Dynamics of PDE, Vol.13, No.1, 75-90, 2016

Well-posedness and global attractor of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions

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Communicated by Sergey Zelik, received December 19, 2014.

Abstract. Our aim in this paper is to study the well-posedness and the long-time behavior of solutions for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions. We prove the well-posedness of solutions and the existence of a global attractor in $H^1(\bar{\Omega}, d\nu)$ for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions by using Aubin-Lions compactness Theorem.

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1991 *Mathematics Subject Classification.* Primary 35, 37.

Key words and phrases. Global attractor, Cahn-Hilliard-Brinkman system, Dynamic boundary conditions, Dissipativity, Aubin-Lions compactness Theorem.

The authors of this paper would like to express their sincere thanks to the reviewer for valuable comments and suggestions. This work was supported by the National Science Foundation of China Grant (11401459) and the China Postdoctoral Science Foundation funded project(No.2013M532026).

1. Introduction

In this paper, we consider the following Cahn-Hilliard-Brinkman system:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (u\phi) = \nabla \cdot (M\nabla\mu), \ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$\mu = -\epsilon\Delta\phi + \frac{1}{\epsilon} f(\phi), \ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.2}$$

$$-\nu\Delta u + \eta u = -\nabla p - \gamma\phi\nabla\mu, \ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.3}$$

$$\nabla \cdot u = 0, \ \ (x,t) \in \Omega \times \mathbb{R}^+. \tag{1.4}$$

Equation (1.1)-(1.4) is subject to the following dynamic boundary conditions

$$u(x,t) = 0, \ \ (x,t) \in \Gamma \times \mathbb{R}^+, \tag{1.5}$$

$$\frac{\partial \mu}{\partial \vec{n}} = 0, \ \ (x,t) \in \Gamma \times \mathbb{R}^+, \tag{1.6}$$

$$\frac{1}{d}\frac{\partial \phi}{\partial t} = \alpha\Delta_\Gamma\phi - \frac{\partial \phi}{\partial \vec{n}} - \beta\phi, \ \ (x,t) \in \Gamma \times \mathbb{R}^+ \tag{1.7}$$

and initial conditions

$$\phi(x,0) = \phi_0(x), \ \ x \in \Omega, \tag{1.8}$$

$$\phi(x,0) = \theta_0(x), \ \ x \in \Gamma, \tag{1.9}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ and $\mathbb{R}^+ = [0, +\infty)$, $\nu > 0$ is the viscosity, $\eta > 0$ is the fluid permeability, $M > 0$ stands for the mobility, $\epsilon > 0$ is related to the diffuse interface thickness, $\gamma > 0$ is a surface tension parameter,$d > 0$ $\alpha > 0$ $\beta \geq 0$ are constants, p is the fluid pressure, $\vec{n}$ is the normal vector on Γ, Δ_Γ is the Laplace-Beltrami operator on the surface Γ of Ω and f is the derivative of a double well potential $F(s) = \frac{1}{4}(s^2 - 1)^2$ describing phase separation.

Dynamic boundary conditions were recently proposed by physicists to describe spinodal decomposition of binary mixtures where the effective interaction between the wall (i.e., the boundary) and two mixture components is short-ranged, and this type of boundary conditions is very natural in many mathematical models such as heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, heat transfer in two medium, problems in fluid dynamics. The well-posedness and long-time behavior of solutions for many equations with dynamical boundary conditions have been studied extensively(see [**3, 4, 5, 9, 10, 13, 16, 22, 23, 24, 29, 32, 33, 34, 36, 39, 43**]). For example, the global well-posedness of solutions for the non-isothermal Cahn-Hilliard equation with dynamic boundary conditions was proved in [**21**]. In [**22**], the author proved the existence and uniqueness of a global solution for a Cahn-Hilliard model in bounded domains with permeable walls. The global existence and uniqueness of solutions for the Cahn-Hilliard equation with highest-order boundary conditions were proved in [**36**]. In [**33**], the authors proved the maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions. The fact that any global weak/strong solution of the Cahn-Hilliard equation with dynamic boundary conditions converges to a single steady state as time $t \to +\infty$ was proved in [**13**]. In [**25**], the author proved the existence of a global attractor and an exponential attractor in $H^1(\Omega)$ for a homogeneous two-phase flow model and established any global weak/strong solution converges to a single steady state as time $t \to +\infty$, and provided its convergence rate. Under the assumptions that the potential function is real analytic and satisfies certain growth conditions, the authors proved each solution of a Cahn-Hilliard equation with Wentzell boundary conditions and mass conservation converges to a steady state as time goes to ∞ and its convergence rate was obtained as well in [**39**]. Meanwhile, they recalled some results about the existence of global and exponential attractors and gave their properties. In [**23**], the author proved the existence of an exponential attractor for a Cahn-Hilliard model in bounded domains with permeable walls. The existence of a global attractor for the reaction-diffusion equation with dynamical boundary conditions was proved in [**16**]. In [**29**], the authors proved the existence of an exponential attractor for the Cahn-Hilliard equation with dynamical boundary conditions. The smooth effect of the process $\{U(t,\tau)\}_{t\geq\tau}$ associated with a non-autonomous homogeneous two-phase flow model was proved in [**30**] by verifying $\mathcal{A}^{\mathbb{Y}} = \mathcal{A}^{\mathbb{V}}$. In [**42**], the authors proved the existence of a global attractor for p-Laplacian equations with dynamical boundary conditions by using asymptotical a priori estimates.

