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Stability of solutions to nonlinear wave equations with switching time delay

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Abstract. In this paper we study well–posedness and asymptotic stability for a class of nonlinear second-order evolution equations with intermittent delay damping. More precisely, a delay feedback and an undelayed one act alternately in time. We show that, under suitable conditions on the feedback operators, asymptotic stability results are available. Concrete examples included in our setting are illustrated. We give also stability results for an abstract model with alternate positive–negative damping, without delay.

Contents

1. Introduction

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|$, and let $A : \mathcal{D}(A) \to H$ be a linear self–adjoint coercive operator on H with dense domain. Let $V := \mathcal{D}(A^{\frac{1}{2}})$, the domain of $A^{\frac{1}{2}}$ with norm $\|v\|_V = \|A^{\frac{1}{2}} v\|_H$, be such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V',$$

with dense embeddings. Then, there exists $\lambda_1 > 0$ such that

$$\lambda_1 \|u\|_H^2 \le \|u\|_V^2, \quad \forall\, u \in V. \tag{1.1}$$

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Moreover, let $U_i,\ i = 1, 2$, be real Hilbert spaces with norm and inner product denoted respectively by $\|\cdot\|_{U_i}$ and $\langle\cdot,\cdot\rangle_{U_i}$ and let $B_i(t) \in \mathcal{L}(U_i, H)$, $i = 1, 2$, be time–dependent operators satisfying

$$B_1^*(t)B_2^*(t) = 0, \quad \forall\, t > 0.$$

Let us consider the problem

$$u_{tt}(t) + Au(t) + B_1(t)B_1^*(t)u_t(t) + B_2(t)B_2^*(t)u_t(t-\tau) = f(u), \quad t > 0, \tag{1.2}$$

$$u(0) = u_0 \in V \quad \text{and} \quad u_t(0) = u_1 \in H, \tag{1.3}$$

where the constant $\tau > 0$ is the time delay.

The example we have in mind is, for $p \geq 0$,

$$u_{tt}(x,t) - \Delta u(x,t) + b_1(t)u_t(x,t) + b_2(t)u_t(x,t-\tau) = -|u|^p u,\ \Omega \times (0,+\infty), \tag{1.4}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \tag{1.5}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.6}$$

with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, where Ω is a bounded and smooth domain of $\mathbb{R}^N$ and b_1, b_2 in $L^\infty(0,+\infty)$ are such that

$$b_1(t)b_2(t) = 0, \quad \forall\, t > 0.$$

In this case $H = U_i = L^2(\Omega)$, $B_i^*(\varphi) = \sqrt{b_i}\varphi$ for all $\varphi \in H$ and $i = 1, 2$, $V = H_0^1(\Omega)$ and λ_1 in (1.1) is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, being (1.1) the usual Poincaré's inequality.

Time delay is often present in applications and practical problems and it is by now well–known that even an arbitrarily small delay in the feedback may destabilize a system wich is uniformly exponentially stable in absence of delay. See, e.g., [**6, 7, 21, 27**] where examples of the destabilization effect due to time delay are given.

The idea is then to use a stabilizing feedback in order to contrast the instability due to the presence of a delay term. In [**21, 27**] a standard damping and a delayed one act simultaneously and the stability of the systems is guaranteed if the coefficient of the undelayed damping is bigger than the one of the delay feedback. Then, in [**22, 23**], the authors consider the case of delayed–undelayed feedback acting in alternate time intervals and give sufficient conditions on the feedback operators in order to have stability.

A similar problem has been considered in [**3**] for the one dimensional wave equation but with a completely different strategy. Indeed, in [**3**] stability results are obtained, only for particular values of the time delays related to the length of the domain (cfr. [**12**]), by using the D'Alembert formula.

Here we extend the results of [**22, 23**] to nonlinear models but also we significantly improve some results there proved. Indeed, we are able to remove an assumption (see (2.14) below) on the feedback bounds obtaining more general results also in the linear case.

By using suitable observability inequalities for the model with only the undelayed feedback and through the definition of a suitable energy (see (2.13)), we obtain sufficient conditions ensuring asymptotic stability.

Some concrete examples falling in our abstract setting are also illustrated.

With the same approach we also consider nonlinear second order evolution equations without delay but with positive–negative dampings acting alternately.

This kind of problem was first considered in [**13**] in the linear case and then extended to nonlinear models in [**10**]. In both papers only the case of distributed damping was considered. Here, we meaningfully generalize these results by considering the case of local damping. More precisely, in concrete examples, the positive (stabilizing) damping and the negative (destabilizing) one may be localized in whatever subsets of the domain. The only geometric requirement is, of course, that the positive damping has to be localized in a region satisfying a control geometric property (see [**4**]).

The interest for models with intermittent delayed–undelayed damping or positive –negative dampings is motivated by various applications. For example, the presence of positive–negative damping can be found in aerodynamics: nose wheel shimmy of an airplane is the consequence of a negative damping, which is controlled by a suitable hydraulic shimmy damper which induces a positive damping ([**26**]). Another example of sign–changing damping comes from Quantum Field Theory and Landau instability (see [**14**]) and from mesodynamics with the laser driven pendulum (see[**8**]). Actually, negative damping may appear in every–day–life, for example Gunn diodes, used as source of microwave power, and suspension bridges ([**15**], [**20**], [**18**], [**19**]), which may experience negative damping in a catastrophic way, like Takoma Bridge. Observe also that the recent results given in [**11**] show that dampings with pulsating coefficients are more effective, with respect to the ones with constant coefficients, in order to stabilize second order evolution equations. This is a further motivation for our study.

The paper is organized as follows. In section 2 we introduce our abstract setting and give a well–posedness result. In section 3 we prove the asymptotic stability results. We consider first distributed dampings, then the localized case and finally, for the linear model, we give the results under more explicit conditions. Finally, in section 4 we consider the model without delay and positive–negative dampings.

2. The abstract setting

In order to deal with the well-posedness of (1.2) – (1.3), first we consider the abstract problem

$$u_{tt}(t) + Au(t) + B(t)u_t(t) = f(u)$$

and its associated Cauchy problem

$$u_{tt}(t) + Au(t) + B(t)u_t(t) = f(u), \quad t > 0, \tag{2.1}$$

$$u(0) = u_0 \in V \quad \text{and} \quad u_t(0) = u_1 \in H. \tag{2.2}$$

Here H and V are as before and $B = B_1B_1^* : V \to V'$. We recall the next definition:

Definition 2.1. A function u is a weak solution of (2.1) – (2.2) if for any $T > 0$ we have

$$u \in L^2(0, T; V) \cap H^1(0, T; H) \cap H^2(0, T; V')$$

with $B(t)u_t(t) \in H$ for any t, $\langle Bu_t, u_t\rangle_H \in L^2(0, T)$ and

$$Au \in L^2(0, T; V'), \quad Bu_t \in L^2(0, T; V'), \quad f(u) \in L^2(0, T; H).$$

Moreover, u is such that $u(0) = u_0,\ u_t(0) = u_1$ and

$$u_{tt}(t) + Au(t) + B(t)u_t(t) = f(u) \quad \text{in } L^2(0, T; V').$$

Now, rewrite (2.1) – (2.2) as

$$U_t + LU + C(U) = 0, \tag{2.3}$$

$$U(0) = U^0, \tag{2.4}$$

where $U = (u, u_t)$, $L = \begin{pmatrix} 0 & -I \\ A & B \end{pmatrix}$, $C(U) := \begin{pmatrix} 0 \\ -f(u) \end{pmatrix}$, and $U^0 = (u_0, u_1)$.

