight $w_\beta(x) = (1 + x^2)^{-\beta/2}$, we are lead to estimate quantities such that $w_\beta(x) - w_\beta(y)$. In order to estimate $w_\beta(x) - w_\beta(y)$, we shall distinguish three areas that we will call $\Delta_1(x), \Delta_2(x)$ and $\Delta_3(x)$. Those areas are defined as follows,

$$\Delta_1(x) = \{y \ / \ |x - y| < 2\}$$
$$\Delta_2(x) = \{y \ / \ |x - y| \ge 2\} \cap \{y \ / |x - y| \le \frac{1}{2} \max(|x|, |y|)\}$$
$$\Delta_3(x) = \{y \ / \ |x - y| \ge 2\} \cap \{y \ / |x - y| > \frac{1}{2} \max(|x|, |y|)\}.$$

Note that we have $\mathbb{R} = \Delta_1(x) \cup \Delta_2(x) \cup \Delta_3(x)$. In the sequel, we shall also use the notation $w_\beta(x) \approx w_\beta(y)$ if there exists two positive constants c and C such that $c \le \frac{w(x)}{w(y)} \le C$. In those different areas, we will need to use the following estimates :

- A straightforward computation gives that $|\partial_x w_\beta(x)| + |\partial_x^2 w_\beta(x)| \le C w_\beta(x)$
- On $\Delta_1(x)$, we have that $w_\beta(x) \approx w_\beta(y)$ and moreover

$$|w_\beta(x) - w_\beta(y)| \le |x - y| \sup_{z \in [x,y]} |\partial_x w_\beta(z)| \le C|x - y| w_\beta(x)$$

On the other hand, if α is an even, smooth and compactly supported function such that $\alpha(x) = 1$ if $|x| < 1$ and $\alpha(x) = 0$ if $|x| > 2$, then

$$|w_\beta(y) - w_\beta(x) + \alpha(x-y)(x-y)\partial_x w_\beta(x)| \leq C|x-y|^2 w_\beta(x) \tag{3.1}$$

- On $\Delta_2(x)$, we shall only use that $w_\beta(x) \approx w_\beta(y)$
- On $\Delta_3(x)$, we have $1 \leq w_\beta(x)^{-1} \leq C|x-y|^\beta$ and $1 \leq w_\beta(y)^{-1} \leq C|x-y|^\beta$

Remark 3.1. Obviously, similar estimates hold for $\gamma_\beta(x) = w_\beta(x)^{1/2}$. Indeed, it suffices to replace w_β with γ_β and β with $\beta/2$.

The purpose of the following subsections is to prove that we can indeed control those commutators and that we have a nice bound for Λw_β.

3.1. Two commutator estimates involving the weight w_β. In the next lemma, we obtain two commutator estimates that are crucial in the proof of the energy inequality.

Lemma 3.2. *Let $w_\beta(x) = (1+x^2)^{-\beta/2}$, $0 < \beta < 1$, then we have the two following estimates*

- *Let $p \geq 2$ be such that $\frac{3}{2} - \beta(1-\frac{1}{p}) > 1$, then the commutator $\frac{1}{w_\beta}[\Lambda^{1/2}, w_\beta]$ is bounded from $L^p(w_\beta dx)$ to $L^p(w_\beta dx)$.*
- *Let $2 \leq p < \infty$, then the commutator $\frac{1}{\sqrt{w_\beta}}[\Lambda, \sqrt{w_\beta}]$ is bounded from $L^p(w_\beta dx)$ to $L^p(w_\beta dx)$.*

Proof of lemma 3.2.

Let us prove the first commutator estimate. We first write

$$\Lambda^{1/2} f(x) = c_0 \int \frac{f(x) - f(y)}{|x-y|^{3/2}}\, dy$$

so that

$$\frac{1}{w_\beta(x)}[\Lambda^{1/2}, w_\beta] f(x) = c_0 \frac{1}{w_\beta(x)^{\frac{1}{p}}} \int \frac{w_\beta(x) - w_\beta(y)}{w_\beta(x)^{1-\frac{1}{p}} w_\beta(y)^{\frac{1}{p}} |x-y|^{3/2}} w_\beta(y)^{\frac{1}{p}} f(y)\, dy$$

Let us set

$$K(x,y) \equiv \frac{w_\beta(x) - w_\beta(y)}{w_\beta(x)^{1-\frac{1}{p}} w_\beta(y)^{\frac{1}{p}} |x-y|^{3/2}}$$

On $\Delta_1(x)$ we have

$$|K(x,y)| \leq C \frac{1}{|x-y|^{1/2}}$$

On $\Delta_2(x)$, since $w_\beta(x) \approx w_\beta(y)$, we get

$$|K(x,y)| \leq C \frac{1}{|x-y|^{3/2}}$$

On $\Delta_3(x)$, we have the following estimate

$$|K(x,y)| \leq C \frac{w_\beta(x)^{\frac{1}{p}-1} + w_\beta(y)^{-\frac{1}{p}}}{|x-y|^{3/2}} \leq C' \frac{1}{|x-y|^{\frac{3}{2}-\beta(1-\frac{1}{p})}}$$

Note that, for $0 < \beta < 1$ we have $\frac{3}{2} - \beta(1-\frac{1}{p}) > 1$ if $p \geq 2$. Therefore, if we introduce the function $x \mapsto \Phi(x)$ as follows

$$\Phi(x) \equiv \min\left(\frac{1}{|x|^{1/2}}, \frac{1}{|x|^{\frac{3}{2}-\beta(1-\frac{1}{p})}}\right),$$

we find that Φ belongs to $L^1(\mathbb{R})$ and that

$$\left|\frac{1}{w_\beta(x)}[\Lambda^{1/2}, w_\beta]f(x)\right| \leq C\frac{1}{w_\beta(x)^{\frac{1}{p}}}\int \Phi(x-y)w_\beta(y)^{\frac{1}{p}}|f(y)|\ dy$$

The integral appearing in the right hand side is nothing but the convolution of $x \mapsto \Phi(x) \in L^1(\mathbb{R})$ with $x \mapsto w_\beta(x)^{\frac{1}{p}}|f(x)| \in L^p(\mathbb{R})$. To finish the proof, we just have to take the power p in both side then to integrate with respect to x and by Young's inequality for convolution, we get

$$\int\left|\frac{1}{w_\beta(x)}[\Lambda^{1/2}, w_\beta]f(x)\right|^p\ wdx \leq C\int\left|(\Phi * w_\beta^{1/p}f)(x)\right|^p\ dx \leq C\|\Phi\|_{L^1}^p\|w_\beta^{1/p}f\|_{L^p}^p$$

and therefore,

$$\left\|\frac{1}{w_\beta(x)}[\Lambda^{1/2}, w_\beta]f(x)\right\|_{L^p(w_\beta dx)} \leq C\|f\|_{L^p(w_\beta dx)}$$

Let us prove the second commutator estimate. Let us denote $\gamma_\beta = \sqrt{w_\beta}$, recall that

$$\Lambda f(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{\epsilon<|x-y|<\frac{1}{\epsilon}}\frac{f(x)-f(y)}{|x-y|^2}\ dy$$

Therefore,

$$\frac{1}{\sqrt{w_\beta(x)}}[\Lambda, \sqrt{w_\beta(x)}]f = \frac{1}{\pi\gamma_\beta(x)}\lim_{\epsilon\to 0}\int_{\epsilon<|x-y|<\frac{1}{\epsilon}}\frac{\gamma_\beta(y)-\gamma_\beta(x)}{|x-y|^2}\ f(y)\ dy$$

Then, as we did before, we split the integral into three pieces. In other to deal with the integration in $\Delta_1(x)$, we need to introduce a even, smooth and compactly supported function α such that $\alpha(x) = 1$ if $|x| < 1$ and $\alpha(x) = 0$ if $|x| > 2$. By doing so, we get an extra term which is nothing but the truncated Hilbert transform of f (see 2.1) times another controlled term. More precisely, we write the commutators as follows

$$\begin{aligned} &\frac{1}{\sqrt{w_\beta(x)}}\left[\Lambda, \sqrt{w_\beta(x)}\right]f \\ = \ &\frac{1}{\pi\gamma_\beta(x)}\int_{\Delta_1(x)}\frac{\gamma_\beta(y)-\gamma_\beta(x)-(y-x)\alpha(x-y)\partial_x\gamma_\beta(x)}{|x-y|^2}f(y)\ dy \\ - \ &\frac{\partial_x\gamma_\beta(x)}{\gamma_\beta(x)}\mathcal{H}_\# f(x) + \frac{1}{\pi\gamma_\beta(x)}\int_{\Delta_2(x)\cup\Delta_3(x)}\frac{\gamma_\beta(y)-\gamma_\beta(x)}{|x-y|^2}\ f(y)\ dy \end{aligned}$$

Then, observe that on $\Delta_1(x)$ we have (see 3.1)

$$\frac{1}{\gamma_\beta(x)^{1-\frac{2}{p}}\gamma_\beta(y)^{\frac{2}{p}}}\frac{|w_\beta(y)-w_\beta(x)+\alpha(x-y)(x-y)\partial_x w_\beta(x)|}{|x-y|^2} \leq C$$

On $\Delta_2(x)$, we have

$$\frac{1}{\gamma_\beta(x)^{1-\frac{2}{p}}\gamma_\beta(y)^{\frac{2}{p}}}\frac{|\gamma_\beta(y)-\gamma_\beta(x)|}{|x-y|^2} \leq C\frac{1}{|x-y|^2}$$

Here, we used the property that on $\Delta_1(x)$ and $\Delta_2(x)$ we have $\gamma_\beta(x) \approx \gamma_\beta(y)$ and therefore $\gamma_\beta(x)^{1-\frac{2}{p}}\gamma_\beta(y)^{\frac{2}{p}} \approx \gamma_\beta(x)$.