A diffuse interface variant of Brinkman equation has been proposed to model phase separation of incompressible binary fluids in a porous medium (see [**31**]). The coupled system consists of a convective Cahn-Hilliard equation for the phase field ϕ, i.e., the difference of the relative concentrations of the two phases, coupled with a modified Darcy equation proposed by H. C. Brinkman [**7**] in 1947 for the fluid velocity u. This equation incorporates a diffuse interface surface force proportional to $\phi\nabla\mu$, where μ is the so-called chemical potential which is the variational derivative of the free energy functional

$$E(\phi) = \int_\Omega \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{\epsilon}F(\phi)\,dx.$$

For this reason, equation (1.1)-(1.4) has been called Cahn-Hilliard-Brinkman system. Such a system belongs to a class of diffuse interface models which are used to describe the behavior of multi-phase fluids. The Cahn-Hilliard-Navier-Stokes system has been investigated from the numerical and analytical viewpoint in several papers (see, e.g., [**1, 2, 6, 11, 12, 17, 18, 19, 28, 37, 44, 45**]). The long-time behavior and well-posedness of solutions for the Cahn-Hilliard-Hele-Shaw system were proved in [**40, 41**]. In [**26**], the authors have considered the well-posedness and long-time behavior of solutions for a non-autonomous Cahn-Hilliard-Darcy system with mass source modeling tumor growth which is more complicated than the Cahn-Hilliard-Brinkman system thanks to the compressibility of the fluid. Meanwhile, they established the existence of a pullback attractor in $H^2(\Omega)$, and proved any global weak/strong solution converges to a single steady state as time $t \to +\infty$ and obtained its convergence rate.

Cahn-Hilliard-Brinkman system (1.1)-(1.4) with M, ν, and η possibly depending on ϕ has been analyzed from the numerical point of view in [**11, 15**]. From the analytical point of view, the authors in [**8**] have considered the well-posedness of solutions and the existence of a global attractor in $H^1(\Omega)$ for the Cahn-Hilliard-Brinkman system (1.1)-(1.4) with positive constants M, ν, η, ϵ and more general $f(u)$ when the system is subject to the Neumann boundary conditions, and established the convergence of a given weak solution to a single equilibrium via Łojasiewicz-Simon inequality and gave its convergence rate. Furthermore, the authors have also studied the behavior of the solutions as the viscosity goes to zero, i.e., the existence of a weak solution to the Cahn-Hilliard-Hele-Shaw system was proved as the limit of solutions to the Cahn-Hilliard-Brinkman system when the Cahn-Hilliard-Brinkman system approaches the Cahn-Hilliard-Hele-Shaw system. In contrast to the Cahn-Hilliard-Brinkman system with Neumman boundary conditions, not so much is known on well-posedness and long-time behavior of solutions for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions. In this paper, we study the existence of a global attractor for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions (1.1)-(1.9).

The rest of this paper is organized as follows: In the next Section, we introduce some notations and lemmas used in the sequel, and give the definition of weak solutions for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions (1.1)-(1.9). In Section 3, we prove the well-posedness of solutions for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions (1.1)-(1.9). Section 4 is devoted to prove the existence of a global attractor for the Cahn-Hilliard-Brinkman system with dynamical boundary conditions (1.1)-(1.9).

Throughout this paper, for the sake of simplicity, we assume $M = \epsilon = \gamma = \nu = \eta = 1$. Let C be a generic constant that is independent of the initial datum for ϕ. Define the average of function $\phi(x)$ over Ω as

$$m\phi = \frac{1}{|\Omega|}\int_\Omega \phi(x)\,dx.$$

2. Preliminaries

In order to study the problem (1.1)-(1.9), we introduce the space of divergence-free functions defined by

$$\mathcal{V} = \{u \in (\mathcal{C}_c^\infty(\Omega))^3 : \nabla\cdot u = 0\}.$$

Denote by H and V the closure of $\mathcal{V}$ with respect to the norms in $(L^2(\Omega))^3$ and $(H^1(\Omega))^3$, respectively, and let $\Omega_T = \Omega\times(0,T)$ and $\Gamma_T = \Gamma\times(0,T)$.

We define the Lebesgue spaces as follows

$$L^p(\Gamma) = \left\{v : \|v\|_{L^p(\Gamma)} < \infty\right\},$$

where

$$\|v\|_{L^p(\Gamma)} = \left(\int_\Gamma |v|^p\,dS\right)^{\frac{1}{p}}$$

for $p\in[1,\infty)$. Moreover, we have

$$L^p(\Omega)\oplus L^p(\Gamma) = L^p(\bar\Omega, d\nu),\quad p\in[1,\infty)$$

and

$$\|U\|_{L^p(\bar\Omega,d\nu)} = (\int_\Omega |u|^p\,dx + \int_\Gamma |v|^p\,dS)^{\frac{1}{p}}$$

for any $U=(u,v)\in L^p(\bar\Omega,d\nu)$, where the measure $d\nu = dx|_\Omega \oplus dS|_\Gamma$ on $\bar\Omega$ is defined by $\nu(A) = |A\cap\Omega| + S(A\cap\Gamma)$ for any measurable set $A\subset\bar\Omega$.

We also define the Sobolev space $H^1(\bar\Omega,d\nu)$ as the closure of $C^1(\overline{\Omega})$ with respect to the norm given by

$$\|\phi\|^2_{H^1(\bar\Omega,d\nu)} = \left(\int_\Omega |\nabla\phi|^2\,dx + \int_\Gamma \alpha|\nabla_\Gamma\phi|^2 + \beta|\phi|^2\,dS\right)^{\frac{1}{2}}$$

for any $\phi\in C^1(\overline{\Omega})$, denote by X^* the dual space of X and let $H^s(\Omega)$, $H^s(\Gamma)$ $(s\in\mathbb{R})$ be the usual Sobolev spaces. In general, any vector $\theta\in L^p(\bar\Omega,d\nu)$ will be of the form (θ_1, θ_2) with $\theta_1 \in L^p(\Omega, dx)$ and $\theta_2 \in L^p(\Gamma, dS)$, and there need not be any connection between θ_1 and θ_2.

Let the operator $A : H^1(\bar{\Omega}, d\nu) \to (H^1(\bar{\Omega}, d\nu))^*$ be associated with the bilinear form defined by

$$\begin{aligned} a(\phi, \psi) =& \langle A(\phi), \psi\rangle \\ =& \int_\Omega \nabla\phi \cdot \nabla\psi \, dx + \int_\Gamma \alpha \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi + \beta\phi\psi \, dS. \end{aligned} \tag{2.1}$$

REMARK 2.1. ([**20**]) $C(\bar{\Omega})$ is a dense subspace of $L^2(\bar{\Omega}, d\nu)$ and a closed subspace of $L^\infty(\bar{\Omega}, d\nu)$.