On the nonlinear term f we assume

$$\begin{aligned} &f \text{ is locally Lipschitz continuous, i.e.} \\ &\quad \forall\, K > 0 \;\; \exists\, L(K) \;\; \text{such that} \;\; \|f(u) - f(v)\|_H \le L(K)\|u - v\|_V, \end{aligned} \tag{2.5}$$

provided $\|u\|_V, \|v\|_V \le K$;

$$sf(s) \le 0, \quad \forall\, s \in \mathbb{R}, \tag{2.6}$$

which implies

$$F(s) := \int_0^s f(r)dr \le 0, \quad \forall\, s \in \mathbb{R}$$

or

$$sf(s) - F(s) \le 0, \quad \forall\, s \in \mathbb{R}\,. \tag{2.7}$$

As prototype, we can consider the function $f(u) = -|u|^p u$, $p \ge 0$. Clearly f is locally Lipschitz continuous. Moreover, we remark that the sign assumptions on f is quite reasonable and hard to to relax. Indeed, Levin, Park and Serrin in [**16**] and [**17**] proved that the solutions of $u_{tt} - \Delta u + a(x,t)u_t = |u|^p u$ in Ω with $p > 0$ and $a(x,t) \ge 0$ can blow up in finite time.

Observe that, setting $\mathcal{H} := V \times H$, (2.5) implies that $C : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz continuous. Hence, defining

$$D(L) := \{(u,v) \in V \times V : Au + Bv \in H\},$$

we can apply [**5**, Theorem 7.2], obtaining the following existence result:

THEOREM 2.2. *Suppose that L is a maximal monotone mapping, $L0 = 0$ and $U^0 \in D(L)$. Then there exists T_M such that problem (2.3) – (2.4) has a unique strong solution U on the interval $[0, T_M)$, i.e. $U \in W^{1,\infty}(0, T_M; \mathcal{H})$. Furthermore, if we assume only $U^0 \in \mathcal{H}$ we obtain a unique weak solution $U \in C([0, T_M); \mathcal{H})$.*
In both cases we have

$$\lim_{t \to T_M} \|u(t)\|_V = \infty,$$

provided $T_M < \infty$.

Observe that if A is a self-adjoint, positive and coercive operator with dense domain in H and if $B \in \mathcal{L}(V, V')$ is such that $\langle Bv, v\rangle_H \ge 0$ for all $v \in V$, then L is a maximal monotone operator with dense domain in $\mathcal{H}$ (see [**1**], [**2**]). Hence, as a consequence of Theorem 2.2, we have:

COROLLARY 2.3. Assume that A is a self-adjoint, positive and coercive operator with dense domain in H and $B \in \mathcal{L}(V, V')$ is such that $\langle Bv, v\rangle_H \ge 0$ for all $v \in V$. If $(u_0, u_1) \in D(L)$ then there exists T_M such that problem (2.1) – (2.2) has a unique strong solution u on the interval $[0, T_M)$, i.e. $u \in W^{1,\infty}(0, T_M; V)$. Furthermore, if we assume only $(u_0, u_1) \in \mathcal{H}$ we obtain a unique weak solution $(u, u_t) \in C([0, T_M); \mathcal{H})$.
In both cases we have

$$\lim_{t \to T_M} \|u(t)\|_V = \infty,$$

provided $T_M < \infty$.

Now, for any solution of problem (2.1) – (2.2), we consider the energy associated to such a solution:

$$E_S(t) = E_S(u;t) := \frac{1}{2}\Big(\|u(t)\|_V^2 + \|u_t(t)\|_H^2\Big) - \mathcal{F}(u), \tag{2.8}$$

where $\mathcal{F}$ is a real–valued functional such that $\mathcal{F}(0) = 0$ and $\mathcal{F}'(u)(v) = \langle f(u), v\rangle_{V',V}$ for all $u, v \in V$. Of course, in problem (1.4) – (1.6),

$$\mathcal{F}(u) = \int_\Omega F(u)dx,$$

where $F(s) = \int_0^s f(t)dt$, i.e. $F(s) = -\dfrac{|s|^{p+2}}{p+2}$ for the model case. The following existence result holds

Theorem 2.4. *Assume that A is a self–adjoint, positive and coercive operator with dense domain in H, $B \in \mathcal{L}(V, V')$ is such that $\langle Bv, v\rangle_H \geq 0$ for all $v \in V$ and $\mathcal{F} \leq 0$. Moreover, assume that there exists a positive constant C such that $E_S(T) \leq CE_S(0)$ for all $T \in (0, T_M)$. If $(u_0, u_1) \in D(L)$ then problem* (2.1) – (2.2) *has a unique strong solution u on the interval* $[0, \infty)$*. Furthermore, if $(u_0, u_1) \in \mathcal{H}$ we obtain a unique weak solution $(u, u_t) \in C([0, \infty); \mathcal{H})$.*

Proof. Thanks to Corollary 2.3, we know that there exists a unique solution in $[0, T_M)$. Assume, by contradiction, that $T_M < \infty$. Then

$$\lim_{t\to T_M} \|u(t)\|_V = \infty. \tag{2.9}$$

By definition of $E_S(t)$ and since $\mathcal{F} \leq 0$, it follows that

$$\|u(t)\|_V^2 \leq 2E_S(T) \leq 2CE_S(0).$$

Hence (2.9) cannot happen. ∎

Clearly, if (2.6) is satisfied, then $\mathcal{F} \leq 0$. Moreover, observe that in the linear case, i.e. $f \equiv 0$, the existence and uniqueness of a solution in $[0, \infty)$ is guaranteed, for example, by [**5**, Theorem 7.1].

Now, we assume that for all $n \in \mathbb{N}$, there exists $t_n > 0$, with $t_n < t_{n+1}$, such that

$$B_2(t) = 0 \;\forall\; t \in I_{2n} = [t_{2n}, t_{2n+1}),$$
$$B_1(t) = 0 \;\forall\; t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),$$

with $B_1 \in C^1([t_{2n}, t_{2n+1}]; \mathcal{L}(U_1, H))$ and $B_2 \in C^1([t_{2n+1}, t_{2n+2}]; \mathcal{L}(U_2, H))$. We further assume

$$\tau \leq T_{2n}, \quad \forall\, n \in \mathbb{N}, \tag{2.10}$$

where T_n denotes the length of the interval I_n, that is

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}. \tag{2.11}$$

Let W be an Hilbert space such that H is continuously embedded into W, i.e.

$$\|u\|_W^2 \leq C\|u\|_H^2, \quad \forall\, u \in H \text{ with } \; C > 0 \; \text{ independent of } u. \tag{2.12}$$

We assume that, for all $n \in \mathbb{N}$, there exist three positive constants m_{2n}, M_{2n} and M_{2n+1}, with $m_{2n} \le M_{2n}$, such that for all $u \in H$ we have

i) $m_{2n}\|u\|_W^2 \le \|B_1^*(t)u\|_{U_1}^2 \le M_{2n}\|u\|_W^2$ for $t \in I_{2n} = [t_{2n}, t_{2n+1}),\ \forall\, n \in \mathbb{N}$;

ii)$\|B_2^*(t)u\|_{U_2}^2 \le M_{2n+1}\|u\|_W^2$ for $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),\ \forall\, n \in \mathbb{N}$.

Let us introduce the energy functional

(2.13)

$$E(t) = E(u;t) := \frac{1}{2}\Big(\|u(t)\|_V^2 + \|u_t(t)\|_H^2\Big) + \frac{1}{2}\int_{t-\tau}^t \|B_2^*(s+\tau)u_t(s)\|_{U_2}^2 ds - \mathcal{F}(u).$$

Note that (2.13) is the usual energy $E_S(\cdot)$, for wave–type equation in presence of the nonlinearity f, plus an integral term (see [**22**], cfr. also [**21**]) due to the presence of the time delay.