Finally, on $\Delta_3(x)$ we use the fact that $\gamma_\beta(x) \leq 1, \gamma_\beta(x)^{-1} \leq C|x-y|^{\beta/2}$. We also have that $\gamma_\beta(y) \leq 1, \gamma_\beta(y)^{-1} \leq C|x-y|^{\beta/2}$, therefore

$$\begin{aligned}\frac{1}{\gamma_\beta(x)^{1-\frac{2}{p}}\gamma_\beta(y)^{\frac{2}{p}}}\frac{|\gamma_\beta(y)-\gamma_\beta(x)|}{|x-y|^2} &\leq C\frac{\gamma_\beta(x)^{\frac{2}{p}-1}+\gamma_\beta(y)^{-\frac{2}{p}}}{|x-y|^2}\\ &\leq C'\frac{1}{|x-y|^{2-\beta\max(\frac{1}{2}-\frac{1}{p},\frac{1}{p})}}\end{aligned}$$

Now, let us introduce the function $x \mapsto \Theta(x)$ as follows

$$\Theta(x) \equiv \min\left(1, \frac{1}{|x|^{2-\beta(\frac{1}{2}-\frac{1}{p},\frac{1}{p})}}\right)$$

Thus, we have proved that

$$\left|\frac{1}{\sqrt{w_\beta(x)}}[\Lambda, \sqrt{w_\beta(x)}]f\right| \leq C\frac{1}{w_\beta(x)^{\frac{1}{p}}}\int \Theta(x-y)w_\beta(y)^{\frac{1}{p}}|f(y)|\ dy + C|\mathcal{H}^{\#}f(x)|$$

Since $2-\beta\max(\frac{1}{2}-\frac{1}{p},\frac{1}{p}) > \frac{3}{2}$, then the function Θ is an integrable function on $\mathbb{R}$. Taking the power p in both side, multiplying by w and then integrating with respect to x give the following

$$\int |\frac{1}{\sqrt{w_\beta(x)}}[\Lambda, \sqrt{w_\beta(x)}]f|^p\ w_\beta dx \leq C\int (\Theta * G)(x)\ dx + C'\int |\mathcal{H}^{\#}f(x)|^p\ w_\beta dx$$

where we set $G(y) = w_\beta(y)^{1/p}|f(y)|$. Therefore, since $\Theta \in L^1(\mathbb{R})$ and $G \in L^p(\mathbb{R})$, Young's inequality for the convolution gives

$$\left\|\frac{1}{\sqrt{w(x)}}[\Lambda, \sqrt{w_\beta(x)}]f\right\|^p_{L^p(w_\beta dx)} \leq C''\int |f(x)|^p\ w_\beta\ dx$$

where, in the second part of the inequality, we have used that the truncated Hilbert transform of f is a Calderón-Zygmund type operator and as such is bounded on $L^p(w_\beta dx)$ (by the $L^p(w_\beta dx)$ norm of f) since $w_\beta \in \mathcal{A}_p$ for all $p \in [2,\infty)$. This concludes the proof of the second commutator estimate. □

3.2. Bounds for Λw_β. We have used in the previous subsection the bound $|\partial_x w_\beta(x)| \leq Cw_\beta(x)$. A similar estimate holds for the nonlocal operator Λ :

LEMMA 3.3. *For all $\beta \in (0,1)$, we have $|\Lambda w_\beta(x)| \leq Cw_\beta(x)$*

Proof of lemma 3.3. We need to estimate the following singular integral

$$\Lambda w_\beta(x) = \frac{P.V.}{\pi}\int \frac{w_\beta(x)-w_\beta(y)}{|x-y|^2}\ dy$$

To do so, we split the integral in three pieces

$$\frac{P.V.}{\pi}\int \frac{w_\beta(x)-w_\beta(y)}{|x-y|^2}\ dy = \frac{P.V.}{\pi}\sum_{i=1}^{3}\int_{\Delta_i(x)} \frac{w_\beta(x)-w_\beta(y)}{|x-y|^2}\ dy \equiv \sum_{i=1}^{3} I_i$$

The domains of integrations $\Delta_i(x)$ with $i = 1, 2, 3$ are the same ones as those introduced in the previous subsection. Using (3.1), we get the following estimate for the integration in $\Delta_1(x)$

$$\begin{aligned} I_1 &\leq \frac{P.V.}{\pi} \int_{\Delta_1(x)} \frac{|w_\beta(x) - w_\beta(y)|}{|x-y|^2}\, dy \\ &\leq \frac{P.V.}{\pi} \int_{\Delta_1(x)} \frac{|w_\beta(y) - w_\beta(x) + \alpha(x-y)(x-y)\partial_x w_\beta(x)|}{|x-y|^2}\, dy \\ &\leq C w_\beta(x) \end{aligned}$$

For the integral over $\Delta_2(x)$, we have

$$I_2 \leq \frac{P.V.}{\pi} \int_{\Delta_2(x)} \frac{|w_\beta(x) - w_\beta(y)|}{|x-y|^2}\, dy \leq \frac{P.V.}{\pi} \int_{\Delta_2(x)} \frac{|w_\beta(x)|}{|x-y|^2}\, dy < C w_\beta(x)$$

The last integral can be estimated as follows

$$\begin{aligned} \frac{P.V.}{\pi} \int_{\Delta_3(x)} \frac{|w_\beta(x) - w_\beta(y)|}{|x-y|^2}\, dy &\leq C \int_{\Delta_3(x)} \frac{|w_\beta(x)|}{|x-y|^2}\, dy + C \int_{\Delta_3(x)} \frac{1}{|x-y|^{2+\beta}}\, dy \\ &\leq C' w_\beta(x) \end{aligned}$$

This concludes the proof of the lemma. □

4. A priori estimates in weighted Sobolev spaces

In order to prove the theorems, we approximate our initial data by data which vanish at infinity, so that we may use the existence and regularity results obtained in the last section (see section 6). For a solution θ in H^s, $s = 0$, $1/2$ or 1, we have obviously $\theta \in H^s(w_\beta\, dx)$. This will allow us to estimate the norm of θ in $H^s(w_\beta\, dx)$; we shall show that those estimates do not depend on the $H^s(dx)$ norm of θ_0, but only on the norm of θ_0 in $H^s(w_\beta\, dx)$ and thus we shall be able to relax the approximation.

In the sequel, we shall just write w instead of w_β for the sake of readibility.

4.1. Estimates for the $L^2(wdx)$ norm. In this subsection, we consider the solution $\theta \in H^1$ associated to some initial value $\theta_0 \in H^1$ and try to estimate its $L^2(wdx)$ norm.

As usually, we multiply the transport equation by $w\theta$ and we integrate with respect to the space variable. We obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int \theta^2 w\ dx\right) &= \int \theta \partial_t \theta\ w\ dx \\ &= -\int \theta \Lambda \theta\ w dx - \int \theta \mathcal{H}\theta \partial_x \theta w\ dx. \end{aligned}$$

When integrating by parts, we take into account the weight w and get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int \theta^2\, dx\right) = &-\int |\Lambda^{1/2}\theta|^2\, w\, dx - \frac{1}{2}\int \theta^2\, \Lambda\theta\, w\, dx \\ &- \int \Lambda^{1/2}\theta[\Lambda^{1/2}, w]\theta\ dx + \frac{1}{2}\int \theta^2 \mathcal{H}\theta\ \partial_x w\ dx. \end{aligned}$$

Using lemma 3.2

$$\begin{aligned}\int \Lambda^{1/2}\theta \, \frac{1}{w}[\Lambda^{1/2}, w]\theta \, w \, dx &\leq \|\Lambda^{1/2}\theta\|_{L^2(w\ dx)} \left\|\frac{1}{w}[\Lambda^{1/2}, w]\theta\right\|_{L^2(wdx)} \\ &\leq C\|\Lambda^{1/2}\theta\|_{L^2(wdx)}\|\theta\|_{L^2(wdx)} \\ &\leq \frac{1}{2}\int |\Lambda^{1/2}\theta|^2 \ w \ dx \ + \ \frac{C^2}{2}\int \theta^2 \ w \ dx.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\frac{1}{2}\int \theta^2 \mathcal{H}\theta \ \partial_x w \ dx &\leq C\|\theta\|_\infty \int |\theta||\mathcal{H}\theta| w \ dx \\ &\leq C'\|\theta_0\|_\infty \|\theta\|^2_{L^2(wdx)}\end{aligned}$$

Thus, we find that

$$\frac{d}{dt}\left(\int \theta^2 w \ dx\right) + \int |\Lambda^{1/2}\theta|^2 \ w \ dx \leq C(1+\|\theta_0\|_\infty)\int \theta^2 \ w \ dx - \int \theta^2 \Lambda\theta \, w \, dx$$

If θ_0 is nonnegative, then the maximum principle gives us that $\theta \geq 0$. Then, using the pointwise Córdoba and Córdoba inequality [**12**] (valid for $\theta \geq 0$)

$$\Lambda(\theta^3) \leq 3\theta^2 \Lambda\theta$$

and using lemma 3.3, we get

$$\frac{1}{2}\int \theta^2 \partial_x \mathcal{H}\theta \ w \ dx \leq -\frac{1}{6}\int \Lambda(\theta^3) \ w \ dx = -\frac{1}{6}\int \theta^3 \Lambda w \ dx \leq C\|\theta_0\|_\infty \int \theta^2 w \ dx$$

Integrating in time $s \in [0, T]$ we conclude thanks to Gronwall's lemma that we have a global control of both $\|\theta\|_{L^\infty([0,T],L^2(wdx))}$ and $\|\Lambda^{1/2}\theta\|_{L^2([0,T],L^2(wdx))}$ by $\|\theta_0\|_{L^2(wdx)}$ and $\|\theta_0\|_\infty$.