Next, we recall briefly some lemmas used to prove the well-posedness of weak solutions for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9).

LEMMA 2.2. *(*[**35**]*)* Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ and let $1 < q < \infty$. Assume that $\{g_n\} \subset L^q(\mathcal{O})$ with $\|\{g_n\}\|_{L^q(\mathcal{O})} \leq C$, where C is independent of n and there exists $g \in L^q(\mathcal{O})$ such that $\{g_n\} \to g$, as $n \to \infty$, almost everywhere in $\mathcal{O}$. Then $g_n \to g$, as $n \to \infty$ weakly in $L^q(\mathcal{O})$.

LEMMA 2.3. *(*[**21, 29**]*) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. Consider the following linear problem*

$$\begin{cases} -\Delta\phi = j_1, & x \in \Omega, \\ -\alpha\Delta_\Gamma \phi + \frac{\partial\phi}{\partial\vec{n}} + \beta\phi = j_2, & x \in \Gamma. \end{cases}$$

Assume that $(j_1, j_2) \in H^s(\bar{\Omega}, d\nu)$, $s \geq 0$, $s + \frac{1}{2} \notin \mathbb{N}$. Then the following estimate holds

$$\|\phi\|_{H^{s+2}(\bar{\Omega}, d\nu)} \leq C(\|j_1\|_{H^s(\Omega)} + \|j_2\|_{H^s(\Gamma)})$$

for some constant $C > 0$.

LEMMA 2.4. *(*[**38**]*) Let V, H, V^* be three Hilbert spaces such that $V \subset H = H^* \subset V^*$, where H^* and V^* are the dual spaces of H and V, respectively. Suppose $u \in L^2(0, T; V)$ and $\frac{\partial u}{\partial t} \in L^2(0, T; V^*)$. Then u is almost everywhere equal to a function continuous from $[0, T]$ into H.*

Finally, we give the definition of weak solutions for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9).

DEFINITION 2.5. Assume that $(\phi_0, \theta_0) \in H^1(\bar{\Omega}, d\nu)$. For any fixed $T > 0$, a function (u, ϕ) is called a weak solution of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) on $(0, T)$, if

$$\mu \in L^2(0, T; H^1(\Omega)) \text{ is given by (1.2)}$$

and

$$\begin{aligned} &\phi \in \mathcal{C}([0, T]; H^1(\bar{\Omega}, d\nu)) \cap L^2(0, T; H^2(\bar{\Omega}, d\nu)), \\ &u \in L^2(0, T; V) \end{aligned}$$

satisfy

$$\int_{\Omega_T} \phi_t \psi \, dx + \int_{\Omega_T} \nabla \cdot (u\phi)\psi \, dx + \int_{\Omega_T} \nabla \mu \cdot \nabla \psi \, dx = 0,$$
$$\int_{\Omega_T} \nabla u \cdot \nabla v + uv \, dx + \int_{\Omega_T} (v\phi) \cdot \nabla \mu \, dx = 0,$$
$$\frac{1}{d}\int_{\Gamma_T} \phi_t \theta \, dS + \int_{\Omega_T} \nabla \phi \cdot \nabla \theta + f(\phi)\theta \, dx + \int_{\Gamma_T} \alpha \nabla_\Gamma \phi \cdot \nabla_\Gamma \theta + \beta \phi \theta \, dS = \int_{\Omega_T} \mu \theta \, dx$$

for all test functions $v \in V$ and ψ, $\theta \in H^1(\bar{\Omega}, d\nu)$.

3. The well-posedness of weak solutions

In this section, we prove the well-posedness of weak solutions for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9). Now, we state it as follows.

THEOREM 3.1. *Assume that $(\phi_0, \theta_0) \in H^1(\bar{\Omega}, d\nu)$. Then there exists a unique weak solution $(u(t), \phi(t))$ for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions* (1.1)-(1.9) *such that $m\phi(t) = m\phi_0$, which depends continuously on the initial data (ϕ_0, θ_0) with respect to the norm in $H^1(\bar{\Omega}, d\nu)$.*

Proof. We first prove the existence of weak solutions for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) by the Faedo-Galerkin method (see [**38**]). We consider the eigenvalue problems $A\psi = \lambda\psi$ and $A_1\omega = \kappa\omega$, where $A_1 = -P\Delta$ is the Stokes operator and P is the Leray-Helmotz projection from $L^2(\Omega)$ onto H. It is well-known that there exist two sequences of non-decreasing numbers $\{\lambda_n\}_{n=1}^{\infty}$, $\{\kappa_n\}_{n=1}^{\infty}$ and two sequences of functions $\{\psi_n\}_{n=1}^{\infty}$, $\{\omega_n\}_{n=1}^{\infty}$, which are orthonormal and complete in $H^1(\bar{\Omega}, d\nu)$, H, respectively, such that for every $k \geq 1$, we have

$$A\psi_k = \lambda_k \psi_k,$$
$$A_1 \omega_k = \kappa_k \omega_k$$

and

$$\lim_{k \to +\infty} \lambda_k = +\infty,$$
$$\lim_{k \to +\infty} \kappa_k = +\infty.$$

For any $n \geq 1$, we introduce two finite-dimensional spaces $W_n = span\{\psi_1, ..., \psi_n\}$ and $H_n = span\{\omega_1, ..., \omega_n\}$. Let P_n be the orthogonal projection from $H^1(\bar{\Omega}, d\nu)$ to W_n and let Π_n be the orthogonal projection from H to H_n.

