Dynamics of PDE, Vol.12, No.4, 379-403, 2015

Nonnegative solutions of a fractional sub-Laplacian differential inequality on Heisenberg group

Y. Liu, Y. Wang, and J. Xiao

Communicated by Y. Charles Li, received May 8, 2015.

Abstract. In this paper we study nonnegative solutions of

$$(\dagger) \qquad |g|^{\gamma}_{\mathbb{H}^n} u^p \le (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u \quad \text{on} \quad \mathbb{H}^n,$$

where $\mathbb{H}^n$ is the Heisenberg group; $|\cdot|_{\mathbb{H}^n}$ is the homogeneous norm; $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian; $(p, \alpha, \gamma) \in (1, \infty) \times (0, 2) \times [0, (p-1)Q)$; and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. In particular, we prove that any nonnegative solution of $(\dagger)$ is zero if and only if $p \le \frac{Q+\gamma}{Q-\alpha}$.

Contents

1. Introduction 380
2. Preliminaries 382
3. A weak solution for $|g|^{\gamma}_{\mathbb{H}^n} u^p(g) \le (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u(g)$ 385
4. Proof of Theorem 1.1 388
5. Proof of Lemma 4.1 394
References 402

1991 *Mathematics Subject Classification.* Primary 35R03; Secondary 35R11.

Key words and phrases. Heisenberg group, nonnegative weak solution, fractional sub-Laplacian.

YL was supported by the National Natural Science Foundation of China (No.10901018, 11471018), the Fundamental Research Funds for the Central Universities (No.FRF-TP-14-005C1), Program for New Century Excellent Talents in University and the Beijing Natural Science Foundation under Grant (No.1142005).

YW was supported by NSFC No. 1120143 and AARMS Postdoctoral Fellowship; JX was supported by NSERC of Canada (FOAPAL # 202979463102000) and URP of Memorial University (FOAPAL # 208227463102000).

1. Introduction

The Heisenberg group $\mathbb{H}^n$ is a Lie group which can be identified with $\mathbb{R}^{2n} \times \mathbb{R}$ under the group action

$$gh = (x,t)(x',t') = \big(x + x',\, t + t' + 2\sum_{j=1}^{n}(x_{n+j}x'_j - x_j x'_{n+j})\big)$$

for any $g = (x,t), h = (x',t') \in \mathbb{H}^n$. $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$ and $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on $\mathbb{H}^n$ defined by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2,$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, \cdots, n.$$

The homogeneous norm $|g|_{\mathbb{H}^n}$ is defined as

$$|g|_{\mathbb{H}^n} = \big(|x|^4 + |t|^2\big)^{\frac{1}{4}} \quad \forall \quad g = (x,t) \in \mathbb{H}^n.$$

For $0 < \alpha < 2$ let $(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}$ be the fractional sub-Laplacian on the Heisenberg group which is defined as follows (cf. [**10**, Theorem 3.11]): if u belongs to the Schwartz class $\mathcal{S}(\mathbb{H}^n)$, then

$$(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u(g) = \text{P.V.} \int_{\mathbb{H}^n} (u(h) - u(g))\widetilde{R}_{-\alpha}(h^{-1}g)\, dh,$$

where

$$\widetilde{R}_{-\alpha}(g) = \frac{\frac{-\alpha}{2}}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty s^{\frac{-\alpha}{2}-1} h(s,g)\, ds$$

and $h(s,\cdot)$ is the heat kernel associated with $-\Delta_{\mathbb{H}^n}$.

In this paper, we consider the following fractional sub-Laplacian differential inequality

$$|g|^{\gamma}_{\mathbb{H}^n} u^p(g) \le (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u(g), \qquad g \in \mathbb{H}^n. \tag{1.1}$$

The inequality will be understood in the weak sense, whose exact meaning is deferred to next section. Now we state our main result.

Theorem 1.1. *Assume that $1 < p < \infty$, $0 < \alpha < 2$, and $0 \le \gamma < (p-1)Q$. Let u be a nonnegative solution of the fractional sub-Laplacian differential inequality (1.1). Then $u \equiv 0$ if and only if $p \le \frac{Q+\gamma}{Q-\alpha}$.*

For the Laplacian differential inequality in $\mathbb{R}^n$,

$$u^p \le -\Delta u \tag{1.2}$$

with $p > 1$ and $n \ge 2$, it is well known that a nonnegative weak solution to (1.2) equals to 0 if and only if $p \le n/(n-2)$ (cf. [**14**]). This result has been extended to more general differential inequalities in $\mathbb{R}^n$, Heisenberg groups and even Riemannian manifolds, see [**5, 6, 23, 17, 30, 1, 19, 26, 27, 28**] and references therein.

In Heisenberg groups, it is proved that

$$|g|^{\gamma}_{\mathbb{H}^n} u^p \le -\Delta_{\mathbb{H}^n} u \tag{1.3}$$

with $p \in (1, \infty)$ and $\gamma > -2$, admits only trivial nonnegative solution if and only if $p \leq \frac{Q+\gamma}{Q-2}$, as showed in [**1**]. The method of [**1**] is based on a local argument that uses carefully the chosen test functions for (1.3) and conducts integration-by-parts argument – however, the argument fails to be applied to (1.1) where one may encounter the non-local property of the fractional operator $(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}$, which rules out the possibility of applying any local analysis.

It is observed by Caffarelli and Silvestre [**3**] that the fractional Laplacian can be reduced to a local problem through bringing one more variable into play. To be more precise, the fractional Laplacian can be characterized as a Dirichlet-Neumann operator for an appropriate differential equation of divergence form, to which a local argument may be applied. Based on this crucial observation, the paper [**31**] by the second and third authors extends the results for (1.2) to its fractional counterpart, where the main difficulties came from choosing suitable test functions to the extension problem and boundedness of the extended solutions, but were successfully overcame by proving a new extension estimate and a mixed trace estimate.

Recently, Ferrari and Franchi [**10**] reduced the fractional sub-Laplacian $(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}$ into a degenerate local elliptic problem as [**3**] (see Section 2.3 of [**10**] for details). Based on their work and [**31**], it is very natural to study the fractional differential inequality (1.1) on $\mathbb{H}^n$. However, though our strategy relies more or less on [**31**], we may not be able to apply their argument to the current situation directly. On the one hand, it is the sub-Laplacian under consideration rather than the Laplacian. On the other hand, we study the weighted nonlinearity $|g|^{\gamma}_{\mathbb{H}^n} u^p$ in this paper. Since [**31**] only addresses the case of $\gamma = 0$, new ingredients are needed.

Among others, the main difficulty in proving Theorem 1.1 is the extension estimate in a weighted $L^p(\mathbb{H}^n)$ space; see Lemma 4.1 below. The paper [**31**] manages to bound the extension operator on $L^p(\mathbb{R}^n)$ by interpolating between $L^\infty(\mathbb{R}^n)$ and weak-$L^1(\mathbb{R}^n)$. However, such a weak estimate fails if $\gamma > 0$ even for locally supported function. We first observe that the extension operator may be well-defined in $L^{\frac{Q+\gamma}{Q}}(\mathbb{H}^n)$, rather than $L^1(\mathbb{H}^n)$, provided that the function is dyadically supported. By putting all dyadic pieces together, it only diverges logarithmically, which may be circumvent by a carefully duality argument. See Lemma 4.1 and its proof for more details on this issue.

Here, it is worth recording two consequences of our main theorem which are new to our best knowledge (except some endpoint cases). Let us first consider the following fractional sub-Laplacian equation

$$(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u = |g|^{\gamma}_{\mathbb{H}^n} u^p \quad \text{in} \quad \mathbb{H}^n, \tag{1.4}$$

where $1 < p < \infty$, $0 < \alpha \leq 2$, and $0 \leq \gamma < (p-1)Q$. It is widely believed that the nonnegative solution to (1.4) is trivial if and only if $p < \frac{Q+\gamma+\alpha}{Q-\alpha}$. However, the known results in this direction are far from complete. When $\alpha = 2$ and $\gamma = 0$, Birindelli and Prajapat [**2**] prove the nonexistence result under the assumption of smoothness and cylindrical symmetry on the solution u. When $\alpha = 2$, the results in [1] imply that for $1 < p \leq \frac{Q+\gamma}{Q-2}$, the only nonnegative solutions of (1.4) are the trivial ones. Recently, Cinti and Tan in [**8**] solve the problem similar to ours when $\alpha = \frac{1}{2}$ involving a CR square root of the sub-Laplacian, under the assumption of boundedness, cylindrical symmetry and smoothness. Since any nonnegative solution to (1.4) solves the inequality (1.1) as well, it follows from Theorem 1.1 that

Corollary 1. *Assume that* $1 < p \leq \frac{Q+\gamma}{Q-\alpha}$, $0 < \alpha < 2$ *and* $0 \leq \gamma < (p-1)Q$. *Let* u *be a nonnegative solution of the nonlinear equation (1.4). Then* $u \equiv 0$.

Remark 1.2. Unfortunately, we can't handle the cases when $\frac{Q+\gamma}{Q-\alpha} < p < \frac{Q+\gamma+\alpha}{Q-\alpha}$ at this moment. Other than moving plane method, more new ingredients may be needed to completely solve this problem, see [**2, 8**] for more discussion.

Nevertheless, it should be noted that our main result can be extended to other Carnot groups with similar arguments, once there is a Caffarelli-Silvestre type local characterization of the fractional Laplacian. Especially, in $\mathbb{R}^n$, the results in [**31**] could be extended to $\gamma > 0$. More precisely, we have:

Corollary 2. *Assume that* $1 < p < \infty$, $0 < \alpha < 2$ *and* $0 \leq \gamma < (p-1)n$. *Let* u *be a nonnegative solution of the fractional differential inequality*

$$|x|^\gamma u^p \leq (-\Delta)^{\frac{\alpha}{2}} u \quad in \quad \mathbb{R}^n. \tag{1.5}$$

Then $u \equiv 0$ *if and only if* $p \leq \frac{n+\gamma}{n-\alpha}$.

This article is organized as follows. In Section 2, we collect some known facts and results of Heisenberg groups $\mathbb{H}^n$ and Carnot group $\mathbb{H}^n \times \mathbb{R}$. Section 3 recalls the extension problem related to the fractional sub-Laplacian and gives the local version of a weak solution to (1.1). In Section 4 we list the main lemmas and prove Theorem 1.1 by using these lemmas. Section 5 is designed to verify Lemma 4.1.

Notation. Throughout this paper, unless otherwise indicated, we will use C and c to denote constants, which are not necessarily the same at each occurrence. By $\mathsf{A} \lesssim \mathsf{B}$, we mean that there is a constant $C > 0$ such that $\mathsf{A} \leq C\mathsf{B}$. By $\mathsf{A} \sim \mathsf{B}$, we mean that there exist $C > 0$ and $c > 0$ such that $c \leq \frac{\mathsf{A}}{\mathsf{B}} \leq C$.

2. Preliminaries

As a basic reference for the Heisenberg group $\mathbb{H}^n$ we refer the reader to Stein's book [**24**]. We recall that the Heisenberg group $\mathbb{H}^n$ is a Lie group with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and the group action

$$gh = (x,t)(x',t') = \big(x + x',\, t + t' + 2\sum_{j=1}^{n}(x_{n+j}x'_j - x_j x'_{n+j})\big)$$

for any $g = (x,t), h = (x',t') \in \mathbb{H}^n$. Clearly, $g^{-1} = (-x,-t)$ and $(0,\cdots,0,0)$ is its unit element, which can be simply written as o. A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_{2n+1} = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \quad j = 1,\cdots,n.$$

All non-trivial commutators are $[X_j, X_{n+j}] = -4X_{2n+1}$, $j = 1,\cdots,n$. So the vector fields $X_1,\cdots,X_{2n}$ satisfy the so-called Hörmander condition (see [**18**]). Denote by ^{n} the Heisenberg Lie algebra. It can admits a vector space decomposition $^n = V_1 \bigoplus V_2$ such that $V_1 = \text{span}\{X_1,\cdots,X_{2n}\}, V_2 = \text{span}\{X_{2n+1}\}$ and $[V_1, V_2] = 0$. Then $\mathbb{H}^n$ is nilpotent Lie group of step 2. The first layer V_1, the so-called horizontal layer, plays a key role in the theory, since it generates ^{n} by commutation.

The sub-Laplacian $\Delta_{\mathbb{H}^n}$ is defined by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2.$$

For any function $f : \mathbb{H}^n \to \mathbb{R}$ for which the partial derivatives $X_j f$ exist for $j = 1, \cdots, 2n$, the horizontal gradient of f, denoted by $\nabla_{\mathbb{H}^n} f$, as the horizontal section

$$\nabla_{\mathbb{H}^n} f := \sum_{i=1}^{2n} (X_i f) X_i,$$

whose coordinates are $(X_1 f, ..., X_{2n} f)$. Moreover, if $\phi = (\phi_1, \ldots, \phi_{2n})$ is an horizontal section such that $X_j \phi_j \in L^1_{\text{loc}}(\mathbb{H}^n)$ for $j = 1, \ldots, 2n$, we define $\text{div}_{\mathbb{H}^n} \phi$ as the real valued function

$$\text{div}_{\mathbb{H}^n}(\phi) := -\sum_{j=1}^{2n} X_j^* \phi_j = \sum_{j=1}^{2n} X_j \phi_j.$$

The dilations on $\mathbb{H}^n$ have the form

$$\delta_\lambda(x, t) = (\lambda x, \lambda^2 t), \ \lambda > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. We denote the measure of any measurable set E by $|E|$. Then $|\delta_\lambda E| = \lambda^Q |E|$, where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$.