REMARK 4.1. If no assumption is made on the sign of θ_0, we just obtain

(4.1)
$$\frac{d}{dt}\left(\int \theta^2 w \ dx\right) + \int |\Lambda^{1/2}\theta|^2 \ w \ dx \leq C(1+\|\theta_0\|_\infty)\int \theta^2 \ w \ dx + \|\theta_0\|_\infty \int |\theta\Lambda\theta| \, w \, dx,$$

which requires a control on $\|\Lambda\theta\|_{L^2(wdx)}$.

4.2. Estimates for the $H^{1/2}(wdx)$ norm. In this subsection, we consider the evolution norm of θ in $H^{1/2}(wdx)$. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int |\Lambda^{1/2}\theta|^2 \ wdx\right) &= \int \partial_t\theta \Lambda^{1/2}(w\Lambda^{1/2}\theta) dx \\ &= -\int \Lambda\theta \Lambda^{1/2}(w\Lambda^{1/2}\theta) dx - \int \mathcal{H}\theta \partial_x \theta \Lambda^{1/2}(w\Lambda^{1/2}\theta) \ dx.\end{aligned}$$

Then, we get the weight w outside from the differential terms

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int |\Lambda^{1/2}\theta|^2 \ wdx\right) &= -\int |\Lambda\theta|^2 \ wdx - \int \mathcal{H}\theta \partial_x \theta \Lambda\theta \ wdx \\ &+ \int \Lambda\theta (w\Lambda^{1/2}\Lambda^{1/2}\theta - \Lambda^{1/2}(w\Lambda^{1/2}\theta)) \ dx \\ &+ \int \Big(w\Lambda^{1/2}\Lambda^{1/2}\theta - \Lambda^{1/2}(w\Lambda^{1/2}\theta)\Big) \mathcal{H}\theta \partial_x \theta \ dx\end{aligned}$$

Finally, we distribute in the second term the weight $w=\gamma^2$ equally into the ∂_x and the Λ term, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int|\Lambda^{1/2}\theta|^2\ wdx\right) &= -\int|\Lambda\theta|^2\ wdx-\int\mathcal{H}\theta\partial_x(\gamma\theta)\Lambda(\gamma\theta)\ dx\\ &\quad-\int\mathcal{H}\theta\gamma\Lambda\theta\ (\gamma\partial_x\theta-\partial_x(\gamma\theta))\ dx\\ &\quad-\int\partial_x(\gamma\theta)\mathcal{H}\theta(\gamma\Lambda\theta-\Lambda(\gamma\theta))\ dx\\ &\quad+\int\Lambda\theta(w\Lambda\theta-\Lambda^{1/2}(w\Lambda^{1/2}\theta))\ dx\\ &\quad+\int(w\Lambda\theta-\Lambda^{1/2}(w\Lambda^{1/2}\theta))\mathcal{H}\theta\partial_x\theta\ dx\\ &= -\int|\Lambda\theta|^2\ wdx+J_1+J_2+J_3+J_4+J_5\end{aligned}$$

Let us estimate J_1. Using the $\mathcal{H}^1$-BMO duality, we write

$$J_1\leq C_1'\|\mathcal{H}\theta\|_{BMO}\|\partial_x(\gamma\theta)\Lambda(\gamma\theta)\|_{\mathcal{H}^1}$$

Now, we shall use the fact that if a function $f\in L^2$ then the function $g=f\mathcal{H}f$ belongs to the Hardy space $\mathcal{H}^1$: indeed, we have

$$2\mathcal{H}(f\mathcal{H}f)(x)=(\mathcal{H}f(x))^2-f(x)^2 \tag{4.2}$$

so that $f\mathcal{H}f$ belongs to $\mathcal{H}^1$ and we have

$$\|f\mathcal{H}f\|_{\mathcal{H}^1}=\|f\mathcal{H}f\|_1+\|\mathcal{H}\big(f\mathcal{H}f\big)\|_1\leq C\|f\|_{L^2}^2$$

From formula (4.2), we get the following estimate

$$J_1\lesssim\|\theta_0\|_\infty\|\partial_x(\gamma\theta)\|_{L^2}^2\lesssim\|\theta_0\|_\infty\left(\|\theta\|_{L^2(wdx)}^2+\|\Lambda\theta\|_{L^2(wdx)}^2\right).$$

To estimate J_2, we use the fact that $|\partial_x\gamma|<C_2'\gamma$ and that $w\in\mathcal{A}_4$, we obtain

$$\begin{aligned}J_2=\int\mathcal{H}\theta\ \gamma\Lambda\theta\ \theta\partial_x\gamma\ dx &\lesssim \int\left|w^{1/2}\mathcal{H}\theta\ w^{1/2}\Lambda\theta\ \theta\right|\ dx\\ &\lesssim C\|\mathcal{H}\theta\|_{L^2(wdx)}\|\Lambda\theta\|_{L^2(wdx)}\|\theta\|_{L^\infty}.\end{aligned}$$

Therefore,

$$J_2\lesssim\|\theta_0\|_\infty\|\theta\|_{L^2(wdx)}\|\Lambda\theta\|_{L^2(wdx)}.$$

In order to estimate J_3, we take p_1 and q_1 with $2<p_1<\infty$ and $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{2}$ and using lemma 3.2 we obtain

$$\begin{aligned}J_3 &\leq \|\partial_x(\gamma\theta)\|_2\|\mathcal{H}\theta(\gamma\Lambda\theta-\Lambda(\gamma\theta))\|_2\\ &\lesssim \|\partial_x(\gamma\theta)\|_2\|\mathcal{H}\theta\|_{L^{q_1}(wdx)}\left\|\frac{1}{\gamma}(\gamma\Lambda\theta-\Lambda(\gamma\theta)\right\|_{L^{p_1}(wdx)}\\ &\lesssim \|\partial_x(\gamma\theta)\|_2\|\theta\|_{L^{q_1}(wdx)}\|\theta\|_{L^{p_1}(wdx)}\end{aligned}$$

Then, using

$$\|\theta\|_{L^r(wdx)}\leq C\|\theta\|_\infty^{1-\frac{2}{r}}\|\theta\|_{L^2(wdx)}^{\frac{2}{r}}$$

with $r=p_1$ and $r=q_1$, we find,

$$J_3\lesssim\|\theta_0\|_\infty\|\theta\|_{L^2(wdx)}\left(\|\theta\|_{L^2(wdx)}+\|\Lambda\theta\|_{L^2(wdx)}\right)$$

The estimation of J_4 is easy, it suffices to use lemma 3.2

$$J_4 \leq \|\Lambda\theta\|_{L^2(wdx)} \left\| \frac{1}{w}[\Lambda^{1/2}, w]\Lambda^{1/2}\theta \right\|_{L^2(wdx)} \leq C_4 \|\Lambda\theta\|_{L^2(wdx)} \|\Lambda^{1/2}\theta\|_{L^2(wdx)}$$

It remains to estimate J_5. We first write $1 = w^{1/2}w^{-1/2}$ and use Cauchy-Schwarz inequality. Then, we use Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. We take p and q with $2 < p < 4$, and we assume p to be close enough to 2 to grant that $\frac{3}{2} - \beta(1 - \frac{1}{p}) > 1$ so that we may apply lemma 3.2. Therefore, we get

$$\begin{aligned} J_5 &\leq \|w^{1/2}\partial_x\theta\|_{L^2} \left\| w^{-1/2}\mathcal{H}\theta[\Lambda^{1/2}, w]\Lambda^{1/2}\theta \right\|_{L^2} \\ &\lesssim \|\partial_x\theta\|_{L^2(wdx)} \|\mathcal{H}\theta\|_{L^q(wdx)} \left\| \frac{1}{w}[\Lambda^{1/2}, w]\Lambda^{1/2}\theta \right\|_{L^p(wdx)} \\ &\lesssim \|\partial_x\theta\|_{L^2(wdx)} \|\theta\|_{L^q(wdx)} \|\Lambda^{1/2}\theta\|_{L^p(wdx)} \end{aligned}$$