Consider the approximate solution $(\phi_n(t), u_n(t))$ in the form

$$\phi_n(t) = \sum_{i=1}^{n} g_i(t)\psi_i,$$
$$u_n(t) = \sum_{i=1}^{n} h_i(t)\omega_i,$$

we obtain $(\phi_n(t), u_n(t))$ from solving the following problem

$$\int_{\Omega_T} \frac{\partial \phi_n}{\partial t}\psi_k\,dx + \int_{\Omega_T} \nabla\cdot(u_n\phi_n)\psi_k\,dx + \int_{\Omega_T} \nabla\mu_n\cdot\nabla\psi_k\,dx = 0, \tag{3.1}$$

$$\int_{\Omega_T} \nabla u_n\cdot\nabla\omega_k + u_n\omega_k\,dx + \int_{\Omega_T} \omega_k\cdot\phi_n\nabla\mu_n\,dx = 0, \tag{3.2}$$

$$\begin{aligned}\frac{1}{d}\int_{\Gamma_T} \frac{\partial \phi_n}{\partial t}\psi_k\,dS &+ \int_{\Omega_T} \nabla\phi_n\cdot\nabla\psi_k + f(\phi_n)\psi_k\,dx \\ &+ \int_{\Gamma_T} \alpha\nabla_\Gamma\phi_n\cdot\nabla_\Gamma\psi_k + \beta\phi_n\psi_k\,dS \\ = &\int_{\Omega_T} \mu_n\psi_k\,dx,\end{aligned} \tag{3.3}$$

$$\mu_n = -\Delta\phi_n + P_n f(\phi_n), \tag{3.4}$$

$$\int_\Omega \phi_n(0)\psi_k\,dx = \int_\Omega \phi_0\psi_k\,dx,\ k = 1,\cdots,n, \tag{3.5}$$

$$\int_\Gamma \phi_n(0)\psi_k\,dS = \int_\Gamma \theta_0\psi_k\,dS,\ k = 1,\cdots,n. \tag{3.6}$$

Since f is continuous, using the Peano theorem, we obtain the local (in time) existence of $(\phi_n(t), u_n(t))$. Next, we will establish some a priori estimates for $(\phi_n(t), u_n(t))$. We have

$$\begin{aligned}&\frac{1}{d}\|\frac{\partial\phi_n(t)}{\partial t}\|^2_{L^2(\Gamma)} + \frac{d}{dt}\left(\frac{1}{2}\|\phi_n\|^2_{H^1(\bar\Omega,d\nu)} + \int_\Omega F(\phi_n)\,dx\right) + \|\nabla\mu_n\|^2_{L^2(\Omega)} \\ =& -\int_\Omega \nabla\cdot(u_n\phi_n)\mu_n, \\ &\|u_n\|^2_{H^1(\Omega)} = -\int_\Omega (u_n\phi_n)\cdot\nabla\mu_n, \\ &\frac{1}{2d}\frac{d}{dt}\|\phi_n(t)\|^2_{L^2(\Gamma)} + \|\phi_n\|^2_{H^1(\bar\Omega,d\nu)} + \int_\Omega f(\phi_n)\phi_n\,dx = \int_\Omega \mu_n\phi_n\,dx,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{d}\|\frac{\partial\phi_n(t)}{\partial t}\|^2_{L^2(\Gamma)} + \frac{d}{dt}\left(\frac{1}{2}\|\phi_n\|^2_{H^1(\bar\Omega,d\nu)} + \frac{1}{2d}\|\phi_n(t)\|^2_{L^2(\Gamma)} + \int_\Omega F(\phi_n)\,dx\right) \\ &+ \|\nabla\mu_n\|^2_{L^2(\Omega)} + \|u_n\|^2_{H^1(\Omega)} + \|\phi_n\|^2_{H^1(\bar\Omega,d\nu)} + \int_\Omega f(\phi_n)\phi_n\,dx \\ =& \int_\Omega \mu_n\phi_n\,dx \\ \le& \|\mu_n - m\mu_n\|_{L^2(\Omega)}\|\phi_n\|_{L^2(\Omega)} + |\Omega| m\mu_n m\phi_n \\ \le& C\|\nabla\mu_n\|_{L^2(\Omega)}\|\phi_n\|_{L^2(\Omega)} + |\Omega| m\mu_n m\phi_0.\end{aligned} \tag{3.7}$$

Thanks to (3.4), we obtain

$$\int_\Omega \mu_n\,dx \le \frac{1}{d}\|\frac{\partial\phi_n(t)}{\partial t}\|_{L^1(\Gamma)} + \beta\|\phi_n(t)\|_{L^1(\Gamma)} + \|\phi_n(t)\|^3_{L^3(\Omega)} - |\Omega| m\phi_0, \tag{3.8}$$

which implies that

$$|\Omega| m\mu_n m\phi_0 \le C\|\frac{\partial\phi_n(t)}{\partial t}\|_{L^2(\Gamma)} + C\|\phi_n(t)\|_{L^2(\Gamma)} + C\|\phi_n(t)\|^3_{L^3(\Omega)}. \tag{3.9}$$

Note that

$$f(s)s \geq 2F(s), \tag{3.10}$$

$$s^4 \leq 8F(s) + 4. \tag{3.11}$$

Combining (3.8)-(3.11) with Young inequality, we find

$$\begin{aligned}(3.12)\quad &\frac{1}{d}\|\frac{\partial\phi_n(t)}{\partial t}\|^2_{L^2(\Gamma)} + \frac{d}{dt}\left(\|\phi_n\|^2_{H^1(\bar{\Omega},d\nu)} + \frac{1}{d}\|\phi_n(t)\|^2_{L^2(\Gamma)} + \int_\Omega 2F(\phi_n)\,dx\right)\\ &+ \|\nabla\mu_n\|^2_{L^2(\Omega)} + 2\|u_n\|^2_{H^1(\Omega)} + \|\phi_n\|^2_{H^1(\bar{\Omega},d\nu)} + \int_\Omega 3F(\phi_n)\,dx\\ \leq& C.\end{aligned}$$

From Sobolev trace embedding theorem, Young inequality and the classical Gronwall inequality, we deduce that there exist two positive constants δ and ρ_1 such that

$$\begin{aligned}(3.13)\quad &\|\phi_n\|^2_{H^1(\bar{\Omega},d\nu)} + \frac{1}{d}\|\phi_n(t)\|^2_{L^2(\Gamma)} + \int_\Omega 2F(\phi_n)\,dx\\ \leq& C\left(\|\phi_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{H^1(\Gamma)} + \int_\Omega 2F(\phi_0)\,dx\right)e^{-\delta t} + \rho_1.\end{aligned}$$