We can define a homogeneous norm function on $\mathbb{H}^n$ by

$$|g|_{\mathbb{H}^n} = \left(|x|^4 + |t|^2\right)^{\frac{1}{4}}, \quad g = (x, t) \in \mathbb{H}^n.$$

Moreover, for all $g \in \mathbb{H}^n$, $r > 0$, $|\delta_r g|_{\mathbb{H}^n} = r|g|_{\mathbb{H}^n}$, $|g^{-1}|_{\mathbb{H}^n} = |g|_{\mathbb{H}^n}$, and $|g|_{\mathbb{H}^n} > 0$ if $g \neq 0$. This norm satisfies the triangular inequality and leads to a left-invariant distance function $d(g, h) = |g^{-1} h|_{\mathbb{H}^n}$. Then the ball of radius r centered at g is given by

$$B(g, r) = \{h \in \mathbb{H}^n : \ d(g, h) < r\}.$$

Because the ball $B(g, r)$ is the left translation by g of $B(o, r)$, although the shape of $B(g, r)$ much varies with the position of the center g, we have

$$|B(g, r)| = \mu_1 r^Q,$$

where μ_1 is a positive constant. Moreover, there exists another distance d_c which is called Carnot-Carathéodory on $\mathbb{H}^n$ and it is globally equivalent to the metric d (see e.g. [**11**]).

In what follows we recall some facts for the Riesz potential on the Heisenberg group. Denote by $h = h(s, g)$ the fundamental solution of $(-\Delta_{\mathbb{H}^n}) + \frac{\partial}{\partial s}$ on $\mathbb{H}^n \times (0, \infty)$ (see [**13**, Proposition 3.3], or [**12**]), which is also called the heat kernel associated with $-\Delta_{\mathbb{H}^n}$. By [**29**] the following estimates for the heat kernel $h(s, \cdot)$ hold true: there exists a positive constant C such that for all $g \in \mathbb{H}^n, s > 0$,

$$h(s, g) \approx s^{-\frac{Q}{2}} \exp\{-\frac{|g|^2_{\mathbb{H}^n}}{Cs}\}. \tag{2.1}$$

Via Theorem 3.15, Proposition 3.17 and 3.18 in [**13**] we have the following lemma.

LEMMA 2.1. *Suppose* $0 < \alpha < Q$. *Then*

$$R_\alpha(g) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty s^{\frac{\alpha}{2} - 1} h(s, g) \, ds$$

converges absolutely for $g \neq o$. *In addition,* R_α *is a kernel of type* α *and the kernels* R_α *admit the following convolution rule: if* $\alpha > 0$, $\beta > 0$ *and* $g \neq o$, *then*

$$R_{\alpha+\beta}(g) = R_\alpha(g) * R_\beta(g).$$

The Riesz potential on the Heisenberg group is defined as follows:

$$(-\Delta_{\mathbb{H}^n})^{-\frac{\alpha}{2}}u(g) = u * R_\alpha(g)$$

for any $u \in Dom((-\Delta_{\mathbb{H}^n})^{-\frac{\alpha}{2}})$*. Then the Riesz potential is the inverse of the fractional Laplacian* $(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}$ *in the sense of:*

$$(-\Delta_{\mathbb{H}^n})^{-\frac{\alpha}{2}}((-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}u) = (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}((-\Delta_{\mathbb{H}^n})^{-\frac{\alpha}{2}}u) = u.$$

Next, we recall the Carnot group $\hat{\mathbb{H}}^n := \mathbb{H}^n \times \mathbb{R}$ (see [**10**]). Its Lie algebra $\hat{\ }^n$ admits the stratification:

$$\hat{\ }^n = \hat{V}_1 \oplus V_2,$$

where $\hat{V}_1 = \text{span}\{Y, V_1\}$ and $Y = \frac{\partial}{\partial y}$ for $y \in \mathbb{R}$. Since the basis $\{X_1, \ldots, X_{2n}\}$ of V_1 has been already fixed once and for all, the associated basis for $\hat{V}_1$ will be $\{X_1, \ldots, X_{2n}, Y\}$. Moreover, the horizontal layer defines a fiber bundle $H\hat{\mathbb{H}}^n$ over $\hat{\mathbb{H}}^n$ (the *horizontal bundle*) by left translation. Its sections are the *horizontal vector fields*.

The corresponding dilations on $\hat{\mathbb{H}}^n$ are given by

$$\hat{\delta}_r(g, y) = (\delta_r g, ry), \ r > 0.$$

The homogeneous dimension of $\hat{\mathbb{H}}^n$ is $Q + 1$. Then we introduce a homogeneous norm function $|\cdot|_{\hat{\mathbb{H}}^n}$ on $\hat{\mathbb{H}}^n$ defined as follows:

$$|(g, y)|_{\hat{\mathbb{H}}^n} = (|g|^2_{\mathbb{H}^n} + |y|^2)^{1/2},$$

which satisfies $|\hat{\delta}_r(g, y)|_{\hat{\mathbb{H}}^n} = r|(g, y)|_{\hat{\mathbb{H}}^n}$ for all $(g, y) \in \hat{\mathbb{H}}^n$, $r > 0$. Let $\hat{d}$ be a function on $\hat{\mathbb{H}}^n \times \hat{\mathbb{H}}^n$ defined by

$$\hat{d}((g, y), (g', y')) := |(g^{-1}g', y' - y)|_{\hat{\mathbb{H}}^n}.$$

It is easy to check that $\hat{d}$ satisfies the triangle inequality, that is, $\hat{d}$ is a metric on $\hat{\mathbb{H}}^n$. The corresponding ball is denoted by $\hat{B}((g, y), r)$ for any $(g, y) \in \hat{\mathbb{H}}^n$ and $r > 0$.

Similarly, for a smooth function u on $\hat{\mathbb{H}}^n$, the horizontal gradient of u is defined as

$$\nabla_{\hat{\mathbb{H}}^n} u := \sum_{i=1}^{2n} (X_i u) X_i + (Yu)Y,$$

whose coordinates are $(X_1 u, ..., X_{2n}u, Yu)$. Moreover, if $\Phi = (\phi_1, \ldots, \phi_{2n+1})$ is an horizontal section such that $X_j\phi_j, Y\phi_{2n+1} \in L^1_{\text{loc}}(\hat{\mathbb{H}}^n)$ for $j = 1, \ldots, 2n$, we define $\text{div}_{\hat{\mathbb{H}}^n}\Phi$ as follows:

$$\text{div}_{\hat{\mathbb{H}}^n}(\Phi) := -\sum_{j=1}^{2n} X_j^* \phi_j - Y^*\phi_{2n+1} = \sum_{j=1}^{2n} X_j\phi_j + Y\phi_{2n+1}.$$

If $E \subset \hat{\mathbb{H}}^n$ is a measurable set, the notion of measure $|\partial E|_{\hat{\mathbb{H}}^n}$ has been introduced in [**16**]. We say that E has locally finite $\hat{\mathbb{H}}^n$-perimeter (or, is also called a $\hat{\mathbb{H}}^n$-Caccioppoli set) if for any bounded open set $\Omega \subseteq \hat{\mathbb{H}}^n$

$$|\partial E|_{\hat{\mathbb{H}}^n}(\Omega) := \sup\left\{ \int_E \text{div}_{\hat{\mathbb{H}}^n}\Psi dgdy : \Psi \in C^1_0(\Omega, H\hat{\mathbb{H}}^n), |\Psi(g, y)| \le 1 \right\} < \infty.$$

Therefore, $|\partial E|_{\hat{\mathbb{H}}^n}$ is a Radon measure in $\hat{\mathbb{H}}^n$, invariant under group translations and the homogeneous of degree Q. Moreover, the following representation theorem holds (see [**4**]).

LEMMA 2.2. *Let $\Omega \subseteq \hat{\mathbb{H}}^n$ be an open set. If E is a $\hat{\mathbb{H}}^n$-Caccioppoli set with Euclidean C^1 boundary, then there is an explicit representation of the $\hat{\mathbb{H}}^n$-perimeter in terms of the Euclidean $2n+1$-dimensional Hausdorff measure $\mathcal{H}^{2n+1}$*

$$|\partial E|_{\hat{\mathbb{H}}^n}(\Omega) = \int_{\partial E\cap\Omega} \left(\sum_{j=1}^{2n} \langle X_j, n\rangle^2_{\mathbb{R}^{2n+2}} + \langle Y, n\rangle^2_{\mathbb{R}^{2n+2}} \right)^{1/2} d\mathcal{H}^{2n+1},$$

where $n = n(x)$ is the Euclidean unit outward normal to ∂E.

We have also the following divergence theorem on $\hat{\mathbb{H}}^n$ by using the Riesz theorem (see e.g. [**10, 11**]).

LEMMA 2.3. *If E is a regular bounded open set in $\hat{\mathbb{H}}^n$ with Euclidean C^1 boundary and $\Psi \in C^1(\overline{E}, H\hat{\mathbb{H}}^n)$, then*

$$-\int_E \operatorname{div}_{\hat{\mathbb{H}}^n} \Psi \, dx = \int_{\partial E} \langle \Psi, \nu_E\rangle d|\partial E|_{\hat{\mathbb{H}}^n},$$

where ν_E is the intrinsic horizontal outward normal to ∂E.

3. A weak solution for $|g|^{\gamma}_{\mathbb{H}^n} u^p(g) \leq (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u(g)$

We begin with reviewing the extension results related to the fractional sub-Lapacian. The extension problem related to the fractional Laplacian on the Euclidean space has been investigated by Caffarelli and Silvestre (cf. [**3**]). Recently, Ferrari and Franchi obtain the analogous result on the Carnot group in [**10**], where they turn the fractional sub-Laplacian into a Dirichlet-Neumann operator for an appropriate differential equation of divergence form: if $\alpha \in (0, 2)$, $u = u(g)$ is a function defined in $\mathbb{H}^n$, and $\mathcal{P} = \mathcal{P}(g, y)$ is a solution to the boundary value problem

$$\begin{cases} -\operatorname{div}_{\hat{\mathbb{H}}^n}(y^{1-\alpha}\nabla_{\hat{\mathbb{H}}^n}\mathcal{P}) = 0 \quad \text{in} \quad \hat{\mathbb{H}}^n_+ := \mathbb{H}^n \times (0, \infty); \\ \mathcal{P}(g, 0) = u(g) \quad \forall \quad g \in \mathbb{H}^n, \end{cases} \tag{3.1}$$

then there is a constant $\tilde{C}_\alpha$ depending on α such that

$$-\lim_{y\to 0^+} y^{1-\alpha}\mathcal{P}_y(g, y) = \tilde{C}_\alpha(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u(g). \tag{3.2}$$

Clearly, the divergence form in (3.1) can be rewritten as

$$(-\Delta_{\mathbb{H}^n})\mathcal{P} + (1-\alpha)y^{-1}\mathcal{P}_y + \mathcal{P}_{yy} = 0.$$

Following from [**10**] we know that the Poisson kernel $P_{\mathbb{H}^n}(\cdot, y)$ in the half-space $\mathbb{H}^n \times (0, \infty)$ defined below:

$$P_{\mathbb{H}^n}(\cdot, y) := C_\alpha \, y^\alpha \int_0^\infty s^{(-\alpha-2)/2} e^{-\frac{y^2}{4s}} h(s, \cdot) \, ds,$$

where

$$C_a = 2^{a-1}\Gamma(\frac{\alpha}{2})^{-1}.$$

Therefore, via (2.1)

$$P_{\mathbb{H}^n}(g, y) \approx \frac{y^\alpha}{\left(|g|^2_{\mathbb{H}^n} + |y|^2\right)^{\frac{Q+\alpha}{2}}}.$$

As done in [**25**] we can conclude that

$$(-\Delta_{\mathbb{H}^n})P_{\mathbb{H}^n}(g,y) + (1-\alpha)y^{-1}\frac{\partial P_{\mathbb{H}^n}(g,y)}{\partial y} + \frac{\partial^2 P_{\mathbb{H}^n}(g,y)}{\partial y^2} = 0.$$

Consequently,

$$\mathcal{P}(g,y) = \int_{\mathbb{H}^n} u(h)P_{\mathbb{H}^n}(h^{-1}g,y)\,dh = u * P_{\mathbb{H}^n}(\cdot,y)(g) \tag{3.3}$$

satisfies (3.1). The solution $\mathcal{P}(g,y)$ of (3.1) is also called the α-extension of $u(g)$ due to (3.3). Naturally, $\mathcal{P}$ can be extended to $\hat{\mathbb{H}}^n$ via putting

$$\tilde{\mathcal{P}}(g,y) = \begin{cases} \mathcal{P}(g,y), & \forall\ \ g \in \mathbb{H}^n \ \ \& \ \ y \geq 0; \\ \mathcal{P}(g,-y), & \forall\ \ g \in \mathbb{H}^n \ \ \& \ \ y < 0. \end{cases}$$