Now, we give the following definition:

Definition 2.5. A solution of problem (1.2) – (1.3) is a function u such that for any $T > 0$

$$u \in L^2(0,T;V) \cap H^1(0,T;H) \cap H^2(0,T;V')$$

with $\|B_1^* u_t\|_{U_1} \in L^2(0,T)$, $\|B_2^* u_t(\cdot - \tau)\|_{U_2} \in L^2(0,T)$ and

$Au \in L^2(0,T;V'),\ B_1^*u_t \in L^2(0,T;V'),\ B_2^*u_t(\cdot{-}\tau) \in L^2(0,T;V'),\ f(u) \in L^2(0,T;H)$.

Moreover, u is such that $u(0) = u_0$, $u_t(0) = u_1$ and

$$u_{tt}(t) + Au(t) + B_1(t)B_1^*(t)u_t(t) + B_2(t)B_2^*(t)u_t(t-\tau) = f(u) \text{ in } L^2(0,T;V').$$

Observe that if $H = U_i = L^2(\Omega)$, $i = 1,2$, and $V = H_0^1(\Omega)$, then the condition $f(u) = -|u|^p u \in L^2(0,T;H)$ is clearly satisfied when $p \ge 0$ if $N = 1,2$ or $0 < p \le \frac{2}{N-2}$ if $N \ge 3$.

Remark 2.6. Our assumptions do not ensure that the energy $E(\cdot)$ is decreasing on the time intervals I_{2n} where only the standard frictional damping acts, i.e. $B_2 \equiv 0$, as of course it happens for the standard energy $E_S(\cdot)$. In order to have a decay estimate for $E(\cdot)$ in the intervals I_{2n}, we should assume, as in [**23**],

$$\inf_{n\in\mathbb{N}} \frac{m_{2n}}{M_{2n+1}} > 0, \tag{2.14}$$

and define $E(\cdot)$ as

$$E(t) = E(u;t) := \frac{1}{2}\Big(\|u(t)\|_V^2 + \|u_t(t)\|_H^2\Big) + \frac{\xi}{2}\int_{t-\tau}^t \|B_2^*(s+\tau)u_t(s)\|_{U_2}^2 ds - \mathcal{F}(u).$$

where ξ is a positive number satisfying

$$\xi < \inf_{n\in\mathbb{N}} \frac{m_{2n}}{M_{2n+1}}.$$

However, here we do not need E decreasing in the time intervals without delay I_{2n}, since in these time intervals we will work with the standard energy $E_S(\cdot)$. Consequently, we do not assume (2.14) to obtain our stability results.

Proposition 2.7. Assume i), ii), (2.6) and (2.10). For any regular solution of problem (1.2) – (1.3), the energy $E(t)$ satisfies

$$E'(t) \le M_{2n+1}\|u_t\|_W^2, \tag{2.15}$$

for $t \in I_{2n+1}$, $n \in \mathbb{N}$.

Proof: Differentiating the energy functional, we have

$$E'(t) = \langle u_t, u\rangle_V + \langle u_{tt}, u_t\rangle_H + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_{U_2}^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_{U_2}^2 - \langle f(u), u_t\rangle_H.$$

Then, from equation (1.2),

$$\begin{aligned} E'(t) &= \langle u_t, u_{tt} + Au - f(u)\rangle_{V,V'} + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_{U_2}^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_{U_2}^2 \\ &= -\langle u_t, B_1(t)B_1^*(t)u_t(t) + B_2(t)B_2^*(t)u_t(t-\tau)\rangle_{V,V'} \\ &\quad + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_{U_2}^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_{U_2}^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} E'(t) = &-\|B_1^*(t)u_t(t)\|_{U_1}^2 - \langle B_2^*(t)u_t, B_2^*(t)u_t(t-\tau)\rangle_{U_2} \\ &+ \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_{U_2}^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_{U_2}^2. \end{aligned}$$

For $t \in I_{2n+1}$, it is $B_1(t) = 0$ and so the previous identity gives

$$E'(t) = -\langle B_2^*(t)u_t, B_2^*(t)u_t(t-\tau)\rangle_{U_2} + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_{U_2}^2 - \frac{1}{2}\|B_2^*(t)u_t(t-\tau)\|_{U_2}^2.$$

By using Young's inequality we have

$$E'(t) \le \frac{1}{2}\|B_2^*(t)u_t(t)\|_{U_2}^2 + \frac{1}{2}\|B_2^*(t+\tau)u_t(t)\|_{U_2}^2.$$

This proves (2.15) using assumption ii) because $t+\tau$ belongs either to I_{2n+1}, or to I_{2n+2} and in the last case $B_2^*(t+\tau) = 0$. ■

Proceeding analogously to [**22**] and using Theorem 2.4 we can prove the following existence result.

THEOREM 2.8. *Under the assumptions of Theorem 2.4, if $(u_0, u_1) \in V \times H$, for any $T > 0$ we obtain a unique weak solution*

$$u \in C([0,T]; V) \cap C^1([0,T]; H).$$

Proof. We can combine analogous lemma in [**9**] with the well–posedness result in [**22**]. We can argue on the interval $[0, t_2)$ which is the union of the first time interval $[0, t_1)$, where the delay term is no present, and the second time interval $[t_1, t_2)$, where on the contrary only the delay feedback B_2 is present. First, on $[0, t_1]$, since $B_2 \equiv 0$, we are in the situation of [**9**]. Thus, for initial data $u_0 \in V$ and $u_1 \in H$, the solution u belongs to $C([0, t_1]; V) \cap C([0, t_1]; H)$. Then, we decompose the second interval $[t_1, t_2)$ into the intervals $(t_1 + l\tau, t_1 + (l+1)\tau)$, for $l = 0, \dots, L$, where L is the first value such that $t_1 + (L+1)\tau \ge t_2$. The last interval is then $(t_1 + L\tau, t_2)$. Now, we look at the interval $(t_1, t_1 + \tau)$. In this time interval problem (1.2) – (1.3) can be rewritten as

$$\begin{aligned} &u_{tt}(t) + Au(t) = g_1(t) + f(u), \quad t \in (t_1, t_1+\tau), \\ &u(t_1+) = u(t_1-) \text{ and } u_t(t_1+) = u_t(t_1-), \end{aligned} \tag{2.16}$$

where $g_1(t) = -B_2(t)B_2^*(t)u_t(t-\tau)$ belongs to $C([t_1, t_1+\tau); H)$ from the first step. Indeed, for $t \in (t_1, t_1+\tau)$, it is $t - \tau \in (0, t_1)$. Then, since $(u(t_1-), u_t(t_1-))$ belongs to $V \times H$, the existence of local solution $u \in C^1([t_1, t_1+\delta]; H) \cap C([t_1, t_1+\delta]; V)$, $\delta \le \tau$, follows from [**24**, Theorems 1.4 and 1.5, Ch. 6]. Now observe that, from (2.15),

$$E(t) \le e^{2M_1\tau}E(t_1), \quad \forall\, t \in [t_1, t_1+\delta],$$

then $\delta = \tau$, namely there exists a solution $u \in C^1([t_1, t_1+\tau]; H) \cap C([t_1, t_1+\tau]; V)$. By iterating this procedure we find $u \in C^1([t_1+\tau, t_1+2\tau]; H) \cap C([t_1+\tau, t_1+2\tau]; V)$ and then on the whole interval (t_1, t_2). ■

3. Stability results

In this section, we give sufficient conditions ensuring stability results in case of distributed/localized damping. More explicit conditions are given in the linear case, improving previous results given in [**23**].