Moreover, using the following weighted Gagliardo-Nirenberg inequality (see inequality 2.4)

$$\|\Lambda^{1/2}\theta\|_{L^4(wdx)} \lesssim \|\theta\|_\infty^{1/2} \|\Lambda\theta\|_{L^2(wdx)}^{1/2},$$

we get

$$\begin{aligned} \|\Lambda^{1/2}\theta\|_{L^p(wdx)} &\leq \|\Lambda^{1/2}\theta\|_{L^4(wdx)}^{2-\frac{4}{p}} \|\Lambda^{1/2}\theta\|_{L^2(wdx)}^{\frac{4}{p}-1} \\ &\lesssim \|\theta_0\|_\infty^{1-\frac{2}{p}} \|\Lambda\theta\|_{L^2(wdx)}^{1-\frac{2}{p}} \|\Lambda^{1/2}\theta\|_{L^2(wdx)}^{\frac{4}{p}-1}. \end{aligned}$$

Then, since

$$\|\theta\|_{L^q(wdx)} \leq \|\theta_0\|_\infty^{\frac{2}{p}} \|\theta\|_{L^2(wdx)}^{1-\frac{2}{p}},$$

we get

$$J_5 \lesssim \|\theta_0\|_{L^\infty} \|\theta\|_{L^2(wdx)}^{1-\frac{2}{p}} \|\Lambda\theta\|_{L^2(wdx)}^{2-\frac{2}{p}} \|\Lambda^{1/2}\theta\|_{L^2(wdx)}^{\frac{4}{p}-1}.$$

Using the fact that

$$\|\Lambda^{1/2}\theta\|_{L^2(wdx)}^{\frac{4}{p}-1} \leq \|\theta\|_{L^2(wdx)}^{\frac{2}{p}-\frac{1}{2}} \|\Lambda\theta\|_{L^2(wdx)}^{\frac{2}{p}-\frac{1}{2}},$$

we obtain

$$J_5 \leq C_5 \|\theta_0\|_{L^\infty} \|\theta\|_{L^2(wdx)}^{1/2} \|\Lambda\theta\|_{L^2(wdx)}^{3/2}.$$

Using Young's inequality, we finally find that there exists constants $C_6 > 0$ and $C_7 > 0$ (where C_7 depends on $\|\theta_0\|_\infty$), such that

$$\begin{aligned} \frac{d}{dt}\int |\Lambda^{1/2}\theta|^2 \; wdx \leq &(C_6\|\theta_0\|_\infty - 1)\int |\Lambda\theta|^2 \; wdx \\ &+ C_7\left(\int \theta^2 \; wdx + \int |\Lambda^{1/2}\theta|^2 \; wdx\right). \end{aligned} \tag{4.3}$$

Combining (4.1) and (4.3), we finally obtain

$$\begin{aligned} \frac{d}{dt}\left(\int |\theta|^2 + |\Lambda^{1/2}\theta|^2 \; wdx\right) \leq &(C_8\|\theta_0\|_\infty - 1)\int |\Lambda\theta|^2 \; wdx \\ &+ C_9\left(\int \theta^2 \; wdx + \int |\Lambda^{1/2}\theta|^2 \; wdx\right) \end{aligned} \tag{4.4}$$

By Gronwall's lemma, we conclude that we have a control of $\|\theta\|_{L^\infty L^2(wdx)}$, of $\|\Lambda^{1/2}\theta\|_{L^\infty L^2(wdx)}$ and of $\|\Lambda\theta\|_{L^2 L^2(wdx)}$ by $\|\theta_0\|_\infty$, $\|\theta_0\|_{L^2(wdx)}$ and $\|\Lambda^{1/2}\theta_0\|_{L^2(wdx)}$ (if $\|\theta_0\|_\infty < \frac{1}{C_8}$, where $C_8 > 0$ is a constant depending only on β).

4.3. Estimates for the $H^1(wdx)$ norm. In this subsection, we estimate the norm of θ in $H^1(wdx)$.

In order to study the evolution of the $H^1(wdx)$ norm of θ, we shall study the evolution of the semi-norm $\|\partial_x\theta\|_{L^2(wdx)}$ instead of $\|\Lambda\theta\|_{L^2(wdx)}$ since they are equivalent (see Remark 2). Therefore, we write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int |\partial_x\theta|^2\ wdx\right) &= -\int \partial_t\theta\ \partial_x(w\partial_x\theta)\ dx\\ &= \int (\partial_x\theta)^2\ \mathcal{H}\theta\ \partial_x w\ dx + \int \partial_x\theta\ \Delta\theta\ \mathcal{H}\theta\, w\ dx\\ &\quad + \int \Lambda\theta\ \partial_x\theta\ \partial_x w\ dx + \int \Lambda\theta\Delta\theta\ w\ dx\end{aligned}$$

The last term which come from the linear part of the equation can be rewritten as

$$\int \Lambda\theta\Delta\theta\ w\ dx = -\int \Lambda\theta\Lambda^2\theta\ w\ dx = -\int \Lambda^{3/2}\theta[\Lambda^{1/2}, w]\Lambda\theta - \int |\Lambda^{3/2}\theta|^2\ w\ dx$$

Moreover, an integration by parts gives

$$\frac{1}{2}\int (\partial_x\theta)^2\ \mathcal{H}\theta\ \partial_x w\ dx = -\int \partial_x\theta\ \Delta\theta\ \mathcal{H}\theta\ w\ dx - \frac{1}{2}\int (\partial_x\theta)^2\ \Lambda\theta\ w\ dx$$

So that, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int |\partial_x\theta|^2\ wdx\right)\\ =\ & -\int |\Lambda^{3/2}\theta|^2\ w\ dx - \int \Lambda^{3/2}\theta[\Lambda^{1/2}, w]\Lambda\theta - \frac{1}{2}\int (\partial_x\theta)^2\Lambda\theta\ w\ dx\\ +\ & \frac{1}{2}\int (\partial_x\theta)^2\ \mathcal{H}\theta\ \partial_x w\ dx + \int \partial_x\theta\Lambda\theta\ \partial_x w\ dx\\ =\ & -\int |\Lambda^{3/2}\theta|^2\ w\ dx + J_1 + J_2 + J_3 + J_4\end{aligned}$$

To estimate J_1 we write

$$J_1 = -\int w(x)\Lambda^{3/2}\theta\ \frac{1}{w(x)}[\Lambda^{1/2}, w]\Lambda\theta\, dx \le \|\Lambda^{3/2}\theta\|_{L^2(wdx)}\|\frac{1}{w(x)}[\Lambda^{1/2}, w]\Lambda\theta\|_{L^2(wdx)}$$

Therefore, using the second part of 3.2, we conclude that

$$J_1 \le C_1\|\Lambda^{3/2}\theta\|_{L^2(wdx)}\|\Lambda\theta\|_{L^2(wdx)}$$

For J_2, using Holder's inequality together with the fact that $w_\beta \in \mathcal{A}_3$ allows us to get

$$J_2 = -\frac{1}{2}\int (\partial_x\theta)^2\Lambda\theta\ w\ dx = -\frac{1}{2}\int w^{\frac{1}{3}}\partial_x\theta\ w^{\frac{1}{3}}\partial_x\theta\ w^{\frac{1}{3}}\mathcal{H}\partial_x\theta\ dx \le C\|\partial_x\theta\|^3_{L^3(wdx)}$$

Then, using the following weighted Gagliardo-Nirenberg inequality

$$\|\partial_x\theta\|_{L^3(wdx)} \le C_2\|\theta\|_\infty^{1/3}\|\Lambda^{3/2}\theta\|^{2/3}_{L^2(wdx)}$$

we get

$$J_2 \leq C_2 \|\theta\|_\infty \|\Lambda^{3/2}\theta\|^2_{L^2(wdx)}$$

The estimation of J_3 and J_4 are quite similar to the estimation of J_2. Indeed, we have

$$J_3 \leq C'_3 \int (\partial_x \theta)^2 \ |\mathcal{H}\theta| \ w \ dx \leq C_3 \|\partial_x \theta\|^2_{L^3(wdx)} \|\theta\|_{L^3(wdx)}$$

Then, using the interpolation inequality

$$\|\theta\|_{L^3(wdx)} \leq \|\theta\|^{1/3}_\infty \|\theta\|^{2/3}_{L^2(wdx)},$$

together with the Gagliardo-Nirenberg inequality previously recalled, we get

$$J_3 \leq C_3 \|\theta\|_\infty \|\Lambda^{3/2}\theta\|^{4/3}_{L^2(wdx)} \|\theta\|^{2/3}_{L^2(wdx)}$$

For J_4, we write

$$J_4 \leq C'_4 \int w^{\frac{1}{2}} |\partial_x \theta| \ w^{\frac{1}{2}} |\mathcal{H}\partial_x \theta| \ \ dx \leq C_4 \|\partial_x \theta\|^2_{L^2(wdx)}$$