By Theorem 4.6 in [**10**], we know that $\tilde{\mathcal{P}}(g,y)$ is a weak solution of the equation

$$\operatorname{div}_{\mathbb{H}^n}\left(|y|^{1-\alpha}\nabla_{\hat{\mathbb{H}}^n}\mathcal{P}\right) = 0 \quad \text{in } \mathbb{H}^n \times (-1,1)$$

whenever $u \in H^\alpha(\mathbb{H}^n)$ is a solution to $(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}u = 0$, where $H^\alpha(\mathbb{H}^n)$ is the Sobolev space introduced in [**13**] under the graph norm

$$\| u \|_{H^\alpha(\mathbb{H}^n)} = \| u \|_{L^2} + \| (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u \|_{L^2} .$$

Via (3.1) and (3.2), the inequality (1.1) can be rewritten as

$$\begin{cases} -\operatorname{div}_{\hat{\mathbb{H}}^n}(y^{1-\alpha}\nabla_{\hat{\mathbb{H}}^n}\mathcal{P}) = 0 \quad \forall \quad (g,y) \in \mathbb{H}^n \times (0,\infty); \\ \lim\limits_{y\to 0^+} y^{1-\alpha}\dfrac{\partial \mathcal{P}}{\partial y}(g,y) + |g|^\gamma_{\mathbb{H}^n} u^p(g) \leq 0 \quad \forall \quad g \in \mathbb{H}^n. \end{cases} \tag{3.4}$$

LEMMA 3.1. *Assume that* $1 < p < \infty$ *and* $\alpha \in (0,2)$. *Suppose that* u *is a nonnegative solution to* (1.1) *and* ω *is its* α*-extension. If* $|y|^{\frac{1-\alpha}{2}}|\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)| \in L^2(\hat{\mathbb{H}}^n)$, *then for any nonnegative continuous function* ψ *satisfying* $|y|^{\frac{1-\alpha}{2}}|\nabla_{\hat{\mathbb{H}}^n}\psi(g,y)| \in L^2(\hat{\mathbb{H}}^n)$, *one has*

$$\int_{\mathbb{H}^n} |g|^\gamma_{\mathbb{H}^n} u^p(g)\psi(g,0)\,dg \leq \iint_{\hat{\mathbb{H}}^n} \nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)\cdot\nabla_{\hat{\mathbb{H}}^n}\psi(g,y)|y|^{1-\alpha}\,dgdy. \tag{3.5}$$

PROOF. Without losing of generality, we may assume that ψ supports in the ball $\hat{B}_R := \hat{B}((o,0),R)$ centered at the origin point $(o,0) \in \hat{\mathbb{H}}^n$ with radius $R > 0$.

For any $\varepsilon > 0$, via the equation $\operatorname{div}_{\hat{\mathbb{H}}^n}(|y|^{1-\alpha}\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)) = 0$ we get

$$\begin{aligned}
&\iint_{\hat{B}_R} \nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)\cdot\nabla_{\hat{\mathbb{H}}^n}\psi(g,y)|y|^{1-\alpha}dgdy \\
&\qquad = \iint_{\hat{B}_R\setminus\{|y|<\varepsilon\}} + \iint_{\hat{B}_R\cap\{|y|<\varepsilon\}} \nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)\cdot\nabla_{\hat{\mathbb{H}}^n}\psi(g,y)|y|^{1-\alpha}dgdy \\
&\qquad = \iint_{\hat{B}_R\setminus\{|y|<\varepsilon\}} \operatorname{div}_{\hat{\mathbb{H}}^n}(|y|^{1-\alpha}\psi\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}})dgdy \\
&\qquad\quad - \iint_{\hat{B}_R\setminus\{|y|<\varepsilon\}} \psi\operatorname{div}_{\hat{\mathbb{H}}^n}(|y|^{1-\alpha}\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}})dgdy \\
&\qquad\quad + \iint_{\hat{B}_R\cap\{|y|<\varepsilon\}} \nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)\cdot\nabla_{\hat{\mathbb{H}}^n}\psi(g,y)|y|^{1-\alpha}dgdy \\
&\qquad = \iint_{\hat{B}_R\setminus\{|y|<\varepsilon\}} \operatorname{div}_{\hat{\mathbb{H}}^n}(|y|^{1-\alpha}\psi\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}})dgdy \\
&\qquad\quad + \iint_{\hat{B}_R\cap\{|y|<\varepsilon\}} \nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)\cdot\nabla_{\hat{\mathbb{H}}^n}\psi(g,y)|y|^{1-\alpha}dgdy \\
&\qquad =: I + II.
\end{aligned}$$

The term II goes to 0 as $\varepsilon \to 0^+$ due to $|y|^{\frac{1-\alpha}{2}}|\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}}(g,y)| \in L^2_{loc}(\hat{\mathbb{H}}^n)$

Denote $E = \hat{B}_R\cap\{|y| < \varepsilon\}$. Applying the divergence theorem on the Heisenberg group (cf. Lemma 2.3) to the term I we have

$$\begin{aligned}
I &= -\int_{\partial E} |y|^{1-\alpha}\psi\langle\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}},\nu\rangle d|\partial E|_{\hat{\mathbb{H}}^n} \\
&= -\iint_{\hat{B}_R\cap\{|y|=\varepsilon\}} \varepsilon^{1-\alpha}\psi(g,\varepsilon)\frac{\partial\tilde{\mathcal{P}}}{\partial y}(g,\varepsilon)dg \\
&\quad - \iint_{\partial\hat{B}_R\cap\{|y|<\varepsilon\}} |y|^{1-\alpha}\psi\langle\nabla_{\hat{\mathbb{H}}^n}\tilde{\mathcal{P}},\nu\rangle d|\partial E|_{\hat{\mathbb{H}}^n} \\
&= -\iint_{\hat{B}_R\cap\{|y|=\varepsilon\}} \varepsilon^{1-\alpha}\psi(g,\varepsilon)\frac{\partial\tilde{\mathcal{P}}}{\partial y}(g,\varepsilon)dg.
\end{aligned}$$

The inequality in (3.4) implies

$$|g|^{\gamma}_{\mathbb{H}^n}u^p(g) \le -\lim_{\varepsilon\to 0^+}\varepsilon^{1-\alpha}\frac{\partial\mathcal{P}}{\partial y}(g,\varepsilon),$$

thereby completing the proof of (3.5). □

REMARK 3.2. By the above proof of Lemma 3.1, we conclude that, when $\psi(g,y)$ is the α-extension of $\psi(g,0)$, the inequality (3.5) is equivalent to

$$\int_{\mathbb{H}^n} |g|^{\gamma}_{\mathbb{H}^n}u^p(g)\psi(g,0)\,dg \le \int_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}u(g)\psi(g,0)dg,$$

whence implying that $u \in \dot{H}^{\frac{\alpha}{2}}(\mathbb{H}^n)\cap L^{p+1}(\mathbb{H}^n, |g|^{\gamma}_{\mathbb{H}^n}dg)$, where $\dot{H}^s(\mathbb{H}^n)$ is defined as the class of the functions u with the property that $(-\Delta_{\mathbb{H}^n})^{\frac{s}{2}}u \in L^2(\mathbb{H}^n)$ (cf. [**10**] and [**13**]).

Based on the above arguments we give the definition of a weak solution of (1.1).

DEFINITION 3.3. A function u defined on $\mathbb{H}^n$ is called a nonnegative weak solution of (1.1) provided that u is a nonnegative function and its extension $\mathcal{P}$ satisfies both $|y|^{\frac{1-\alpha}{2}}|\nabla_{\hat{\mathbb{H}}^n}\mathcal{P}(g,y)| \in L^2(\hat{\mathbb{H}}^n)$ and

$$\int_{\mathbb{H}^n} |g|^{\gamma}_{\mathbb{H}^n} u^p(g)\phi(g,y)|_{y=0}\, dg \leq \iint_{\hat{\mathbb{H}}^n_+} \nabla_{\hat{\mathbb{H}}^n}\mathcal{P}\cdot\nabla_{\hat{\mathbb{H}}^n}\phi \;\; y^{1-\alpha}\, dgdy, \tag{3.6}$$

for any compactly supported nonnegative function ϕ satisfying $|y|^{\frac{1-\alpha}{2}}|\nabla_{\hat{\mathbb{H}}^n}\phi(g,y)| \in L^2(\hat{\mathbb{H}}^n)$.

4. Proof of Theorem 1.1

To verify Theorem 1.1, we still need two more lemmas.

LEMMA 4.1. *Let $1 < p < \infty$ and $\alpha \in (0,2)$. If $u \in L^p(\mathbb{H}^n, |g|^{\gamma}_{\mathbb{H}^n} dg)$, then*

$$\left(\iint_{\hat{\mathbb{H}}^n_+} |\mathcal{P}(g,y)|^{\frac{(Q+2-\alpha)p}{Q+\gamma}} y^{1-\alpha}\, dgdy\right)^{\frac{Q+\gamma}{(Q+2-\alpha)p}} \lesssim \|u\|_{L^p(\mathbb{H}^n,|g|^{\gamma}_{\mathbb{H}^n}dg)} \tag{4.1}$$

for $0 \leq \gamma < (p-1)Q$, where $\mathcal{P}(g,y)$ is given by (3.3).

REMARK 4.2. If $\gamma = 0$, we can prove Lemma 4.1 by the well-known interpolation theorem similar to the method in [**31**]. If $\gamma \neq 0$ and assume that u is supported in the set $B(g, 2^j r)\backslash B(g, 2^{j-1}r)$ for $j > 1$ and $r > 0$, we can translate this case into the case of $\gamma = 0$ by scaling. For the general case, see Section 4 for the detail of its proof.

REMARK 4.3. If $\gamma < 0$, the inequality (4.1) isn't valid. In fact, let $u(g) = \chi_{B(g_0,1)}$ for any $g_0 \in \mathbb{H}^n$ with $|g_0|_{\mathbb{H}^n} \geq 2$. Then

$$|g|^{\gamma}_{\mathbb{H}^n} \leq (|g_0|_{\mathbb{H}^n} - 1)^{\gamma} \;\;\forall\;\; g \in B(g_0,1).$$

Then

$$\| u \|_{L^p(\mathbb{H}^n,|g|^{\gamma}_{\mathbb{H}^n}dg)} = \left(\int_{B(g_0,1)} |g|^{\gamma}_{\mathbb{H}^n} dg\right)^{\frac{1}{p}} \lesssim (|g_0|_{\mathbb{H}^n} - 1)^{\frac{\gamma}{p}} \to 0 \;\text{ as }\; |g_0|_{\mathbb{H}^n} \to \infty.$$

While for $q = \frac{(Q+2-\alpha)p}{Q+\gamma}$ one has that

$$\begin{aligned}
&\left(\iint_{\hat{\mathbb{H}}^n_+} |\int_{\mathbb{H}^n} \frac{y^{\alpha}\chi_{B(g_0,1)}(h)}{(|h^{-1}g|^2_{\mathbb{H}^n} + y^2)^{\frac{Q+\alpha}{2}}}\, dh|^q y^{1-\alpha}\, dgdy\right)^{\frac{1}{q}} \\
&\gtrsim \left(\int_0^{\infty}\int_{B(g_0,1)} (\int_{B(g_0,1)} \frac{y^{\alpha}}{(2+y)^{Q+\alpha}} dh)^q y^{1-\alpha} dgdy\right)^{\frac{1}{q}} \\
&\gtrsim \left(\int_0^{\infty} \frac{y^{1-\alpha+\alpha q}}{(2+y)^{(Q+\alpha)q}} dy\right)^{\frac{1}{q}} \\
&\gtrsim \left(\int_1^{\infty} \frac{1}{(2+y)^{(Q+\alpha)q}} dy\right)^{\frac{1}{q}} \\
&\gtrsim \frac{1}{[(Q+\alpha)q-1]^{\frac{1}{q}}} 3^{-(Q+\alpha)+\frac{1}{q}}.
\end{aligned}$$

Consequently, the inequality (4.1) doesn't hold.