From Young inequality and Hölder inequality, we infer that there exist two positive constants C_1, C_2 such that

$$\begin{aligned}(3.14)\quad \|\phi\|^2_{H^1(\bar{\Omega},d\nu)} - C_1 \leq& \|\phi\|^2_{H^1(\bar{\Omega},d\nu)} + \frac{1}{d}\|\phi(t)\|^2_{L^2(\Gamma)} + \int_\Omega 2F(\phi)\,dx\\ \leq& C_2\left(\|\phi\|^4_{H^1(\Omega)} + \|\phi\|^2_{H^1(\Gamma)} + 1\right).\end{aligned}$$

By virtue of (3.13)-(3.14), we obtain

$$\|\phi_n\|^2_{H^1(\bar{\Omega},d\nu)} \leq C_3\left(\|\phi_0\|^4_{H^1(\Omega)} + \|\theta_0\|^2_{H^1(\Gamma)} + 1\right)e^{-\delta t} + \rho_1 + C_1. \tag{3.15}$$

Integrating (3.12) from 0 to t, we obtain

$$\begin{aligned}(3.16)\quad &\frac{1}{d}\int_0^t \|\frac{\partial\phi_n(s)}{\partial t}\|^2_{L^2(\Gamma)}\,ds + \int_0^t \|\nabla\mu_n(s)\|^2_{L^2(\Omega)}\,ds + 2\int_0^t \|u_n(s)\|^2_{H^1(\Omega)}\,ds\\ \leq& C(1+T) + C_4\left(\|\phi_0\|^4_{H^1(\Omega)} + \|\theta_0\|^2_{H^1(\Gamma)} + 1\right)\end{aligned}$$

for any $t \in (0, T]$.

Due to (3.15)-(3.16), we find

$\{\phi_n\}_{n=1}^\infty$ *is uniformly bounded in* $L^\infty(0,T;H^1(\bar{\Omega},d\nu))$,
$\{\mu_n\}_{n=1}^\infty$ *is uniformly bounded in* $L^2(0,T;H^1(\Omega))$,
$\{u_n\}_{n=1}^\infty$ *is uniformly bounded in* $L^2(0,T;V)$,
$\{\frac{\partial\phi_n(t)}{\partial t}\}_{n=1}^\infty$ *is uniformly bounded in* $L^2(0,T;L^2(\Gamma))$.

Therefore, one can extract subsequences $\{\phi_{n_j}\}_{j=1}^{\infty}$, $\{u_{n_j}\}_{j=1}^{\infty}$, $\{\mu_{n_j}\}_{j=1}^{\infty}$, $\{\frac{\partial \phi_{n_j}(t)}{\partial t}\}_{j=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$, $\{\frac{\partial \phi_n(t)}{\partial t}\}_{n=1}^{\infty}$, respectively, such that

$$\begin{aligned}
&\phi_{n_j} \rightharpoonup \phi \textit{ weakly star in } L^\infty(0,T;H^1(\bar{\Omega},d\nu)),\\
&u_{n_j} \rightharpoonup u \textit{ weakly in } L^2(0,T;V),\\
&\mu_{n_j} \rightharpoonup \chi \textit{ weakly in } L^2(0,T;H^1(\Omega)),\\
&\frac{\partial \phi_{n_j}(t)}{\partial t} \rightharpoonup \frac{\partial \phi(t)}{\partial t} \textit{ weakly in } L^2(0,T;L^2(\Gamma)).
\end{aligned}$$

From (3.15), we obtain

$$\{f(\phi_n)\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^2(0,T;L^2(\Omega)). \tag{3.17}$$

We infer from Lemma 2.3 and (3.16)-(3.17) that

$$\{\phi_n\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^2(0,T;H^2(\bar{\Omega},d\nu)), \tag{3.18}$$

which implies one can extract a subsequence $\{\phi_{n_j}\}_{j=1}^{\infty}$ of $\{\phi_n\}_{n=1}^{\infty}$ such that

$$\phi_{n_j} \rightharpoonup \phi \textit{ weakly in } L^2(0,T;H^2(\bar{\Omega},d\nu)).$$

For any $\psi \in H^1(\bar{\Omega},d\nu)$, set $\psi_n = P_n\psi$, we have

$$\begin{aligned}
|\int_\Omega \frac{\partial \phi_n}{\partial t}\psi_n\,dx| \le& \int_\Omega |u_n\phi_n||\nabla\psi_n|\,dx + \int_\Omega |\nabla\mu_n||\nabla\psi_n|\,dx\\
\le& \|u_n\|_{L^3(\Omega)}\|\phi_n\|_{L^6(\Omega)}\|\nabla\psi_n\|_{L^2(\Omega)} + \|\nabla\mu_n\|_{L^2(\Omega)}\|\nabla\psi_n\|_{L^2(\Omega)},
\end{aligned}$$

which implies that

$$\{\frac{\partial \phi_n}{\partial t}\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^2(0,T;(H^1(\bar{\Omega},d\nu))^*).$$

Therefore, we can extract a subsequence such that

$$\frac{\partial \phi_{n_j}}{\partial t} \rightharpoonup \frac{\partial \phi}{\partial t} \textit{ weakly in } L^2(0,T;(H^1(\bar{\Omega},d\nu))^*).$$

By virtue of the Aubin-Lions compactness theorem, we can extract a further subsequence (still denote by $\{\phi_{n_j}\}_{j=1}^{\infty}$) such that additionally

$$\phi_{n_j} \longrightarrow \phi \textit{ strongly in } L^2(0,T;H^1(\bar{\Omega},d\nu)). \tag{3.19}$$

From (3.17), (3.19) and Lemma 2.2, we obtain

$$f(\phi_{n_j}) \rightharpoonup f(\phi) \textit{ weakly in } L^2(0,T;L^2(\Omega)). \tag{3.20}$$