Therefore, by the maximum principle for the L^∞ norm and Young's inequality, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int |\partial_x\theta|^2 \ wdx\right) &\leq (C_2\|\theta_0\|_\infty - 1) \int |\Lambda^{3/2}\theta|^2 \ w \ dx \\ &+ C_1 \|\Lambda^{3/2}\theta\|^2_{L^2(wdx)} \|\theta\|_{L^2(wdx)} + C_4 \|\partial_x\theta\|^2_{L^2(wdx)} \\ &+ C_3 \|\theta\|_\infty \|\Lambda^{3/2}\theta\|^{4/3}_{L^2(wdx)} \|\theta\|^{2/3}_{L^2(wdx)} \\ &\leq (C'_2\|\theta_0\|_\infty - 1) \int |\Lambda^{3/2}\theta|^2 \ wdx \\ &\quad +C_5 \left(\|\theta\|^2_{L^2(wdx)} + \|\partial_x\theta\|^2_{L^2(wdx)} \right), \end{aligned}$$

where the constant C_5 depends on $\|\theta_0\|_\infty$. Then, integrating in time $s \in [0, T]$ gives

$$\begin{aligned} \|\theta(T, .)\|^2_{H^1(wdx)} &\leq (C'_2\|\theta_0\|_\infty - 1) \int_0^T \|\Lambda^{3/2}\theta\|^2_{L^2(wdx)} \ ds \\ &\quad + C_5 \int_0^T \|\theta(s, .)\|^2_{H^1(wdx)} \ ds \end{aligned} \tag{4.5}$$

Therefore, Grönwall's lemma allows us to conclude that we have a global control of $\|\partial_x\theta\|_{L^\infty L^2(wdx)}$ and $\|\Lambda^{3/2}\theta\|_{L^2 L^2(wdx)}$ by $\|\theta_0\|_\infty$ and $\|\theta_0\|_{H^1(wdx)}$, provided that $\|\theta_0\|_\infty < \frac{1}{C'_2}$. Note that $C'_2 > 0$ is a constant that depends only on β.

5. Proof of the theorems

5.1. The truncated initial data. We shall approximate θ_0 by $\theta_{0,R} = \theta_0(x)\psi(\frac{x}{R})$, where ψ satisfies the following assumptions :

- $\psi \in \mathcal{D}(\mathbb{R})$
- $0 \leq \psi \leq 1$
- $\psi(x) = 1$ for $x \in [-1, 1]$ and $= 0$ for $|x| \geq 2$

This approximation neither alters the non-negativity of the data, nor increases its L^∞ norm. We have obviously the strong convergence, when $R \to +\infty$, of $\theta_{0,R}$ to θ_0 in $H^s(w\,dx)$ if $\theta_0 \in H^s(w\,dx)$ and $s = 0$ or $s = 1$. The only difficult case is

$s = 1/2$. This could be dealt with through an interpolation argument. But we shall give a direct proof that

$$\lim_{R\to+\infty} \|\Lambda^{1/2}(\theta_0 - \theta_{0,R})\|_{L^2(w\,dx)} = 0.$$

As we have the strong convergence of $\psi_R\Lambda^{1/2}\theta_0$ to $\Lambda^{1/2}\theta_0$ in $L^2(w\,dx)$, we must estimate the norm of the commutator $[\Lambda^{1/2}, \psi_R]\theta_0$ in $L^2(w\,dx)$, where we write $\psi_R(x) = \psi(\frac{x}{R})$. We just write

$$\Big|[\Lambda^{1/2}, \psi_R]\theta_0\Big| \le C \int \frac{|\psi_R(x) - \psi_R(y)|}{|x-y|^{3/2}} |\theta_0(y)| \quad dy$$

with

$$\frac{|\psi_R(x) - \psi_R(y)|}{|x-y|^{3/2}} \le \min\left(\frac{\|\partial_x\psi\|_\infty}{R|x-y|^{1/2}}, \frac{2\|\psi\|_\infty}{|x-y|^{3/2}}\right) = \frac{1}{R^{3/2}} K(\frac{x-y}{R})$$

where the kernel K is integrable, nonnegative and radially decreasing; thus, from inequality (2.2), we find that

$$\Big|[\Lambda^{1/2}, \psi_R]\theta_0\Big| \le \|K\|_1 R^{-1/2} \mathcal{M}\theta_0$$

which gives

$$\|[\Lambda^{1/2}, \psi_R]\theta_0\|_{L^2(w\,dx)} \le CR^{-1/2}\|\theta_0\|_{L^2(w\,dx)}.$$

5.2. Proof of theorem 1.1. We consider the sequence $\theta_{0,N}$, $N \in \mathbb{N}$ and $N \ge 1$. We have the convergence of $\theta_{0,N}$ to θ_0 in $H^{1/2}(w\,dx)$. Moreover, if $\|\theta_0\|_\infty$ is small enough we know that we have a solution θ_N of our transport equation $\mathcal{T}$ with initial value $\theta_{0,N}$. Using the *a priori* estimates of the previous section, we get (uniformly with respect to N) that the sequence θ_N is bounded in the space $L^\infty([0,T], H^{1/2}(wdx))$ and $L^2([0,T], H^1(wdx))$ for every $T \in (0,\infty)$. Now, let $\psi(x,t) \in \mathcal{D}((0,\infty] \times \mathbb{R})$, then $\psi\theta_N$ is bounded in $L^2([0,T], H^1)$. Moreover, we have

$$\partial_t(\psi\theta_N) = \theta_N\partial_t\psi + \psi\partial_t\theta_N = (I) + (II)$$

Obviously, (I) is bounded in $L^2([0,T], L^2)$. For (II), we write

$$\psi\partial_t\theta_N = -\psi\partial_x\theta_N\mathcal{H}\theta_N - \psi\Lambda\theta_N = -\psi\partial_x(\theta_N\mathcal{H}\theta_N) + \psi\theta_N\Lambda\theta_N - \psi\Lambda\theta_N$$

Since θ_N is bounded in $L^2([0,T], L^2(w\,dx))$ then by the continuity of the Hilbert transform on L^2, the sequence $\mathcal{H}\theta_N$ is bounded in $L^2([0,T], L^2(w\,dx))$ therefore, since θ_N is bounded in $L^\infty([0,T], L^\infty)$, we get that $\psi\partial_x(\theta_N\mathcal{H}\theta_N)$ $(=\partial_x(\psi\theta_N\mathcal{H}\theta_N) - (\partial_x\psi)\theta_N\mathcal{H}\theta_N\ )$ is bounded in $L^2([0,T], H^{-1})$. Therefore, since $\psi(1-\theta_N)\Lambda\theta_N$ is bounded in $L^2([0,T], L^2))$ we conclude that $\partial_t(\psi\theta_N)$ is bounded in $L^2([0,T], H^{-1})$. By Rellich compactness theorem [**25**], there exists a subsequence θ_{N_k} and a function θ such that

$$\theta_{N_k} \xrightarrow[N_k\to+\infty]{} \theta \text{ strongly in } L^2_{loc}((0,\infty)\times\mathbb{R}),$$

Futhermore, since the sequence θ_{N_k} is bounded in spaces whose dual space are separable Banach spaces, we get the two following *-weak convergences, for all $T < \infty$

$$\theta_{N_k} \xrightarrow[N_k\to+\infty]{} \theta \quad \text{*-weakly in } L^\infty([0,T], H^{1/2}(wdx)),$$

and,

$$\theta_{N_k} \xrightarrow[N_k\to+\infty]{} \theta \quad \text{*-weakly in } L^2([0,T], H^1(wdx)),$$

It remains to check that θ is a solution of the transport equation $\mathcal{T}$. Let Φ be a compactly supported smooth function, we need to prove the equality

$$\int\int_{t>0} \theta\, \partial_t \Phi\ dx\ dt = \int\int_{t>0} \Phi\left(\mathcal{H}\theta\partial_x\theta + \Lambda\theta\right)\ dx\ dt - \int \Phi(0,x)\theta_0(x)\ dx.$$

To prove this equality, it suffices to prove that we can pass to the weak limit in the following equality

$$\int\int_{t>0} \theta_{N_k}\ \partial_t\Psi\ dx\ dt = \int\int_{t>0} \Psi\left(\mathcal{H}\theta_{N_k}\partial_x\theta_{N_k} + \Lambda\theta_{N_k}\right)\ dx\ dt - \int \Psi(0,x)\theta_{N_k,0}(x)\ dx.$$

The *-weak convergence of θ_{N_k} toward θ in $L^\infty((0,T),L^2))$ implies the convergence in $\mathcal{D}'([0,T]\times\mathbb{R})$ and therefore

$$\partial_t\theta_{N_k} \xrightarrow[N_k\to+\infty]{} \partial_t\theta \ \text{ in } \mathcal{D}'([0,T]\times\mathbb{R}).$$

Moreover, since $\Lambda\theta_{N_k}$ is a (uniformly) bounded sequence on $L^2([0,\infty]\times\mathbb{R})$ therefore we also have convergence in the sense of distribution

$$\Lambda\theta_{N_k} \xrightarrow[n_k\to+\infty]{} \Lambda\theta \ \text{ in } \mathcal{D}'([0,T]\times\mathbb{R}).$$

It remains to treat the nonlinear term, we rewrite it as

$$\int\int_{t>0} \Psi\mathcal{H}\theta_{N_k}\ \partial_x\theta_{N_k}\ dx\ dt = -\int\int_{t>0} \theta_{N_k}\mathcal{H}\theta_{N_k}\partial_x\Psi - \int\int_{t>0} \Psi\theta_{N_k}\partial_x\mathcal{H}\theta_{N_k}\ dt\ dx.$$

Using the strong convergence of θ_{N_k} on $L^2_{loc}((0,\infty)\times\mathbb{R})$ and the *-weak convergence of $\mathcal{H}\theta_{N_k}$ in $L^2([0,T],L^2)$, we conclude that the products $\theta_{N_k}\mathcal{H}\theta_{N_k}$ converge weakly in $L^1_{loc}((0,\infty)\times\mathbb{R})$ toward $\theta\mathcal{H}\theta$. For the second term, we also use the strong $L^2_{loc}((0,\infty)\times\mathbb{R})$ convergence of θ_{N_k} and the weak convergence of $\partial_x\mathcal{H}\theta$ on $L^2((0,\infty)\times\mathbb{R})$, we conclude that the product converges in $L^1_{loc}((0,\infty)\times\mathbb{R})$. □

5.3. Proof of theorem 1.2. The proof of Theorem 1.2 is similar to the proof of Theorem 1.1, using *a priori* estimates on the $H^1(w\,dx)$ norm instead of the $H^{1/2}(w\,dx)$ norm.