3.1. Distributed damping.

First of all, consider the case $U_1 = W = H$, that is the case of distributed feedback $B_1(t)$. In this case, of course, the constant C in the estimate (2.12) is 1 . The following result holds (see [**9**, Theorem 4.1]; cfr. [**13**]).

THEOREM 3.1. *Assume* i) *and* (2.7). *Then, any solution* u *of* (1.2) – (1.3) *satisfies*

$$E_S(t_{2n+1}) \le \frac{1}{1 + \frac{T_{2n}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2n}} + \frac{3T_{2n}^2}{32 m_{2n}} + \frac{M_{2n} T_{2n}^2}{16 \lambda_1}}} E_S(t_{2n}), \ n \in \mathbb{N}, \tag{3.1}$$

where λ_1 *is the constant in* (1.1).

THEOREM 3.2. *Assume* i), ii), (1.1) *and* (2.6) – (2.10). *If*

$$\sum_{n=0}^{\infty} (2M_{2n+1} T_{2n+1} + \ln \tilde{c}_n) = -\infty, \tag{3.2}$$

where

$$\tilde{c}_n = \frac{1}{1 + \frac{T_{2n}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2n}} + \frac{3T_{2n}^2}{32 m_{2n}} + \frac{M_{2n} T_{2n}^2}{16 \lambda_1}}} + M_{2n+1} T_{2n+1}, \tag{3.3}$$

then system (1.2) – (1.3) *is asymptotically stable, that is any solution* u *of* (1.2) – (1.3) *satisfies* $E_S(t) \to 0$ *as* $t \to +\infty$.

For some comments on (3.2) we refer to the next Remark 3.3.

Proof of Theorem 3.2. Observe that (2.15) implies

$$E'(t) \le 2M_{2n+1} E(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}), \ n \in \mathbb{N}.$$

Then we deduce

$$E(t_{2n+2}) \le e^{2M_{2n+1} T_{2n+1}} E(t_{2n+1}), \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Now, note that

$$E(t_{2n+1}) = E_S(t_{2n+1}) + \frac{1}{2} \int_{t_{2n+1}-\tau}^{t_{2n+1}} \|B_2^*(s+\tau) u_t(s)\|_{U_2}^2 ds,$$

and then, as $|I_{2n}| \geq \tau$, $n \in \mathbb{N}$, and $B_2(t)$ is null on the intervals I_{2n},

$$\begin{aligned} E(t_{2n+1}) &\leq E_S(t_{2n+1}) + \frac{1}{2} M_{2n+1} \int_{t_{2n+1}-\tau}^{\min\{t_{2n+1}, t_{2n+2}-\tau\}} \|u_t(s)\|_H^2 ds \\ &\leq E_S(t_{2n+1}) + M_{2n+1} \int_{t_{2n+1}-\tau}^{\min\{t_{2n+1}, t_{2n+2}-\tau\}} E_S(s) ds \\ &\leq E_S(t_{2n+1}) + M_{2n+1} T_{2n+1} E_S(t_{2n+1}-\tau) \\ &\leq E_S(t_{2n+1}) + M_{2n+1} T_{2n+1} E_S(t_{2n}). \end{aligned} \tag{3.5}$$

Then, from Theorem 3.1 and (3.5) we deduce

$$E(t_{2n+1}) \leq \left(\frac{1}{1 + \frac{T_{2n}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2n}} + \frac{3T_{2n}^2}{32 m_{2n}} + \frac{M_{2n} T_{2n}^2}{16\lambda_1}}} + M_{2n+1} T_{2n+1} \right) E_S(t_{2n}), \tag{3.6}$$

and therefore, by (3.4),

$$\begin{aligned} E_S(t_{2n+2}) &\leq e^{2M_{2n+1}T_{2n+1}} E(t_{2n+1}) \\ &\leq e^{2M_{2n+1}T_{2n+1}} \left(\frac{1}{1 + \frac{T_{2n}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2n}} + \frac{3T_{2n}^2}{32 m_{2n}} + \frac{M_{2n} T_{2n}^2}{16\lambda_1}}} + M_{2n+1} T_{2n+1} \right) E_S(t_{2n}). \end{aligned} \tag{3.7}$$

Since (3.7) holds for any $n \in \mathbb{N}$ we conclude

$$E_S(t_{2n+2}) \leq \Pi_{p=0}^n e^{2M_{2p+1}T_{2p+1}} \left(\frac{1}{1 + \frac{T_{2p}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2p}} + \frac{3T_{2p}^2}{32 m_{2p}} + \frac{M_{2p} T_{2p}^2}{16\lambda_1}}} + M_{2p+1} T_{2p+1} \right) E_S(0). \tag{3.8}$$

Now observe that the standard energy $E_S(\cdot)$ is not decreasing in general. However, it is decreasing for $t \in [t_{2n}, t_{2n+1})$, when only the standard dissipative damping acts and so

$$E_S(t) \leq E_S(t_{2n}), \quad \forall\, t \in [t_{2n}, t_{2n+1}). \tag{3.9}$$

Moreover, for $t \in [t_{2n+1}, t_{2n+2})$, it results

$$E_S(t) \leq E(t) \leq e^{2M_{2n+1}T_{2n+1}} E(t_{2n+1}), \tag{3.10}$$

where in the second inequality we have used (2.15).

Then, by (3.8), (3.9), (3.10) and (3.6), asymptotic stability occurs if (3.2) is satisfied. ■

Remark 3.3. Observe that (3.2) holds true if the following easier conditions are satisfied:

$$\sum_{n=0}^{\infty} M_{2n+1} T_{2n+1} < +\infty \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \ln\left(1+\frac{T_{2n}^3}{30}\frac{1}{\frac{4}{\lambda_1 m_{2n}}+\frac{3T_{2n}^2}{32m_{2n}}+\frac{M_{2n}T_{2n}^2}{16\lambda_1}}\right)=+\infty. \tag{3.12}$$

Indeed, it is easy to see (cfr. [**25**]) that (3.11) and

$$\sum_{n=0}^{\infty} \ln \tilde{c}_n = -\infty \tag{3.13}$$

with $\tilde{c}_n$, $n \in \mathbb{N}$, as in (3.3), imply (3.2). Now it is sufficient to observe that, under assumption (3.11), the conditions (3.12) and (3.13) are equivalent. Indeed if (3.13) holds true then

$$-\ln\left(1+\frac{T_{2n}^3}{30}\frac{1}{\frac{4}{\lambda_1 m_{2n}}+\frac{3T_{2n}^2}{32m_{2n}}+\frac{M_{2n}T_{2n}^2}{16\lambda_1}}\right)$$
$$=\ln\left(\frac{1}{1+\frac{T_{2n}^3}{30}\frac{1}{\frac{4}{\lambda_1 m_{2n}}+\frac{3T_{2n}^2}{32m_{2n}}+\frac{M_{2n}T_{2n}^2}{16\lambda_1}}}\right)<\ln\tilde{c}_n$$

and therefore also (3.12) is satisfied. Assume now that (3.11) and (3.12) are satisfied. Then, by (3.11),

$$M_{2n+1}T_{2n+1}\to 0, \quad n\to\infty. \tag{3.14}$$

If (3.13) does not hold then it has to be

$$\ln\tilde{c}_n \to 0, \quad n\to\infty.$$

But then, by (3.14), it results

$$\tilde{c}_n \sim \frac{1}{1+\frac{T_{2n}^3}{30}\frac{1}{\frac{4}{\lambda_1 m_{2n}}+\frac{3T_{2n}^2}{32m_{2n}}+\frac{M_{2n}T_{2n}^2}{16\lambda_1}}},$$

in contradiction with (3.12).