Hence, we have

$$\chi = -\Delta\phi + f(\phi) = \mu.$$

Thanks to

$$\int_{\Omega_T} \nabla\cdot(u_n\phi_n - u\phi)\psi\,dx = -\int_{\Omega_T} \phi(u_n-u)\cdot\nabla\psi\,dx - \int_{\Omega_T}(\phi_n-\phi)u_n\cdot\nabla\psi\,dx$$

for any $\psi \in H^1(\bar{\Omega},d\nu)$ and

$$\int_{\Omega_T} v\cdot(\phi_n\nabla\mu_n - \phi\nabla\mu)\,dx = \int_{\Omega_T} v\cdot\nabla\mu_n(\phi_n-\phi)\,dx + \int_{\Omega_T} v\phi\cdot(\nabla\mu_n-\nabla\mu)\,dx$$

for any $v \in V$, which imply

$$\nabla \cdot (u_n\phi_n) \rightharpoonup \nabla \cdot (u\phi) \textit{ weakly in } L^2(0,T;(H^1(\bar{\Omega},d\nu))^*),$$
$$\phi_n\nabla\mu_n \rightharpoonup \phi\nabla\mu \textit{ weakly in } L^2(0,T;V^*).$$

Therefore, a weak solution (u,ϕ) for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) has been proved and we obtain

$$\phi(t) \in \mathcal{C}(\mathbb{R}^+;H^1(\bar{\Omega},d\nu))$$

from lemma 2.4.

Finally, we prove the uniqueness and the continuous dependence on the initial data of the solutions. Let (u_1,ϕ_1,p_1), (u_2,ϕ_2,p_2) be two solutions for the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) with the initial data $(\phi_{1_0},\theta_{1_0})$, $(\phi_{2_0},\theta_{2_0})$, respectively, and $m\phi_{1_0} = m\phi_{2_0}$. Let $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$, $p = p_1 - p_2$, then (u,ϕ,p) satisfies the following equations

$$\frac{\partial\phi}{\partial t} + u\cdot\nabla\phi_1 + u_2\cdot\nabla\phi = \Delta\mu,\ \ (x,t)\in\Omega\times\mathbb{R}^+, \tag{3.21}$$

$$\mu = \mu_1 - \mu_2 = -\Delta\phi + f(\phi_1) - f(\phi_2),\ \ (x,t)\in\Omega\times\mathbb{R}^+, \tag{3.22}$$

$$-\Delta u + u = -\nabla p - \phi_1\nabla\mu - \phi\nabla\mu_2,\ \ (x,t)\in\Omega\times\mathbb{R}^+, \tag{3.23}$$

$$\nabla\cdot u = 0,\ \ (x,t)\in\Omega\times\mathbb{R}^+. \tag{3.24}$$

Equation (3.21)-(3.24) is subject to the following boundary conditions

$$u(x,t) = 0,\ \ (x,t)\in\Gamma\times\mathbb{R}^+, \tag{3.25}$$

$$\frac{\partial\mu}{\partial\vec{n}} = 0,\ \ (x,t)\in\Gamma\times\mathbb{R}^+, \tag{3.26}$$

$$\frac{1}{d}\frac{\partial\phi}{\partial t} = \alpha\Delta_\Gamma\phi - \frac{\partial\phi}{\partial\vec{n}} - \beta\phi,\ \ (x,t)\in\Gamma\times\mathbb{R}^+ \tag{3.27}$$

and initial conditions

$$\phi(x,0) = \phi_{1_0} - \phi_{2_0},\ \ x\in\Omega, \tag{3.28}$$

$$\phi(x,0) = \theta_{1_0} - \theta_{1_0},\ \ x\in\Gamma. \tag{3.29}$$

Taking the inner product of the equation (3.21) with $-\Delta\phi$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2_{H^1(\bar{\Omega},d\nu)} + \frac{1}{d}\|\phi_t(t)\|^2_{L^2(\Gamma)} \\ &+ \int_\Omega (u\phi_1)\cdot\nabla\Delta\phi + (u_2\phi)\cdot\nabla\Delta\phi + |\nabla\Delta\phi|^2\,dx \\ = &\int_\Omega \nabla(f(\phi_1) - f(\phi_2))\cdot\nabla\Delta\phi\,dx. \end{aligned} \tag{3.30}$$

Multiplying (3.23) by u and integrating by parts, we find

$$\begin{aligned} \|u\|^2_{H^1(\Omega)} &= -\int_\Omega (u\phi_1)\cdot\nabla\mu + (u\phi)\cdot\nabla\mu_2\,dx \\ &= -\int_\Omega (u\phi)\cdot\nabla\mu_2\,dx + \int_\Omega (u\phi_1)\cdot\nabla\Delta\phi\,dx \\ &\quad - \int_\Omega (u\phi_1)\cdot\nabla(f(\phi_1) - f(\phi_2))\,dx. \end{aligned} \tag{3.31}$$

Combining (3.30) with (3.31), we obtain

$$\begin{aligned}
(3.32)\qquad & \frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2_{H^1(\bar{\Omega},d\nu)} + \frac{1}{d}\|\phi_t(t)\|^2_{L^2(\Gamma)} + \int_\Omega |\nabla\Delta\phi|^2\,dx + \|u\|^2_{H^1(\Omega)} \\
&= \int_\Omega \nabla(f(\phi_1)-f(\phi_2))\cdot\nabla\Delta\phi\,dx - \int_\Omega (u\phi)\cdot\nabla\mu_2\,dx - \int_\Omega (u_2\phi)\cdot\nabla\Delta\phi\,dx \\
&\quad - \int_\Omega (u\phi_1)\cdot\nabla(f(\phi_1)-f(\phi_2))\,dx \\
&\le \|\nabla(f(\phi_1)-f(\phi_2))\|_{L^2(\Omega)}\|\nabla\Delta\phi\|_{L^2(\Omega)} \\
&\quad + \|u\|_{L^3(\Omega)}\|\phi_1\|_{L^6(\Omega)}\|\nabla(f(\phi_1)-f(\phi_2))\|_{L^2(\Omega)} \\
&\quad + \|u\|_{L^3(\Omega)}\|\phi\|_{L^6(\Omega)}\|\nabla\mu_2\|_{L^2(\Omega)} + \|u_2\|_{L^3(\Omega)}\|\phi\|_{L^6(\Omega)}\|\nabla\Delta\phi\|_{L^2(\Omega)}.
\end{aligned}$$