LEMMA 4.4. *Assume that* $1 < p < \infty$ *and* $\alpha \in (0, 2)$. *Let* φ *be a smooth function in* $\hat{\mathbb{H}}^n_+$ *with the compact support supp*(φ) *such that* $0 \leq \varphi \leq 1$ *and* $\varphi = 1$ *in an nonempty open subset of supp*(ϕ). *If* u *is a nonnegative weak solution to* (1.1) *and* $\mathcal{P}$ *is its* α*-extension, then for any* $0 < l \ll 1$ *and* $s \gg 1$ *there is a constant* $C_{s,Q} > 0$ *depending on* s *and* Q *such that*

$$\begin{aligned} &\int_{\mathbb{H}^n} \varphi^s|_{y=0}\ |g|^{\gamma}_{\mathbb{H}^n} u^{p-l} dg \\ &\qquad \leq C_{s,Q} \left(\iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}^{p'-l'} \varphi^s\, y^{1-\alpha} dg dy \right)^{\frac{1-l}{p'-l'}} \\ &\qquad \times \left(l^{-\frac{p'-l'}{p'-1+l-l'}} \iint_{\hat{\mathbb{H}}^n_+} |\nabla \varphi|^{\frac{2p'-2l'}{p'-1+l-l'}} y^{1-\alpha} dg dy \right)^{\frac{p'-1+l-l'}{p'-l'}}, \end{aligned} \tag{4.2}$$

where

$$p' = \frac{(Q+2-\alpha)p}{Q+\gamma} \quad \& \quad l' = \frac{(Q+2-\alpha)l}{Q+\gamma} \quad \& \quad 0 \leq \gamma < (p-1)Q.$$

PROOF. The nonnegativity of u implies that $\mathcal{P}$ is also nonnegative. Choose a small number $0 < \delta \ll 1$, and let

$$\mathcal{P}_\delta = \mathcal{P} + \delta \quad \& \quad \psi(g, y) = \varphi(g, y)^s \mathcal{P}_\delta(g, y)^{-l},$$

where $0 < l \ll 1$, $s \gg 1$. We firstly claim that the function ψ can be chosen as a test function for (3.6). For $j = 1, 2, \cdots, 2n$,

$$X_j \psi = -l \mathcal{P}_\delta^{-1-l} \varphi^s X_j \mathcal{P} + s \mathcal{P}_\delta^{-l} \varphi^{s-1} X_j \varphi$$

and

$$Y \psi = -l \mathcal{P}_\delta^{-1-l} \varphi^s Y \mathcal{P} + s \mathcal{P}_\delta^{-l} \varphi^{s-1} Y \varphi.$$

Consequently,

$$\nabla_{\hat{\mathbb{H}}^n} \psi = -l \mathcal{P}_\delta^{-1-l} \varphi^s \nabla_{\hat{\mathbb{H}}^n} \mathcal{P} + s \mathcal{P}_\delta^{-l} \varphi^{s-1} \nabla_{\hat{\mathbb{H}}^n} \varphi. \tag{4.3}$$

Clearly, $\psi \in L^{p+1}(\hat{\mathbb{H}}^n)$. Since $\mathcal{P}_\delta$ is uniformly away from 0, $\mathcal{P}_\delta^{-1}$ is uniformly bounded from above, then for the fixed constants $l > 0$ and $s > 1$

$$|\nabla_{\hat{\mathbb{H}}^n} \psi(g, y)| y^{\frac{1-\alpha}{2}}\ dg dy \in L^2(\hat{\mathbb{H}}^n_+),$$

which proves the claim. Hence, via (3.6), (4.3), the Hölder inequality and the Young inequality we have

$$\begin{aligned}
& l \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{-1-l} \varphi^s |\nabla_{\hat{\mathbb{H}}^n} \mathcal{P}|^2 \, y^{1-\alpha} dg dy + \int_{\mathbb{H}^n} \varphi^s|_{y=0} |g|^\gamma_{\mathbb{H}^n} (u+\delta)^{p-l} \, dg \\
& \le s \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{-l} \varphi^{s-1} (\nabla_{\hat{\mathbb{H}}^n} \mathcal{P} \cdot \nabla_{\hat{\mathbb{H}}^n} \varphi) \, y^{1-\alpha} dg dy \\
& = \iint_{\hat{\mathbb{H}}^n_+} (l \mathcal{P}_\delta^{-1-l} \varphi^s)^{\frac{1}{2}} (\nabla_{\hat{\mathbb{H}}^n} \mathcal{P} \cdot \nabla_{\hat{\mathbb{H}}^n} \varphi) s l^{-1/2} \mathcal{P}^{\frac{1-l}{2}} \varphi^{\frac{s}{2}-1} \, y^{1-\alpha} dg dy \\
& \le \frac{\left(l \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{-1-l} \varphi^s |\nabla_{\hat{\mathbb{H}}^n} \mathcal{P}|^2 \, y^{1-\alpha} dg dy \right)^{1/2}}{\left(s^2 l^{-1} \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{1-l} \varphi^{s-2} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^2 \, y^{\alpha} dg dy \right)^{-1/2}} \\
& \le \frac{l}{2} \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{-1-l} \varphi^s |\nabla_{\hat{\mathbb{H}}^n} \mathcal{P}|^2 \, y^{1-\alpha} dg dy + \frac{s^2 l^{-1}}{2} \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{1-l} \varphi^{s-2} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^2 \, y^{1-\alpha} dg dy.
\end{aligned}$$

By moving the first term in the right side to the left side we get

$$\begin{aligned}
& \frac{l}{2} \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{-1-l} \varphi^s |\nabla_{\hat{\mathbb{H}}^n} \mathcal{P}|^2 \, y^{1-\alpha} dg dy + \int_{\mathbb{H}^n} \varphi^s|_{y=0} |g|^\gamma_{\mathbb{H}^n} (u+\delta)^{p-l} \, dg \\
& \qquad \le \frac{s^2 l^{-1}}{2} \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{1-l} \varphi^{s-2} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^2 \, y^{1-\alpha} dg dy. \tag{4.4}
\end{aligned}$$

By the Hölder inequality again, the right side of (4.4) can be bounded by

$$\begin{aligned}
& 2^{-1} s^2 l^{-1} \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{1-l} \varphi^{s-2} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^2 \, y^{1-\alpha} dg dy \\
& \quad = \iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{1-l} \varphi^{\frac{1-l}{p'-l'} s} \left(2^{-1} s^2 l^{-1} \varphi^{s-2-\frac{1-l}{p'-l'} s} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^2 \right) y^{1-\alpha} dg dy \\
& \quad \le \| \mathcal{P}_\delta^{1-l} \varphi^{\frac{1-l}{p'-l'} s} \|_{L^{p_1}(\hat{\mathbb{H}}^n_+, y^{1-\alpha} dg dy)} \\
& \qquad \times \| 2^{-1} s^2 l^{-1} \varphi^{s-2-\frac{1-l}{p'-l'} s} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^2 \|_{L^{p_1'}(\hat{\mathbb{H}}^n_+, y^{1-\alpha} dg dy)},
\end{aligned} \tag{4.5}$$

where

$$p_1 = \frac{p'-l'}{1-l}.$$

And for $s \gg 1$ and $0 < l \ll 1$, we have

$$\varphi^{s-2-\frac{1-t}{p'-l'} s} \le 1.$$

Hence, by using (4.4) and (4.5) we finally obtain

$$\begin{aligned}
& \int_{\mathbb{H}^n} \varphi^s|_{y=0} |g|^\gamma_{\mathbb{H}^n} (u+\delta)^{p-l} \, dg \\
& \qquad \lesssim \left(\iint_{\hat{\mathbb{H}}^n_+} \mathcal{P}_\delta^{p'-l'} \varphi^s \, y^{1-\alpha} dg dy \right)^{\frac{1-l}{p'-l'}} \\
& \qquad \quad \times \left(l^{-\frac{p'-l'}{p'-1+l-l'}} \iint_{\hat{\mathbb{H}}^n_+} |\nabla_{\hat{\mathbb{H}}^n} \varphi|^{\frac{2p'-2l'}{p'-1+l-l'}} \, y^{1-\alpha} dg dy \right)^{\frac{p'-1+l-l'}{p'-l'}}.
\end{aligned}$$

Letting $\delta \to 0$, we obtain (4.2), which arrives at the desired proof. $\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Firstly, we consider the case $\frac{Q+\gamma}{Q} < p \leq \frac{Q+\gamma}{Q-\alpha}$. Assume that u is a nonnegative weak solution to (1.1). Similar to the method in [**31**], the next goal is to estimate the second factor of right side of (4.2) by selecting a series of appropriate test functions. More precisely, for a large number $R > 0$ consider the following function

$$\varphi(g,y) = \begin{cases} 1, & |(g,y)|_{\hat{\mathbb{H}}^n} < R, \\ \left|\hat{\delta}_{R^{-1}}(g,y)\right|_{\hat{\mathbb{H}}^n}^{-l}, & |(g,y)|_{\hat{\mathbb{H}}^n} \geq R. \end{cases}$$

By the expression of the vector fields $X_j's$ and Y we have

$$\begin{aligned} X_j\varphi(g,y) &= -\frac{R^l l}{(|g|^2_{\mathbb{H}^n} + y^2)^{\frac{l}{2}+1}} \left(\frac{x_j(\sum_{i=1}^{2n} x_i^2)}{\left[(\sum_{i=1}^{2n} x_i^2)^2 + t^2\right]^{\frac{1}{2}}} + \frac{2x_{n+j}t}{\left[(\sum_{i=1}^{2n} x_i^2)^2 + t^2\right]^{\frac{1}{2}}} \right) \\ &= -\frac{R^l l}{|(g,y)|^{l+2}_{\hat{\mathbb{H}}^n} |g|^2_{\mathbb{H}^n}} \left(x_j|x|^2 + 2x_{n+j}t \right), \end{aligned}$$

$$\begin{aligned} X_{n+j}\varphi(g,y) &= -\frac{R^l l}{(|g|^2_{\mathbb{H}^n} + y^2)^{\frac{l}{2}+1}} \left(\frac{x_{n+j}(\sum_{i=1}^{2n} x_i^2)}{\left[(\sum_{i=1}^{2n} x_i^2)^2 + t^2\right]^{\frac{1}{2}}} - \frac{2x_j t}{\left[(\sum_{i=1}^{2n} x_i^2)^2 + t^2\right]^{\frac{1}{2}}} \right) \\ &= -\frac{R^l l}{|(g,y)|^{l+2}_{\hat{\mathbb{H}}^n} |g|^2_{\mathbb{H}^n}} \left(x_{n+j}|x|^2 - 2x_j t \right) \end{aligned}$$

and

$$Y\varphi(g,y) = -\frac{R^l l y}{(|g|^2_{\mathbb{H}^n} + y^2)^{\frac{l}{2}+1}} = -\frac{R^l l y}{|(g,y)|^{l+2}_{\hat{\mathbb{H}}^n}} \quad \text{where} \quad j = 1, 2, \cdots, n.$$

Then we can get

$$|\nabla_{\hat{\mathbb{H}}^n}\varphi| = \Big(\sum_{j=1}^{2n}(X_j\varphi)^2 + (Y\varphi)^2\Big)^{\frac{1}{2}} \lesssim R^l l |(g,y)|^{-l-1}_{\hat{\mathbb{H}}^n}. \tag{4.6}$$

Since R is big enough, $l = (\ln R)^{-1}$ is sufficiently small. For any $k \in \mathbb{Z}_+$ we define a cutoff function η_k by

$$\eta_k(g,y) = \begin{cases} 1, & 0 \leq |(g,y)|_{\hat{\mathbb{H}}^n} \leq kR; \\ 2 - |\hat{\delta}_{(kR)^{-1}}(g,y)|_{\hat{\mathbb{H}}^n}, & kR \leq |(g,y)|_{\hat{\mathbb{H}}^n} \leq 2kR; \\ 0, & |(g,y)|_{\hat{\mathbb{H}}^n} \geq 2kR. \end{cases}$$

In a similar manner to establish (4.6), we have

$$|\nabla_{\hat{\mathbb{H}}^n}\eta_k| \lesssim (kR)^{-1}. \tag{4.7}$$

In what follows we consider the function

$$\varphi_k(g,y) = \varphi(g,y)\eta_k(g,y)$$

with $\varphi_k(g,y)$ tending to $\varphi(g,y)$ from below as $k \to \infty$. It is easy to see that

$$\nabla_{\hat{\mathbb{H}}^n}\varphi_k = \varphi\nabla_{\hat{\mathbb{H}}^n}\eta_k + \eta_k\nabla_{\hat{\mathbb{H}}^n}\varphi.$$

Thus, for any $b \geq 2$ we have

$$|\nabla_{\hat{\mathbb{H}}^n}\varphi_k|^b \lesssim |\varphi\nabla_{\hat{\mathbb{H}}^n}\eta_k|^b + |\eta_k\nabla_{\hat{\mathbb{H}}^n}\varphi|^b. \tag{4.8}$$

Now we are in a position to estimate

$$I_k(b) := \iint_{\hat{\mathbb{H}}^n_+} |\nabla_{\hat{\mathbb{H}}^n}\varphi_k|^b y^{1-\alpha}\, dgdy,$$

where $b = \frac{2p'-2l'}{p'-1+l-l'}$. Denote by $\hat{B}_{kR} = \hat{B}((o,0),kR)$ for any $k \in \mathbb{Z}_+$. Via (4.8) we have

$$\begin{aligned} I_k(b) &\lesssim \iint_{\hat{\mathbb{H}}^n_+} \eta_k^b|\nabla_{\hat{\mathbb{H}}^n}\varphi|^b y^{1-\alpha}\, dgdy + \iint_{\hat{\mathbb{H}}^n_+} \varphi^b|\nabla_{\hat{\mathbb{H}}^n}\eta_k|^b y^{1-\alpha}\, dgdy \\ &\lesssim \iint_{\hat{\mathbb{H}}^n_+\setminus\hat{B}_R} |\nabla_{\hat{\mathbb{H}}^n}\varphi|^b y^{1-\alpha} dgdy + \iint_{(\hat{B}_{2kR}\setminus\hat{B}_{kR})\cap\hat{\mathbb{H}}^n_+} \varphi^b|\nabla_{\hat{\mathbb{H}}^n}\eta_k|^b y^{1-\alpha}\, dgdy \\ &= J_1 + J_2. \end{aligned} \tag{4.9}$$