Remark 3.4. Observe that, under the assumptions of Theorem 3.2, one can prove that also

$$E(t)\to 0 \quad \text{as } t\to+\infty, \tag{3.15}$$

for every solution u of (1.2) – (1.3). Indeed, recall that

$$E(t)=E_S(t)+\frac{1}{2}\int_{t-\tau}^{t}\|B_2^*(s+\tau)u_t(s)\|_{U_2}^2 ds$$

and that we are assuming $T_{2n}\ge\tau$, for all $n\in\mathbb{N}$, and B_2 null in the intervals I_{2n}. Then, if $t\in\bar{I}_{2n}=[t_{2n},t_{2n+1}]$,

$$\begin{aligned} E(t) &\le E_S(t)+\frac{1}{2}\int_{t_{2n+1}-\tau}^{\min\{t,t_{2n+2}-\tau\}}\|B_2^*(s+\tau)u_t(s)\|_{U_2}^2 ds \\ &\le E_S(t)+M_{2n+1}\int_{t_{2n+1}-\tau}^{\min\{t,t_{2n+2}-\tau\}}E_S(s)ds \le E_S(t)+M_{2n+1}T_{2n+1}E_S(t_{2n}); \end{aligned} \tag{3.16}$$

if $t \in I_{2n+1}$, by using (2.15) and (3.16), we have

(3.17)
$$E(t) \le e^{2M_{2n+1}T_{2n+1}} E(t_{2n+1}) \le e^{2M_{2n+1}T_{2n+1}} \left[E_S(t_{2n+1}) + M_{2n+1}T_{2n+1}E_S(t_{2n})\right].$$

Therefore, observing that by (3.2) one has

$$\sup_n M_{2n+1}T_{2n+1} < +\infty,$$

(3.15) follows when Theorem 3.2 applies.

As an example of application of Theorem 3.2 one can consider problem (1.4) – (1.6) assuming

i_w) $0 < m_{2n} \le b_1(t) \le M_{2n}$, $b_2(t) = 0$, for all $t \in I_{2n} = [t_{2n}, t_{2n+1})$ and $b_1 \in C^1(\bar{I}_{2n})$ for all $n \in \mathbb{N}$;

ii_w) $|b_2(t)| \le M_{2n+1}$, $b_1(t) = 0$ for all $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$ and $b_2 \in C^1(\bar{I}_{2n+1})$ for all $n \in \mathbb{N}$.

The previous result can be extended to a more general situation. Indeed, consider the nonlinear wave system

(3.18) $$u_{tt}(t) + Au(t) + B_1(t)B_1^*(t)g(u_t) + B_2(t)B_2^*(t)u_t(t-\tau) = f(u), \quad t > 0,$$

(3.19) $$u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1,$$

with $(u_0, u_1) \in V \times H$. On the functions g and f we make the following assumptions:

$$(A) \qquad \begin{cases} g : \mathbb{R} \longrightarrow \mathbb{R} \text{ is a } C^1 \text{ function with } g(0) = 0, \\ \exists\, B \ge A > 0 \text{ such that } 0 < A \le g'(v) \le B \ \forall\, v \in \mathbb{R}, \\ f \text{ satisfies (2.6) and (2.7).} \end{cases}$$

Moreover, on B_1 we assume, in place of i), that, for all $n \in \mathbb{N}$, there exist positive constants m_{2n}, M_{2n}, with $m_{2n} \le M_{2n}$, such that, for all $u \in H$, we have

i') $m_{2n}\|u\|_W^2 \le \langle B_1^*(t)u, B_1^*(t)g(u)\rangle_{U_1} \le M_{2n}\|u\|_W^2$ for $t \in I_{2n} = [t_{2n}, t_{2n+1})$, $\forall\, n \in \mathbb{N}$.

As prototype, one can think to the problem

(3.20) $$u_{tt}(x,t) - \Delta u(x,t) + b_1(t)g(u_t) + b_2(t)u_t(x,t-\tau) = -|u|^p u, \ \ \Omega \times (0, +\infty),$$

(3.21) $$u(x,t) = 0 \text{ in } \partial\Omega \times (0, +\infty),$$

(3.22) $$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega,$$

where Ω, u_0, u_1, b_1, b_2 and p are as before.

For (3.18) – (3.19), if $B_2(t) = 0$ for all $t \in (0, +\infty)$, Theorem 3.1 becomes (see [**9**, Theorem 5.1])

Theorem 3.5. *Assume* i') *and suppose that also* (A) *holds. Then, any solution* u *of* (3.18) – (3.19) *satisfies*

(3.23) $$E_S(t_{2n+1}) \le \frac{1}{1 + \frac{T_{2n}^3}{30} \frac{1}{\frac{4}{\lambda_1 m_{2n}} + \frac{3T_{2n}^2}{32 m_{2n}} + \frac{M_{2n}T_{2n}^2}{16\lambda_1}}} E_S(t_{2n}), \ \ n \in \mathbb{N}.$$

Therefore, observe that, since $B_1(t) = 0$ for all $t \in I_{2n+1}$, (2.15) still holds. Thus, as Theorem 3.1 implies Theorem 3.2, Theorem 3.5 immediately gives, for global defined solutions, the following fundamental application via (2.15):

THEOREM 3.6. *Assume* i') *and* ii). *Moreover suppose that* (2.10) *and* (A) *are satisfied. If* (3.2) *holds, then system* (3.18) – (3.19) *is asymptotically stable, that is any solution* u *of* (3.18) – (3.19) *satisfies* $E_S(t) \to 0$ *as* $t \to +\infty$.

REMARK 3.7. Observe that problems (1.2) – (1.3) and (3.18) – (3.19) with $B_2(t) \equiv 0$ correspond to the models with on-off damping considered in [**9**]. Hence Theorems 3.2 and 3.6 give stability results also in these situations.

Of course, the abstract setting of the previous theorems let us deal with higher order problems in bounded and smooth domain of $\mathbb{R}^N$. For example, Theorem 3.6 can be applied to the problem

(3.24) $u_{tt}(x,t) + \Delta^{2m}u(x,t) + b_1(t)g(u_t) + b_2(t)u_t(x,t-\tau) = f(u),\ \Omega \times (0,+\infty),$

(3.25) $Cu(x,t) = 0 \in \mathbb{R}^{2m}$ in $\partial\Omega \times (0,+\infty),$

(3.26) $u(x,0) = u_0(x) \in D(\Delta^m) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega,$

where $m \in N$, f, g, b_1, b_2 and p are as before and C is a boundary operator such that *the first eigenvalue of* Δ^{2m} *under the boundary conditions* $Cu(x,t) = 0 \in \mathbb{R}^{2m}$ *in* $\partial\Omega \times (0,+\infty)$ *is strictly positive.* For example, one can consider as C the Dirichlet operator, while the case of Neumann boundary conditions must be excluded since the first eigenvalue is 0.

3.2. Localized damping.

In this section we consider the more general situation $U_1 \neq W$. In practice, for concrete models, the feedback operators B_1 and B_2 may be localized in subregions of Ω.

PROPOSITION 3.8. Assume i), ii), (2.6) and (2.10). For any regular solution of problem (1.2) – (1.3) the energy E_S is decreasing on the intervals I_{2n}, $n \in \mathbb{N}$. In particular,

$$E'_S(t) = -\|B_1^*(t)u_t(t)\|^2_{U_1}. \tag{3.27}$$

Moreover, on the intervals I_{2n+1}, $n \in \mathbb{N}$, the estimate (2.15) holds.