5.4. The case of data in $L^2(dx)$ or $L^2(wdx)$. When $\theta_0 \in L^2\cap L^\infty$ and is non-negative, we have a priori estimates on the L^2 norm of θ that involves only $\|\theta_0\|_2$ and $\|\theta_0\|_\infty$, but this is not sufficient to grant existence of the solution θ, as we have not enough regularity to control the nonlinear term $\mathcal{H}\theta\partial_x\theta$.

Indeed, we have a control of $\mathcal{H}\theta$ in $L^2H^{1/2}$ and of $\partial_x\theta$ in $L^2H^{-1/2}$. But to pass to the limit in our use of the Rellich theorem, we should have (local) strong convergence of θ_{η_k} to θ in $L^2H^{1/2}$ while we may establish only the *-weak convergence. This can be seen as follows : if θ_n is a bounded sequence in $L^2H^{1/2}$ that converge locally strongly in L^2L^2 to a limit θ and if $\mathcal{H}\theta_n\partial_x\theta_n$ converges in $\mathcal{D}'$, we write

$$\begin{aligned}\mathcal{H}\theta_n\partial_x\theta_n =& \partial_x(\theta_n\mathcal{H}\theta_n) - \theta_n\partial_x\mathcal{H}\theta_n\\ =&\partial_x(\theta_n\mathcal{H}\theta_n) + \theta_n\Lambda\theta_n\\ =&\partial_x(\theta_n\mathcal{H}\theta_n) + \frac{1}{2}\Lambda(\theta_n^2) + C\int \frac{(\theta_n(t,x)-\theta_n(t,y))^2}{|x-y|^2}\,dy.\end{aligned}$$

While we have the convergence in $\mathcal{D}'$ of $\partial_x(\theta_n \mathcal{H}\theta_n) + \frac{1}{2}\Lambda(\theta_n^2)$ to $\partial_x(\theta\mathcal{H}\theta) + \frac{1}{2}\Lambda(\theta^2)$, we can only write

$$\lim_{n\to+\infty} \int \frac{(\theta_n(t,x)-\theta_n(t,y))^2}{|x-y|^2}\,dy = \int \frac{(\theta(t,x)-\theta(t,y))^2}{|x-y|^2}\,dy + \mu,$$

where μ is a non-negative measure.

6. The construction of regular enough solutions revisited

The global existence results of Córdoba, Córdoba and Fontelos in [**14**] and of Dong in [**17**] correspond to Theorems 1.1 to 1.2 in the case $\beta = 0$: they are mainly based on the maximum principle (if θ_0 is bounded, then θ remains bounded and if θ_0 is non-negative, θ remains non-negative) along with the use of some useful identities or inequalities involving the nonlocal operators Λ and $\mathcal{H}$. We do not know whether our solutions become smooth (this is known in the case $\beta = 0$ for Theorem 1.1, this is proved by Kiselev [**20**]). Another interesting question is whether we have eventual regularity in the sense of [**31**] for our solutions.

In this section, for conveniency, we sketch a complete proof of Theorems 1.1 and 1.2 in the case $\beta = 0$, under a smallness assumption on $\|\theta_0\|_\infty$ (although this latter case is treated in [**14**], we shall give a slightly different proof for the *a priori* estimates). Before starting the *a priori* estimates, one has to deal with the existence issue, namely, proving the existence of at least one solution. This step is rather important for this model since for instance one can derive a nice energy estimate for the L^2 (resp weighted L^2) norm (see [**14**], resp see section 4.1) whereas the existence of such a solution is not clear in both cases (see section 6.2). Since we aim at proving global existence results and not only *a priori* estimates, we need to give a proof of the existence of regular enough solutions. This is done in six steps and is based on classical arguments.

First step : regularizations of the equation and of the data

We use a nonnegative smooth compactly supported function φ (with $\int \varphi(x)\,dx = 1$) and for positive parameters ϵ, η we consider the parabolic approximation of equation $(\mathcal{T}_1)$:

$$(\mathcal{T}_1^{\epsilon,\eta}) \;:\; \begin{cases} \partial_t\theta + \theta_x\mathcal{H}\theta + \nu\Lambda\theta = \epsilon\Delta\theta \\ \theta(0,x) = \theta_0 * \varphi_\eta(x). \text{ (with } \varphi_\eta(x) \equiv \frac{1}{\eta}\varphi(\frac{x}{\eta})\text{)} \end{cases}$$

Recall that $\Delta\theta = \partial_x^2\theta$, then we can rewrite the problem into an integral form as follows

$$\theta = e^{\epsilon t\Delta}(\theta_0 * \varphi_\eta) - \int_0^t e^{\epsilon(t-s)\Delta}(\theta_x\mathcal{H}\theta + \nu\Lambda\theta)\,ds.$$

We may solve this equation in $\mathcal{C}([0,T_{\epsilon,\eta}],H^3)\cap L^2((0,T_{\epsilon,\eta}),H^4)$, for some small enough time $T_{\epsilon,\eta}$. Indeed, we have, for $T>0$ and for a constant C_ϵ independent of T, for all $\gamma_0 \in H^3$, $u, v \in \mathcal{C}([0,T],H^3)\cap L^2((0,T),\dot{H}^4)$ and $w \in L^2([0,T],H^2)$:

- $\sup_{0<t<T} \|e^{\epsilon t\Delta}\gamma_0\|_{H^3} \le \|\gamma_0\|_{H^3}$ and $\|\Delta e^{\epsilon t\Delta}\theta_0\|_{L^2((0,T),L^2)} \le C_\epsilon\|\theta_0\|_{H^3}$
- $\int_0^t e^{\epsilon(t-s)\Delta} w\,ds \in \mathcal{C}([0,T],H^3)\cap L^2([0,T],H^4)$

- $\sup_{0<t<T} \left\| \int_0^t e^{\epsilon(t-s)\Delta} w \, ds \right\|_2 \leq C_\epsilon T^{1/2} \|w\|_{L^2 H^2}$
- $\sup_{0<t<T} \left\| \partial_x^3 \int_0^t e^{\epsilon(t-s)\Delta} w \, ds \right\|_2 \leq C_\epsilon \|w\|_{L^2 H^2}$
- $\left\| \Delta \int_0^t e^{\epsilon(t-s)\Delta} w \, ds \right\|_{L^2((0,T),L^2)} \leq C_\epsilon \|w\|_{L^2 H^2}$
- $\|\Lambda u\|_{L^2 H^2} \leq C T^{1/2} \|u\|_{L^\infty H^3}$
- $\|u_x \mathcal{H} v\|_{L^2 H^2} \leq C T^{1/2} \|u\|_{L^\infty H^3} \|v\|_{L^\infty H^3}$

Thus, using Picard's iterative scheme, we find a solution

$$\theta = e^{\epsilon t \Delta} \gamma_0 - \int_0^t e^{\epsilon(t-s)\Delta} (\theta_x \mathcal{H}\theta + \nu \Lambda \theta) \, ds$$

on an interval $[0, T_{\epsilon,\eta}]$, where $T_{\epsilon,\eta}$ depends only on ϵ and $\|\gamma_0\|_2$. If $\|\theta\|_{H^3}$ remains bounded, we may bootstrap the estimates to get an extension to a larger interval. Thus, if $T^*_{\epsilon,\eta}$ is the maximal existence time, we must have

$$T^*_{\epsilon,\eta} < +\infty \Rightarrow \sup_{0<t<T^*_{\epsilon,\eta}} \|\theta_{\epsilon,\eta}(t,.)\|_{H^3} = +\infty.$$

The strategy is then to have a criterion on θ_0 to ensure that $T^*_{\epsilon,\eta} = +\infty$ for every $\epsilon > 0$ and to get uniform controls on the solutions $\theta_{\epsilon,\eta}$ to allow to get a limit when ϵ and η go to 0.