At first, by (4.7) we get

$$\begin{aligned} J_2 \lesssim& (kR)^{-b}\int_{(\hat{B}_{2kR}\setminus\hat{B}_{kR})\cap\hat{\mathbb{H}}^n_+} \varphi^b y^{1-\alpha}\, dgdy \\ \lesssim& (kR)^{-b}\left(\sup_{(\hat{B}_{2kR}\setminus\hat{B}_{kR})\cap\hat{\mathbb{H}}^n_+} \varphi^b\right)\int_{B(o,2kR)}\int_0^{2kR} y^{1-\alpha} dydg \\ \lesssim& (kR)^{-b}\left(\frac{kR}{R}\right)^{-bl}(kR)^{2-\alpha}(2kR)^Q \\ \approx& k^{Q+2-\alpha-b-bl}R^{Q+2-\alpha-b}. \end{aligned} \tag{4.10}$$

Secondly, using the polar coordinate formula on the Heisenberg group and (4.6), we obtain

$$\begin{aligned} J_1 &\lesssim \int_{|g|>R}\int_0^\infty |\nabla_{\hat{\mathbb{H}}^n}\varphi|^b y^{1-\alpha}\, dgdy + \int_{\mathbb{H}^n}\int_R^\infty |\nabla_{\hat{\mathbb{H}}^n}\varphi|^b y^{1-\alpha}\, dgdy \\ &\lesssim R^{bl}l^b\int_{|g|>\frac{R}{2}}\int_0^\infty |(g,y)|_{\hat{\mathbb{H}}^n}^{-bl-b} y^{1-\alpha}\, dgdy + R^{bl}l^b\int_{\mathbb{H}^n}\int_{\frac{R}{2}}^\infty |(g,y)|_{\hat{\mathbb{H}}^n}^{-bl-b} y^{1-\alpha}\, dgdy \\ &\lesssim R^{bl}l^b\int_0^\infty \left(\frac{R}{2}+y\right)^{-b-bl+Q} y^{1-\alpha} dy + R^{bl}l^b\int_{\frac{R}{2}}^\infty y^{-b-bl+Q+1-\alpha} dy \\ &\approx l^b R^{Q+2-b-\alpha} \quad \text{if} \quad b+bl > Q+2-\alpha. \end{aligned} \tag{4.11}$$

By (4.10), (4.11) and (4.9) we get

$$I_k(b) \lesssim k^{Q+2-\alpha-b-bl}R^{Q+2-\alpha-b} + l^b R^{Q+2-\alpha-b}.$$

Consequently,

$$I_k(b) \lesssim k^{-bl} + l^b \quad \text{under} \quad b \geq Q+2-\alpha. \tag{4.12}$$

Here it should be noted that $I_k(b)$ is uniformly bounded in R and k.

Now, it follows from (4.2) and (4.12) that

$$\int_{\mathbb{H}^n} \varphi_k^s u^{p-l} |g|_{\mathbb{H}^n}^{\gamma} dg \lesssim \frac{\left(l^{-\frac{p'-l'}{p'-1+l-l'}}[k^{-bl}+l^b]\right)^{\frac{p'-1+l-l'}{p'-l'}}}{\left(\iint_{\hat{\mathbb{H}}_+^n} \mathcal{P}^{p'-l'} \varphi_k^s y^{1-\alpha} dgdy\right)^{-\frac{1-l}{p'-l'}}}.$$

Upon taking $k \to \infty$ in the above, we gain

$$\int_{\mathbb{H}^n} \varphi^s u^{p-l} |g|_{\mathbb{H}^n}^{\gamma} dg \lesssim l \left(\iint_{\hat{\mathbb{H}}_+^n} \mathcal{P}^{p'-l'} \varphi^s y^{1-\alpha} dgdy\right)^{\frac{1-l}{p'-l'}}. \tag{4.13}$$

Note that

$$u \in L^{p+1}(\mathbb{H}^n, |g|_{\mathbb{H}^n}^{\gamma} dg) \Rightarrow \mathcal{P} \in L^{\frac{(p+1)(Q+2-\alpha)}{Q+\gamma}}(\hat{\mathbb{H}}_+^n,\ y^{1-\alpha} dgdy)$$

(due to Lemma 4.1) and that the integral of right side of (4.13) is uniformly bounded in l. So, by letting $l \to 0+$, we have

$$\begin{aligned}\int_{\mathbb{H}^n} \varphi^s u^p |g|_{\mathbb{H}^n}^{\gamma} dg &= \lim_{l\to 0+} \int_{\mathbb{H}^n} \varphi^s u^{p-l} |g|_{\mathbb{H}^n}^{\gamma} dg \\ &\le \lim_{l\to 0+} l \left(\iint_{\hat{\mathbb{H}}_+^n} \mathcal{P}^{p'-l'} \varphi^s y^{1-\alpha} dgdy\right)^{\frac{1-l}{p'-l'}} = 0,\end{aligned}$$

whence reaching $u = 0$ provided $b + bl > Q + 2 - \alpha$.

Also, note that

$$b = \frac{2(p'-l')}{p'-1+l-l'}.$$

Thus, choosing l to be sufficiently small, we get

$$b + bl = \frac{2(p'-l')}{p'-1+l-l'} + \frac{2l(p'-l')}{p'-1+l-l'} > \frac{2p'}{p'-1}.$$

Because

$$\frac{2p'}{p'-1} \ge Q+2-\alpha, \text{ i.e., } (Q-\alpha)p' \le Q+2-\alpha,$$

and

$$p' = \frac{p(Q+2-\alpha)}{Q+\gamma},$$

we have

$$\frac{p(Q+2-\alpha)(Q-\alpha)}{Q+\gamma} \le Q+2-\alpha, \text{ i.e., } \frac{p(Q-\alpha)}{Q+\gamma} \le 1.$$

Moreover, Lemma 4.1 implies $p > \frac{Q+\gamma}{Q}$.

Secondly, we prove the existence of a positive solution of (1.1) when $p > \frac{Q+\gamma}{Q-\alpha}$. The main method is to perturb the fundamental solution $R_\alpha(g)$ by the following smooth cut-off function:

$$\rho(g) = \begin{cases} 1, & |g|_{\mathbb{H}^n} \le 1; \\ \text{smooth and radially decreasing}, & 1 \le |g|_{\mathbb{H}^n} \le 2; \\ 0, & |g|_{\mathbb{H}^n} \ge 2. \end{cases}$$

For $0 < \delta < Q - \alpha$, define u_δ by

$$u_\delta(g) = \rho * R_{\delta+\alpha}(g) \quad \forall \quad g \in \mathbb{H}^n.$$

In what follows we show that u_δ satisfies the inequality (1.1) under $p > \frac{Q+\gamma}{Q-\alpha}$ when $|g|_{\mathbb{H}^n} \geq 4$. By Lemma 2.1 we see

$$R_{\delta+\alpha}(g) \approx |g|_{\mathbb{H}^n}^{\delta+\alpha-Q}.$$

For any $g, h \in \mathbb{H}^n$ with $|g|_{\mathbb{H}^n} \geq 4$ and $|h|_{\mathbb{H}^n} \leq 2$, it is easy to see that $|h^{-1}g|_{\mathbb{H}^n} \approx |g|_{\mathbb{H}^n}$, and so

$$|h^{-1}g|_{\mathbb{H}^n}^{\alpha+\delta-Q} \approx |g|_{\mathbb{H}^n}^{\alpha+\delta-Q} \ \ \& \ \ u_\delta(g) = \int_{|h|_{\mathbb{H}^n} \leq 2} \rho(h) R_{\delta+\alpha}(h^{-1}g)dh \approx \ |g|_{\mathbb{H}^n}^{\alpha+\delta-Q}.$$

Then, for any $g \in \mathbb{H}^n$ with $|g|_{\mathbb{H}^n} \geq 4$ one has

$$|g|_{\mathbb{H}^n}^{\gamma} u_\delta^p(g) \approx |g|_{\mathbb{H}^n}^{\alpha p+\delta p+\gamma-Qp}. \tag{4.14}$$

By Lemma 2.1 again we get

$$(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u_\delta(g) = (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} (\rho * R_\delta(g) * R_\alpha(g)) = \rho * R_\delta(g).$$

Similarly, we obtain that

$$(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u_\delta(g) \approx |g|_{\mathbb{H}^n}^{\delta-Q} \tag{4.15}$$

holds for all $g \in \mathbb{H}^n$ with $|g|_{\mathbb{H}^n} \geq 4$. If $p > \frac{Q+\gamma}{Q-\alpha}$, then there exists a sufficiently small constant δ such that $(Q-\alpha-\delta)p > Q-\delta+\gamma$. By using this inequality, (4.14) and (4.15) we conclude that for all $g \in \mathbb{H}^n$ with $|g|_{\mathbb{H}^n} \geq 4$,

$$u_\delta^p(g) \leq (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u_\delta(g).$$

Finally, let

$$m = \max\left\{ \frac{\max\limits_{|g|_{\mathbb{H}^n} \leq 4} |g|_{\mathbb{H}^n}^{\gamma} u_\delta^p(g)}{\min\limits_{|g|_{\mathbb{H}^n} \leq 4} (-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u_\delta(g)}, 1 \right\}.$$

The previous arguments imply that u_δ solves the inequality

$$|g|_{\mathbb{H}^n}^{\gamma} u^p \leq m(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} u \quad \text{in} \quad \mathbb{H}^n.$$

Therefore, $u = m^{\frac{1}{1-p}} u_\delta$ is the positive solution to (1.1) in $\mathbb{H}^n$, whence completing the proof. ☐

5. Proof of Lemma 4.1

Before proving Lemma 4.1, we recall the dyadic decomposition of homogeneous space, which is valid on the Heisenberg and Carnot groups (cf. [**7**] and [**21**]).

LEMMA 5.1. *Let $\mathcal{X}$ be a homogeneous space equipped with a quasi-metric d and a doubling measure μ. There exists $\lambda > 1$ so that for any (large negative) integer $m \in \mathbb{Z}$, there exist points $\{g_j^k\}$ in $\mathcal{X}$ and a family $\mathcal{D}_m = \{E_j^k\}$ of sets for $k = m, m+1, \cdots$ and $j = 1, 2, \cdots$ such that*

(i) *$B(g_j^k, \lambda^k) \subseteq E_j^k \subseteq B(g_j^k, \lambda^{k+1})$,*

(ii) *For each $k = m, m+1, \cdots$, $\mathcal{X} = \bigcup_j E_j^k$ and $\{E_j^k\}$ is pairwise disjoint in j,*

(iii) *If $k < l$ then either $E_j^k \bigcap E_j^l = \emptyset$ or $E_j^k \subseteq E_j^l$.*

The family $\mathcal{D} = \bigcup_{m\in\mathbb{Z}} \mathcal{D}_m$ is called a dyadic cube decomposition of $\mathcal{X}$ and the sets in $\mathcal{D}$ are called dyadic cubes. If $\tilde{Q} = E_j^k \in \mathcal{D}_m$ for some $m \in \mathbb{Z}$, we say $\tilde{Q}$ is centered at g_j^k and define the sidelength of $\tilde{Q}$ to be $l(\tilde{Q}) = 2\lambda^k$. We also denote by $\tilde{Q}^*$ the containing ball $B(g_j^k, \lambda^{k+1})$ of $\tilde{Q}$.

Moreover, the following is valid for the homogeneous spaces (cf. [**9**]).

LEMMA 5.2. (Covering Lemma). *If $\{B\}$ is a family of balls with bounded radii in a homogeneous space $\mathcal{X}$, then there is a countable pairwise disjoint subfamily $\{B_i\}$ of balls so that each ball in $\{B\}$ is contained in one of the balls θB_i, where $\theta = \kappa(3\kappa + 2)$ and κ is the constant in the quasi-triangle inequality for the quasi-metric in the homogeneous space $\mathcal{X}$.*

It should be noted that $\kappa = 1$ whenever $\mathcal{X} = \mathbb{H}^n$ or $\mathcal{X} = \hat{\mathbb{H}}^n_+$.