Proof. By differentiating $E_S(\cdot)$, we have

$$E'_S(t) = \langle u_t, u\rangle_V + \langle u_{tt}, u_t\rangle_H - \langle f(u), u_t\rangle_H.$$

Then, recalling that $B_2(t) = 0$ in I_{2n}, from equation (1.2) it follows that

$$E'_S(t) = \langle u_t, u_{tt} + Au - f(u)\rangle_{V,V'} = -\langle u_t, B_1(t)B_1^*(t)u_t(t)\rangle_{V,V'},$$

for all $t \in I_{2n}$. Thus, identity (3.27) holds. ∎

Consider now the system

$$w_{tt}(t) + Aw(t) + B_1(t)B_1^*(t)w_t = f(w), \quad t \in (t_{2n}, t_{2n+1}),\ n \in \mathbb{N}, \tag{3.28}$$

$$w(t_{2n}) = w_0^n \quad \text{and} \quad w_t(t_{2n}) = w_1^n \tag{3.29}$$

with $(w_0^n, w_1^n) \in V \times H$. For our stability result we need that the next observability type inequality holds. Namely we assume that, for every n, there exists a time $\overline{T}_n$ such that

$$T_{2n} > \overline{T}_n, \tag{3.30}$$

and that, for every n and every time T, with $T_{2n} \geq T > \overline{T}_n$, there is a constant d_n, depending on T but independent of (w_0^n, w_1^n), such that

$$E_S(t_{2n}+T) \leq d_n \int_{t_{2n}}^{t_{2n}+T} \|B_1^*(t)w_t(t)\|_{U_1}^2\, dt, \tag{3.31}$$

for every weak solution of problem (3.28) – (3.29) with initial data $(w_0^n, w_1^n) \in V \times H$.

Remark 3.9. The observability inequality above is satisfied for solutions of wave–type equations when the nonlinearity f satisfies some requirements. For instance in [**28**] Zuazua proved (3.31) if f is globally Lipschitz, as a perturbation of the well–known linear case, or also when f satisfies

$$(2+\delta)F(s) \geq sf(s), \tag{3.32}$$

for some $\delta > 0$.

Proposition 3.10. Assume i). Moreover, we assume that there is a sequence $\{\overline{T}_n\}_n$, such that (3.30) is satisfied and the observability estimate (3.31) holds for every $T \in (\overline{T}_n, T_{2n}]$, $\forall\, n \in \mathbb{N}$. Then, for any solution of system (1.2) – (1.3) we have

$$E_S(t_{2n+1}) \leq \hat{d}_n E_S(t_{2n}), \quad \forall\, n \in \mathbb{N}, \tag{3.33}$$

where

$$\hat{d}_n = \frac{d_n}{d_n+1}, \tag{3.34}$$

d_n being the observability constant in (3.31) corresponding to the time T_{2n}.

Proof. To prove (3.33) it suffices to use the estimate (3.27) in (3.31), reminding that $B_2(t) = 0$ on (t_{2n}, t_{2n+1}). Indeed, (3.31) gives

$$E_S(t_{2n+1}) \leq d_n \int_{t_{2n}}^{t_{2n+1}} \|B_1^*(t)u_t(t)\|_{U_1}^2\, dt. \tag{3.35}$$

By integrating (3.27) on the interval $[t_{2n}, t_{2n+1}]$, we have

$$E_S(t_{2n+1}) - E_S(t_{2n}) = -\int_{t_{2n}}^{t_{2n+1}} \|B_1^*(t)u_t(t)\|_{U_1}^2\, dt,$$

and therefore, using (3.35),

$$E_S(t_{2n+1}) - E_S(t_{2n}) \leq -\frac{1}{d_n} E_S(t_{2n+1}).$$

Thus,

$$E_S(t_{2n+1})\Big(\frac{d_n+1}{d_n}\Big) \leq E_S(t_{2n}). \qquad \blacksquare$$

THEOREM 3.11. *Assume hypotheses of Proposition* 3.10, ii), (2.6) *and* (2.10). *If*

$$\sum_{n=0}^{\infty}[2CM_{2n+1}T_{2n+1}+\ln(\hat{d}_n+CM_{2n+1}T_{2n+1})]=-\infty, \tag{3.36}$$

where C *is the constant in the norm embedding* (2.12) , *then system* (1.2) – (1.3) *is asymptotically stable, that is any solution* u *of* (1.2) – (1.3) *satisfies* $E_S(t)\to 0$ *as* $t\to+\infty$.

Proof. The proof is analogous to the one of Theorem 3.2. Simply we use now the inequality (3.33) in place of (3.1) on the intervals I_{2n}. ∎

REMARK 3.12. As in Remark 3.3, one can show that (3.36) is verified if, in particular,

$$\sum_{n=0}^{\infty}M_{2n+1}T_{2n+1}<+\infty \quad \text{and} \quad \sum_{n=0}^{\infty}\ln\hat{d}_n=-\infty. \tag{3.37}$$

Observe also that d_n depends on n since, by hypothesis, B_1 may depend on the time variable. However, if B_1 is independed of t, then by a translation of t_{2n} the constant d_n becomes independent of n. But, if $d_n=d>0$ for all n, then the condition

$$\sum_{n=0}^{\infty}\ln\hat{d}_n=-\infty$$

is clearly satisfied. On the other hand, the first condition in (3.37) depends only on the length of the intervals I_{2n+1} and on the boundedness constant of B_2^* on the same intervals, hence (3.37) can be easily checked.

As an example of model for which this result holds, we can consider

$$\begin{aligned} &u_{tt}(x,t)-\Delta u(x,t)+b_1(t)\chi_\omega u_t(x,t) \\ &\qquad\qquad +b_2(t)\chi_{\tilde\omega}u_t(x,t-\tau)=f(u), \quad \text{in } \Omega\times(0,+\infty) \end{aligned} \tag{3.38}$$

$$u(x,t)=0 \quad \text{on} \quad \partial\Omega\times(0,+\infty), \tag{3.39}$$

$$u(x,0)=u_0(x) \quad \text{and} \quad u_t(x,0)=u_1(x) \quad \text{in} \quad \Omega, \tag{3.40}$$

with initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, b_1,b_2 as before and the nonlinearity f as in [**28**]. Moreover, we assume that the set $\omega\subset\Omega$ satisfies a control geometric property (see [**4**]) and that $\tilde\omega\subset\omega$.

As for the distributed damping, the previous result can be extended to a more general situation. Indeed, consider again the nonlinear wave system (3.18) – (3.19) with $(u_0,u_1)\in V\times H$. On the functions g and f we assume (A).

Proposition 3.8 becomes

PROPOSITION 3.13. Assume i'), ii), (2.6) and (2.10). For any regular solution of problem (3.18) – (3.19) the energy E_S is such that

$$E_S'(t)=-\langle B_1^*(t)u_t(t),B_1^*(t)g(u_t)\rangle_{U_1}, \tag{3.41}$$

for all $t\in I_{2n}$, $n\in\mathbb{N}$. Moreover, on the intervals I_{2n+1}, $n\in\mathbb{N}$, the estimate (2.15) holds.

As before, consider now the system

$$w_{tt}(t) + Aw(t) + B_1(t)B_1^*(t)g(w_t) = f(w), \quad t \in (t_{2n}, t_{2n+1}),\ n \in \mathbb{N}, \tag{3.42}$$

$$w(t_{2n}) = w_0^n \quad \text{and} \quad w_t(t_{2n}) = w_1^n, \tag{3.43}$$

with $(w_0^n, w_1^n) \in V \times H$. For our stability result we need that the next inequality holds. Namely we assume that, for every n, there exists a time $\overline{T}_n$ such that (3.30) holds and that, for every n and every time T, with $T_{2n} \geq T > \overline{T}_n$, there is a constant d_n, depending on T but independent of (w_0^n, w_1^n), such that

$$E_S(t_{2n} + T) \leq d_n E_S(t_{2n}), \tag{3.44}$$

for every weak solution of problem (3.42) – (3.43) with initial data $(w_0^n, w_1^n) \in V \times H$.