Due to

$$\begin{aligned}
(3.33)\qquad & \|\nabla(f(\phi_1)-f(\phi_2))\|_{L^2(\Omega)} \\
&\le \|f'(\phi_1)\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} + 3\|\phi_1+\phi_2\|_{L^6(\Omega)}\|\phi\|_{L^6(\Omega)}\|\nabla\phi_2\|_{L^6(\Omega)},
\end{aligned}$$

we infer from (3.32)-(3.33) and Young inequality that

$$(3.34)\qquad \frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2_{H^1(\bar{\Omega},d\nu)} + \|u\|^2_{H^1(\Omega)} \le \mathbb{L}(t)\|\phi(t)\|^2_{H^1(\bar{\Omega},d\nu)},$$

where

$$\begin{aligned}
(3.35)\qquad \mathbb{L}(t) =& C(1+\|\phi_1\|^2_{L^6(\Omega)})(\|f'(\phi_1)\|^2_{L^\infty(\Omega)} + \|\phi_1+\phi_2\|^2_{L^6(\Omega)}\|\nabla\phi_2\|^2_{L^6(\Omega)}) \\
&+ C(\|u_2\|^2_{L^3(\Omega)} + \|\nabla\mu_2\|^2_{L^2(\Omega)}).
\end{aligned}$$

Thanks to

$$\begin{aligned}
(3.36)\qquad \|f'(\phi_1)\|^2_{L^\infty(\Omega)} =& \|3\phi_1^2-1\|^2_{L^\infty(\Omega)} \\
\le& C(\|\phi_1\|^4_{L^\infty(\Omega)}+1) \\
\le& C(\|\phi_1\|^2_{H^1(\bar{\Omega},d\nu)}\|\phi_1\|^2_{H^2(\bar{\Omega},d\nu)}+1),
\end{aligned}$$

we conclude from (3.15)-(3.16), (3.18), (3.35)-(3.36) that

$$\int_0^T \mathbb{L}(s)\,ds = \mathcal{M}(T) < \infty.$$

From the classical Gronwall inequality, we obtain

$$\begin{aligned}
&\|\phi(t)\|^2_{H^1(\bar{\Omega},d\nu)} + \int_0^t \|u(s)\|^2_{H^1(\Omega)}\,ds \\
&\le \left(\|\phi_{1_0}-\phi_{2_0}\|^2_{H^1(\Omega)} + \|\theta_{1_0}-\theta_{2_0}\|^2_{H^1(\Gamma)}\right)e^{\mathcal{M}(T)},
\end{aligned}$$

Therefore, $(u_1(x,t),\phi_1(x,t)) = (u_2(x,t),\phi_2(x,t))$ a.e. in $\overline{\Omega_T}$, if $\phi_{1_0}(x) = \phi_{2_0}(x)$ in Ω and $\theta_{1_0}(x) = \theta_{2_0}(x)$ in Γ, and $(u(x,t),\phi(x,t))$ depends continuously on the initial data (ϕ_0,θ_0) with respect to the norm in $H^1(\bar{\Omega},d\nu)$. This is the end of the proof. □

For every fixed $I\in\mathbb{R}$, let $V_I = \{\phi\in H^1(\bar{\Omega},d\nu) : m\phi = I\}$, by Theorem 3.1, we can define the operator semigroup $\{S_I(t)\}_{t\ge 0}$ in V_I by

$$S_I(t)(\phi_0,\theta_0) = \phi(t) = \phi(t;(\phi_0,\theta_0))$$

for all $t \geq 0$, which is (V_I, V_I)-continuous, where $\phi(t)$ is the solution of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) with

$$\phi(x,0) = (\phi_0, \theta_0) \in H^1(\bar{\Omega}, d\nu).$$

4. The existence of a global attractor

In this section, we prove the existence of a global attractor in V_I for the semigroup $\{S_I(t)\}_{t\geq 0}$ generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9). First of all, we prove the existence of an absorbing set in V_I for the semigroup $\{S_I(t)\}_{t\geq 0}$ generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9).

THEOREM 4.1. *Let $\{S_I(t)\}_{t\geq 0}$ be a semigroup generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions* (1.1)-(1.9). *Then there exists an absorbing set in V_I. That is, there exists a positive constant $\mathcal{R}_1 = \rho_2 + 1$ satisfying for any bounded subset $B \subset V_I$, there is a positive time $T_1 = T_1(B)$ depending on the V_I-norm of B such that*

$$\|\phi(t)\|_{V_I}^2 \leq \mathcal{R}_1^2$$

for any $t \geq T_1$, where ρ_2 is specified in (4.2).

Proof. Repeating the proof of Theorem 3.1, we find

$$\begin{aligned}(4.1)\qquad &\frac{1}{d}\|\frac{\partial\phi(t)}{\partial t}\|_{L^2(\Gamma)}^2 + \frac{d}{dt}\left(\|\phi\|_{H^1(\bar{\Omega},d\nu)}^2 + \frac{1}{d}\|\phi(t)\|_{L^2(\Gamma)}^2 + \int_\Omega 2F(\phi)\,dx\right)\\ &+ \|\nabla\mu\|_{L^2(\Omega)}^2 + 2\|u\|_{H^1(\Omega)}^2 + \|\phi\|_{H^1(\bar{\Omega},d\nu)}^2 + \int_\Omega 3F(\phi)\,dx\\ \leq& C.\end{aligned}$$

It is similar with the proof of (3.15), we obtain

$$(4.2)\qquad \|\phi\|_{H^1(\bar{\Omega},d\nu)}^2 \leq C_3\left(\|\phi_0\|_{H^1(\Omega)}^4 + \|\theta_0\|_{H^1(\Gamma)}^2 + 1\right)e^{-\delta t} + \rho_2,$$

which implies that there exists a positive constant $\mathcal{R}_1 = \rho_2 + 1$ satisfying for any bounded subset $B \subset V_I$, there is a positive time $T_1 = T_1(B)$ depending on the V_I-norm of B such that

$$\|\phi(t)\|_{V_I}^2 \leq \mathcal{R}_1^2$$

for any $t \geq T_1$.