PROOF OF LEMMA 4.1. Via modifying the argument in [**22**], we prove Lemma 4.1 according to three steps. But, at first, we are required to introduce some notations. For a measure μ on $\hat{\mathbb{H}}^n_+$ we denote $|\hat{E}|_\mu$ the measure of a measurable set $\hat{E}$ in $\hat{\mathbb{H}}^n_+$. For a ball B in $\mathbb{H}^n$ or a ball $\hat{B}$ in $\hat{\mathbb{H}}^n_+$ denote by $r(B)$ or $r(\hat{B})$ the corresponding radii, respectively. We introduce a measure σ on $\mathbb{H}^n$ with $d\sigma(g) = |g|_{\mathbb{H}^n}^{-\frac{\gamma}{p-1}} dg$ and a measure ω on $\hat{\mathbb{H}}^n_+$ with $d\omega(g, y) = y^{1-\alpha+\alpha q} dgdy$. Denote by $d\hat{\sigma}(g, y) = \delta_0(y)d\sigma(g)$, where $\delta_0(g)$ is the Dirac delta function. Let K be the kernel on $\hat{\mathbb{H}}^n_+ \times \hat{\mathbb{H}}^n_+$ which is defined by

$$K((g, y), (h, y')) = \frac{1}{(|h^{-1}g|^2_{\mathbb{H}^n} + (y - y')^2)^{\frac{Q+\alpha}{2}}}. \tag{5.1}$$

For $(g, y) \in \hat{\mathbb{H}}^n_+$ set

$$\begin{aligned} T\hat{\phi}(g, y) &= \int_{\hat{\mathbb{H}}^n_+} K((g, y), (h, y'))\hat{\phi}(h, y')\, d\hat{\sigma}(h, y') \\ &= \int_{\mathbb{H}^n} \frac{\phi(h)}{(|h^{-1}g|^2_{\mathbb{H}^n} + y^2)^{\frac{Q+\alpha}{2}}} |h|_{\mathbb{H}^n}^{-\frac{\gamma}{p-1}} dh, \end{aligned}$$

where $\hat{\phi}(h, 0) = \phi(h)$ is a function defined on $\mathbb{H}^n$. Let

$$T^*\hat{\psi}(g, y) = \int_{\hat{\mathbb{H}}^n_+} K((g, y), (h, y'))\hat{\psi}(h, y')\, d\omega(h, y'),\ (g, y) \in \hat{\mathbb{H}}^n_+,$$

where $\hat{\psi}$ is a function defined on $\hat{\mathbb{H}}^n_+$. The next goal is to verify

$$\left(\int_{\hat{\mathbb{H}}^n_+} |T\hat{\phi}(g, y)|^q d\omega\right)^{\frac{1}{q}} \lesssim \left(\int_{\hat{\mathbb{H}}^n_+} |\hat{\phi}|^p d\hat{\sigma}\right)^{\frac{1}{p}} \quad \text{where} \quad q = \frac{(Q + 2 - \alpha)p}{Q + \gamma}. \tag{5.2}$$

Here, it should be mentioned that (4.1) can be obtained by choosing $\hat{\phi}(g, 0) = u(g)|g|_{\mathbb{H}^n}^{\frac{\gamma}{p-1}}$ in (5.2).

Step 1: For $p > \frac{Q+\gamma}{Q}$ and $q = \frac{(Q+2-\alpha)p}{Q+\gamma}$, we need to show that the weak type inequality

$$\sup_\rho \rho\, |\{(g, y) \in \hat{\mathbb{H}}^n_+ : T^*\hat{\psi}(g, y) > \rho\}|_\sigma^{\frac{1}{p'}} \lesssim \left(\int_{\hat{\mathbb{H}}^n_+} \hat{\psi}(g, y)^{q'} d\omega(g, y)\right)^{\frac{1}{q'}}$$

holds for $\hat{\psi} \geq 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Without loss of generality, we may assume that $\hat{\psi}$ is nonnegative and bounded with support in some ball. For $\rho > 0$, we denote $\hat{\Omega}_\rho = \{(g, y) \in \hat{\mathbb{H}}^n_+ : T^*\hat{\psi}(g, y) > \rho\}$. Let $\hat{\mathcal{D}}$ be a dyadic decomposition of $\hat{\mathbb{H}}^n_+$ (cf. Lemma 5.1) and let $\hat{\mathcal{D}}_{\Omega_\rho}$ denote the dyadic cubes $\hat{\tilde{Q}} \in \hat{\mathcal{D}}$ with the property that $R\hat{\tilde{Q}}^* \subseteq \hat{\Omega}_\rho$ for a fixed constant R which can be chosen later. For each $m \in \mathbb{Z}$, we denote $\hat{\mathcal{D}}_{\rho,m} = \hat{\mathcal{D}}_{\Omega_\rho} \bigcap \hat{\mathcal{D}}_m$ and $\hat{\Omega}_{\rho,m} = \bigcup_{\hat{\tilde{Q}} \in \hat{\mathcal{D}}_{\rho,m}} \tilde{Q}$, where $\hat{\mathcal{D}}_m$ can be similarly constructed as Lemma 5.1.

Let $A > 1$ be a constant which will be chosen later. It is easy to check that $\hat{\Omega}_\rho \subseteq \hat{\Omega}_{\frac{\rho}{A}}$ and $\hat{\Omega}_{\rho,m} \subseteq \hat{\Omega}_{\frac{\rho}{A},m}$ for all $m \in \mathbb{Z}$. Following from [**20**] and [**21**], we conclude that the following covering property of Whitney-type for $\hat{\Omega}_{\frac{\rho}{A},m}$ is also valid: there exists a constant R, independent of ρ, m and A, such that the sequence of maximal (with respect to inclusion) dyadic cubes $\{\hat{\tilde{Q}}_j\}$ in $\hat{\mathcal{D}}_{\frac{\rho}{A},m}$ satisfies:

a. $\hat{\Omega}_{\frac{\rho}{A},m} = \bigcup_j \hat{\tilde{Q}}_j$ and $\hat{\tilde{Q}}_i \bigcap \hat{\tilde{Q}}_j = \emptyset$ for $i \neq j$;

b. $R\hat{\tilde{Q}}_j^* \subseteq \hat{\mathcal{D}}_{\frac{\rho}{A},m}$ and $2R\lambda\hat{\tilde{Q}}_j^* \bigcap \hat{\Omega}^c_{\frac{\rho}{A}} \neq \emptyset$ for all j;

c. $\sum_j \chi_{30\hat{\tilde{Q}}_j^*} \lesssim \chi_{\hat{\Omega}_{\frac{\rho}{A}}}$.

In what follows we need to show that there is a positive constant C, independent of f, ρ, m, j and A, so that

$$T^*(\chi_{(2\hat{\tilde{Q}}_j^*)^c}\hat{\psi})(g,y) \leq C\left(\frac{\rho}{A}\right), \quad (g,y) \in \hat{\tilde{Q}}_j. \tag{5.3}$$

In fact, for $(g,y) \in \hat{\tilde{Q}}_j$, $(h',y') \in (2\hat{\tilde{Q}}_j^*)^c$ and $(g',z) \in 2R\lambda\hat{\tilde{Q}}_j^* \bigcap \hat{\Omega}^c_{\frac{\rho}{A}}$, we have

$$\hat{d}((g',z),(h',y')) \lesssim \hat{d}((g,y),(h',y')).$$

so

$$T^*(\chi_{(2\hat{\tilde{Q}}_j^*)^c}\hat{\psi})(g,y) \lesssim T^*(\chi_{(2\hat{\tilde{Q}}_j^*)^c}\hat{\psi})(g',z) \leq C\left(\frac{\rho}{A}\right).$$

Upon choosing $A = 2C$ with C as in (5.3), we get that if $(g,y) \in \hat{\tilde{Q}}_j \bigcap \hat{\Omega}_{\rho,m}$ then

$$\int_{2\hat{\tilde{Q}}_j^*} K((g,y),(h,y'))\frac{d\omega(h,y')}{\left(\hat{\psi}(h,y')\right)^{-1}} = T\hat{\psi}(g,y) - T^*(\chi_{(2\hat{\tilde{Q}}_j^*)^c}\hat{\psi})(g,y) > \frac{\rho}{2}. \tag{5.4}$$

Denote by $\mathrm{supp}(\hat{\sigma})$ the support of the measure $\hat{\sigma}$. Let

$$j \ \ \& \ \ (g,y) \in \hat{\tilde{Q}}_j \bigcap \hat{\Omega}_{\rho,m} \bigcap \mathrm{supp}(\hat{\sigma})$$

be temporarily fixed. Choose a decreasing sequence of balls $\hat{B}_j^0 \supseteq \hat{B}_j^1 \supseteq \hat{B}_j^2 \supseteq \cdots$ such that $\hat{B}_j^0 = \hat{B}((g,y), 4r(\hat{\tilde{Q}}_j^*))$, and

$$\begin{cases} \hat{B}_j^k = \hat{B}((g,y), 2^{-k}r(\hat{B}_j^0)) & \text{if} \quad \hat{B}_j^k \bigcap \mathrm{supp}(\hat{\sigma}) \neq \emptyset; \\ \hat{B}_j^k = \emptyset \quad \text{if} \quad \hat{B}_j^k \bigcap \mathrm{supp}(\hat{\sigma}) = \emptyset. \end{cases}$$

Therefore, $\hat{B}_j^k \bigcap \mathrm{supp}\hat{\sigma} \neq \emptyset$ and $\hat{B}_j^k \bigcap \mathrm{supp}\hat{\omega} \neq \emptyset$ if $\hat{B}_j^k \neq \emptyset$. Note that

$$2\hat{\tilde{Q}}_j^* \subseteq \hat{B}_j^0 \subseteq 6\hat{\tilde{Q}}_j^*$$

and

$$\frac{1}{2}r(\hat{B}_j^k) \leq \hat{d}((g,y),(g',y')) \leq r(\hat{B}_j^k) \ \text{ for } \ (g',y') \in \hat{B}_j^k \backslash \hat{B}_j^{k+1} \ \text{ if } \ \hat{B}_j^k \neq \emptyset.$$

So, via (5.1) we get

$$\int_{\hat{B}_j^0} K((g,y),(g',y'))\hat{\psi}(g',y')\,d\omega(g',y') \lesssim \sum_{k:\hat{B}_j^k \neq \emptyset} \frac{\int_{\hat{B}_j^k \backslash \hat{B}_j^{k+1}} \hat{\psi}(g',y')\,d\omega(g',y')}{(r(\hat{B}_j^k))^{Q+\alpha}}.$$

Next we select a subsequence $\{\hat{B}_i = \hat{B}_j^{k_i}\}$ of $\{\hat{B}_j^k\}$ satisfying the following properties:

d. $\hat{B}_0 = \hat{B}_j^0$;

e. $| 5\hat{B}_{i+1} |_{\hat{\sigma}} < \frac{1}{2} | 5\hat{B}_i |_{\hat{\sigma}}$;

f. $| 5\hat{B}_j^k |_{\hat{\sigma}} \geq \frac{1}{2} | 5\hat{B}_i |_{\hat{\sigma}}$, if $\hat{B}_i' \subseteq \hat{B}_j^k \subseteq \hat{B}_i$ for $i = 1, 2, \cdots$, where $\hat{B}_i' = \hat{B}_j^{k_{i+1}-1}$.

Then

$$\frac{1}{r(\hat{B}_j^k)^{Q+\alpha}} \lesssim \min\{\frac{1}{r(\hat{B}_i')^{Q+\alpha}}, K((g,y),(g',y'))\}$$

holds for all $(g',y') \in \hat{B}_j^k$ and all k with $\hat{B}_i' \subseteq \hat{B}_j^k \subseteq \hat{B}_i$. Hence,

$$\begin{aligned}
&\sum_{k:\hat{B}_i' \subseteq \hat{B}_j^k \subseteq \hat{B}_i} \frac{1}{r(\hat{B}_j^k)^{Q+\alpha}} \chi_{\hat{B}_j^k \setminus \hat{B}_j^{k+1}}(g',y') \\
&\qquad \lesssim \min\{\frac{1}{r(\hat{B}_i')^{Q+\alpha}}, K((g,y),(g',y'))\} \sum_{k:\hat{B}_i' \subseteq \hat{B}_j^k \subseteq \hat{B}_i} \chi_{\hat{B}_j^k \setminus \hat{B}_j^{k+1}}(g',y') \\
&\qquad \lesssim \min\{\frac{1}{r(\hat{B}_i')^{Q+\alpha}}, K((g,y),(g',y'))\} \chi_{\hat{B}_i}(g',y').
\end{aligned}$$

Since $p < q$ and $q' < p'$, via (5.4) we have

$$\begin{aligned}
&\rho \sum_i \left(\frac{|5\hat{B}_i|_{\hat{\sigma}}}{|5\hat{B}_0|_{\hat{\sigma}}} \right)^{\frac{1}{q'} - \frac{1}{p'}} \\
&\leq \rho \sum_i \left(\frac{1}{2^i} \right)^{\frac{1}{q'} - \frac{1}{p'}} \\
&\lesssim \int_{2\hat{\tilde{Q}}_j^*} K((g,y),(g',y')) \hat{\psi}(g',y')\, d\omega(g',y') \\
&\lesssim \int_{\hat{B}_j^0} K((g,y),(g',y')) \hat{\psi}(g',y')\, d\omega(g',y') \\
&\lesssim \sum_i \sum_{k:\hat{B}_i' \subseteq \hat{B}_j^k \subseteq \hat{B}_i} \frac{1}{r(\hat{B}_j^k)^{Q+\alpha}} \int_{\hat{B}_j^k \setminus \hat{B}_j^{k+1}} \hat{\psi}(g',y')\, d\omega(g',y') \\
&\lesssim \sum_i \int_{\mathbb{H}_+^n} \left(\sum_{k:\hat{B}_i' \subseteq \hat{B}_j^k \subseteq \hat{B}_i} \frac{1}{r(\hat{B}_j^k)^{Q+\alpha}} \chi_{\hat{B}_j^k \setminus \hat{B}_j^{k+1}}(g',y') \right) \hat{\psi}(g',y') d\omega(g',y') \\
&\lesssim \sum_i \int_{\hat{B}_i} \min\{\frac{1}{r(\hat{B}_i')^{Q+\alpha}}, K((g,y),(g',y'))\} \hat{\psi}(g',y') d\omega(g',y').
\end{aligned}$$