The inequality above is satisfied for solutions of wave–type equations when the nonlinearities f and g satisfy some requirements. In [**28**], for example, (3.44) is proved if f is globally Lipschitz or when f satisfies (3.32) for some $\delta > 0$ and g is globally Lipschitz (hence if g is as in (A)) and there exists $c > 0$ such that

$$g(s)s \geq c|s|^2, \quad \forall s \in \mathbb{R}.$$

Theorem 3.11 becomes

Theorem 3.14. *Assume* i'), ii), (2.6) *and* (2.10). *Moreover, we assume that there is a sequence* $\{\overline{T}_n\}_n$, *such that* (3.30) *is satisfied and* (3.44) *holds for every* $T \in (\overline{T}_n, T_{2n}]$, $\forall\, n \in \mathbb{N}$. *If* (3.36) *holds then system* (3.18) – (3.19) *is asymptotically stable, that is any solution* u *of* (3.18) – (3.19) *satisfies* $E_S(t) \to 0$ *as* $t \to +\infty$.

Remark 3.15. One can make the same considerations made in Remark 3.7, obtaining stability results also for the localized on-off damping. These results are then more general than the ones proved in [**9**].

3.3. Localized damping: the linear case.

In the linear case (i.e. $f \equiv 0$) we can improve previous results given in [**23**] by removing the assumption (2.14) on the coefficients. As in [**23**] we can determine more explicitely, in terms of the coefficients T_{2n}, m_{2n}, M_{2n}, the constant $\hat{d}_n$ of Proposition 3.10, for all $n \in \mathbb{N}$.

Consider now the conservative system associated with (1.2) – (1.3)

$$w_{tt}(t) + Aw(t) = 0, \quad t > 0, \tag{3.45}$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1, \tag{3.46}$$

with $(w_0, w_1) \in V \times H$.

To prove stability results we need that a suitable observability inequality holds. Then, we assume that there exists a time $\overline{T} > 0$ such that, for every time $T > \overline{T}$, there is a constant c, depending on T but independent of the initial data, such that

$$E_S(0) \leq c \int_0^T \|w_t(s)\|_W^2 ds, \tag{3.47}$$

for every weak solution of problem (3.45) – (3.46) with initial data $(w_0, w_1) \in V \times H$.

The following result is proved in [**23**]:

Proposition 3.16. Assume i) and $f \equiv 0$. Moreover, we assume that the observability inequality (3.47) holds for every time $T > \overline{T}$ and that, setting $T^* := \inf_n\{T_{2n}\}$,

$$T^* > \overline{T}. \tag{3.48}$$

Then, for any solution of system (1.2) − (1.3) we have

$$E_S(t_{2n+1}) \leq \hat{c}_n E_S(t_{2n}), \quad \forall\, n \in \mathbb{N}, \tag{3.49}$$

where

$$\hat{c}_n = \frac{2c(1+4C^2T_{2n}^2M_{2n}^2)}{m_{2n}+2c(1+4C^2T_{2n}^2M_{2n}^2)}, \tag{3.50}$$

c being the observability constant in (3.47) corresponding to the time T^* and C the constant in the norm embedding (2.12) between W and H.

Combining the previous proposition with estimate (2.15) one can obtain the following theorem.

Theorem 3.17. *Assume hypotheses of Proposition* 3.16, ii), (2.6) *and* (2.10). *If*

$$\sum_{n=0}^{\infty}[2CM_{2n+1}T_{2n+1} + \ln(\hat{c}_n + CM_{2n+1}T_{2n+1})] = -\infty, \tag{3.51}$$

where $\hat{c}_n$ *is as in* (3.50) *and* C *is the constant in the norm embedding* (2.12), *then system* (1.2)−(1.3) *is asymptotically stable, that is for every solution of* (1.2)−(1.3) $E_S(t) \to 0$ *as* $t \to +\infty$.

Proof. The proof is analogous to the one of Theorem 3.2. Simply we use now the inequality (3.49) in place of (3.1) on the intervals I_{2n}. ∎

Remark 3.18. As in Remark 3.3 we can show that (3.51) is verified in particular if

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty, \quad \text{and} \quad \sum_{n=0}^{\infty} \ln \hat{c}_n = -\infty. \tag{3.52}$$

Now, it is easy to see that the second condition of (3.52) is equivalent (see the proof of [**23**, Theorem 3.3] for details) to

$$\sum_{n=0}^{\infty} \frac{m_{2n}}{1+4C^2T_{2n}^2M_{2n}^2} = +\infty. \tag{3.53}$$

which is, together with the first condition of (3.52) on the intervals with delay, the assumption of [**23**, Theorem 3.3]. Actually, as clearly appears from the proof, Theorem 3.3 of [**23**] holds true under the more general condition (3.51). The authors there preferred, for sake of clairness, to formulate the assumption in an easier but less general form.

Remark 3.19. Observe that Theorem 3.17 significantly improve [**23**, Theorem 3.3]. Indeed it allows to obtain the same stability result by removing the assumption (2.14) on the coefficients, which is crucial in the proof of [**23**, Theorem 3.3].

4. Stability result: localized positive–negative damping without delay

In this section we want to generalize the results given in [**10**] to the localized situation. In particular, in order to deal with a positive–negative damping, we consider the problem

$$u_{tt}(t) + Au(t) + B_1(t)B_1^*(t)u_t(t) - B_3(t)B_3^*(t)u_t(t) = f(u), \quad t > 0, \tag{4.1}$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1, \tag{4.2}$$

where $B_i(t) \in \mathcal{L}(U_i, H)$, $i = 1, 3$. Here H and U_i, $i = 1, 3$, are real Hilbert spaces as before. On the time–dependent operators B_i we assume

$$B_1^*(t)B_3^*(t) = 0, \quad \forall\, t > 0$$

and, for all $n \in \mathbb{N}$, there exists $t_n > 0$, with $t_n < t_{n+1}$, such that

$$B_3(t) = 0, \ \forall\, t \in I_{2n} = [t_{2n}, t_{2n+1}),$$
$$B_1(t) = 0, \ \forall\, t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),$$

with $B_1 \in C^1([t_{2n}, t_{2n+1}]; \mathcal{L}(U_1, H))$ and $B_3 \in C([t_{2n+1}, t_{2n+2}]; \mathcal{L}(U_3, H))$. We further assume that there exist two Hilbert spaces W_1, W_3 such that, for $i = 1, 3$,

$$\|u\|_{W_i}^2 \le C_i \|u\|_H^2, \quad \forall\, u \in H \text{ with } C_i > 0 \text{ independent of } u, \tag{4.3}$$

and, for all $n \in \mathbb{N}$, there exist three positive constants m_{2n}, M_{2n} and M_{2n+1}, with $m_{2n} \le M_{2n}$, such that for all $u \in H$ we have

j) $m_{2n}\|u\|_{W_1}^2 \le \|B_1^*(t)u\|_{U_1}^2 \le M_{2n}\|u\|_{W_1}^2$ for $t \in I_{2n} = [t_{2n}, t_{2n+1})$, $\forall\, n \in \mathbb{N}$;
jj) $\|B_3^*(t)u\|_{U_3}^2 \le M_{2n+1}\|u\|_{W_3}^2$ for $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, $\forall\, n \in \mathbb{N}$.