Second step : applying the maximum principle

This point is classical. If θ is the solution of equation $(\mathcal{T}^{\epsilon,\eta})$, we define $M(t) = \sup_{x\in\mathbb{R}^3} \theta(t,x)$ and $m(t) = \inf_{x\in\mathbb{R}^3} \theta(t,x)$. For $t = t_0$, if $M(t_0) > 0$ then the supremum is attained at some point x_0, and we have $\partial_t \theta(t_0, x_0) \leq 0$, since $\Lambda\theta(t_0,x_0) \geq 0$, $\Delta\theta(t_0,x_0) \leq 0$ and $\partial_x \theta(t_0,x_0) = 0$ (recall that $\theta(t_0,.)$ is $\mathcal{C}^2$); now, we have, for $t < t_0$, $\frac{\theta(t,x_0)-\theta(t_0,x_0)}{t-t_0} \geq \frac{M(t)-M(t_0)}{t-t_0}$ so that $\limsup_{t\to t_0^-} \frac{M(t)-M(t_0)}{t-t_0} \leq 0$. We see that this is enough to get that M is non-inecreasing on the set $\{t \ / \ M(t) > 0\}$, and thus to get $M(t) \leq M(0)$; a similar argument gives $m(t) \geq m(0)$. This gives us that $\|\theta\|_\infty \leq \|\theta_0 * \varphi_\eta\|_\infty \leq \|\theta_0\|_\infty$ and, if $\theta_0 \geq 0$, then $\theta(x,t) \geq 0$ for all $t > 0$.

Third step : global existence for the regularized problem

In order to show that the H^3 norm of a solution θ to equation $(\mathcal{T}^{\epsilon,\eta})$ does not blow up, we now compute $\partial_t(\|\theta\|_2^2 + \|\partial_x^3\theta\|_2^2)$. As $\partial_x^3\theta$ belongs (locally in time on

$[0,T^*_{\epsilon,\eta}))$ to $L^2([0,T^*_{\epsilon,\eta}),H^1)$ and $\partial_t\partial_x^3\theta$ to L^2H^{-1}, therefore we may write

$$\begin{aligned}\partial_t(\|\theta\|_2^2+\|\partial_x^3\theta\|_2^2) =&2\int \partial_t\theta(\theta-\partial_x^6\theta)\,dx\\ =&-2\|\Lambda^{1/2}\theta\|_2^2-2\|\Lambda^{7/2}\theta\|_2^2-2\epsilon\|\partial_x\theta\|_2^2-2\epsilon\|\partial_x^4\theta\|_2^2\\ &-2\int\theta\mathcal{H}\theta\partial_x\theta\,dx+2\int\partial_x^3\theta\partial_x^3(\mathcal{H}\theta\partial_x\theta)\,dx\\ =&-2\|\Lambda^{1/2}\theta\|_2^2-2\|\Lambda^{7/2}\theta\|_2^2-2\epsilon\|\partial_x\theta\|_2^2-2\epsilon\|\partial_x^4\theta\|_2^2\\ &-2\int\theta\mathcal{H}\theta\partial_x\theta\,dx+2\int\partial_x^3\theta\partial_x^3(\mathcal{H}\theta)\ \partial_x\theta\,dx\\ &+6\int\partial_x^3\theta\partial_x^2(\mathcal{H}\theta)\ \partial_x^2\theta\,dx+5\int\partial_x^3\theta\partial_x(\mathcal{H}\theta)\ \partial_x^3\theta\,dx\\ \leq&-2\|\Lambda^{1/2}\theta\|_2^2-2\|\Lambda^{7/2}\theta\|_2^2-2\epsilon\|\partial_x\theta\|_2^2-2\epsilon\|\partial_x^4\theta\|_2^2\\ &+2\|\theta\|_\infty\|\theta\|_2\|\partial_x\theta\|_2+(2\|\partial_x\theta\|_7+5\|\mathcal{H}\partial_x\theta\|_7)\|\partial_x^3\theta\|_{7/3}^2\\ &+6\|\partial_x^2\theta\|_3^2\|\mathcal{H}\partial_x^2\theta\|_3\end{aligned}$$

We then use the boundedness of the Hilbert transform on L^3 and L^7 and the Gagliardo–Nirenberg inequalities

$$\|\partial_x^2\theta\|_3\leq\|\theta\|_\infty^{1/3}\|\partial_x^3\theta\|_2^{2/3}$$

$$\|\partial_x\theta\|_7\leq\|\theta\|_\infty^{5/7}\|\Lambda^{7/2}\theta\|_2^{2/7}$$

$$\|\partial_x^3\theta\|_{7/3}\leq\|\theta\|_\infty^{1/7}\|\Lambda^{7/2}\theta\|_2^{6/7}$$

and we find, for a constant C_0 (that does not depend on θ_0 nor on ϵ),

(6.1)

$$\partial_t(\|\theta\|_2^2+\|\partial_x^3\theta\|_2^2)\leq C_0\|\theta_0\|_\infty(\|\theta\|_2^2+\|\partial_x^3\theta\|_2^2)+2(C_0\|\theta_0\|_\infty-1)\|\Lambda^{7/2}\theta\|_2^2-2\epsilon\|\partial_x^4\theta\|_2^2$$

Thus, if $C_0\|\theta_0\|_\infty<1$, we find that, on $[0,T^*_{\epsilon,\eta})$, we have

$$\|\theta\|_2^2+\|\partial_x^3\theta\|_2^2\leq e^{C_0\|\theta_0\|_\infty t}(\|\theta_0*\varphi_\eta\|_2^2+\|\theta_0*\partial_x^3\varphi_\eta\|_2^2)$$

and thus $T^*_{\epsilon,\eta}=+\infty$.

Fourth step : relaxing ϵ

From inequality 6.1, we get that $\theta_{\epsilon,\eta}$ is controlled, on each bounded interval of time $[0,T]$, uniformly with respect to ϵ, in the following ways :

- $\sup_{\epsilon>0}\sup_{0<t<T}\|\theta_{\epsilon,\eta}(t,.)\|_{H^3}<+\infty$
- $\sup_{\epsilon>0}\int_0^T\|\theta_{\epsilon,\eta}\|_{H^{7/2}}^2\,dt<+\infty$

and we get from equation $(\mathcal{T}_1^{\epsilon,\eta})$, that

- $\sup_{0<\epsilon<1}\int_0^T\|\partial_t\theta_{\epsilon,\eta}\|_{H^{1/2}}^2\,dt<+\infty$

We then use the Rellich theorem [**25**] to get that there exists a sequence $\epsilon_k\to 0$ so that $\theta_{\epsilon_k,\eta}$ converges strongly in $L^2_{\mathrm{loc}}((0,+\infty)\times\mathbb{R})$ to a limit θ_η. As $\theta_{\epsilon,\eta}$ is (locally) bounded in $L^2H^{7/2}$, the strong convergence holds as well in $(L^2H^1)_{\mathrm{loc}}$, so that θ_η

is a solution of $(\mathcal{T}_1)$, with initial value $\theta_0 * \varphi_\eta$.

Moreover, we know that $\|\theta_\eta\|_\infty \leq \|\theta_0\|_\infty$ and that, for every finite $T > 0$,

$$\sup_{0<t<T} \|\theta_\eta(t,.)\|_{H^3} < +\infty \text{ and } \int_0^T \|\theta_\eta\|^2_{H^{7/2}}\,dt < +\infty.$$

Fifth step : uniform estimates in $H^{1/2}$ and H^1

- control of the L^2 norm :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int \theta_\eta^2\,dx\right) &= \int \theta_\eta\,\partial_t\theta_\eta\,dx = -\int \theta_\eta\Lambda\theta_\eta\,dx - \int \theta_\eta(\mathcal{H}\theta)\partial_x\theta_\eta\,dx \\ &\leq -\int |\Lambda^{1/2}\theta_\eta|^2\,dx + \|\theta_0\|_\infty\|\theta_\eta\|_2\|\Lambda\theta_\eta\|_2\end{aligned} \tag{6.2}$$

- control of the $\dot{H}^{1/2}$ norm :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int |\Lambda^{1/2}\theta_\eta|^2\,dx\right) &= \int \Lambda\theta_\eta\,\partial_t\theta_\eta\,dx \\ &= -\int |\Lambda\theta_\eta|^2\,dx - \int (\mathcal{H}\theta_\eta)\,(\Lambda\theta_\eta\,\partial_x\theta_\eta)\,dx \\ &= -\int |\Lambda\theta_\eta|^2\,dx + \int \theta_\eta\,\mathcal{H}(\Lambda\theta_\eta\,\partial_x\theta_\eta)\,dx\end{aligned}$$

We now use the identity, valid for every $f \in L^2$,

$$2\mathcal{H}(f\mathcal{H}f)(x) = (\mathcal{H}f(x))^2 - f(x)^2, \tag{6.3}$$

along with,

$$\partial_x\theta_\eta = \mathcal{H}\Lambda\theta_\eta,$$

to get,

$$\|\mathcal{H}(\Lambda\theta_\eta\,\partial_x\theta_\eta)\|_1 \leq \|\Lambda\theta_\eta\|_2^2,$$

and finally obtain

$$\frac{d}{dt}\int |\Lambda^{1/2}\theta_\eta|^2\,dx + 2(1 - \|\theta_0\|_\infty)\int |\Lambda\theta_\eta|^2\,dx \leq 0. \tag{6.4}$$