Therefore, there exists i_0 such that

$$
\begin{aligned}
&\rho\left(\frac{|5\hat{B}_{i_0}|_{\hat{\sigma}}}{|5\hat{B}_0|_{\hat{\sigma}}}\right)^{\frac{1}{q'}-\frac{1}{p'}}\\
&\lesssim \int_{\hat{B}_{i_0}} \min\{\frac{1}{r(\hat{B}'_{i_0})^{Q+\alpha}}, K((g,y),(g,y'))\}\hat{\psi}(g',y')d\omega(g',y')\\
&\lesssim \left(\int_{\hat{\mathbb{H}}^n_+} \min\{\frac{1}{r(\hat{B}'_{i_0})^{Q+\alpha}}, K((g,y),(g',y'))\}^q d\omega(g',y')\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\
&\lesssim \big[\big(\int_{2\hat{B}'_{i_0}} \min\{\frac{1}{r(\hat{B}'_{i_0})^{Q+\alpha}}, K((g,y),(g',y'))\}^q d\omega(g',y')\big)^{\frac{1}{q}}\\
&\quad+\big(\int_{(2\hat{B}'_{i_0})^c} \min\{\frac{1}{r(\hat{B}'_{i_0})^{Q+\alpha}}, K((g,y),(g',y'))\}^q d\omega(g',y')\big)^{\frac{1}{q}}\big]\\
&\quad\times\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\
&\lesssim \big[\big(\int_{2\hat{B}'_{i_0}} \frac{1}{r(\hat{B}'_{i_0})^{(Q+\alpha)q}} y'^{1-\alpha+\alpha q}dg'dy'\big)^{\frac{1}{q}}\\
&\quad+\big(\int_{(2\hat{B}'_{i_0})^c} \frac{1}{\hat{d}((g,y),(g',y'))^{(Q+\alpha)q}} y'^{1-\alpha+\alpha q}dg'dy'\big)^{\frac{1}{q}}\big]\\
&\quad\times\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\
&\lesssim \big[r(\hat{B}'_{i_0})^{-Q+\frac{Q+2-\alpha}{q}} + \sum_{j=1}^{\infty} \frac{\big(\int_{\hat{d}((g,y),(g',y'))<2^{j+1}r(\hat{B}'_{i_0})} y'^{1-\alpha+\alpha q}dg'dy'\big)^{\frac{1}{q}}}{(2^j r(\hat{B}'_{i_0}))^{(Q+\alpha)q}}\big]\\
&\quad\times\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\
&\lesssim \big[r(\hat{B}'_{i_0})^{-Q+\frac{Q+2-\alpha}{q}}\\
&\quad+\big(\sum_{j=1}^{\infty}(2^j r(\hat{B}'_{i_0}))^{-(Q+\alpha)q+Q+2-\alpha+\alpha q}\big)^{\frac{1}{q}}\big]\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\
&\lesssim | 5\hat{B}'_{i_0}|_{\hat{\sigma}}^{-\frac{1}{p'}}\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\
&\lesssim | 5\hat{B}_{i_0}|_{\hat{\sigma}}^{-\frac{1}{p'}}\left(\int_{\hat{B}_{i_0}} \hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}},
\end{aligned}
$$

where we have used the fact that

$$\begin{cases} \mid 5\hat{B}'_{i_0} \mid_{\hat{\sigma}}^{\frac{1}{p'}} = \int_{\theta B'_{i_0}} |g|_{\mathbb{H}^n}^{-\frac{\gamma}{p-1}} dg \lesssim r(\hat{B}'_{i_0})^{\frac{Q}{p'}-\frac{\gamma}{p}}; \\ \mid 5\hat{B}'_{i_0} \mid_{\hat{\sigma}} \geq \frac{1}{2} \mid 5\hat{B}_{i_0} \mid_{\hat{\sigma}}; \\ \hat{B}_{i_0} \subseteq \hat{B}_0 = \hat{B}^0_j \subseteq 6\hat{\hat{Q}}^*_j. \end{cases}$$

Moreover, $\gamma < (p-1)Q$ ensures that $Q(1-q)+2-\alpha < 0$ and $p > \frac{Q+\gamma}{Q}$. Subsequently, we have shown that for any $(g,y) \in \hat{\hat{Q}}_j \bigcap \hat{\Omega}_{\rho,m} \bigcap \mathrm{supp}(\hat{\sigma})$, there is a ball $\hat{B}_{(g,y)} \subseteq 6\hat{\hat{Q}}^*_j$ with center (g,y) such that

$$\rho^{q'} \mid 5\hat{B}_{(g,y)} \mid_{\hat{\sigma}} \lesssim \mid 30\hat{\hat{Q}}^*_j \mid_{\hat{\sigma}}^{1-\frac{q'}{p'}} \int_{\hat{B}_{(g,y)}} \hat{\psi}(g',y')^{q'} d\omega(g',y').$$

Via the covering lemma (cf. Lemma 5.2), there exists a pairwise disjoint sequence $\{\hat{B}^i_j\}$ of balls in the family $\{\hat{B}_{(g,y)} : (g,y) \in \hat{\hat{Q}}_j \bigcap \hat{\Omega}_{\rho,m} \bigcap \mathrm{supp}(\hat{\sigma})\}$ such that $\hat{\hat{Q}}_j \bigcap \hat{\Omega}_{\rho,m} \bigcap \mathrm{supp}(\hat{\sigma}) \subseteq \bigcup_i \hat{B}^i_j$. Therefore, $\mid \hat{\mathbb{H}}^n_+ \backslash \mathrm{supp}(\hat{\sigma}) \mid_{\hat{\sigma}} = 0$ and $5\hat{B}^0_j \subseteq 30\hat{\hat{Q}}^*_j$ imply

$$\begin{aligned} \rho^{p'} \mid \hat{\hat{Q}}_j \bigcap \hat{\Omega}_{\rho,m} \mid_{\hat{\sigma}} &\leq \rho^{p'} \sum_i |5\hat{B}^i_j|_{\hat{\sigma}} \\ &\lesssim \rho^{p'-q'} \mid 30\hat{\hat{Q}}^*_j \mid_{\hat{\sigma}}^{1-\frac{q'}{p'}} \sum_i \int_{\hat{B}^i_j} \hat{\psi}(g',y')^{q'} d\omega(g',y') \\ &\lesssim \rho^{p'-q'} \mid 30\hat{\hat{Q}}^*_j \mid_{\hat{\sigma}}^{1-\frac{q'}{p'}} \int_{\eta\hat{\hat{Q}}^*_j} \hat{\psi}(g',y')^{q'} d\omega(g',y'). \end{aligned}$$

By summing this inequality over the family of all maximal cubes $\hat{\hat{Q}}_j$ in $\hat{\mathcal{D}}_{\frac{\rho}{A},m}$ and noting both $\hat{\Omega}_\rho \subseteq \hat{\Omega}_{\frac{\rho}{A}}$ and $q' < p'$, we obtain

$$\begin{aligned} \rho^{p'} \mid \hat{\Omega}_{\rho,m} \mid_{\hat{\sigma}} &\lesssim \sum_j \rho^{p'-q'} \mid 30\hat{\hat{Q}}^*_j \mid_{\hat{\sigma}}^{1-\frac{q'}{p'}} \int_{30\hat{\hat{Q}}^*_j} \hat{\psi}(g',y')^{q'} d\omega(g',y') \\ &\lesssim \rho^{p'-q'} \left(\sum_j \mid 30\hat{\hat{Q}}^*_j \mid_{\hat{\sigma}} \right)^{1-\frac{q'}{p'}} \left(\sum_j \left(\int_{30\hat{\hat{Q}}^*_j} \hat{\psi}(g',y')^{q'} d\omega(g',y') \right)^{\frac{p'}{q'}} \right)^{\frac{q'}{p'}} \\ &\lesssim \left(\rho^{p'} \mid \hat{\Omega}_{\frac{\rho}{A}} \mid_{\hat{\sigma}} \right)^{1-\frac{q'}{p'}} \int_{\hat{\mathbb{H}}^n_+} \hat{\psi}(g',y')^{q'} d\omega(g',y'), \end{aligned}$$

where we have utilized the properties (a) and (c) of the family of maximal cubes $\hat{\hat{Q}}_j$ in $\hat{\Omega}_{\frac{\rho}{A},m}$ in the last step. Thus, letting $m \to \infty$ and taking the supremum in ρ for $0 < \rho < N$, we conclude that there exists a constant C independent of N such that

$$\sup_{0<\rho<N} \rho^{p'} \mid \hat{\Omega}_\rho \mid_{\hat{\sigma}} \leq C \left(\sup_{0<\rho<N} \rho^{p'} \mid \hat{\Omega}_\rho \mid_{\hat{\sigma}} \right)^{1-\frac{q'}{p'}} \int_{\hat{\mathbb{H}}^n_+} \hat{\psi}(g',y')^{q'} d\omega(g',y').$$

If $\sup_{0<\rho<N}\rho^{p'}\mid\hat{\Omega}_\rho\mid_{\hat{\sigma}}$ is finite, we can derive the desired weak type inequality by dividing both sides of the above inequality by $\left(\sup_{0<\rho<N}\rho^{p'}\mid\hat{\Omega}_\rho\mid_{\hat{\sigma}}\right)^{1-\frac{q'}{p'}}$ and taking $N\to\infty$. Let $\hat{B}$ be the ball containing the support of $\hat{\psi}$. Fix the constant $\kappa>1$. It is obvious that $\rho^{p'}\mid\kappa\hat{B}\mid_{\hat{\sigma}}\le N^{p'}\mid\kappa\hat{B}\mid_{\hat{\sigma}}<\infty$, provided $N>\rho$. In what follows we only need to show that

$$\sup_{\rho>0}\rho^{p'}\mid\hat{\Omega}_\rho\backslash\kappa\hat{B}\mid_{\hat{\sigma}}\le\left(\int_{\hat{\mathbb{H}}^n_+}\hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{p'}{q'}},\tag{5.5}$$

thereby implying $\sup_{\rho>0}\rho^{p'}\mid\hat{\Omega}_\rho\backslash\kappa\hat{B}\mid_{\hat{\sigma}}<\infty$. Given a constant $\beta>1$. When $(g,y)\in\hat{\Omega}_\rho\backslash\kappa\hat{B}$, upon setting

$$\hat{D}_{(g,y)}=\hat{B}((g,y)_{\hat{B}},\beta\hat{d}((g,y),(g,y)_{\hat{B}})),$$

where $(g,y)_{\hat{B}}$ is the center of $\hat{B}$, one has $(g,y)\in\hat{D}_{(g,y)}$ due to

$$\hat{B}\subseteq\hat{D}_{(g,y)}\ \&\ \hat{d}((g,y),(g,y)_{\hat{B}})\ge\kappa r(\hat{B}).$$

If $(g',y')\in\hat{B}$, then

$$\hat{d}((g,y),(g,y)_{\hat{B}})\le\hat{d}((g,y),(g',y'))+\kappa^{-1}\hat{d}((g,y),(g,y)_{\hat{B}}),$$

and hence

$$\hat{d}((g,y),(g',y'))\ge(1-\kappa^{-1})\hat{d}((g,y),(g,y)_{\hat{B}})=(1-\kappa^{-1})\beta^{-1}r(\hat{D}_{(g,y)}).$$

Consequently, one has

$$K((g,y),(g',y'))\lesssim\frac{1}{[r(\hat{D}_{(g,y)})]^{Q+\alpha}}.$$

Since

$$\mid\hat{D}_{(g,y)}\mid_{\omega}^{\frac{1}{q}}=\left(\int_{\hat{D}_{(g,y)}}y^{1-\alpha+\alpha q}dydg\right)^{\frac{1}{q}}\approx r(\hat{D}_{(g,y)})^{\frac{Q+2-\alpha}{q}+\alpha}$$

and

$$\mid\hat{D}_{(g,y)}\mid_{\hat{\sigma}}^{\frac{1}{p'}}\lesssim r(\hat{D}_{(g,y)})^{\frac{Q}{p'}-\frac{\gamma}{p}},$$

one finds

$$\frac{1}{[r(\hat{D}_{(g,y)})]^{Q+\alpha}}\mid\hat{D}_{(g,y)}\mid_{\omega}^{\frac{1}{q}}\mid\hat{D}_{(g,y)}\mid_{\hat{\sigma}}^{\frac{1}{p'}}\lesssim 1.$$

Utilizing both Hölder's inequality and $\hat{B}\subseteq\hat{D}_{(g,y)}$ gives

$$\begin{aligned}\rho&\le T^*\hat{\psi}(g,y)\\&=\int_{\hat{B}}K((g,y),(g',y'))\hat{\psi}(g',y')d\omega(g',y')\\&\lesssim\frac{1}{[r(\hat{D}_{(g,y)})]^{Q+\alpha}}\mid\hat{D}_{(g,y)}\mid_{\omega}^{\frac{1}{q}}\left(\int_{\hat{B}}\hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}}\\&\lesssim\mid\hat{D}_{(g,y)}\mid_{\hat{\sigma}}^{-\frac{1}{p'}}\left(\int_{\hat{B}}\hat{\psi}(g',y')^{q'}d\omega(g',y')\right)^{\frac{1}{q'}},\end{aligned}$$

whence

$$\rho^{p'} \mid \hat{D}_{(g,y)} \mid_{\hat{\sigma}} \lesssim \left(\int_{\hat{B}} \hat{\psi}(g',y')^{q'} d\omega(g',y')\right)^{\frac{p'}{q'}}. \tag{5.6}$$

Because $\hat{\Omega}_\rho \backslash \kappa\hat{B} \subseteq \bigcup \hat{D}_{(g,y)}$ and $\hat{D}_{(g,y)}$'s are all balls with common center, via (5.6) and the monotone convergence theorem we immediately have

$$\rho^{p'} \mid \hat{\Omega}_\rho \backslash \kappa\hat{B} \mid_{\hat{\sigma}} \leq \left(\int_{\hat{B}} \hat{\psi}(g',y')^{q'} d\omega(g',y')\right)^{\frac{p'}{q'}},$$

whence (5.5).