The energy functional $E(t)$ coincide in this case with $E_S(t)$ and the next result holds.

PROPOSITION 4.1. Assume (2.6). Then, for any regular solution of problem (4.1) – (4.2) the energy is decreasing on the intervals I_{2n} and increasing on I_{2n+1}, $n \in \mathbb{N}$. In particular,

$$E_S'(t) = -\|B_1^*(t)u_t(t)\|_{U_1}^2, \quad \forall\, t \in I_{2n} \tag{4.4}$$

and

$$E_S'(t) = \|B_3^*(t)u_t(t)\|_{U_3}^2, \quad \forall\, t \in I_{2n+1} \tag{4.5}$$

Proof. Proceeding as in Proposition 3.8, one has

$$E_S'(t) = \langle u_t, u\rangle_V + \langle u_{tt}, u_t\rangle_H - \langle f(u), u_t\rangle_H.$$

Then, recalling that $B_3(t) = 0$ in I_{2n} and $B_1(t) = 0$ in I_{2n+1}, from equation (4.1) it follows that

$$E_S'(t) = \langle u_t, u_{tt} + Au - f(u)\rangle_{V,V'} = -\langle u_t, B_1(t)B_1^*(t)u_t(t)\rangle_{V,V'}$$

for all $t \in I_{2n}$ and

$$E_S'(t) = \langle u_t, u_{tt} + Au - f(u)\rangle_{V,V'} = \langle u_t, B_3(t)B_3^*(t)u_t(t)\rangle_{V,V'}$$

for all $t \in I_{2n+1}$. Thus, identities (4.4) and (4.5) hold. ■

As in the previous section we consider the system (3.28) – (3.29) for which we assume that the observability inequality (3.31) holds.

Setting again $T_n := t_{n+1} - t_n$, we have:

PROPOSITION 4.2. Assume i) and suppose that there is a sequence $\{\overline{T}_n\}_n$, such that (3.30) and (3.31) hold for every $T \in (\overline{T}_n, T_{2n}]$, $\forall\, n \in \mathbb{N}$. Then, for any solution of system (1.2) – (1.3) we have

$$E_S(t_{2n+1}) \leq \hat{d}_n E_S(t_{2n}), \quad \forall\, n \in \mathbb{N}, \tag{4.6}$$

where

$$\hat{d}_n = \frac{d_n}{d_n + 1}, \tag{4.7}$$

d_n being the observability constant in (3.31) corresponding to the time T_{2n}.

The proof of the previous Proposition is similar to the one of Proposition 3.10, so we omit it.

Using Proposition 4.2, one can give an asymptotic stability result.

THEOREM 4.3. *Assume hypotheses of Proposition* 4.2, ii), (2.6) *and* (2.10). *If*

$$\sum_{n=0}^{\infty} [2C_3 M_{2n+1} T_{2n+1} + \ln \hat{d}_n] = -\infty, \tag{4.8}$$

where C_3 is the constant in the norm embedding (4.3) *between W_3 and H, then system* (4.1) – (4.2) *is asymptotically stable, that is any solution u of* (4.1) – (4.2) *satisfies $E_S(t) \to 0$ as $t \to +\infty$.*

Proof. From (4.5) and (4.3) we obtain

$$E_S(t) \leq e^{2C_3 M_{2n+1} T_{2n+1}} E_S(t_{2n+1}), \quad \forall\, t \in I_{2n+1} = [t_{2n+2}, t_{2n+1}].$$

Therefore, by using (4.6), we have

$$E_S(t_{2n+2}) \leq e^{2C_3 M_{2n+1} T_{2n+1}} \hat{d}_n E_S(t_{2n}).$$

Now, we can conclude proceeding as in the proof of Theorem 3.2. ∎

REMARK 4.4. In particular (4.8) is satisfied if

$$\sum_{n=0}^{\infty} M_{2n+1} T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln \hat{d}_n = -\infty.$$

As an example of model for which the previous result holds, we can consider

$$\begin{aligned} &u_{tt}(x,t) - \Delta u(x,t) + b_1(t)\chi_\omega u_t(x,t) - b_3(t)\chi_{\tilde{\omega}} u_t(x,t) = f(u) \quad \text{in } \Omega \times (0,+\infty),\\ &u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty),\\ &u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \end{aligned}$$

with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, b_1, b_3 in $L^\infty(0,+\infty)$ such that

$$b_1(t) b_3(t) = 0, \quad \forall\, t > 0,$$

and the nonlinearity f as in [**28**]. Moreover, we assume that the set $\omega \subset \Omega$ satisfies a control geometric property.

On the coefficients b_1 and b_3 we assume

$j_w)$ $0 < m_{2n} \leq b_1(t) \leq M_{2n}$, $b_3(t) = 0$ for all $t \in I_{2n} = [t_{2n}, t_{2n+1})$ and $b_1 \in C^1(\bar{I}_{2n})$ for all $n \in \mathbb{N}$;

$jj_w)$ $|b_3(t)| \leq M_{2n+1}$, $b_1(t) = 0$ for all $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$ and $b_3 \in C(\bar{I}_{2n+1})$ for all $n \in \mathbb{N}$.

We emphasize that in the case without delay, since we deal only with the standard energy $E_S(\cdot)$, the set $\tilde{\omega}$ where the negative damping is localized may be any subset of Ω, not necessarily a subset of ω.

REMARK 4.5. Combining the results and the methods used so far, we can obtain stability results for problems with distributed or localized positive-negative damping with delay. We recall that the case of distributed positive-negative damping without delay was studied in [**10**].

4.1. Positive–negative damping without delay: the linear case.

In the linear case, as for the case with delay feedback, we can use a more explicit observability constant in the interval I_{2n} where only the positive damping is present.

Consider problem (3.45) – (3.46) and assume that the observability inequality (3.47) holds (with now W_1 instead of W).

We can restate Proposition 3.16.

PROPOSITION 4.6. Assume j) and $f \equiv 0$. Moreover, we assume that the observability inequality (3.47) holds for every time $T > \overline{T}$ and that, denoting $T^* := \inf_n\{T_{2n}\}$,

$$T^* > \overline{T}. \tag{4.9}$$

Then, for any solution of system (4.1) – (4.2), we have

$$E_S(t_{2n+1}) \le \hat{c}_n E_S(t_{2n}), \quad \forall\, n \in \mathbb{N}, \tag{4.10}$$

where

$$\hat{c}_n = \frac{2c(1+4{C_1}^2T_{2n}^2M_{2n}^2)}{m_{2n}+2c(1+4{C_1}^2T_{2n}^2M_{2n}^2)}, \tag{4.11}$$

c being the observability constant in (3.47) corresponding to the time T^* and C_1 the constant in the norm embedding (4.3) between W_1 and H.

Combining the previous proposition with (4.5) we can obtain the following stability result.

THEOREM 4.7. *Assume hypotheses of Proposition* 4.6, jj)*,* (2.6) *and* (2.10)*. If*

$$\sum_{n=0}^{\infty}[2C_3M_{2n+1}T_{2n+1} + \ln \hat{c}_n] = -\infty, \tag{4.12}$$

where $\hat{c}_n$ *is as in* (4.11) *and* C_3 *is the constant in the norm embedding* (4.3) *between* W_3 *and* H*, then system* (4.1) – (4.2) *is asymptotically stable, that is any solution* u *of* (4.1) – (4.2) *satisfies* $E_S(t) \to 0$ *as* $t \to +\infty$.

REMARK 4.8. As in Remark 3.18, one can prove that (4.12) is satisfied if

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty, \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{m_{2n}}{1+4C_1^2T_{2n}^2M_{2n}^2} = +\infty.$$

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