- control of the $\dot{H}^1$ norm : we write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\Lambda\theta_\eta|^2\,dx &= \int \Lambda^2\theta_\eta\,\partial_t\theta_\eta\,dx \\ &= -\int |\Lambda^{3/2}\theta_\eta|^2\,dx - \frac{1}{2}\int \partial_x(\mathcal{H}\theta_\eta)\,(\partial_x\theta_\eta)^2\,dx.\end{aligned}$$

Using a Gagliardo–Nirenberg inequality, we get

$$\frac{1}{2}\left|\int \partial_x(\mathcal{H}\theta_\eta)\,(\partial_x\theta_\eta)^2\,dx\right| \leq C\|\partial_x\theta\|_3^3 \leq C_1\|\theta\|_\infty\|\Lambda^{3/2}\theta_\eta\|_2^2,$$

and finally obtain,

$$\frac{d}{dt}\left(\int |\Lambda\theta_\eta|^2\,dx\right) + 2(1 - C_1\|\theta_0\|_\infty)\int |\Lambda^{3/2}\theta_\eta|^2\,dx \leq 0. \tag{6.5}$$

Sixth step : relaxing η

From inequalities (6.2) and (6.4), we get that, for $\theta_0 \in H^{1/2}$, (when $\|\theta_0\|_\infty$ is small enough) θ_η is controlled, on each bounded interval of time $[0,T]$, uniformly with respect to η, in the following ways :

- $\sup_{\eta>0} \sup_{0<t<T} \|\theta_\eta(t,.)\|_{H^{1/2}} < +\infty$,
- $\sup_{\eta>0} \int_0^T \|\theta_\eta\|_{H^1}^2 \, dt < +\infty$

and we get from equation $(\mathcal{T}_1)$, that

- $\sup_{\eta>0} \int_0^T \|\partial_t \theta_{\epsilon,\eta}\|_{H^{-1/4}}^2 \, dt < +\infty$.

We may then use the Rellich theorem [**25**] and get that there exists a sequence $\eta_k \to 0$ so that θ_{η_k} converges strongly in $L^2_{\text{loc}}((0,+\infty)\times\mathbb{R})$ to a limit θ. As θ_η is (locally) bounded in L^2H^1,we have weak convergence in L^2H^1; we then write $\mathcal{H}\theta_\eta\partial_x\theta_\eta = \partial_x(\theta_\eta\mathcal{H}\theta_\eta) - \theta_\eta\mathcal{H}\partial_x\theta_\eta$ and find that θ is a solution of $(\mathcal{T}_1)$, with initial value θ_0.

Moreover, we find that we have

- $\|\theta\|_\infty \le \|\theta_0\|_\infty$
- $\sup_{t>0} \|\Lambda^{1/2}\theta(t,.)\|_2 \le \|\Lambda^{1/2}\theta_0\|_2$
- $\int_0^{+\infty} \|\Lambda\theta\|_2^2 \, ds \le \frac{1}{2(1-\|\theta_0\|_\infty)} \|\Lambda^{1/2}\theta_0\|_2^2$
- $\|\theta(t,.)\|_2 \le \|\theta_0\|_2 + \|\theta_0\|_\infty \int_0^t \|\Lambda\theta(s,.)\|_2 \, ds$.

Similarly, if $\theta_0 \in H^1$ (with $\|\theta_0\|_\infty$ small enough) , then inequality (6.5) will give a control of the H^1 norm of θ_η uniformly with respect to η, and thus, we find for the limit θ that,

- $\sup_{t>0} \|\Lambda\theta(t,.)\|_2 \le \|\Lambda\theta_0\|_2$,
- $\int_0^{+\infty} \|\Lambda^{3/2}\theta\|_2^2 \, ds \le \frac{1}{2(1-C_1\|\theta_0\|_\infty)} \|\Lambda\theta_0\|_2^2$.

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References

1. G.R. Baker, X. Li, A.C. Morlet. *Analytic structure of two 1D-transport equations with nonlocal fluxes.* Physica D: Nonlinear Phenomena, **91**(4):349-375, 1996.
2. H. Bae, R. Granero-Belinchón. *Global existence for some transport equations with nonlocal velocity.* Advances in Mathematics, vol. **269**, pp 197-219, 2015.
3. L.A. Caffarelli, A. Vasseur. *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation.* Annals of Math. (2), **171**(3):1903-1930, 2010.
4. A. Castro, D. Córdoba. *Global existence, singularities and Ill-posedness for a non-local flux.* Advances in Mathematics. **219** (2008), 6, 1916-1936.
5. A. Castro, D. Córdoba. *Infinite energy solutions of the surface quasi-geostrophic equation.* Advances in Mathematics. **225** (2010) 1820-1829.

6. D. Chae, A. Córdoba, D. Córdoba, M. A. Fontelos. *Finite time singularities in a 1D model of the quasi-geostrophic equation.* Advances in Mathematics. **194** (2005), 203-223.
7. C. H. Chan, M. Czubac, L. Silvestre. *Eventual regularization of the slightly supercritical fractional Burgers equation. Discrete and Continuous Dynamical Systems*, Volume: **27**, Number: 2, June 2010, Pages 847-861.
8. R. Coifman, Y. Meyer. *Wavelets: Calderón-Zygmund and Multilinear Operators*, Cambridge University Press, 336 pages
9. P. Constantin, P. Lax, A. Majda. *A simple one-dimensional model for the three dimensional vorticity*, Comm. Pure Appl. Math. **38** (1985), 715-724.
10. P. Constantin, A.J. Majda, E. Tabak. *Formation of strong fronts in the* 2*-D quasigeostrophic thermal active scalar.* Nonlinearity, **7** (1994), pp. 1495-1533.
11. P. Constantin, V. Vicol. *Nonlinear maximum principles for dissipative linear nonlocal operators and applications.* Geometric And Functional Analysis, **22**(5):1289-1321, 2012.
12. A. Córdoba, D. Córdoba. *A maximum principle applied to quasi-geostrophic equations,* Comm. Math. Phys. **249** (2004), pp. 511-528.
13. A. Córdoba, D. Córdoba, M.A. Fontelos. *Integral inequalities for the Hilbert transform applied to a nonlocal transport equation.* J. Math. Pures Appl. (9), **86**, 6:529-540, 2006
14. A. Córdoba, D. Córdoba, M.A. Fontelos. *Formation of singularities for a transport equation with nonlocal velocity,* Annals of Math. 162 (2005) **3**, 1375-1387.
15. S. De Gregorio. *On a one-dimensional model for the three-dimensional vorticity equation*, J. Statist. Phys. **59** (1990), 1251-1263.
16. T. Do. *On a 1d transport equation with nonlocal velocity and supercritical dissipation.* Journal of Differential Equations, **256**-9:3166-3178, 2014.
17. H. Dong. *Well-posedness for a transport equation with nonlocal velocity.* J. Funct. Anal, **255**:3070-3097, (2008).
18. L. Hedberg. *On certain convolution inequalities.* Proc. Amer. Math. Soc. **10** (1972), 505-510.
19. R. Hunt, B. Muckenhoupt, R. Wheeden. *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227-251.
20. A. Kiselev. *Regularity and blow up for active scalars.* Math. Model. Nat. Phenom, 5(4):225-255, 2010
21. A. Kiselev. *Nonlocal maximum principles for active scalars.* Advances in Mathematics, 227,**5**:1806-1826, 2011.
22. A. Kiselev, F. Nazarov and R. Shterenberg. *On blow up and regularity in dissipative Burgers equation*, Dynamics of PDEs, **5** (2008), 211-240
23. A. Kiselev, F. Nazarov, and A. Volberg. *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation.* Invent. Math., **167**(3):445-453, 2007
24. O. Lazar. *On a 1D nonlocal transport equation with nonlocal velocity and subcritical or supercritical diffusion*, preprint.
25. P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem.* Chapman & Hall/CRC (2002).
26. D. Li, J. L. Rodrigo. Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation. *Advances in Mathematics*, **217**, no. 6, 2563-2568 (2008).
27. D. Li, J. L. Rodrigo. *On a One-Dimensional Nonlocal Flux with Fractional Dissipation*, SIAM J. Math. Anal. **43** (2011), 507-526.
28. B. Muckenhoupt. *Weighted norm inequalities for the Hardy maximal function.* Transactions of the American Mathematical Society, vol. **165**: 207-226. (1972).
29. H. Okamoto, T. Sakajo, M. Wunsch. On a generalization of the Constantin-Lax-Majda equation. Nonlinearity, **21**(10): 2447-2461 (2008).
30. E. Stein. Harmonic Analysis : Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Math. Series **43**, Princeton Univ. Press, Princeton, NJ, 1993.
31. L. Silvestre. Eventual regularization for the slightly supercritical quasi-geostrophic equation. Ann. Inst. H. Poincaré Anal. Non Linéaire, **27**(2):693-704, 2010
32. L. Silvestre, V. Vicol. Transactions of the American Mathematical Society **368** (2016), no. 9, 6159-6188..

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