Step 2: We show that T is of restricted type (p,q), namely, is, for $\frac{Q+\gamma}{Q} < p < q < \infty$, one has

$$\left(\int_{\hat{\mathbb{H}}^n_+} T(\chi_{\hat{E}})(g,y)^q d\omega(g,y)\right)^{\frac{1}{q}} \lesssim \left(\int_{\hat{\mathbb{H}}^n_+} \chi_{\hat{E}} d\hat{\sigma}(g,y)\right)^{\frac{1}{p}} \tag{5.7}$$

for all measurable set $\hat{E}$ in $\hat{\mathbb{H}}^n_+$. Without loss of generality, we may assume that the measurable set $\hat{E}$ has finite measure. Let $\hat{\phi}$ be a nonnegative function on $\hat{\mathbb{H}}^n_+$ such that $\| \hat{\phi} \|_{L^{q'}(\omega)} \leq 1$.

$$\begin{aligned}
&\int_{\hat{\mathbb{H}}^n_+} T(\chi_{\hat{E}})(g,y)\hat{\phi}(g,y)d\omega(g,y) \\
&\quad = \int_{\hat{E}} T^*(\hat{\phi})(g',y')d\hat{\sigma}(g',y') \\
&\quad = \int_0^\infty \mid \hat{E} \bigcap \{(g',y') \in \hat{\mathbb{H}}^n_+ : T^*(\hat{\phi})(g',y') > \rho \mid_{\hat{\sigma}} d\rho \\
&\quad \leq \int_0^\infty \min\{\mid \hat{E} \mid_{\hat{\sigma}}, \mid \{(g',y') \in \hat{\mathbb{H}}^n_+ : T^*(\hat{\phi})(g',y') > \rho \mid_{\hat{\sigma}}\} d\rho \\
&\quad \lesssim \int_0^\infty \min\{\mid \hat{E} \mid_{\hat{\sigma}}, \frac{\| \hat{\phi} \|^{p'}_{L^{q'}(\omega)}}{\rho^{p'}}\} d\rho \\
&\quad \lesssim \mid \hat{E} \mid^{\frac{1}{p}}_{\hat{\sigma}}.
\end{aligned}$$

By taking the supremum over all such $\hat{\phi}$, we conclude that (5.7) is valid.

Step 3: In what follows we consider the strong type (p,q) estimate for $\mathcal{P}(g,y)$. Define the operator $\mathbf{P}$ as

$$\begin{aligned}
\mathbf{P}(\phi d\sigma)(g,y) &= C_\alpha y^\alpha T\hat{\phi}(g,y) \\
&= C_\alpha y^\alpha \int_{\mathbb{H}^n} \frac{\phi(h)}{(|h^{-1}g|^2_{\mathbb{H}^n} + y^2)^{\frac{Q+\alpha}{2}}} |h|^{-\frac{\gamma}{p-1}}_{\mathbb{H}^n} dh \quad \forall \quad (g,y) \in \hat{\mathbb{H}}^n_+,
\end{aligned}$$

where $\hat{\phi}(g,0) = \phi(g)$. For any measurable set $E \subseteq \mathbb{H}^n$, let $\hat{E} = \{(g,y) \in \hat{\mathbb{H}}^n_+ : g \in E, y \in T\}$, where T is a measurable set in $[0,\infty)$. Note that $\chi_{\hat{E}}(g,0) = \chi_E(g)$. By

(5.7) we have

$$
\begin{aligned}
\left(\int_{\hat{\mathbb{H}}^n_+} \mathbf{P}(\chi_E d\sigma)(g,y)^q y^{1-\alpha} dgdy\right)^{\frac{1}{q}} &= \left(\int_{\hat{\mathbb{H}}^n_+} T(\chi_{\hat{E}})(g,y)^q d\omega(g,y)\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{\hat{\mathbb{H}}^n_+} \chi_{\hat{E}} d\hat{\sigma}(g,y)\right)^{\frac{1}{p}} \\
&\lesssim \left(\int_{\mathbb{H}^n} \chi_E d\sigma(g)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Furthermore, we prove that the operator $\mathbf{P}$ can be uniquely extended to be a bounded linear operator from $L^p(d\sigma)$ into $L^q(\hat{\mathbb{H}}^n_+, y^{1-\alpha}dgdy)$. In fact, let $\phi = \sum_{j=1}^N a_j \chi_{E_j}$, where $a_1 > a_2 > \cdots > a_N > 0$, $E_j, j = 1, \cdots, N$, are mutually disjoint and $|E_j|_\sigma < \infty$ for $j = 1, \cdots, N$. Let $F_j = E_1 \bigcup \cdots \bigcup E_j$, $B_0 = 0$ and $B_j = |F_j|_\sigma$ for $j \geq 1$. Rewrite ϕ as $\phi = \sum_{j=1}^N (a_j - a_{j+1})\chi_{F_j}$, where a_{N+1}=0. Then

$$
\begin{aligned}
\| \mathbf{P}(\phi d\sigma) \|_{L^q(\hat{\mathbb{H}}^n_+, y^{1-\alpha}dgdy)} &\leq \sum_{j=1}^N (a_j - a_{j+1}) \| \mathbf{P}(\chi_{F_j} d\sigma) \|_{L^q(\hat{\mathbb{H}}^n_+, y^{1-\alpha}dgdy)} \\
&\lesssim \| \phi \|_{L^{p,1}(\mathbb{H}^n, d\sigma)},
\end{aligned}
$$

where $L^{p,1}(\mathbb{H}^n, d\sigma)$ is the Lorentz space (cf. Section 1.4 in [**15**]). Since simple functions are dense in $L^{p,1}(\mathbb{H}^n, d\sigma)$ and $L^{p,1}(\mathbb{H}^n, d\sigma) \hookrightarrow L^p(\mathbb{H}^n, d\sigma)$ for $p > 1$, we have

$$
\| \mathbf{P}(\phi d\sigma) \|_{L^q(\hat{\mathbb{H}}^n_+, y^{1-\alpha}dgdy)} \lesssim \| \phi \|_{L^p(\mathbb{H}^n, d\sigma)} \quad \forall \ \phi \in L^p(\mathbb{H}^n, d\sigma).
$$

Furthermore, utilizing $u(g) = \phi(g)|g|_{\mathbb{H}^n}^{\frac{\gamma}{p-1}}$ in the above inequality, we obtain

$$
\| \mathbf{P}(u) \|_{L^q(\hat{\mathbb{H}}^n_+, y^{1-\alpha}dgdy)} \lesssim \| u \|_{L^p(\mathbb{H}^n, |g|^{\gamma}_{\mathbb{H}^n})},
$$

whence completing the proof of (4.1).

□

References

[1] I. Birindelli, I.C. Dolcetta and A. Cutri, Liouville theorems for semilinear equations on the Heisenberg group, *Ann. I.H.P. Analyse non linéaire* 14(1997), 295-308.

[2] I. Birindelli and J. Prajapat, Nonlinear Liouville theorems in the Heisenberg group via the moving plane method, *Comm. Partial Diff. Equ.* 24(1999), 1875-1890.

[3] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Diff. Equ.* 32(2007), 1245-1260.

[4] L. Capogna, D. Danielli and N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequality, *Comm. Anal. Geom.* 2(1994), 203-215.

[5] G. Caristi, L. D'Ambrosio and E. Mitidieri, Liouville theorems for some nonlinear inequalities, *Proc. Steklov Inst. Math.* 260(2008), 90-111.

[6] G. Caristi and E. Mitidieri, Liouville theorems for quasilinear elliptic inequalities, *Doklady Math.* 79(2009), 118-124.

[7] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* 60/61(1990), 601-628.

[8] E. Cinti and J. Tan, A nonlinear Liouville theorem for fractional equations in the Heisenberg group, arXiv:1504. 03122.

[9] R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83(1977), 569-645.

[10] F. Ferrari and B. Franchi, Harnack inequality for fractional sub-Laplacians in Carnot groups, *Math. Z.* 279(2015), 435-458.

[11] B. Franchi, R. Serapioni and F.S. Cassano, Rectifiability and perimeter in the Heisenberg group, *Math. Ann.* 321(2001), 479-531.

[12] G.B. Folland and E.M. Stein, Hardy Spaces on Homogeneous Groups, Vol. 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1982.

[13] G.B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.* 13(1975), 161-207.

[14] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 34(1981), 525-598.

[15] L. Grafakos, Classical Fourier Analysis, Grad. Texts in Math. (second edition), 249(2008), Springer, New York.

[16] N. Garofalo and D.M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, *Commun. Pure Appl. Math.* 49(1996), 1081-1144.

[17] A. Grigor'yan and Y. Sun, On non-negative solutions of the inequality $\Delta u + u^\sigma \le 0$ on Riemannian manifolds, *Comm. Pure Appl. Math.* 67(2014), 1336-1352.

[18] L. Hörmander, Hypoelliptic second-order differential equations, *Acta Math.* 121(1968), 147-171.

[19] S.I. Pohozaev and L. Véron, Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group, *Manuscripta Math.* 102(2000), 85-99.

[20] E.T. Sawyer, A characterization of two weight norm inequalities for fractional and Poisson integrals, *Trans. Amer. Math. Soc.* 308(1988), 533-545.

[21] E.T. Sawyer and R.L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* 114(1992), 813-874.

[22] E.T. Sawyer, R.L. Wheeden and S. Zhao, Weighted norm inequalities for operators of potential type and fractional maximal functions, *Potential Anal.* 5(1996), 523-580.

[23] J. Serrin and H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta Math.* 189(2002), 79-142.

[24] E.M. Stein, Harmonic analysis: real-variable method, orthogonality, and oscillatory integrals, Princeton Univ. Press, Princeton, N.J., 1993.

[25] P. R. Stinga and J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Comm. Partial Diff. Equ.* 35(2010), 2092-2122.

[26] Y. Sun, Uniqueness result for non-negative solutions of semi-linear inequalities on Riemannian manifolds, *J. Math. Anal. Appl.* 419(2014), 643-661.

[27] Y. Sun, On nonexistence of positive solutions of quasi-linear inequality on Riemannian manifolds, *Proc. Amer. Math. Soc.*143(2015)2969-2984.

[28] Y. Sun, On the uniqueness of nonegative solutions of differential inequalities with gradient terms on Riemannian manifolds, *Preprint*, 2014.

[29] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, Analysis and Geometry on Groups, Cambridge Univ. Press, Cambridge, UK, 1992.

[30] Y. Wang and J. Xiao, A constructive approach to positive solutions of $\Delta_p u + f(u, \nabla u) \le 0$ on Riemannian manifolds, *Ann. I.H.P. Analyse non linéaire*, doi:10.1016/j.anihpc.2015.06.003.

[31] Y. Wang and J. Xiao, A uniqueness principle for $u^p \le (-\Delta)^{\frac{\alpha}{2}} u$ in Euclidean space, *Preprint*, 2015.

School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

E-mail address: liuyu75@pku.org.cn

Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

Current address: Department of Mathematics and Statistics, Memorial University, St. John's, NL A1C 5S7, Canada

E-mail address: wangyuzhao2008@gmail.com

Department of Mathematics and Statistics, Memorial University, St. John's, NL A1C 5S7, Canada

E-mail address: jxiao@mun.ca