Integrating (4.1) from t to $t+1$, we obtain

$$\begin{aligned}(4.3)\qquad &\frac{1}{d}\int_t^{t+1}\|\frac{\partial\phi(s)}{\partial t}\|_{L^2(\Gamma)}^2\,ds + \int_t^{t+1}\|\nabla\mu(s)\|_{L^2(\Omega)}^2\,ds + \int_t^{t+1}\|u(s)\|_{H^1(\Omega)}^2\,ds\\ \leq&\rho_3\end{aligned}$$

for any $t \geq T_1$. For brevity, we omit writing out explicitly these bounds here.

□

Next, to prove the asymptotical compactness of the semigroup $\{S_I(t)\}_{t\geq 0}$ generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9), we introduce the following lemma which can be referred to [**14, 27**].

LEMMA 4.2. *Let $X_0 \subset X \subset X_1$ be triple Banach spaces such that X_0, X_1 are reflexive and $X_0 \subset X$ is compact. For any $0 < T < \infty$, define*

$$Y := \left\{ u : u \in L^2(0,T;X_0), \frac{du}{dt} \in L^2(0,T;X_1) \right\}.$$

Then Y is a Banach space equipped with the norm $\|u\|_{L^2(0,T;X_0)} + \|\frac{du}{dt}\|_{L^2(0,T;X_1)}$. Moreover, $Y \subset L^2(0,T;X)$ is compact.

Finally, we prove the asymptotical compactness of the semigroup $\{S_I(t)\}_{t\geq 0}$ generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9).

THEOREM 4.3. *The semigroup $\{S_I(t)\}_{t\geq 0}$ generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions* (1.1)-(1.9) *is asymptotically compact in $H^1(\bar{\Omega}, d\nu)$.*

Proof. Thanks to the existence of an absorbing set for the semigroup $\{S_I(t)\}_{t\geq 0}$ generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9), it is sufficient to consider a bounded subset B of $H^1(\bar{\Omega}, d\nu)$ to prove the compactness of the semigroup $\{S_I(t)\}_{t\geq 0}$. For any fixed $T > 0$, define the set C_T as a subset of the function space $L^2(0,T;H^1(\bar{\Omega}, d\nu))$:

$$C_T := \{\phi : (\phi_0, \theta_0) \in B, \phi(t) = S_I(t)(\phi_0, \theta_0), t \in [0,T]\}.$$

Due to the compactness of $H^2(\bar{\Omega}, d\nu) \subset H^1(\bar{\Omega}, d\nu)$, if $(\phi_0, \theta_0) \in B$, then it has been shown in the previous section that for any fixed $T > 0$, the weak solution ϕ of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) satisfies

$$\phi \in L^2(0,T;H^2(\bar{\Omega}, d\nu)),$$
$$\frac{\partial \phi}{\partial t} \in L^2(0,T;(H^1(\bar{\Omega}, d\nu))^*).$$

By Lemma 4.2 with

$$X_0 = H^2(\bar{\Omega}, d\nu), X = H^1(\bar{\Omega}, d\nu), X_1 = (H^1(\bar{\Omega}, d\nu))^*,$$

we know that C_T is compact in $L^2(0,T;H^1(\bar{\Omega}, d\nu))$.

Taking any bounded sequence $\{(\phi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty} \subset B$, we infer from $\{\phi_n\}_{n=1}^{\infty} \subset C_T$ that there is a function $\phi \in L^2(0,T;H^1(\bar{\Omega}, d\nu))$ and a subsequence of

$$\{S_I(t)(\phi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty},$$

still denoted by $\{S_I(t)(\phi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty}$ for simplicity of notation, such that

$$\lim_{n\to\infty} \int_0^T \|S_I(s)(\phi_{0,n}, \theta_{0,n}) - \phi(s)\|^2_{H^1(\bar{\Omega}, d\nu)} ds = 0, \tag{4.4}$$

which implies that there exists a subsequence of $\{S_I(t)(\phi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty}$, still denoted by $\{S_I(t)(\phi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty}$ for simplicity of notation, such that

$$\lim_{n\to\infty} \|S_I(t)(\phi_{0,n}, \theta_{0,n}) - \phi(t)\|^2_{H^1(\bar{\Omega}, d\nu)} = 0, \quad a.e. t \in (0,T). \tag{4.5}$$

Therefore, for any fixed $t \in (0,T)$, we can select a $t_0 \in (0,t)$ such that

$$\lim_{n\to\infty} \|S_I(t_0)(\phi_{0,n}, \theta_{0,n}) - \phi(t_0)\|^2_{H^1(\bar{\Omega}, d\nu)} = 0.$$

Then, it follows from the continuity of the map $S_I(t)$ in $H^1(\bar{\Omega}, d\nu)$ that

$$\begin{aligned} S_I(t)(\phi_{0,n},\theta_{0,n}) =& S_I(t-t_0)S_I(t_0)(\phi_{0,n},\theta_{0,n}) \\ &\to S_I(t-t_0)\phi(t_0), \ \ in\ H^1(\bar{\Omega}, d\nu). \end{aligned}$$

That is, for any $t > 0$, $\{S_I(t)(\phi_{0,n},\theta_{0,n})\}_{n=1}^{\infty}$ contains a subsequence which is convergence in $H^1(\bar{\Omega}, d\nu)$. Therefore, we conclude that the semigroup $\{S_I(t)\}_{t\geq 0}$ associated with the solutions of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions (1.1)-(1.9) is asymptotically compact in $H^1(\bar{\Omega}, d\nu)$.

□

From Theorem 4.1 and Theorem 4.3, we immediately obtain the following result.

THEOREM 4.4. *The semigroup* $\{S_I(t)\}_{t\geq 0}$ *generated by the Cahn-Hilliard-Brinkman system with dynamic boundary conditions* (1.1)-(1.9) *possesses a global attractor in* V_I.

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